OBSERVED STATISTICAL RISKS IN INSPECTION FOR QUALITY & RELIABILITY

Ernest V. Dzirkal
Victor A. Netes

Key Words – Statistical inspection of quality and reliability, Hypothesis testing, Observed risks, Confidence limits.

Summary & Conclusion - This paper presents the concept of observed risks. These risks are determined after the statistical inspection tests of quality or reliability, so they depend on the test results. They allow evaluating probability of erroneous decisions (risks) after the test, not before it as it is traditionally done. We give the main properties of the observed risks. Numerical examples illustrate the suggested concept and demonstrate its usefulness.

study

1. INTRODUCTION

In natural sciences an experiment is usually planned so that its conjectural error does not exceed some chosen value, but after termination of the experiment its actual error is estimated. However in statistical test problems another approach is generally used. The probability of risks is considered as the measure of risk before the test as well as after it. This is strange, but after the test completion and decision making, the risks does not become more accurate.

This paper intends to fill up this gap for the problems of quality and reliability inspection. Besides that the problem of inspection with the use of confidence limits and two levels of a checked index is solved.

This approach was officially admitted in 1987 in the former USSR and the appropriate technique has been included in the National Standard [1]. Nevertheless it did not attract attention of theoreticians and it is not mentioned in the University courses and practical manuals. Therefore the authors would like to attract attention to this approach, which was described in their previous papers [2, 3] and handbook [4].

2. NOTATION

Q – quality or reliability index of some item.
Q_0 – acceptance level of Q.
Q_1 – rejection level of Q.
H_0 – null hypothesis: $H_0 = \{Q \geq Q_0\}$ for the positive index (the larger the value of Q, the higher quality or reliability); $H_0 = \{Q \leq Q_0\}$ for the negative index (the smaller the value of Q, the lower quality or reliability).
H_1 – alternative hypothesis: $H_1 = \{Q \leq Q_1\}$ for the positive index; $H_1 = \{Q \geq Q_1\}$ for the negative index. (We will consider further the positive index).
x – test data.
X_0 – acceptance region.
X_1 – rejection region.
α – (planned) producer's risk: $\alpha = \Pr\{x \in X_1; H_0\}$.
β – (planned) consumer's risk: $\beta = \Pr\{x \in X_0; H_1\}$.
$Q_*(x,\gamma)$ – lower confidence limit for Q under the test data x and confidence coefficient γ.
$Q^*(x,\gamma)$ – upper confidence limit for Q under the test data x and confidence coefficient γ.

$\Pr(n,\lambda) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^n}{n!}$ – Poisson distribution.

3. OBSERVED RISKS IN THE CASE OF SINGLE-SAMPLE INSPECTION WITH THE USE OF ACCEPTING CONSTANT

In this case we use some test statistic *S(x)*, which is a function of the observations, and the accepting constant *C*. Let the test statistic *S(x)* be such that the larger its value, the stronger the evidence of higher quality or reliability of the tested item). The null hypothesis H_0 is accepted and we make the decision that the item conforms to quality or reliability specifications when $S(x) \geq C$. The H_0 is rejected and we make the decision that the item does not conform to the specifications when $S(x) < C$.

Thus

$$X_0 = \{x : S(x) \geq C\}, \tag{1a}$$

$$X_1 = \{x : S(x) < C\}. \tag{1b}$$

Hence

$$\alpha = \Pr\{S(x) < C; H_0\}, \tag{2a}$$

$$\beta = \Pr\{S(x) \geq C; H_1\}. \tag{2b}$$

The observed producer's risk $\widehat{\alpha}(x^*)$ for test data x^* is defined as the probability that the result for the item with the value of index not less than Q_0 will not be better than x^*.

The observed consumer's risk $\widehat{\beta}(x^*)$ for test data x^* is defined as the probability that the result for the item with the value of index not greater than Q_1 will not be worse than x^*.

