ON RELIABILITY OF RENEWAL SYSTEMS WITH FAST REPAIR

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There are discussed the restorable systems with redundancy under condition of fast restoration. It is shown when it is possible to evaluate the indices of their failure-free operation and maintainability with the use of a method of monotonic trajectories. There are obtained the general estimations of reliability indices of systems in steady-state operation and non-stationary regime of their use. An example of using these general estimations for the reliability assessment of a concrete system is adduced.

study

1. Introduction

This article is a generalization of researches fulfilled by author in the area of reliability assessment of asymptotic methods in the last twenty five years. The research results were mostly described in [1-5].

The first works on the asymptotic estimation of the indices of the reliability of the duplicated systems belong to B. Gnedenko [6-7]. Then it appeared an entire series of the works of I. Kovalenko [14-18] and A. Solovyev [9-12], dedicated to the asymptotic analysis of the repairable systems of the practically arbitrary structure. This direction proved to be completely fruitful and it led to the formation of the independent branch of the theory of reliability. Subsequently interesting results in this direction were obtained by V. Koroluk and A. Turbin [19-20].

In parallel with the development of the analytical methods of the investigating of the reliability of renewal systems as long ago as in the 60's appeared the first works of I. Ushakov [21-22], on the heuristic methods of calculation of the indices of the reliability of systems with the restoration. In this approach constructively was used the A. Renyi limit theorem [24] on the rarefaction of a random flow.

Obtaining the two-sided being converging to each other of the estimations of the indices of reliability made it possible to determine the error in the asymptotic estimations. The first works in this direction were executed by A. Solovyev [11-12] for the distribution time to the first failure of system in steady-state operation of its use. Refinement of these estimations is given by V. Kalashnikov and S. Vsekhsvyatskiy [8], that with an accuracy to the second order of smallness coincided with results previously obtained in [5] for the broader class of systems, allowing the steady-state condition of work.

The problem of evaluating the system reliability indices under the wide assumptions related to the regime of maintenance and operation, the criteria of failure, the nature of functions performed by system, the types of used redundancy, and the distribution time of the elements' failure-free operation were obtained in [1-5].

The idea of analysis of highly reliable systems consists in the following. The process of the functioning of any repairable system can be reduced to the alternating random process, in which intervals, where all elements of system are operational (IO), alternate by intervals, when in the system are any failures of elements, possibly not leading to the system failure. Last intervals in complete agreement with the conventional terminology can be named the intervals of the malfunction (IM). It is understandable that the failure of the system can arise only on IM. If the probability of the failure of the system on IM is small it is possible to use asymptotic methods. Indeed, if on each cycle (a cycle consists from two sequential intervals of operation IO and malfunction IM) the probability of the appearance of a system failure to converge to zero (which leads to an increase in the number of cycles to the system failure and between the system failures), the distribution of the number of cycles before the appearance of a cycle with the system failure will be geometric, and the distribution of the sum of the geometric number of random variables with an increase in the number of terms (with the corresponding normalization) will approach exponential distribution.

Let us examine first the case, when IM on the average is much less than IO. In this case with a high degree of accuracy the considered alternating process can be replaced with the simple restoration process with the generating it distribution function (DF) by the corresponding DF of IO. In this case the malfunction of system can with a certain small probability become the failure of system and with the probability close to one to pass for system unnoticed. In other words, the typical picture of the rarefaction of random process is observed. The application of this procedure of rarefaction to the arbitrary flow of restoration with the tendency of the probability of rarefaction toward one according to the theorem of Renyi [24] leads after the corresponding normalization to the Poisson flow. If the probability of rarefaction is different for different cycles, then as was shown by Yu. Belyaev [23] the result of Renyi [24] remains valid provided all probabilities of rarefaction are converged to zero evenly.

