TWO-LEVEL FACTORIAL LIFE TESTING WITH TYPE -II CENSORED DATA

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Abstract

We consider two-level factorial designs with the response being the observed item's lifetime. For each factor combination, we put on test *n* items and stop testing when exactly *r* items have failed. Our principal assumptions are that there exists a monotone transformation of the random response which belongs to a location-scale family, and that only the location parameter linearly depends on the factors involved. We develop a simple time-saving testing scheme which permits an efficient computational procedure for estimating the factor effects.

Key words: Orthogonal design; type - II censoring; location-scale family; order statistics; life testing.

study

1. Introduction

The idea of this paper is simple: we want to adjust the standard factorial experiment, in which the response is the observed lifetime, to the case of type-II censored observations.

Many industrial experiments are aimed' at finding the factor combination that provides the longest lifetime. Since experiments of this sort take usually a long time and are very expensive, it is desirable to use lifetime acceleration methods (e.g., by applying higher stresses than in normal operating conditions , see e.g. Nelson (1990)), and/or to stop the experiment after prescribed time or after prescribed number of failures has been observed. This paper is devoted to the statistical methodology of lifetime testing with type-II censored lifetime observations.

Papers of Hamada (1995), Hamada and Wu (1991) and Bullington et al (1993) present methods and examples of processing incomplete lifetime data in the framework of factorial life testing experiments. Their methodology is based on introducing a parametric model for the logarithms of the observed lifetimes and on using the maximum likelihood method (MLM) for parameter estimation. The MLM is computationally involved, especially when a large number of parameters is present in the model, e.g. in a screening experiment for studying the influence of many factors on lifetime. Even if the MLM software produces a numerical solution to the maximum likelihood equations, it might be not the desired solution, see the discussion in Hamada and Tse (1992). There are also difficulties in establishing the significance of the maximum likelihood estimates because this issue involves asymptotics which might be not accurate. Hamada (1995) demonstrates that the MLM may provide a disappointing result, e.g. all factors involved seem to be significant, see Table 8 in the above paper. Probably, in the experiments with type-I censored data, the MLM remains the only way to extract information from data.

If, however, for all factor combinations the lifetime data are type-II censored, i.e. are censored after observing $r \geq 2$ failures, then under reasonable assumptions, for two-level factorial experiments, there exists an extremely simple and computationally efficient method of estimating the factor effects.

In Section 2 we describe our method and the basic assumptions. If for each factor combination we observe r failures then our method uses as a response an "optimal" convex combination of the appropriately transformed first r order statistics.

In Section 3 we apply our method to the Thermostat Test data described in Burlington et al (1993). In this experiment, n similar thermostats were tested under identical conditions until the appearance of r failures.

We discuss the properties of our method in terms of estimation accuracy and testing duration.

2. Basic Assumptions. Description of the testing procedure. Parameter estimation.

Suppose that the experiment consists of N runs. Each run j, j = 1, 2, ..., N, corresponds to a fixed combination of the factors involved. We denote by capital letters $A, B, C, ..., K$ these factors.

We make the following assumptions.

(i) In the j-th run, a random sample of n items is tested until r, $r \geq 2$ failures are observed. All items in the sample are statistically identical and are tested under the same conditions. Denote by $\tau_i^{(j)}$, i = 1, 2, ...r, the item lifetimes observed in the j-th run.

(ii) There exists a monotone transformation $\psi(\cdot)$ of $\tau^{(j)}$ into $Y^{(j)}$, $Y^{(j)} = \psi(\tau^{(j)})$, such that $Y^{(j)}$ belongs to a location-scale family, in which only the location parameter depends on the factors involved:

$$Y^{(j)} = \alpha^{(j)}(A, B, C, ..., K) + \beta Z. \tag{1}$$

Here Z is a "standard" parameter-free random variable.

(iii) The location term in (1) linearly depends on the factors involved:

$$\alpha^{(j)} = \theta + A \cdot W_A^{(j)} + ... + K \cdot W_K^{(j)} \tag{2}$$

For simplicity, the letters A, B, ..., K in (2) denote the numerical contribution of the corresponding factors; θ is a constant, the same for all runs and all factor combinations.

