REDUNDANCY AND RENEWAL OF SERVERS IN OPENED QUEUING NETWORKS

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An opened queuing network with a redundancy and a renewal of servers is considered. To calculate a stationary regime in this network directly it is necessary to consider sufficiently complicated and large system of linear algebraic equations. A special control of the network route matrix allows to use product theorems and so to simplify these calculations significantly.

At first the network with a variable set of servers is analyzed. Then we suppose that an each server of the network may fail and be restored independently. At last we consider different variants of servers redundancy in the opened network. Considered network models are on a joint of queuing and reliability theories.

Opened network with variable set of servers

Consider opened queuing network G [1, §2] with Poisson input flow with the intensity λ and l nodes. Each node contains single server with exponentially distributed service time with intensity μ_k, $k = 1,...,l$. Input flow comes into the network from the node 0 (internal source) and customers depart from the network to this node too.

Denote by S the set of all subsets (with ordered by values elements) of the set $\{1,...,l\}$. Fix $s \in S$ and suppose that a motion of customers may be only between the nodes of the subset $s \bigcup \{0\}$ and customers do not depart from other nodes and do not arrive in these nodes. A customer motion in s-th state of the network is defined by the indivisible route matrix $\Theta(s) = \left\|\theta_{ij}(s)\right\|_{i,j \in s \bigcup \{0\}}$:

$$\forall i, j \in s \bigcup \{0\} \;\; \exists\; i_1,...,i_n \in s \bigcup \{0\} : \theta_{ii_1} > 0, \theta_{i_1 i_2} > 0, ..., \theta_{i_n j} > 0 .$$

Suppose that for fixed $\lambda_1,...,\lambda_l$, $0 < \lambda_i < \mu_i$, $1 \le i \le l$, the route matrices $\Theta(s)$ satisfy the conditions

$$\left(\lambda, \lambda_i, i \in s\right) = \left(\lambda, \lambda_i, i \in s\right)\Theta(s), \; s \in S, \; s \ne \varnothing . \tag{1}$$

For $s = \varnothing$ customers do not arrive into the network, do not depart it, do not move between internal nodes of the network and are not served in them.

Fix $A(s) > 0$, $s \in S$, $\sum_{s \in S} A(s) = 1$, suppose that the matrix $\left\|\nu\left(s, s^*\right)\right\|_{s, s^* \in S}$ is indivisible and

$$A(s) \sum_{s^* \in S} \nu\left(s, s^*\right) = \sum_{s^* \in S} A\left(s^*\right) \nu\left(s^*, s\right).$$

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Denote $Y=\{\boldsymbol{n}=(n_1,\ldots,n_l):n_1\ge 0,\ldots,n_l\ge 0\}$ and designate $\boldsymbol{e}_j$ - l-dimensional vector in which j-th component equals 1 and all others equal zero.

The network G with so controlled set of nodes is defined by Markov process $x(t)=(s(t),y(t))$ (*s(t)* characterizes a set of working nodes, *y(t)* characterizes numbers of customers in the network G nodes) with the state set $X=S\times Y$ and transition intensities ($I(A)$ is and an indicator function of an event A)

$$\Lambda\left((s,\boldsymbol{n}),(s^*,\boldsymbol{n}^*)\right)=\gamma_s\left(\boldsymbol{n},\boldsymbol{n}^*\right)I\left(s=s^*\right)+\nu\left(s,s^*\right)I\left(\boldsymbol{n}=\boldsymbol{n}^*\right).$$

Here for $s\in S$, $s\ne\varnothing$,

$$\gamma_s\left(\boldsymbol{n},\boldsymbol{n}^*\right)=\begin{cases}\lambda\theta_{0k}(s),\boldsymbol{n}^*=\boldsymbol{n}+\boldsymbol{e}_k,k\in s,\\ \min(n_k,1)\mu_k\theta_{k0}(s),\boldsymbol{n}^*=\boldsymbol{n}-\boldsymbol{e}_k,k\in s,\\ \min(n_k,1)\mu_k\theta_{kj}(s),\boldsymbol{n}^*=\boldsymbol{n}-\boldsymbol{e}_k+\boldsymbol{e}_j,k\ne j,k,j\in s,\end{cases} \quad (2)$$

and $\gamma_s\left(\boldsymbol{n},\boldsymbol{n}^*\right)\equiv 0, s=\varnothing$. The process $x(t)$ is ergodic [2, §4] and its limit distribution has the form

$$\prod(s,\boldsymbol{n})=A(s)\pi(\boldsymbol{n}),(s,\boldsymbol{n})\in X, \quad (3)$$

$$\pi(\boldsymbol{n})=C^{-1}\prod_{i=1}^{l}\left(\frac{\lambda_i}{\mu_i}\right)^{n_i},\quad \boldsymbol{n}\in Y,\quad C=\prod_{i=1}^{l}\left(\frac{\mu_i}{\mu_i-\lambda_i}\right).$$

Remark 1. *If* $\lambda_i=\lambda<\mu_i$, $1\le i\le l$, *then arbitrary symmetric route matrices* $\Theta(s),s\in S$, *satisfy the conditions (1).*

As an example consider an opened queuing network with two one-server nodes and a route matrix Θ which has zero diagonal elements and ½ no diagonal elements. Here an input intensity is λ and service intensities are μ_1, μ_2. If both servers work then s={1,2} and this network is described by figure 1.