If we now pass to the examination of the alternating process, in which the duration of IM cannot be disregarded in comparison with the duration of IO, but the probability of the appearance of a system failure on each IM is small, then it is possible to arrive at the estimations, which practically coincide with those, that are obtained for the rarefied point flow.

Let us examine now basic and additional random processes in one probability space. In the basic process, describing the behavior of the system, the appearance of the new IM is possibly only after the end of the previous. In the additional process it is allowed the appearance of IM at any moment of time. Each IM in the basic and additional processes can be failure or no failure IM. Since in the additional process for the same amount of time appears not smaller number of IM than in the basic, then lower limit time to the first system failure is the time before the first appearance in the additional process a failure IM, and by upper limit is the time before the appearance in the additional process of the first failure IM, not crossed with any no failure IM (number of which is not less, than in the basic process), arisen before it, plus the part of the duration of this IM before the appearance on it an event – system failure. With satisfaction of a certain condition for fast restoration both these estimations begin to give increasingly smaller relative error, converging with each other.

Let us note now when the obtained asymptotic estimations of the indices of reliability can be used in practice, and in what cases they give serious errors. Asymptotic estimations give a good approximation in such cases when system is highly reliable in the sense, that the failure of system and any malfunction of it are developed along the monotonic trajectory. By monotonic trajectory is understood a trajectory, in which in the process of the appearance of the system failure does not occur the alternations of failures and restorations of separate elements, that is the chain of failures first appears, and then consecutively conclude only the restorations of elements. Practically this corresponds to a condition, when the product of the maximum mean recovery time of elements to the summary of elements failure rate in the system is small.

Furthermore, these estimations are valid, when the examined intervals of the system failure-free time of the reliable work of system are substantially larger then length of cycle “up state – down state”, and they lead to the errors when used in the initial section of the work of highly reliable system.

In practice frequently the asymptotic results are used for the highly reliable systems generally, without specifying, in what sense this high reliability is understood. Here one should note that in the highly reliable systems with the high degree of redundancy IM may represent the sequence of the no monotonically developing trajectories of the appearance of the malfunction of system. In this case the use of asymptotic formulas of the above-indicated type can lead to the serious errors. And in this case the flow of the moments of the appearance of the system failures forms the flow of the intersections of “high level”, that is it will be approached by Poisson flow. But the parameter of this flow should be calculated by another method, and not with the use of a method of the monotonic trajectories.

Let us note that the asymptotic method is constructive and gives extremely good results for the majority of practical situations. It was successfully used for the reliability design of complex computing systems in nuclear and thermal power engineering, chemistry, petroleum chemistry, metallurgy and in other branches.

study

2. Statement of the problem

There is examined a restorable system, containing n elements and k repair units (RU). Each element of system can be only in the operational or the failed state. Each operational element can be located in the loaded or the unloaded regime (the lightened or the reduced regime for the brevity is omitted). Let $F_i(x)$ and $f_i(x)$ denote the DF and the distribution density (DD) of the time of failure-free operation of i-th element in the system, $i = \overline{1,n}$, and m_i is the the mean value of this time, $m_i < \infty$.

The following types of systems with the arbitrary DF of the elements’ restoration time are examined:

1) systems with exponential DF of time to failure for each unit where the system steady-state operation region exists;
2) systems, working in the nonstationary regime with the variable operating conditions, the time of the failure-free operation of elements of which are described by nonstationary Poisson process, and the distribution of the element time restoration can depend from the moment of its failure;
3) systems, all elements of which have limited DD of the time of failure-free operation; it is required also, that this DD in zero would be different from zero, $f_i(0) = c_i \neq 0, i \in \overline{1,n}$.

Within the framework of this article it is assumed, that with probability 1 the failure of any element in the system is instantly revealed and the switching to the reserve element (in the systems with the standby redundancy) is produced without a delay. Failed elements are restoring.