(iv) The experiment has as a two-level factorial orthogonal design, i.e. the coefficients $W_A^{(j)}$, ..., $W_K^{(j)}$ equal tp ±1, and the column- vectors $\mathbf{W_A} = [W_A^{(1)}, ..., W_A^{(N)}]$, ...,$\mathbf{W_K} = [W_K^{(1)}, ..., W_K^{(N)}]$ are pair wise orthogonal.

Remark 1. If the lifetime for each run has a lognormal distribution with only location parameter depending on the factors involved, then $\psi(x) = \log x$, and Z will have a standard Normal distribution.

Another assumption widely used in practice is that $\tau^{(j)}$ has a Weibull distribution, with only the scale parameter depending on the experimental factors. Then the logarithmic transformation also produces the desired form (1) with Z being distributed according to the standard extreme-value distribution $Extr(0, 1) : P(Z > t) = \exp(-e^t)$. (For the proofs see e.g. Gertsbakh (1989), Ch. 2)

Since (1) is a location-scale family, it follows from our assumptions that for each j we observe the first r order statistics of the corresponding sample, and this is the same as observing, for each j,

$$Y_{i:n}^{(j)} = \alpha^{(j)} + \beta Z_{i:n}, \; i = 1,2,\dots,r. \tag{3}$$

Here $Z_{i:n}$ is the i-th order statistic from a sample of n random variables Z_s, $s = 1, \dots, n$.

Remark 2. Add and subtract from the right-hand side of (3) the term $\beta E[Z_{i:n}] = \beta m_{i:n}$. Then we can assume that (3) takes the form

$$Y_{i:n}^{(j)} = \alpha_0^{(j)} + \beta \varepsilon_{i:n}, \; i = 1,2,\dots,r. \tag{4}$$

where $\varepsilon_{i:n}$ is a zero-mean error term, and $\alpha_0^{(j)}$ differs from $\alpha^{(j)}$ by a constant absorbed into the θ-term (see (2)). Note that this constant is the same for all runs $j = 1, \dots, N$.

In order to use all information observed in the j-th run, we suggest to consider as the response a *convex combination* of the first r order statistics:

$$X^{(j)} = \sum_{i=1}^{r} \alpha_i Y_{i:n}^{(j)}, \tag{5}$$

where $\sum_{i=1}^{r} \alpha_i = 1, \; \alpha_i \geq 0$.

Now the response of the j-th run takes the form:

$$X^{(j)} = a^{(j)} + \beta Z^{j}, \tag{6}$$

where Z_j is a zero-mean error term, and $a^{(j)}$ differs from the expression in (2) by a constant $\beta \sum_{i=1}^{r} \alpha_i m_{i:n}$.

It is desirable to choose the coefficients α_i to provide the *minimal* variance of the response, as the following claim states.

Claim 1.

(i) Minimal variance of $\sum_{i=1}^{r} \alpha_i Z_{i:n}$ subject to $\sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 0$ is attained at

$$[\alpha_1^*, ..., \alpha_r^*] = \mathbf{V}^{-1} \cdot 1 \cdot C , \qquad (7)$$

where $\mathbf{V}$ is the covariance matrix of $[Z_{1:n}, ..., Z_{r:n}]$, $C = (\mathbf{1}^T \cdot \mathbf{V}^{-1} \cdot \mathbf{1})^{-1}$, and $\mathbf{1}$ is a column matrix with all elements being equal 1.

(ii) The minimum of the variance equals to C.

The proof is based on the Extended Cauchy-Schwarz inequality, see Johnson and Wichern (1982), p.66. (Use (2.49) there and put $\mathbf{d} = \mathbf{1}$.) #

Estimation of parameters *A, B, ..., K*.

From now on, let us assume that the α_i values are always equal to the optimal α_i^*. Our model (6) now takes the following form:

$$X^{(1)} = \theta + A \cdot W_A^{(1)} + ... + K \cdot W_K^{(1)} + \beta Z^{(1)}$$

....

$$X^{(N)} = \theta + A \cdot W_A^{(N)} + ... + K \cdot W_K^{(N)} + \beta Z^{(N)}$$

Now multiply the *j*-th row by $W_A^{(j)}$, $j = 1,...,n$ and sum up all rows.

