GENERALIZED BIRTH & DEATH PROCESSES AS DEGRADATION MODELS

Vladimir Rykov, (Moscow, Russia)

Abstract

To model degradation processes in technical and biological objects generalized birth and death processes are introduced and studied.

study

1 Introduction and Motivation

Traditional studies of technical and biological objects reliability mainly deals with their reliability function and steady state probabilities for renewable systems. Nevertheless, because there are no infinitely long leaving objects and any repair is possible only from the state of partial failure, the modelling of degradation process during a life period of an object is a mostly interesting topic. From the mathematical point of view the degradation during object's life period can be described by the Birth & Death (B & D) type process with absorbing state. For this process the conditional state probability distribution given object's life period is a mostly interesting characteristic.

During last years an intensive attention to the aging and degradation models for technical and biological objects has been attracted. The organization of special scientific conferences devoted to this topics testifies it. The aging and degradation models suppose the study of the systems with gradual failures for which multi-state reliability models were elaborated (for the history and bibliography see, for example, [1]). In [2] - [5] the model of complex hierarchical system was proposed and the methods for its steady state and time dependent characteristics investigation was done. Controllable fault tolerance reliability systems were considered in [6] - [8]. In the present paper a generalized B & D process as a model for degradation and aging process for technical and biological objects is proposed. Conditional state probabilities given object's life period and their limiting values when $t \to \infty$ are calculated. The variation of the model parameters allows to consider various problems of aging and degradation control. Some simple example illustrate our approach.

2 A General Degradation Model

Most of up-to-date complex technical systems also as biological objects with sufficiently high organization during their life period pass over different states of evolution and existence. From reliability point of view these states can be divided into three groups: the states of normal functioning, the dangerous (degradation) states and the failure states, see fig.1, at which the states are denoted by the letter $\mathbf{s}$, normal, degradation and failure states are joined into the sets N, D, and F respectively, and possible transitions are shown with arrows.

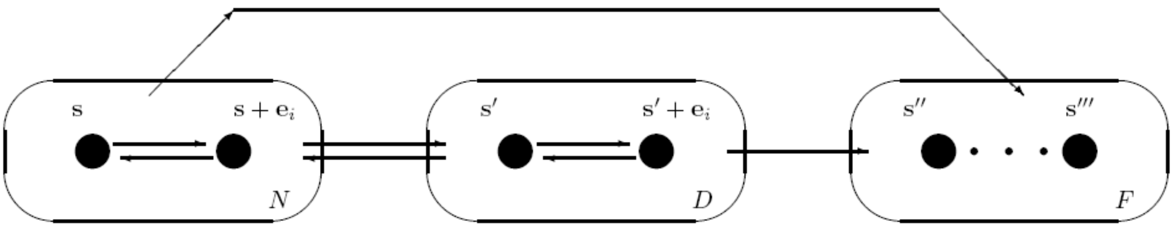

Fig. 1. The structure of the degradation process.

In the simplest case if the nature of the degradation process allows to completely order the states to admit the transition possibilities only to neighboring states it can be modelled by the process of B & D type.

[1]The paper was partially supported by the RFFI Grant No. 04-07-90115.

3 Generalized Birth & Death Process

3.1 Definition. Basic Equalities

Suppose that the states of the object are completely ordered, its transitions only into neighboring states are possible, and their intensities depend on the time spend in the present state. Consider firstly the general case of the process with denumerable set of states $E = \{1, 2, \ldots\}$. To describe the object behavior by a Markov process let us introduce an enlarged states space $\mathcal{E} = E \times [0, \infty)$ and consider two dimensional process $Z(t) = \{S(t), X(t)\}$, where the first component $S(t) \in E$ shows the object's state, and the second one $X(t) \in [0, \infty)$ denotes the time spent in the state since the last entrance into it. Denote by $\alpha_i(x)$ and $\beta_i(x)$ $(i \in E)$ the transition intensities from the state i to the states $i + 1$ and $i - 1$ accordingly under the condition that the time spent at the state i equals to x.

Remark. If the stay time at the state i is considered as a minimum of two independent random variables (r.v): time A_i till to transition into the "next" state $i + 1$ and time B_i till to transition into the "previous" state $i - 1$ with cumulative distribution functions (c.d.f.) $A_i(x)$, $B_i(x)$, probability density functions (p.d.f.) $a_i(x)$, $b_i(x)$, and mean values $a_i = \int[1 - A_i(x\,dx)]$, $b_i = \int[1 - B_i(x)]\,dx$, then the introduced process can be considered as a special case of semi-Markov process (SMP) [9], with conditional transition p.d.f.'s $\alpha_i(x)$ $\beta_i(x)$,

$$\alpha_i(x) = \frac{a_i(x)}{1 - A_i(x)}, \qquad \beta_i(x) = \frac{b_i(x)}{1 - B_i(x)}.$$

The c.d.f. of the stay time at the state i for this process equals

$$Q_i(x) = 1 - (1 - A_i(x))(1 - B_i(x)),$$

and the elements of semi-Markov matrix are

$$Q_{ii+1}(x) = \int_0^x a_i(u)(1 - B_i(u))du, \;\; Q_{ii-1}(x) = \int_0^x b_i(u)(1 - A_i(u))du.$$

Nevertheless, the given formalization open the new possibilities for the investigations and moreover in the degradation models we are studying the conditional probability state distribution on given life period, that did not investigated previously.