The permissible class of the restoration disciplines D is assigned by the following regime of restoration. For each element there is at least one RU, capable to restore it. After restoration the element behaves as a new one. The restoration of element begins immediately, if there is a free RU, capable to restore it. The summary recovery rate of any RU in the interval of the works of this RU is equal to 1 (recovery rate of RU corresponds to time scale, with which this RU produces the restoration) Kozlov, Solovyev [13]. Different interruptions of restoration are permitted, but DF of summary recovery time of *i*-th element that is restored by *j*-th RU is equal to $G_{ij}(x)$ independently of the number and the duration of the interruptions [3].

Let l is the number of failed elements in the system at the moment z. Then class D includs, in particular, disciplines:

- d_1 – discipline FIFO with the straight order of maintenance, with which at the moment z are restoring *min (k, l)* elements, failed the first;
- d_2 - discipline LIFO with the reverse order of maintenance, with which at the moment z are restoring *min (k, l)* elements, failed the last;
- d_3 – discipline in time sharing, with which "simultaneously" are restoring all failed elements with the identical speed, the recovery rate of one element at the moment z is equal one, if $l \le k$, and equal k / l, if $l > k$;
- d_4 – discipline, with which at the moment z are restoring min *(k, l)* elements with the shortest residual recovery time.

There are no limitations to the structure of the system. It is assigned the criterion of the system failure, which can include and the condition of the time redundancy. The state of the elements of the system at the moment z let us assign by the vector $\vec{v}(z) = \{v_1(z),...,v_n(z)\}$, where each component can take the values of {0, 1,…, n}. In this case to failed elements is placed in the correspondence number 0, operational - number from 1 to n. These numbers make it possible to unambiguously assign the order of the replacement of any failed element. Introduced designation $\vec{v}(z)$ may be used for the estimations of the reliability of concrete systems for describing the set of their states, as this is done in the last section of the article.

Let us designate through E the set of the states of system and let us present it in the form $\{\vec{v}(z)\} = E = E_+ \cup E_-$, where E_+ is the area of the operational, and E_- is the area of the defective states of system. System is considered as the defective at the moment z, if $\vec{v}(z) \in E_-$ and failed if its malfunction lasts time, not smaller than $\eta, P\{\eta < x\} = H(x)$. In the absence of the time redundancy ($\eta \equiv 0$) the area of the defective states of system is converted into the area of the system failures. All elements of the system were new at the initial moment of time.

Let $\vec{b}$ is a certain state vector of elements of the system directly before IM, and $\vec{b}^N$ is a state vector of the elements of system on the same IM immediately after the moment of passing the state vector of system from the area E_+ into area E_-. Let us name the path π, leading from the $\vec{b} \in E_+$ in the state $\vec{b}^N \in E_-$ on IM, the sequence of state vectors of elements, beginning from the vector $\vec{b}$, directly preceding the beginning of IM and ending with the vector $\vec{b}^N$, corresponding to the first onset of malfunction of system on this IM; moreover the passage of one state vector into the following occurs only due to that, that exactly one element of the system fails or ends to be restored.

The path length is equal to the number of state vectors, being contained on this path, not including the initial vector $\vec{b}$. Let us name the path monotonic, if on it there are no restorations of elements. Let us name monotonic path minimum for $\vec{b}$, if its path $l(\vec{b})$ is equal to the minimum of path lengths, leading from $\vec{b}$ into E_-. Then the minimum number of elements, failure of which can cause the malfunction of system, equals $s = \min l(\vec{b})$ on $\vec{b} \in E_+$, $b_i > 0$, $i = \overline{1,n}$. There b_i is the *i*-th coordinate of the vector $\vec{b}$, which describes the state of *i*-th element.

It is counted, that the system works under conditions of fast restoration (FR). Practically it means that the mean recovery time of element is substantially lower than the mean time between any two failures of elements in the system [1].

The task is to estimate the indices of failure-free operation and maintainability of a system under the conditions of FR.