Due to the orthogonality of $\mathbf{W_A}, ..., \mathbf{W_K}$, we obtain

$$\sum_{j=1}^{N} W_A^{(j)} X^{(j)} = N \cdot A + \beta\varepsilon , \qquad (8)$$

where ε is a zero-mean error-term. Now the unbiased estimator of *A* equals

$$\hat{A} = \sum_{j=1}^{N} W_A^{(j)} X^{(j)} / N \qquad (9)$$

Similarly we obtain estimators for *B, C, ..., K*. By (9) and (6) their variance equals

$$Var\left[\hat{A}\right] = N^{-1}\beta^2 Var\left[Z^{(j)}\right] = N^{-1}\beta^2 C , \qquad (10)$$

where *C* is determined by (i) in Claim 1.

It is easy to prove that that all estimators of the factor coefficients are pair wise uncorrelated, e.g. $Cov\left[\hat{A}, \hat{R}\right] = 0$. This follows from the properties of $Z^{(j)}$ and from the fact that vectors $\mathbf{W}_{(\cdot)}$ have equal number of positive and negative terms.

Simplified estimator of β.

Let us return to the principal relationship (3), fix two integers *s*, $s \le r$, and $m \le r$, $s < m$, and write (3) for $i = m$ and $i = s$. Subtracting one from another, the $\alpha^{(j)}$ term cancels and we arrive at the formula

$$Y_{m:n}^{(j)} - Y_{s:n}^{(j)} = \beta\left(Z_{m:n} Z_{s:n}\right),\ j = 1, 2, \ldots, N. \tag{11}$$

It follows from (11) that

$$E\left[Y_{m:n}^{(j)}\right] - E\left[Y_{s:n}^{(j)}\right] = \beta\left(E\left[Z_{m:n}\right] - E\left[Z_{s:n}\right]\right) \tag{12}$$

Replace in (12) the expectations in the left-hand side by the corresponding averages and consider the following simplified estimator for β:

$$\beta = \frac{\overline{y}_{m:n} - \overline{y}_{s:n}}{\left(E\left[Z_{m:n}\right] - E\left[Z_{s:n}\right]\right)}. \tag{13}$$

Here $\overline{y}_{s:n} = N^{-1}\sum_{j=1}^{N}\psi\left(\tau_{s:n}^{j}\right)$. (The simplification is in replacing $Z_{m:n} - Z_{s:n}$ by its expectation).

From (12) it follows that

$$Var\left[\hat{\beta}\right] = \frac{Var\left[Y_{m:n}\right] + Var\left[Y_{s:n}\right] - 2Cov\left[Y_{m:n}, Y_{s:n}\right]}{N \cdot \left(E\left[Z_{m:n}\right] - E\left[Z_{s:n}\right]\right)^2}. \tag{14}$$

By (3), $Var[Y_{i:n}] = \beta^2 Var[Z_{i:n}]$, and this results in

$$Var\left[\hat{\beta}\right] = \frac{\beta^2\left[Var\left[Z_{m:n}\right] + Var\left[Z_{s:n}\right] - 2Cov\left[Z_{m:n}, Z_{s:n}\right]\right]}{N \cdot \left(E\left[Z_{m:n}\right] - E\left[Z_{s:n}\right]\right)^2}. \tag{15}$$

We state without proof that the smallest value of $Var\left[\hat{\beta}\right]$ is attained when $m = r$ and $s = 1$.

3. Example: Thermostat Life Cycle Test.

Burlington et al (1993) describe a screening life testing experiment of thermostats aimed at finding the best combination of design parameters (factors) which would provide the longest thermostat life.

Eleven most important design factors *A*, *B*, ..., *K* were selected by an expert team, e.g. the current density *B*, Beryllium Copper grain size *E* and heat treatment *H* . For each factor, two levels were chosen, the lower and the upper, denoted by −1 and +1, respectively - see Table 1. For *E*, for example, the grain sizes of 0.008” and 0.018” were chosen for the low and high level, respectively.