Denote by $\pi_i(t, x)$ the p.d.f. of the process $Z(t)$ at time t,

$$\pi_i(t, x)dx = \mathbf{P}\{S(t) = i, \; x \le X(t) < x + dx\}.$$

These functions satisfy to the Kolmogorov's system of differential equations

$$\frac{\partial \pi_i(t, x)}{\partial t} + \frac{\partial \pi_i(t, x)}{\partial x} = -(\alpha_i(x) + \beta_i(x))\pi_i(t, x), \qquad 0 \le x \le t < \infty, \;\; i \in E \tag{1}$$

with the initial and boundary conditions

$$\begin{cases} \pi_1(t, 0) = \delta(t) + \int_0^t \pi_2(t, x)\beta_2(x)dx, \\ \pi_i(t, 0) = \int_0^t \pi_{i-1}(t, x)\alpha_{i-1}(x)dx + \int_0^t \pi_{i+1}(t, x)\beta_{i+1}(x)dx, \qquad i \in E. \end{cases} \tag{2}$$

In the following we will suppose the process to be non reducible, non degenerated, i.e. it is defined over all time axis. Sufficient condition for non reducibility is

$$a_i b_i > 0 \qquad \text{for all } i \in E.$$

To avoid degeneration of the process it is needed, first of all, to eliminate instant states. For this we suppose that the process possesses the *regularity* property,

$$A_i(0) < 1, \qquad B_i(0) < 1 \qquad \text{for all } i \in E.$$

Nevertheless, the absence of instant states does not yet guarantee non-degeneration of the process because of the possibility of jumps accumulating during the finite time interval, so that the process "can go to infinity" for the finite time interval. One of possible conditions for the non-reducible SMP without instant states to be non-degenerated is the recurrence of its embedded Markov chain [9]. But this conditions are hardly checked for general random walk in terms of its parameters. Therefore, we will use more strong, nevertheless very applicable sufficient condition for the process non-degenerateness. It is the condition of *strong regularity*: there exist constants $c > 0$ and $\epsilon > 0$ such, that

$$\mathbf{P}\{A_i > c\} = 1 - A_i(c) > \epsilon, \quad \text{and} \quad \mathbf{P}\{B_i > c\} = 1 - B_i(c) > \epsilon \qquad \text{for all} \quad i \in E.$$

The proof of sufficiency of the strong regularity of SMP for its non-degenerateness can be found, for example, in [10], [11].

For the non reducible, non degenerated generalized B & D process the Kolmogorov's system of equations (1) with initial and boundary conditions (2) has a unique solution over all time axis. To find it we use the method of characteristics. Accordingly to [14] the characteristics of this system satisfied to the system of equations

$$dt = dx = -\frac{d\pi_i}{(\alpha_i(x) + \beta_i(x))\pi_i}, \qquad i \in E.$$

Thus, two first integrals for the system (1) are

$$t - x = C_1, \qquad \frac{\pi_i(t,x)}{(1 - A(x))(1 - B(x))} = C_2.$$

This means that at the line $t - x = u$ the functions $\pi_i(t,x)$ $(i \in E)$ have the form

$$\pi_i(t,x) = g_i(t-x)(1 - A_i(x))(1 - B_i(x)), \qquad 0 \le x \le t < \infty, \ i \in E, \tag{3}$$

where the functions $g_i(t)$ in accordance with the initial and boundary conditions (2) satisfy to the system of equations

$$\begin{cases} g_1(t) &= \delta(t) + \int\limits_0^t g_2(t-x)(1 - A_2(x)b_2(x))dx, \\ g_i(t) &= \int\limits_0^t g_{i-1}(t-x)a_{i-1}(x))(1 - B_{i-1}(x))dx + \\ & + \int\limits_0^t g_{i+1}(t-x)(1 - A_{i+1}(x)b_{i+1}(x))dx, \qquad i = 2,3,\ldots. \end{cases} \tag{4}$$

The form of these equations shows that their solution should find in terms of its Laplace transforms (LT's). Therefore by passing to the LT's with respect to both variables into relations (3) after the change of the integration order one can get

$$\tilde{\tilde{\pi}}_i(s,v) \equiv \int\limits_0^\infty e^{-st} \int\limits_0^t e^{-vx}\pi(t,x)\,dx\,dt = \tilde{g}_i(s)\tilde{\gamma}_i(s+v), \tag{5}$$

where $\tilde{g}_i(s)$ are the LT of the functions $g_i(t)$, and the functions $\tilde{\gamma}_i(s)$ are

$$\tilde{\gamma}_i(s) = \int\limits_0^\infty e^{-st}(1 - A_i(t))(1 - B_i(t))dt.$$