3. Asymptotic approximation. General model of the system

As shown higher behavior of system is described by the alternating random process, in which alternate IO and IM. System may fail on a certain IM more than one time. The behavior of this system was analyzed in [1].

Let us name a *x*- failure of system such a failure of system, with which the system is in the failure state not less than the time *x*. The concept of *x*- failure will be used for the estimation of the indices of the maintainability of system. Let us say that the system failed (x- failed) on IM along the monotonic way, if from the beginning of IM and to the moment of the failure (x- failure) of system on this IM it had time to end the restoration not of one element.

We will examine highly reliable systems. Results of [1] show, that in this case for the systems of the 1st and 3d type the distribution of the time to the first system failure converges to an exponential function, and for system of 2nd type it converges to $exp\{-\int_0^x \beta(u)du\}$, if the product of the maximum rate of the appearance of intervals of malfunction $\hat{\lambda}$ to the maximum average duration of the interval of malfunction *T* and the probability of the failure of system *q* on IM approach to zero. If in this case and the probability q^* of appearance of more than one failure of system on IM approaches to zero, the DF of time between any two consecutive system failures of the 1st and 3d type converges to exponential function, and the failure rate of 2nd type systems converges to Poisson flow with the variable parameter. Criterion of FR determined below provides conditions, with which $\hat{\lambda}\, T \to 0$, $q \to 0$ и $q^* \to 0$.

4. Refined system model and FR criterion

Let $G(x) = min\ G_{ij}(x)$, $G_*(x) = max\ G_{ij}(x)$, where minimum and maximum are taken according to the numbers *j* of RU, accessible to *i* - th element, and on $i = \overline{1,n}$ (here $G(x)$ and $G_*(x)$ are DF of the correspondingly greatest and shortest recovery time of elements); *s* is the minimum number of elements, failure of which can cause the malfunction of system; $\overline{\Gamma}(\,) = 1 - \Gamma(\,)$ for every DF $\Gamma(\,)$;

$$m_R^{(j)} = j\int_0^\infty x^{j-1}\overline{G}(x)dx,, \qquad m_R = m_R^{(1)}, \qquad m_{R^*}(\eta) = \int_0^\infty\int_0^\infty \overline{G}_*(x+u)\,dx\,dH(u);$$

$\hat{\lambda}$ and $\underline{\lambda}$ are the maximum and minimum failure rates of elements in the operational system.

Let us say, that in the system is satisfied the condition of FR, if $\underline{\lambda} > 0$ and

$$\alpha = [\hat{\lambda}^s \, m_R^{(s)} \,/\, (m_R)^{s-1}] \to 0 \qquad (4.1)$$

and for all DF $F_i(x)$, $i = \overline{1, n}$, there are limited DD.

This condition ensures, that $\hat{\lambda}\, T \to 0$, $q \to 0$ and $q^* \to 0$, and thus it ensures, that DF of the time to the first failure and the time between two adjacent failures of system converges to the exponential function [2].

During the fast restoration almost always the failure of system occurs along the monotonic ways (see section 2), if only the probability of this failure is different from zero. This indicates that the ratio of the probability of the system failure along the no monotonic trajectories to the probability of the failure of system is approached zero.

The following condition unites the condition for the fast restoration (4.1) and the sufficient condition of that, that the probability of the failure of system in the monotonic path is different from zero:

$$\varphi_1 = [\hat{\lambda}^s \, m_R^{(s)} / \underline{\lambda}^{s-1} [m_{R*}(\eta)]^{s-1}] \to 0, \ \underline{\lambda} > 0 \qquad (4.2)$$

Condition

$$\varphi_2 = \hat{\lambda} m_R \to 0 \qquad (4.3)$$

ensures the convergence of DF of time to the first system failure of the 1st and 3d type to the exponential, and the 2nd type to $exp\{-\int_0^x \beta(u)du\}$, that is shown in [5].