Table 1
Thermostat test results, n = 10, r = 2

	Factor												
Run	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	$\tau_{1:10}$	$\tau_{2:10}$
1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	957	2486
2	−1	−1	−1	−1	−1	+1	+1	+1	+1	+1	+1	206	284
3	−1	−1	+1	+1	+1	−1	−1	−1	+1	+1	+1	63	113
4	−1	+1	−1	+1	+1	−1	+1	+1	−1	−1	+1	76	104
5	−1	+1	+1	−1	+1	+1	−1	+1	−1	+1	−1	92	126
6	−1	+1	+1	+1	−1	+1	+1	−1	+1	−1	−1	490	971
7	+1	−1	+1	+1	−1	−1	+1	+1	−1	+1	−1	232	326
8	+1	−1	+1	−1	+1	+1	+1	+1	−1	−1	+1	206	284
9	+1	−1	−1	+1	+1	+1	−1	+1	+1	−1	−1	142	144
10	+1	+1	+1	−1	−1	−1	−1	+1	+1	−1	+1	259	266
11	+1	+1	−1	+1	−1	+1	−1	−1	−1	+1	+1	381	420
12	+1	+1	−1	−1	+1	−1	+1	−1	+1	+1	−1	56	62

The experiment consisted of N = 12 runs with the factors being arranged according to the PLackett-Burman resolution IV design, see Table 1. For each factor combination, n = 10 thermostats were subjected to heating cycles under identical conditions until they fail. The duration of the whole test was 7,342 thermal kilocycles. At the end of the test only two failures were observed in the first and eleventh run (trials), four failures in the fourth trial, and ten failures in all other trials. Table 1 presents the test results for the two smallest observed lifetimes, for each factor combinations.

In our analysis we will ignore the presence of other observed lifetime data. In spite of a seeming loss of information we demonstrate that our results are identical to those obtained by Burlington et al from "complete" data.

In the notation of the previous section, we have r = 2, the observed lifetimes are $\tau_{1:10}$, $\tau_{2:10}$. Our function $\psi(x)$ = log(x) and this means that we assume that thermostat lifetime has either lognormal or Weibull distribution. In our analysis, the observed response in j-th run will be, according to Section 2, $x^j = \alpha_1 \cdot \log\tau_{1:10} + \alpha_2 \cdot \log\tau_{2:10}$.

Our first choice is to assume that the thermostat lifetime has a lognormal distribution. Then Z in (1) is *Normal*(0, 1). Table 2 prescribes to take α_1 = 0.199, α_2 = 0.801. Using (9) and similar formulas, we obtain now the estimates of the effects for all factors. They are:

$\hat{A}$ = −0.32. $\hat{B}$ = −0.09, $\hat{C}$ = −0.12, $\hat{D}$ = 0.06, $\hat{E}$ = −0.92, $\hat{F}$ = 0.005, $\hat{G}$ = −0.25, $\hat{H}$ = −0.21, $\hat{I}$ = −0.27, $\hat{J}$ = −0.27, $\hat{K}$ = −0.31

The normal plot of these estimates, see Fig. 1, clearly indicates that the only significant factor is E, and its sign says that the grain size must be kept on its lower level 0.008". This coincides with Burlington's conclusion.

[For using normal plot to identify the significant factors see e.g. Box and Draper (1987), Ch.4]

Table 2

Optimal weights α_i^* for the convex combinations of the first r order statistics and their variances (n = 10)

Extreme-value distribution | Normal distribution

r	α_1^*	α_2^*	α_3^*	α_4^*	α_5^*	γ	**Var**	α_1^*	α_2^*	α_3^*	α_4^*	α_5^*	γ	**Var**
2	0.148	0.852				0.94	0.61	0.199	0.801				0.96	0.20
3	0.045	0.205	0.750			0.93	0.37	0.161	0.154	0.685			0.92	0.16
4	0.023	0.080	0.0216	0.681		0.92	0.26	0.140	0.135	0.135	0.590		0.88	0.14
5	0.016	0.047	0.096	0.217	0.624	0.90	0.17	0.126	0.124	0.125	0.118	0.507	0.83	0.12