From the other side by passing to the LT's with respect to variable t in the system (4) one get

$$\begin{cases} \tilde{g}_1(s) \equiv \int\limits_0^\infty e^{-st}g_1(t)dt &= 1 + \tilde{g}_2(s)\tilde{\psi}_2(s), \\ \tilde{g}_i(s) \equiv \int\limits_0^\infty e^{-st}g_i(t)dt &= \tilde{g}_{i-1}(s)\tilde{\phi}_{i-1}(s) + \tilde{g}_{i+1}(s)\tilde{\psi}_{i+1}(s), \qquad i = 2,3\ldots, \end{cases} \tag{6}$$

where the functions $\tilde{\phi}_i(s)$ and $\tilde{\psi}_i(s)$ are given by the relations

$$\tilde{\phi}_i(s) = \int\limits_0^\infty e^{-sx}a_i(x)(1 - B_i(x))dx, \qquad \tilde{\psi}_i(s) = \int\limits_0^\infty (1 - A_i(x))b_i(x))dx, \qquad i \in E.$$

The relations (6) can be presented as a system of equations with respect to unknown functions $\tilde{g}_i(s)$

$$\begin{cases} \tilde{g}_1(s) - \tilde{g}_2(s)\tilde{\psi}_2(s) &= 1, \\ -\tilde{g}_{i-1}(s)\tilde{\phi}_{i-1}(s) + \tilde{g}_i(s) - \tilde{g}_{i+1}(s)\tilde{\psi}_{i+1}(s) &= 0, \qquad i = 2,3\ldots. \end{cases} \tag{7}$$

The closed form solution of this system in general case even in the simplest case of usual B & D process does not possible.

In the case of finite number $n+1$ of states in the above system one should put $\tilde{\phi}_{n+1}(s) = 0$.

In spite of the above equations does not possible to solve in closed form they provide calculation different characteristics of the process. Consider some of them.

3.2 Stationary probability distribution

For calculation of the process $Z(t)$ macro-states stationary probabilities

$$\pi_i = \lim_{t\to\infty} \pi_i(t) = \lim_{t\to\infty} \int_0^t \pi(t,x)dx = \lim_{t\to\infty} \int_0^t g_i(t-x)(1-A_i(x))(1-B_i(x))dx$$

we use the connection between asymptotic behavior of functions at infinity and their LT's at zero. Letting $\tilde{\gamma}(0) = \gamma$ and taking into account that accordingly to (5) $\tilde{\pi}_i(s) = \tilde{\tilde{\pi}}_i(s,0)$, we find

$$\lim_{t\to\infty} \pi_i(t) = \lim_{s\to 0} s\tilde{\pi}_i(s) = \gamma_i \lim_{s\to 0} s\tilde{g}_i(s).$$

Thus, for the problem solution it is necessary to calculate the values

$$g_i = \lim_{s\to 0} s\tilde{g}_i(s),$$

for what we use recursive relations (6). By multiplying equalities in these relations by s, and by passing to limit when $s \to 0$ from these recursive relations we get

$$\begin{cases} g_1 = g_2\psi_2, \\ g_i = g_{i-1}\phi_{i-1} + g_{i+1}\psi_{i+1}, \qquad i = 2, 3\ldots, \end{cases} \tag{8}$$

where the following notations $\phi_i = \tilde{\phi}_i(0)$, $\psi_i = \tilde{\psi}_i(0)$ were used. Taking into account that $\phi_i + \psi_i = 1$, rewrite the last expressions in the form

$$g_i(\phi_i + \psi_i) = g_{i-1}\phi_{i-1} + g_{i+1}\psi_{i+1}, \qquad i = 2, 3\ldots,$$

that can be transformed to the following ones

$$g_i\phi_i - g_{i+1}\psi_{i+1} = g_{i-1}\phi_{i-1} - g_i\psi_i, \qquad i = 2, 3\ldots.$$

Now put $\psi_1 = 0$ (because this value does noe defined) and remark that from (8) it follows $g_1\phi_1 - g_2\psi_2 = 0$. Then from the last recursive relation we find

$$g_i\phi_i - g_{i+1}\psi_{i+1} = g_1\phi_1 - g_2\psi_2 = 0, \qquad i = 2, 3, \ldots.$$

The last relation allows to calculate recursively coefficients g_i in the form

$$g_i = \frac{\phi_{i-1}}{\psi_i} g_{i-1} = \ldots = \left(\prod_{1\le j\le i} \frac{\phi_{j-1}}{\psi_j} \right) g_1.$$

Therefore, the stationary probabilities equal

$$\pi_i = \gamma_i G_i g_1, \qquad \text{where} \qquad G_1 = 1, \quad G_i = \prod_{1\le j\le i} \frac{\phi_{j-1}}{\psi_j}, \quad i = 2, 3, \ldots. \tag{9}$$

With the help of normalizing condition $\sum_{i\in E} \pi_i$ we find

$$g_1 = \left(\sum_{1\le i<\infty} \gamma_i G_i. \right)^{-1} \tag{10}$$

Thus, the convergence of the series

$$\sum_{1\le i<\infty} \gamma_i \prod_{1\le j\le i} \frac{\phi_{j-1}}{\psi_j} < \infty \tag{11}$$

is the condition for the stationary regime existence. The same result can be obtained by methods of the SMP theory [9]. We formulate these results as a Theorem.