Therefore

$$\widehat{\alpha}(x^*) = \Pr\left\{S(x) \le S(x^*); H_0\right\}, \tag{3a}$$

$$\widehat{\beta}(x^*) = \Pr\left\{S(x) \ge S(x^*); H_1\right\}. \tag{3b}$$

Thus in determining the observed risks we use the value of test statistic itself and not only the fact that it is greater or less than the acceptance constant C [compare (3a,b) with (2a,b)].

Theoretically the observed risk corresponds to the observed significance level in statistics [5].

Observed risks may be determined in the both cases: acceptance and rejection. If we wish make the consumer's and producer's planned risks equal ($\alpha = \beta$), the above-mentioned decision rule [corresponding with (1a,b)] may be formulated also without using the acceptability constant C by comparing the observed risks as follows:

$\widehat{\alpha} > \widehat{\beta}$ - acceptance, $\widehat{\alpha} < \widehat{\beta}$ - rejection

(see Example 1 and Theorems 3 and 4 below). In other words, we choose the decision, which corresponds to the smaller observed risks.

Example 1

Consider the acceptance sampling with inspection by attributes and let the accepting and rejection levels be $q_0 = 0.05$ and $q_1 = 0.15$, respectively. We assume Poisson distribution for the defective number d:

$$\Pr\left\{d = n; q\right\} = \exp(-Nq)(Nq)^n / n!,$$

where N is sample size, q is actual number of defects.

If $N = 40$, acceptance number $A = 3$, rejection number $R = A + 1 = 4$, then $\alpha = 0.143$ and $\beta = 0.151$.

The observed producer's risk $\widehat{\alpha}$ when the observed number of defects is d^* defines as the probability to have not less defectives than d^* under the fraction defective q_0.

The observed consumer's risk $\widehat{\beta}$ is the probability to have not greater than d^* defectives under the fraction defective q_1.

Therefore $\widehat{\alpha} = 1 - \Pr(d^* - 1, Nq_0)$, $\widehat{\beta} = \Pr(d^*, Nq_1)$.

Table 1 contains the observed risk for this example.

TABLE 1
Observed risks for Example 1

d^*	0	1	2	3	4	5	6	7	8
$\widehat{\alpha}$	1.000	0.865	0.594	0.323	0.143	0.053	0.017	0.005	0.001
$\widehat{\beta}$	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847

The observed producer's risk $\widehat{\alpha}$ equals the planned producer's risk α when $d^* = R = 4$ (R is the minimal number of defects in the rejected lot); when d^* is increasing $\widehat{\alpha}$ is rapidly decreasing.

The observed consumer's risk $\widehat{\beta}$ equals the planned consumer's risk β when $d^* = Ac = 3$ (Ac is the maximum number of defects in the accepted lot); $\widehat{\beta}$ is rapidly decreasing with decreasing d^*.

4. MAIN PROPERTIES OF OBSERVED RISKS

Theorem 1. If $x^* \in X_1$, then $\widehat{\alpha}(x^*) \le \alpha$; if $x^* \in X_0$, then $\widehat{\beta}(x^*) \le \beta$.
Proof. (we provide proofs only for one of the risks, for another risk they are analogous):
Let $x^* \in X_1$. From (1b) $S(x^*) < C$ and $\left\{x : S(x) \le S(x^*)\right\} \subset \left\{x : S(x) < C\right\}$,
so $\Pr\left\{S(x) \le S(x^*)\right\} \le \Pr\left\{S(x) < C\right\}$.
Thus from (3a) and (2a) $\widehat{\alpha}(x^*) \le \alpha$.

Theorem 2. $\sup\limits_{x \in X_1} \widehat{\alpha}(x) = \alpha$, $\sup\limits_{x \in X_0} \widehat{\beta}(x) = \beta$.

Proof: Let $\alpha = \sup\limits_{x \in X_1} S(x)$. If there exists $x^* \in X_1$ so that $S(x^*) = \alpha$, than $X_1 = \left\{x : S(x) \le S(x^*)\right\}$ and

$\alpha = \Pr\left\{X_1; H_0\right\} = \Pr\left\{S(x) \le S(x^*); H_0\right\} = \widehat{\alpha}(x^*)$.