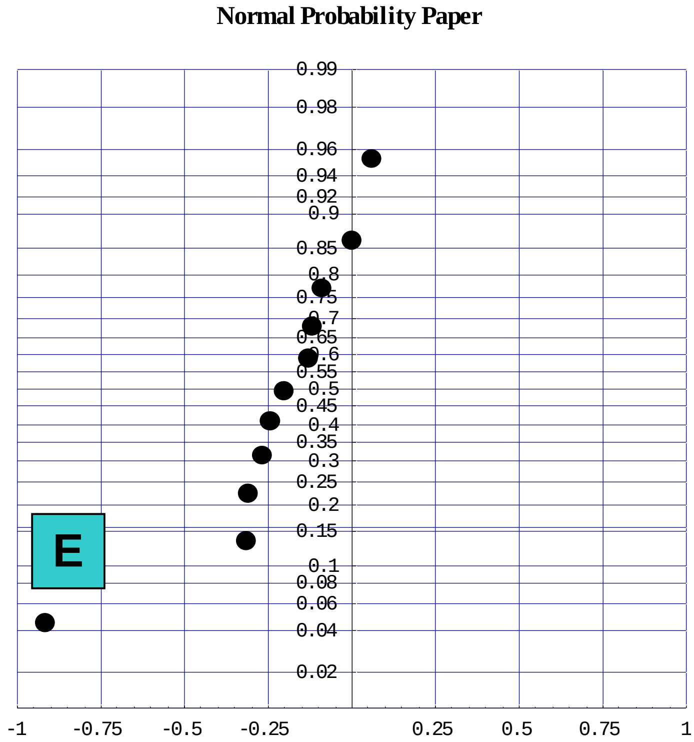

Figure1. Normal plot of factor effects

After some modification of the model by including into it interactions, Hamada (1995) found that that two factors E and H and their interaction EH are significant and that both factors must be kept on their lower level. Our normal plot does not confirm the significance of the H factor. We may, however estimate the interaction effect of EH. Assume that all effects except E equal zero. Multiply the response column [x^1, ..., x^N] by the product of E and H columns. We will find that $\widehat{E \cdot H}$ = 0.42, which may be considered as an evidence that the interaction is significant. Then both factors, E, H should be kept on their upper level. This also coincides with the Hamada's conclusion (1995).

Let us estimate β and the standard error of its estimate. For this purpose we use formulas (13) and (15) and the expected values, and variances and covariances of the order statistics $Z_{1:N}$ and $Z_{2:N}$ given in Tables 3a,b. For the normal case $\hat{\beta}$ = 0.59 and $\sigma_{\hat{\beta}}$ = 0.15.

Table 3a

The covariance matrix of order statistics for n = 10, r = 9, Z ~ Extr(0,1) .

i	1	2	3	4	5	6	7	8	9
1	1.645	0.436	0.275	0.193	0.144	0.111	0.086	0.067	0.051
2		0.646	0.290	0.204	0.152	0.117	0.091	0.071	0.054
3			0.397	0.217	0.162	0.124	0.097	0.076	0.058
4				0.287	0.174	0.137	0.104	0.081	0.062
5					0.227	0.145	0.113	0.088	0.067
6						0.190	0.125	0.098	0.074
7							0.166	0.111	0.085
8								0.152	0.100
9									0.149

Table 3 b

Mean values of the order statistics, n = 10

$m_{(1)}$	$m_{(2)}$	$m_{(3)}$	$m_{(4)}$	$m_{(5)}$
-2.800	-1.826	-1.267	-0.868	-0.544
$m_{(6)}$	$m_{(7)}$	$m_{(8)}$	$m_{(9)}$	$m_{(10)}$
-0.257	-0.012	0.284	0.585	0.990

All the above analysis can be carried out also for the assumption that the lifetime has Weibull distribution. Then $Z \sim Extr(0, 1)$. Practically all results will be very similar to the above normal case. The only difference appears in the estimate of β: now $\hat{\beta}$ = 0.33 and $\sigma_{\hat{\beta}}$ = 0.12.