Theorem 1. *For the generalized B & D process stationary regime existence it is sufficient the convergence of the series (11). In this case the stationary probabilities are given by the formulas (9, 10).* ♡

Moreover, from the form of stationary probabilities it follows the next important corollary

Corollary. *The macro-states stationary probabilities of generalized B & D process are insensitive to the shape of distributions $A_i(x)$, $B_i(x)$ and depend on r.v. A_i, B_i and their distributions only by means of probabilities of jumps embedded random walk up and down and mean time of the process stay in the given state,*

$$\phi_i = \mathbf{P}\{A_i \le B_i\}, \qquad \psi_i = \mathbf{P}\{A_i > B_i\}, \qquad \textit{and} \qquad \gamma_i = \mathbf{E}[\min A_i, B_i]. \qquad \heartsuit \tag{12}$$

For the process with finite number of states $n+1$ in the Kolmogorov's system of equations (1) one should put $\alpha_{n+1}(x) \equiv 0$. In this case the stationary probabilities have the same form (9), but the normalizing constant (10) should be changed by

$$g_1 = \left(\sum_{1 \le i \le n+1} \gamma_i G_i \right)^{-1}.$$

In the case of exponential distributions $A_i(x) = 1 - e^{-\alpha_i t}$ and $B_i(x) = 1 - e^{-\beta_i t}$ one get

$$\phi_i = \frac{\alpha_i}{\alpha_i + \beta_i}, \qquad \psi_i = \frac{\beta_i}{\alpha_i + \beta_i}$$

and the substitution of these expressions into the formulas (9) reduce their to the stationary probabilities of the usual B & D process.

3.3 Distribution of the process states on life period

For many phenomenons especially for degradation processes more appropriate is absorbing process model. For the generalized B & D process with absorbing state $n+1$ in the Kolmogorov's system of equations (1) one should put $\alpha_{n+1}(x) = \beta_{n+1}(x) \equiv 0$. In this case the equation for the function $\pi_{n+1}(t)$ takes the form

$$\frac{\partial \pi_{n+1}(t,x)}{\partial t} + \frac{\partial \pi_{n+1}(t,x)}{\partial x} = 0, \tag{13}$$

with the initial and boundary condition

$$\pi_{n+1}(t,0) = \int_0^t \pi_n(t,x)\alpha_n(x)dx. \tag{14}$$

Thus, all functions $\pi_i(t,x)$ $(i = \overline{1,n})$ have the same solution (3) as before. But the function $\pi_{n+1}(t,x)$ is a constant over the characteristics, which are determined by the equations

$$-\frac{d\pi_{n+1}}{\pi_{n+1}} = dt = dx.$$

This means that

$$\pi_{n+1}(t,x) = g_{n+1}(t-x),$$

and from the boundary conditions (14) it follows that

$$g_{n+1}(t) = \int_0^t g_n(t-x)a_n(x)(1-B_n(x))dx.$$

The solution $\pi_i(t)$ of the system of equations (1, 13, 14) gives the probability of the object stay in some state jointly with its life period T,

$$\pi_i(t) = \mathbf{P}\{S(t) = i,\ t < T\}, \qquad i = 1, 2, \ldots, n.$$

For the degradation problems investigation more useful and adequate characteristic is the conditional state probability distribution on given object's life period

$$\bar{\pi}_i(t) = \mathbf{P}\{S(t) = i \ |t < T\}, \qquad i = 1, 2, \ldots, n.$$

From the above it follows that for $\bar{\pi}_i(t)$ the following representation is true

$$\bar{\pi}_i(t) = \mathbf{P}\{S(t) = i \ |t < T\} = \frac{\pi_i(t)}{R(t)}, \qquad i = 1, 2, \ldots, n$$

where $R(t)$ is the reliability (survival) function of the object, for which the following representation takes place

$$R(t) = 1 - \pi_{n+1}(t).$$

For the LT of the function $\pi_{n+1}(t)$ one can find

$$\tilde{\pi}_{n+1}(s) = \tilde{\tilde{\pi}}_{n+1}(s,0) = \frac{1}{s}\tilde{g}_{n+1}(s) = \frac{1}{s}\tilde{g}_n(s)\tilde{\phi}_n(s). \tag{15}$$

Therefore, for the LT $\tilde{R}(s)$ of the reliability function $R(t) = 1 - \pi_{n+1}(t)$ one has

$$\tilde{R}(s) = \frac{1}{s} - \frac{1}{s}\tilde{g}_n(s)\tilde{\phi}_n(s) = \frac{1}{s}(1 - \tilde{g}_n(s)\tilde{\phi}_n(s)).$$

We will calculate the probability distribution of the object states conditionally given life period in terms of their LT's. It follows from the relation (5) that the LT's $\tilde{\pi}_i(s) = \tilde{\tilde{\pi}}_i(s,0)$ of the macro-states probabilities $\pi_i(t)$ have the form

$$\tilde{\pi}_i(s) = \tilde{g}_i(s)\tilde{\gamma}_i(s). \tag{16}$$

From the initial and boundary conditions it follows that the functions $\tilde{g}_i(s)$ satisfy to the system of equations (7) for $i = 1, 2, \ldots, n$ and

$$\tilde{g}_{n+1}(s) = \tilde{g}_n(s)\tilde{\phi}_n(s).$$

The solution of these equations can be done in terms of Kramer's rule, and it can help to find the limits of the conditional probability states given life period.