If there does not exist such x^*, then there exists a sequence $x_n \in X_1$ so that $S(x_n) \uparrow \alpha$. Then sequence of sets $X'_n = \left\{x : S(x) \le S(x_n)\right\}$ is increasing and $\bigcup X'_n = X_1$, so $\Pr\left\{X'_n\right\} \to \Pr\left\{X_1\right\}$.

Therefore $\widehat{\alpha}(x_n) = \Pr\left\{X'_n; H_0\right\} \to \Pr\left\{X_1; H_0\right\} = \alpha$.

Theorem 3. Let $\alpha = \beta$. Then $\widehat{\beta}(x^*) < \widehat{\alpha}(x^*)$ for $x^* \in X_0$, $\widehat{\alpha}(x^*) \le \widehat{\beta}(x^*)$ for $x^* \in X_1$.
Proof**:**
If $x^* \in X_0$ then $S(x^*) \ge C$ and $\widehat{\beta}(x^*) \le \beta = \alpha = \Pr\{S(x) < C; H_0\} \le \Pr\{S(x) \le S(x^*); H_0\} = \widehat{\alpha}(x^*)$.

Theorem 4. Let $\alpha = \beta$. If $\widehat{\beta}(x^*) < \widehat{\alpha}(x^*)$ then $x^* \in X_0$; if $\widehat{\alpha}(x^*) \le \widehat{\beta}(x^*)$ then $x^* \in X_1$.
Proof**:** The assumption that $x^* \in X_1$ when $\widehat{\beta}(x^*) < \widehat{\alpha}(x^*)$ involves a contradiction, because in this case $\widehat{\alpha}(x^*) \le \widehat{\beta}(x^*) < \widehat{\alpha}(x^*)$. Thus $x^* \in X_0$.

5. INSPECTION WITH THE USE OF CONFIDENCE LIMITS

The decision rule in the case of inspection with the use of confidence limits is [2, 4, 6]:

$Q_*(x, 1-\beta) \ge Q_1,\ Q^*(x, 1-\alpha) > Q_0$ – acceptance; (4a)
$Q_*(x, 1-\beta) < Q_1,\ Q^*(x, 1-\alpha) \le Q_0$ – rejection. (4b)

Under some natural constraints we can choose the extent of test to ensure the fulfillment of one of these two conditions for acceptance or rejection [2, 4, 6].

Usually, the confidence limits are built on the base of some statistic $\xi(x)$ [6] so that

$$Q^*(x,\gamma) = A(\xi(x),\gamma),\ Q_*(x,\gamma) = B(\xi(x),\gamma),$$

where $A(\cdot)$ and $B(\cdot)$ are some functions. In this situation the decision rule (4a,b) is equivalent to the decision rule corresponding (1a,b) with $S(x) = \xi(x)$ and some acceptability constant C [6].

In the case of inspection with the use of confidence limits the observed risks $\widehat{\alpha}(x^*)$ and $\widehat{\beta}(x^*)$ are determined from the equations

$$Q^*(x^*, 1-\widehat{\alpha}) = Q_0, \tag{5a}$$
$$Q_*(x^*, 1-\widehat{\beta}) = Q_1. \tag{5b}$$

Sometimes the extent of the test depends on external circumstances. For example, the duration of field test often equals a standard time period: a month, a quarter, a year. In these cases we can not plan the test to ensure required risks α and β beforehand, so the inspection by means of confidence limits is very suitable.

After obtaining all possible data x we determine the confidence limits $Q^*(x,\gamma_1)$ and $Q_*(x,\gamma_2)$ to satisfy one of the following conditions:

$$Q^*(x,\gamma_1) > Q_0,\ Q_*(x,\gamma_2) = Q_1; \tag{6a}$$
$$Q_*(x,\gamma_2) < Q_1,\ Q^*(x,\gamma_1) = Q_0. \tag{6b}$$

It can be attained by choosing appropriate values of γ_1 and γ_2 with some predetermined relationship between them (it may be recommended $\gamma_1 = \gamma_2$).