Remark 1.

In a preliminary trial to process the data in Table 1, we took as a response only the logarithm of a *single* order statistic $\tau_{2:10}$. The results were essentially the same as for the case of using optimally two first order statistics. An explanation might be the fact that including $\tau_{1:10}$ adds very little to the accuracy of the estimates of factor effects.

The columns named γ in Table 1 show the value of the ratio $\gamma = C/Var[Z_{r:n}]$. Surprisingly, for r 1,2,3,4,5 and n = 10, for $Z \sim N(0, 1)$ and $Z \sim Extr(0, 1)$ the γ-values are quite close to 1. Therefore, the r-th order statistic contains practically the same information as the whole set of the first r order statistics.

Remark 2.

How much we gain in the accuracy of the estimates of factor effects if we increase r? The columns **Var** display the variances of the optimal convex combinations of the first r order statistics. For the normal case, we may gain almost twice in the decrease of the variance by increasing r from 2 to 6. Interestingly, a complete sample of n = 10 for the lognormal case would give the variance equal to 0.1, a reduction of variance by factor of 2, comparing to r = 2.

Assume that Z ~ $Extr(0, 1)$, $r = 5$. Then the variance would decrease by a factor of three, and a complete sample would result in variance reduction by a factor of 3.7, comparing to r=2.

Remark 3: gain in test duration.

In practice, the variance reduction achieved by the *increase* of the r value must be weighted against the increase of the *duration* of the whole life testing experiment.

Let us discuss this issue in more detail. Returning to (1) and using the logarithmic transformation, let us present the k-th smallest observation in the j-th run as

$$\log\tau^j_{k:n} = \alpha^j + \beta Z^j_{k:n}, \tag{16}$$

and, taking expectations,

$$\log E\left[\tau^j_{m:n}\right] = \alpha^j + \beta m^j_{k:n}. \tag{17}$$

Now put in (17) $k = g$ and $k = s$ and approximate the mean of the logarithm by the logarithm of the mean. Then we obtain

$$\log E\left[\tau^j_{g:n}\right] - \log E\left[\tau^j_{s:n}\right] \approx \beta\left(m_{g:n} - m_{s:n}\right) \tag{18}$$

or

$$E\left[\tau^i_{g:n}\right] / E\left[\tau^i_{s:n}\right] \approx \exp\left[\beta\left(m_{g:n} - m_{s:n}\right)\right] \tag{19}$$

Example. Suppose that the logarithm of the observed lifetime is normally distributed, the sample size $n = 10$ and we want compare the increase of test duration arising from the increase of k from 2 to 5. From Table 4 b we see that $[m_{5:10} - m_{3:10}] = -0.123 + 1.00 = 0.877$. For $\beta = 0.59$, we obtain that the means of the test duration increase approximately by a factor of exp[0.59 * 0.877] = 1.68. Suppose now that we observe all $n = 10$ failures, i.e. $g = 10$. Then the test duration increases approximately by a factor exp[0.59(1.54 + 1.00)] ≈ 4.5.

Table 4 a

The covariance matrix of order statistics for $n = 10$, $Z \sim N(0, 1)$

i	1	2	3	4	5	6	7	8	9	10
1	0.344	0.171	0.116	0.088	0.071	0.058	0.049	0.041	0.034	0.027
2		0.214	0.147	0.112	0.090	0.074	0.062	0.052	0.043	0.034
3			0.175	0.134	0.108	0.089	0.075	0.063	0.052	0.041
4				0.158	0.128	0.106	0.089	0.075	0.062	0.049
5					0.151	0.126	0.106	0.089	0.074	0.058
6						0.151	0.128	0.108	0.090	0.071
7							0.158	0.138	0.112	0.088
8								0.175	0.147	0.116
9									0.214	0.171
10										0.344

Table 4 b
Mean values of the order statistics (normal case)

$m_{(1)}$	$m_{(2)}$	$m_{(3)}$	$m_{(4)}$	$m_{(5)}$
−1.5388	−1.0014	−0.6561	−0.3758	−0.1227

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