To calculate their we should evaluate the asymptotic behavior of the functions $\pi_i(t)$ $(i = 1, 2, \ldots, n)$ and $R(t)$ when $t \to \infty$. We will do that with the help of their LT. Denote by $\Delta(s)$ the determinant of the matrix of coefficients of n first equations of the system (7) and by $\Delta_i(s)$ the determinant of the same matrix in which i-th column is changed by the vector-column of the equation right side (vector e_n). Then taking into account the expression (16) and the solution of the system (7) in terms of the Kramer's rule we get

$$\begin{cases} \tilde{\pi}_i(s) &= \tilde{\gamma}_i(s)\tilde{g}_i(s) = \tilde{\gamma}_i(s)\frac{\Delta_i(s)}{\Delta(s)} \qquad (i = \overline{1,n}), \\ \tilde{\pi}_{n+1}(s) &= \frac{\tilde{\phi}_n(s)}{s}\tilde{g}_n(s) = \frac{\tilde{\phi}_n(s)\Delta_n(s)}{s\Delta(s)}. \end{cases}$$

Theorem 2. *Asymptotical behavior of the functions $\pi_i(t)$ and $R(t)$ when $t \to \infty$ coincide and is determined by the maximal non-zero root s_1 of the characteristic equation $\Delta(s) = 0$. This provide the existence of the limit*

$$\bar{\pi}_i = \lim_{t\to\infty} \bar{\pi}_i(t) = \frac{\tilde{\pi}_i(s_1)}{\tilde{R}(s_1)}.$$

Proof. Since the functions $\tilde{\pi}_i(s)$ are analytical in right half-plain, then accordingly to the inverse LT formula the behavior of their originals are determined by their singularities in the left half-plane [16]. Denote by s_1 the maximal non-zero root of the characteristic equation $\Delta(s) = 0$, and suppose that in it neighboring the function $\Delta(s)$ has a pole of the first order. Then the asymptotic behavior of the functions $\pi_i(t)$ when $t \to \infty$ are determined by the value s_1 of maximal root of the characteristic equation, and the coefficients in it are determined by the residuals of the functions $\tilde{\pi}_i(s)$ at this point. Therefore to find the limit of the functions $\bar{\pi}_i(t)$ it is necessary to compare the asymptotic behavior of the functions $\pi_i(t)$ and $R(t)$ and to show that their asymptotic behavior when $t \to \infty$ are coincide.

From the relation (15) for the function $\tilde{\pi}_{n+1}(s)$ it follows that it can be represented in the form

$$\tilde{\pi}_{n+1}(s) = \frac{1}{s}\tilde{g}_n(s)\tilde{\phi}_n(s) = \frac{\tilde{\phi}_n(s)\Delta_n(s)}{s\Delta(s)} = \frac{B}{s} + \frac{\Delta_R(s)}{\Delta(s)}$$

with some coefficient B and some function $\Delta_R(s)$. Let us show that the coefficient B equals to one, $B = 1$. Really, from the definition it follows that

$$B = \lim_{s\to 0} s\tilde{\pi}_{n+1}(s) = \lim_{s\to 0} \tilde{\phi}_n(s)g_n(s) = \tilde{\phi}_n g_n(0) = \tilde{\phi}_n \frac{\Delta_n(0)}{\Delta(0)}.$$

Consider the matrix $\Psi'(s)$ of coefficients of the n first equations of the system (7), denote by $\Psi'_n(s)$ the matrix, obtained from this one by changing its last column by the right hand side equation vector (vector $\vec{e}_n$), put $\Psi' = \Psi'(0)$, $\Psi'_n = \Psi'_n(0)$ and consider the matrix

$$\Psi' - \phi_n \Psi'_n.$$

Since elements of matrix Ψ' are the transition probabilities of absorbing random walk, that is $\phi_i + \psi_i = 1$, then the sum of rows of the matrix Ψ' without its last elements are equal to the zero row. Because the matrix Ψ'_n differs from Ψ' only with the last column, thus the sum of rows of the matrix $\Psi' - \phi_n \Psi'_n$ without elements of its last column are also zero row. Consider now the elements of the last column. At the first place in it is an element $-\phi_n$, at the before last one is an element $-\psi_n$, and at the last one is an element $1 = \phi_n + \psi_n$. Thus, the rows of the matrix $\Psi' - \phi_n \Psi'_n$ are linearly dependent, and, therefore, its determinant equals to zero. It follows from this fact that $\phi_n \Delta_n(0) = \Delta(0)$ and consequently $B = \frac{\phi_n \Delta_n(0)}{\Delta(0)} = 1$. At least it follows from that that LT $\tilde{R}(s)$ of the reliability function $R(t)$ is presented in the form

$$\tilde{R}(s) = \frac{\Delta_R(s)}{\Delta(s)}.$$

Thus, the asymptotic behavior of the reliability function $R(t)$ when $t \to \infty$ also as all others functions $\pi_i(t)$ is determined by the maximal root s_1 of the characteristic equation $\Delta(s) = 0$. In this case the coefficients in asymptotic representation of the functions $\pi_i(t)$ and $R(t)$ when $t \to \infty$ are the residuals of the functions $\tilde{\pi}_i(s)$ and $\tilde{R}(s)$ at the point $s = s_1$,

$$A_{i1} = \lim_{s \to s_1} (s - s_1)\tilde{\pi}_i(s) = \lim_{s \to s_1} (s - s_1)\gamma_i(s)\frac{\Delta_i(s)}{\Delta(s)} = \frac{\gamma_i(s_1)\Delta_i(s_1)}{\dot{\Delta}(s_1)};$$

$$A_{R1} = \lim_{s \to s_1} (s - s_1)\tilde{R}(s) = \lim_{s \to s_1} (s - s_1)\frac{\Delta_R(s)}{\Delta(s)} = \frac{\Delta_R(s_1)}{\dot{\Delta}(s_1)}.$$