In the case of (6a) we make a decision about acceptance with the consumer's risk $\widehat{\beta} = 1-\gamma_2$. In the case of (6b) we make a decision about rejection with the producer's risk $\widehat{\alpha} = 1-\gamma_1$.

Example 2

Consider the field test of an item. We check its MTBF and the acceptance and rejection levels of MTBF are T_0 and $T_1 = 0.5T_0$ respectively. The test duration is limited and equals $t = 4T_0$. Let the distribution of time between failures be exponential. In this case we can not guarantee the planned risks α and β less than 0.2. These risks satisfy neither producer nor consumer. However, the test was carried out and its data was fixed.

The confidence limits for MTBF are [4]

$$T_* = t/\Delta_{1-\gamma_2}(r),\ T^* = t/\Delta_{\gamma_1}(r-1),$$

where $t = 4T_0$ is duration of the test, r is the number of failures during this time and $\Delta_\gamma(n)$ is the percentile of the Poisson distribution, i.e. the root of the equation $\Pr(n,\Delta_\gamma(n)) = \gamma$.

Choosing γ_1 and γ_2 to satisfy (6a) or (6b), we obtain the results presented in Table 2.

The maximum values of observed risks $\widehat{\alpha}$ and $\widehat{\beta}$ equal 0.2, but they correspond only with $r = 5$ and $r = 6$. For other test results, the observed risks are less than 0.2. Therefore if, for example, the number of failures $r = 2$, then the item will be accepted with observed risk $\widehat{\beta} = 0.015$ and the consumer will not be afraid that his risk is too great.

TABLE 2
Decisions and Risks for Example 2

Number of failures	Decision	Observed risk: $\hat{\alpha}$ for acceptance, $\hat{\beta}$ for rejection
0	acceptance	< 0.001
1		0.005
2		0.015
3		0.05
4		0.10
5		0.20
6	rejection	0.20
7		0.13
8		0.05
…		…

6. CONCLUSION

study

It seems to the authors that the suggested approach solves the following problems:

- The problem of experiment's error estimation in statistical inspection is solved in a natural way: after the test completion with taking its results into account. This decision deserves to be included in textbooks, handbooks, standards etc. in order to complete the traditional approach using only planned risks.

- The long-standing question how confidence limits can be used in statistical inspection (i.e. about the connection between determination and check test) is solved for the case of two-level quality or reliability inspection.

The suggested approach allows:

- To determine the observed risks and to make more precise and realistic decision making.

- To check quality and reliability directly using confidence limits of the checked index itself and not using indirect measures connected with this index (number of failures, defects etc.). It enables to check complex indices such as availability and efficiency ratios.

- To introduce into the results of check test some quantitative appraisal of quality, for example to divide accepted items into quality levels according to the values of observed risks fixed, when the appropriate lots were tested.

- In spite of absence of preliminary test planning it is possible to make decision based on all obtained statistical data and indicating the observed risks.

REFERENCES

[1] GOST (State All-Union Standard) 27.410–87. Industrial product dependability. Inspection methods of reliability indices and plans of check tests on reliability (in Russian).

[2] Dzirkal, E.V. "Statistical monitoring with the use of confidence limits in the case of a fixed number of observations", Eng. Cybern., vol. 20, 1982, pp. 61–65.

[3] Netes, V.A. "Risk observed in statistical inspection", Dependability and Quality Control, 1991, No 10, pp. 26–32 (in Russian).

[4] Handbook of Reliability Engineering. Ed. I.A. Ushakov. John Wiley & Sons, Inc., 1994.

[5] Cox, D.R., and D.V. Hinkley, Theoretical Statistics, Chapman & Hall, 1974.

[6] Pavlov, I.V. Statistical Methods of Estimation of Reliability of Complex Systems Based on Experimental Results, Radio i Svyaz Press, 1982 (in Russian).