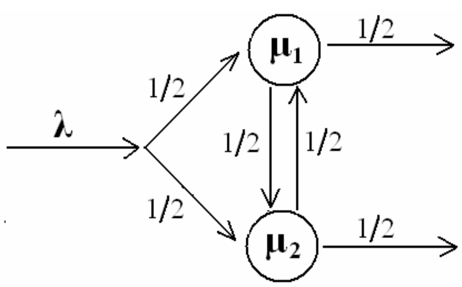

Figure 1. The queuing network with s={1,2}.

Suppose now that the first server works and the second server does not work. Then the initial two nodes network transforms into an one node network with s={1} and with θ_{ij}=1 as for i=0, j=1 so for i=1, j=0, in all other cases θ_{ij}=0.

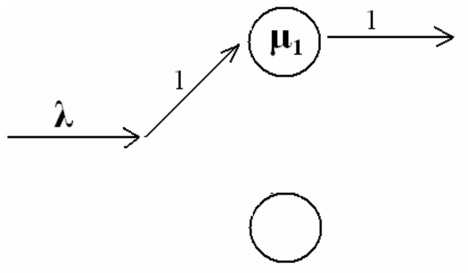

Figure 2. The queuing network with s={1}.

Analogously it is possible to consider the case when s={2} and the second server works but the first server does not work.

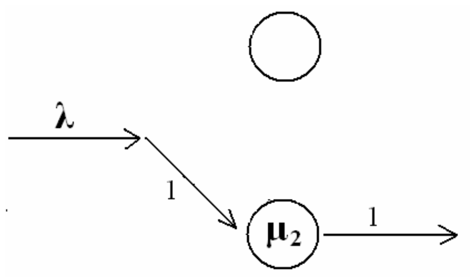

Figure 3. Queuing network with s={2}.

Last case is when both servers does not work and so s= Ø.

Figure 4. Queuing network with s= Ø.

Now define failures and renewals of servers in considered networks as follows .

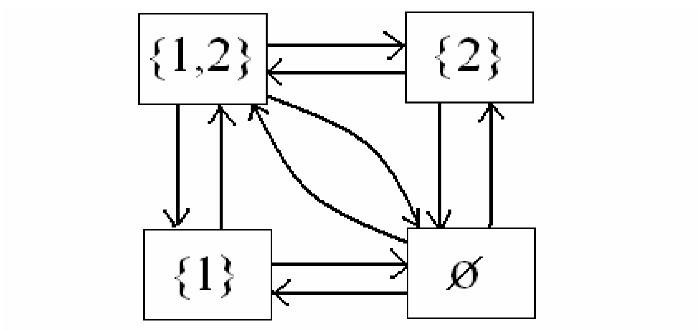

Figure 5. Transitions between queuing networks with different structures.

Here transition intensities in the figure 5 are defined by the matrix $\left\| \nu\left(s, s^*\right) \right\|_{s,s^* \in S}$.

The formula (3) gives limit distribution in the queuing network with so variable structure and with $\lambda_1 = \lambda_2 = \lambda$ in the formula (1). This network characterizes failures and renewals of servers in the initial two node opened queuing network. Figures 1-5 show that even in this simple case a description of the opened queuing network with failures and renewals of servers is sufficiently complicated. To consider a redundancy of servers it is necessary to analyze significantly more complicated scheme. Next section is devoted to an analysis of this scheme.

Opened queuing network with failures and renewals of servers.

Consider closed queuing network $\tilde{G}_k$ with a single working place, a single repair place and m_k customers. The customers move along a route: a working part – a repair part and so on. The working (repair) part consists of a queue before the working (repair) place and of the working (repair) place. Each customer fails with an intensity α_k at the working place and is repaired with an intensity β_k at the repair place. So a service time at the working (at the repair) place may be interpreted as a working (repair) time of a customer and the closed queuing network $\tilde{G}_k$ may be considered as a system of a redundancy (if $m_k > 1$) and a renewal.