Therefore, when $t \to \infty$ there exists the limit

$$\bar{\pi}_i = \lim_{t \to \infty} \frac{\pi_i(t)}{R(t)} = \lim_{s \to s_1} \frac{\tilde{\pi}_i(s)}{\tilde{R}(s)} = \frac{\tilde{\pi}_i(s_1)}{\tilde{R}(s_1)} = \frac{\gamma_i(s_1)\Delta_i(s_1)}{\Delta_R(s_1)} = \frac{A_{i1}}{A_{R1}}. \qquad \heartsuit$$

4 An Example

To illustrate the above results let us consider a system with only three states, which can be considered as an example of the aggregated states model (see [5], [12]), where all states of each group: normal functioning N, degradation D, and failure F are joined into one. Suppose for the simplicity that the failures arise in accordance with the Poisson flow, but the repair times are generally distributed with c.d.f. $B(x)$ and the hazard rate $\beta(x)$. Moreover suppose that the direct transition from normal state into the failure state are also possible with intensity γ. The marked transition graph for the process at the figure 2 is presented.

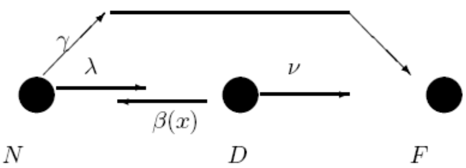

Fig. 2. The marked transition graph of the process with aggregated states.

In accordance with given transition graph the Kolmogorov's system of differential equations for system states probabilities has the form

$$\begin{cases} \frac{d\pi_N(t)}{dt} = -(\lambda + \gamma)\pi_N(t) + \int_0^t \beta(x)\pi_D(t,x)dx, \\ \frac{\partial \pi_D(t,x)}{\partial t} + \frac{\partial \pi_D(t,x)}{\partial x} = -(\nu + \beta(x))\pi_D(t,x), \\ \frac{d\pi_F(t)}{dt} = \gamma\pi_N(t) + \nu\pi_D(t,x) \end{cases} \qquad (17)$$

with the initial and the boundary conditions

$$\begin{cases} \pi_D(t,0) = \lambda\pi_N(t), \\ \pi_N(0) = 1, \ \pi_D(0,0) = \pi_F(0) = 0. \end{cases} \qquad (18)$$

The reliability function of the system is

$$R(t) = 1 - \pi_F(t) = 1 - \int_0^t [\gamma\pi_N(u) + \nu\pi_D(u)]\, du, \tag{19}$$

where the functions $\pi_N(t)$ and $\pi_D(t,x)$ are the solutions of the two first equations of the system (17) and

$$\pi_D(t) = \int_0^t \pi_D(t,x)\, dx. \tag{20}$$

The solution of the second equation from the system (17) accordingly to (3) can be given in the form

$$\pi_D(t,x) = g_D(t-x)e^{-\nu x}(1 - B(x)),$$

where the function $g_D(t)$ is determined from the boundary condition (18). It gives

$$\pi_D(t,x) = \lambda\pi_N(t-x)e^{-\nu x}(1 - B(x)). \tag{21}$$

Substitution of this solution into the first equation of the system (17) gives the following equation

$$\frac{d\pi_N(t)}{dt} = -(\lambda+\gamma)\pi_N(t) + \lambda\int_0^t \beta(x)\pi_N(t-x)e^{-\nu x}(1 - B(x))dx. \tag{22}$$

The best method for its solution is a LT approach. In the terms of LT with the initial condition (18) an equation (22) after the usual order of integration changing takes the form

$$s\tilde{\pi}_N(s) - 1 = -(\lambda+\gamma)\tilde{\pi}_N(s) + \lambda\tilde{b}(s+\nu)\tilde{\pi}_N(s),$$

where $\tilde{b}(s) = \int\limits_0^\infty e^{-st}b(x)dx$ is a LT of the p.d.f.$b(x)$. It follows from here that the solution of the last equation has a form

$$\tilde{\pi}_N(s) = \left[s + \gamma + \lambda(1 - \tilde{b}(s+\nu))\right]^{-1}. \tag{23}$$

Next, the calculation of the LT $\tilde{\pi}_D(s)$ of the function $\pi_D(t)$, given by the equality (20) after substitution into it of the expression (23) gives

$$\begin{aligned} \tilde{\pi}_D(s) &= \int_0^\infty e^{-st}\int_0^t \pi_D(t,x)dxdt = \int_0^\infty e^{-st}\int_0^t \lambda\pi_N(t-x)e^{-\nu x}(1 - B(x))dxdt = \\ &= \frac{\lambda\tilde{\pi}_N(s)(1 - \tilde{b}(s+\nu))}{s+\nu} = \frac{\lambda(1 - \tilde{b}(s+\nu))}{(s+\nu)(s+\gamma+\lambda(1 - \tilde{b}(s+\nu))}. \end{aligned} \tag{24}$$