Describe a current number of customers at the working part of the network $\tilde{G}_k$ by ergodic discrete Markov process $\tilde{y}_k(t)$ with state set $\tilde{Y}_k = \{\tilde{n}_k : 0 \le \tilde{n}_k \le m_k\}$ and transition intensities

$$\tilde{\gamma}_k(\tilde{n}_k, \tilde{n}_k - 1) = \alpha_k \min(1, \tilde{n}_k),\ \tilde{\gamma}_k(\tilde{n}_k, \tilde{n}_k + 1) = \beta_k \min(1, m_k - \tilde{n}_k),\ \ \tilde{n}_k \in \tilde{Y}_k,$$

and limit distribution

$$P_k(\tilde{n}_k) = C_k \left(\frac{\beta_k}{\alpha_k}\right)^{\tilde{n}_k},\ C_k^{-1} = \sum_{\tilde{n}_k=0}^{m_k} \left(\frac{\beta_k}{\alpha_k}\right)^{\tilde{n}_k}.$$

Remark 2. *The transition intensities $\tilde{\gamma}_k$ describe a system of an unloaded redundancy and a renewal. For a system of a loaded redundancy and a renewal [3] the process $\tilde{y}_k(t)$ has transition intensities*

$$\tilde{\gamma}_k(\tilde{n}_k, \tilde{n}_k - 1) = \alpha_k \tilde{n}_k,\ \ \tilde{\gamma}_k(\tilde{n}_k, \tilde{n}_k + 1) = \beta_k (m_k - \tilde{n}_k),\ 0 \le \tilde{n}_k \le m_k,$$

and limit distribution

$$P_k(\tilde{n}_k) = C_k \left(\frac{\beta_k}{\alpha_k}\right)^{\tilde{n}_k} \frac{1}{\tilde{n}_k!},\ C_k^{-1} = \sum_{\tilde{n}_k=0}^{m_k} \left(\frac{\beta_k}{\alpha_k}\right)^{\tilde{n}_k} \frac{1}{\tilde{n}_k!}.$$

Consider l networks $\tilde{G}_k$, $1 \le k \le l$, working independently, and describe them by Markov process $\tilde{y}(t)$ with state set $\tilde{Y} = \{\tilde{\boldsymbol{n}} = (\tilde{n}_1, ..., \tilde{n}_l) : \tilde{n}_k \in \tilde{Y}_k, 1 \le k \le l\}$ and transition intensities

$$\tilde{\gamma}(\tilde{\boldsymbol{n}}, \tilde{\boldsymbol{n}}^*) = \sum_{k=1}^{l} \tilde{\gamma}_k(\tilde{n}_k, \tilde{n}_k^*), \tag{4}$$

and limit distribution

$$P(\tilde{\boldsymbol{n}}) = \prod_{k=1}^{l} P_k(\tilde{n}_k). \tag{5}$$

Suppose that a server in k–th node of the network G may fail and be restored as a customer in the closed network $\tilde{G}_k$, $1 \le k \le l$. Then opened queuing network with a separate redundancy and a renewal of servers in each node may be described by discrete Markov process $(\tilde{y}(t), y(t))$ with state set $\tilde{Y} \times Y$ and transition intensities

$$\Lambda\left((\tilde{\boldsymbol{n}},\boldsymbol{n}),(\tilde{\boldsymbol{n}}^*,\boldsymbol{n}^*)\right)=\gamma_{s(\tilde{\boldsymbol{n}})}\left(\boldsymbol{n},\boldsymbol{n}^*\right)I\left(\tilde{\boldsymbol{n}}=\tilde{\boldsymbol{n}}^*\right)+\tilde{\gamma}\left(\tilde{\boldsymbol{n}},\tilde{\boldsymbol{n}}^*\right)I\left(\boldsymbol{n}=\boldsymbol{n}^*\right). \quad (6)$$

Here $\tilde{\gamma}\left(\tilde{\boldsymbol{n}},\tilde{\boldsymbol{n}}^*\right)$ are defined by the formulas (4), and $\gamma_{s(\tilde{\boldsymbol{n}})}\left(\boldsymbol{n},\boldsymbol{n}^*\right)$- by the formulas (2) with

$$s(\tilde{\boldsymbol{n}})=\{k:\tilde{n}_k>0, 1\le k\le l\}.$$

(7)

If the conditions (1) are true then the process $(\tilde{y}(t), y(t))$ is ergodic and analogously to (3) its limit distribution has the form $P(\tilde{\boldsymbol{n}})\pi(\boldsymbol{n})$, $(\tilde{\boldsymbol{n}},\boldsymbol{n})\in\tilde{Y}\times Y$.

Remark 3. *The formulas (4), (5) describe independent and separate closed networks* $\tilde{G}_k$*,* $1\le k\le l$*, of a redundancy and a renewal of the network* G *servers. But these networks may be aggregated into common closed queuing network* $\tilde{G}$ *with* l *working nodes (places), fixed number of repair nodes (places) and arbitrary indivisible route matrix.*

Suppose that numbers of customers in all nodes (not only working) of so defined network $\tilde{G}$ *is described by some ergodic Markov process* $\tilde{y}(t)$ *with state set* $\tilde{Y}$*, transition intensities* $\tilde{\gamma}\left(\tilde{\boldsymbol{n}},\tilde{\boldsymbol{n}}^*\right)$ *and limit distribution* $P(\tilde{\boldsymbol{n}})$*. Then Markov process* $(\tilde{y}(t), y(t))$ *with transition intensities defined by (6), (7) has limit distribution* $P(\tilde{\boldsymbol{n}})\pi(\boldsymbol{n})$*,* $(\tilde{\boldsymbol{n}},\boldsymbol{n})\in\tilde{Y}\times Y$*.*

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