At least for the LT $\tilde{\pi}_F(s)$ of the function $\pi_F(t)$ from the last of equations (17) one can find

$$s\tilde{\pi}_F(s) = \gamma\tilde{\pi}_N(s) + \nu\tilde{\pi}_D(s) = \frac{\gamma(s+\nu) + \lambda\nu(1 - \tilde{b}(s+\nu))}{(s+\nu)(s+\gamma+\lambda(1 - \tilde{b}(s+\nu)))}. \tag{25}$$

Therefore, for the LT of the reliability function (19) we get

$$\begin{aligned} \tilde{R}(s) &= \frac{1}{s} - \tilde{\pi}_F(s) = \frac{1}{s}\left[1 - \frac{\gamma(s+\nu) + \lambda\nu(1 - \tilde{b}(s+\nu))}{(s+\nu)(s+\gamma+\lambda(1 - \tilde{b}(s+\nu)))}\right] = \\ &= \frac{s + \nu + \lambda(1 - \tilde{b}(s+\nu))}{(s+\nu)(s+\gamma+\lambda(1 - \tilde{b}(s+\nu)))}. \end{aligned} \tag{26}$$

From the last expression one can find the mean life time of the object

$$m_F = \tilde{R}(0) = \frac{\nu + \lambda(1 - \tilde{b}(\nu))}{\nu(\gamma + \lambda(1 - b(\nu)))}. \tag{27}$$

For calculation of the limiting values of the conditional state probabilities on life period we use the above procedure, which is based on the connection between asymptotic behavior of the functions $\pi_N(t)$, $\pi_D(t)$, $R(t)$

at infinity and their LT at the neighboring of the maximal non-zeros root of the system characteristic equation $\Delta(s) = 0$. In the considered case the characteristic equation has a form

$$\Delta(s) = (s+\nu)(s+\gamma+\lambda(1-\tilde{b}(s+\nu))) = 0. \tag{28}$$

One of its roots is $s = -\nu$. The second root is determined by the equation

$$(s+\gamma+\lambda(1-\tilde{b}(s+\nu))) = 0$$

or

$$\tilde{b}(s+\nu)) = 1 + \frac{s+\gamma}{\lambda}. \tag{29}$$

Since the function $\tilde{b}(s+\nu))$ is a quite monotone one [15], i.e. monotonically decreases, concave upward, takes the value 1 at the point $s = -\nu$ and $\tilde{b}(\nu)) < 1 + \frac{\gamma}{\lambda}$, then the equation (29) has a unique negative root, and his value depend on the sign of the difference $\gamma - \nu$. If $\gamma \geq \nu$ the root of this equation, which we denote by s_1 is less than $-\nu$, $s_1 \leq -\nu$. On the other hand if $\gamma < \nu$ the root of this equation s_1 is grater than $-\nu$, $s_1 > -\nu$.

Therefore, when $\gamma \geq \nu$ the maximal root of the equation (29) is $-\nu$, and therefore,

$$\begin{aligned}
\bar{\pi}_N &= \lim_{t\to\infty} \bar{\pi}_N(t) = \lim_{t\to\infty} \frac{\pi_N(t)}{R(t)} = \lim_{s\to-\nu} \frac{\tilde{\pi}_N(s)}{\tilde{R}(s)} = \\
&= \lim_{s\to-\nu} \frac{\nu+s}{s+\nu+\lambda(1-\tilde{b}(s+\nu))} = \frac{1}{1+\lambda m_B}; \\
\bar{\pi}_D &= \lim_{t\to\infty} \bar{\pi}_D(t) = \lim_{t\to\infty} \frac{\pi_D(t)}{R(t)} = \lim_{s\to-\nu} \frac{\tilde{\pi}_D(s)}{\tilde{R}(s)} = \\
&= \lim_{s\to-\nu} \frac{\lambda(1-\tilde{b}(s+\nu)}{s+\nu+\lambda(1-\tilde{b}(s+\nu))} = \frac{\lambda m_B}{1+\lambda m_B},
\end{aligned} \tag{30}$$

i.e. if the death intensity from the normal state is grater than the death intensity resulting by degradation, then the limiting distribution of the conditional state probabilities is determined by the parameter $\rho = \lambda m_B$.

From another side, under condition $\gamma < \nu$ the greatest root of the characteristic equation (28) is the root of the equation (29), and consequently

$$\begin{aligned}
\bar{\pi}_N &= \lim_{t\to\infty} \bar{\pi}_N(t) = \lim_{t\to\infty} \frac{\pi_N(t)}{R(t)} = \lim_{s\to s_1} \frac{\tilde{\pi}_N(s)}{\tilde{R}(s)} = \\
&= \lim_{s\to s_1} \frac{s+\nu}{s+\nu+\lambda(1-\tilde{b}(s+\nu))} = \frac{s_1+\nu}{s_1+\nu+\lambda(1-\tilde{b}(s_1+\nu))}; \\
\bar{\pi}_D &= \lim_{t\to\infty} \bar{\pi}_D(t) = \lim_{t\to\infty} \frac{\pi_D(t)}{R(t)} = \lim_{s\to s_1} \frac{\tilde{\pi}_D(s)}{\tilde{R}(s)} = \\
&= \lim_{s\to s_1} \frac{\lambda(1-\tilde{b}(s+\nu)}{s+\nu+\lambda(1-\tilde{b}(s+\nu))} = \frac{\lambda(1-\tilde{b}(s_1+\nu))}{s_1+\nu+\lambda(1-\tilde{b}(s_1+\nu))}.
\end{aligned} \tag{31}$$

Therefore, if the death intensity from the normal state is less than the same as a degradation result, then the limiting distribution of the conditional probabilities is strongly depend on the value s_1 of the equation ([?]) root. Note that in the case if direct transitions from the normal state to the failure state are impossible, i.e. when $\gamma = 0$ the second case takes place.

5 Conclusion

Generalized Birth & Death Processes, which are special class of Semi-Markov Processes are introduced for modelling the degradation processes. The special parametrization of the processes allows to give more convenient presentation of the results. The special attention is focused to the conditional state probabilities given life cycle, which are the mostly interesting for the degradation processes.

References

[1] A. Lisniansky, G. Levitin (2003) *Multi-State System Reliability. Assessment, Optimization and Application.* World Scientific. New Jersey, London, Singapore, Hong Kong, 358p.

[2] B. Dimitrov, V. Rykov, P. Stanchev (2002) On Multi-State Reliability Systems. In: *Proceedings MMR-2002.* Trondheim (Norway) June 17-21, 2002.

[3] V. Rykov, B. Dimitrov (2002) On Multi-State Reliability Systems. In: *Applied Stochastic Models and Information Processes.* Proceedings of the International Seminar, Petrozavodsk, Sept. 8-13. Petrozavodsk, 2002, pp. 128-135. See also http://www.jip.ru/2002-2-2-2002.htm

[4] V. Rykov, B. Dimitrov, D. Green Jr., P. Snanchev (2004) Reliability of complex hierarchical systems with fault tolerance units. In: *Proceedings MMR-2004.* Santa Fe (U.S.A.) June, 2004. (Printed in CD).

[5] B.Dimitrov, D.Green, V. Rykov and P.Stanchev. Reliability Model for Biological Objects. In: *Longevity, Aging and Degradation Models. Transactions of the First Russian-French Conference (LAD-2004)*, Saint Petersburg , June 7-9, 2004, Ed. by V. Antonov, C. Huber, M. Nikulin, V. Polischook, Saint Petersburg State Politechnical University, SPB, 2004. Vol. 2, pp. 230-240.

[6] V. Rykov, D. Efrosinin. Reliability Control of Fault Tolerance Units. In: *Abstracts of The 4-th International Conference on Mathematical Methods in Reliability (MMR-2004)*, Santa Fe, (USA), 21-25 July, 2004 (Published in CD).

[7] V. Rykov, D. Efrosinin (2004). Reliability Control of of Biological Systems with failures. In: *Longevity, Aging and Degradation Models. Transactions of the First Russian-French Conference (LAD-2004)*, Saint Petersburg , June 7-9, 2004, Ed. V.Antonov, C.Huber, M.Nikulin, V.Polischook, Saint Petersburg State Politechnical University, SPB, 2004. Vol. 2, pp. 241-255.

[8] V. Rykov, E. Buldaeva. On reliability controlof fault tolerance units: regenerative approach. In: *Transactions of XXIV International Siminar on Stability Problems for Stochastic Modes*, September 10-17, 2004, Jurmala, Latvia. Transport and Telecommunication Institute, Riga, Latvia, 2004.

[9] V.S. Korolyuk, A.F. Turbin. *Semi-Markov processes and their applications.* Kiev: "Naukova dumka". 1976, 184p. (in Russian)

[10] D. McDonald. On semi-Markov and semi-regenerative processes. I, II.// *Z. fur Wahrch. verw.Geb.*, 42 (1978), No. 2, pp. 261-377; *Ann. of Prob.*, 6 (1978), No.6, pp. 995-1014.

[11] J.Jacod. Theorems de renouvellement et classification pour les chaines semi-Markoviennes.// *Ann. inst. Hanri Poincare*, sect. B, 7 (1971), No.2, pp.83-129.

[12] V.S. Korolyuk, A.F. Turbin. *Phase aggregation of complex systems.* Kiev: "Vish shkola". 1978, 108p. (in Russian).

[13] V.S. Korolyuk, V.V. Korolyuk (1999) *Stochastic Models of Systems.* Kluver Academic Publishers.

[14] I.G. Penrovsky. *Lectures on the theory of usual differential equations* . M.-L.: GITTL. 1952, 232p. (in Russian).

[15] W. Feller. *An Introduction to Probability Theory and its Applications V. II* John Wiley & Sons, Inc. N.Y.-Lnd.-Sidney, 1966.

[16] M. A. Lavrent'ev, B.V. Shabat. *Methods of the theory of complex variable functions.* M.: Phismatgiz. 1958, 678p.