In the practically important cases $m_R^{(s)} \le C\,(m_R)^s$, where C is a certain constant. In these cases with small s ($s \approx 2 \div 4$), close to each other $G(x)$ и $G_*(x)$, which is reached due to the unification of the procedure of restoration, and the low time reserve ($m_R \approx m_R(\eta)$) condition (4.2) can be replaced with condition (4.1) or condition (4.3).

5. Estimation of the indices of reliability for the general model of the system

Let $\tau_i(t)$ is the interval from the moment t to the first failure of the system after moment t with the condition A_t, where with the examination of $\tau_1(t)$ the event A_t = {(all elements of system are operational at the moment t) $\cap D_t$}, and with the examination of $\tau_i(t), i \ge 2$, the event A_t = {(in the interval (t, $t+dt$) occurred (j - 1) - th system failure) $\cap D_t$}, where D_t = {the state of the elements of system at the moment t and used up to the moment t the recovery times of the elements}. Let $B(t, z)$ = {A_t $\cap$ (in the interval (t, z) system did not fail (without taking into account the (i -1) - th failure), $t \le z$}. It is assumed, that in $t = 0$ all elements of system are new and the system is set in operation.

Let IM z indicates IM that begins in the interval (*z, z + dz*); index (*z*) in the designation $u^{(z)}$ indicates, that the value *u* relates to the IM *z*.

Let us examine the systems of the 1st and 2nd type. Let $\lambda(z)$ is the failure rate of elements in the system with $z \in$ IO, $q^{(z)}$ is the probability of the failure of system on IM z, $\lambda = \sup \lambda(z)$ and $q = \sup q(z)$ on $z \geq t$, $\beta(z) = \lambda(z) q^{(z)}$ is the rate of the appearance of failure IM, T_1 is the average duration of non failure IM, T_2 is the average duration from the beginning of IM to the moment of the first failure of system on this IM, T_3 is the average duration from the moment of the first failure on the IM to the end of this IM, $T = T_1 + T_2 + T_3$.

Then using an approach, described in section 1, we will obtain

<u>Theorem 5.1</u>. For the models of systems in question there are valid the estimations

$$exp\{-\int_t^{t+x} \beta(u) du\} \leq P\{\tau_1(t) \geq x\} \leq exp\{-\int_t^{t+x} \beta(u) e^{-\lambda T_1} du\} + \lambda T_2. \qquad (5.1)$$

and for $i \geq 2$

$$(1-q_*) exp\{-\int_t^{t+x} \beta(u) du\} \leq P\{\tau_i(t) \geq x\} \leq exp\{-\int_t^{t+x} \beta(u) e^{-\lambda T_1} du\} + \lambda T_2 + \lambda T_3. \quad (5.2)$$

The lower estimations of theorem 5.1 are sufficiently precise and intelligible. However, the upper estimations in certain cases can be refined. In [4-5] there is shown that it is carried out

<u>Theorem 5.2</u>. With $\lambda q \to 0$ together with the estimations of theorem 5.1 the following upper estimations are valid

$$P\{\tau_1(t) \geq x\} \leq exp\{-\int_t^{t+x} \beta(u) e^{-\lambda T_1} du\}(1+\lambda T_2), \qquad (5.3)$$

and for $i \geq 2$

$$P\{\tau_i(t) \geq x\} \leq exp\{-\int_t^{t+x} \beta(u) e^{-\lambda T_1} du\}(1+\lambda T_2)(1+\lambda T_3). \qquad (5.4)$$

It is also shown in [4-5] that лower estimations (5.1) and (5.2) and upper estimations (5.3) and (5.4) are asymptotically not improved.

<u>Corollary 5.1</u>. With $\lambda T \to 0$

$$P\{\tau_1(t) \geq x\} \approx exp\{-\int_t^{t+x} \beta(u) du\}. \qquad (5.5)$$

If in this case and $q_* \to 0$, that for $i \geq 2$

$$P\{\tau_i(t) \geq x\} \approx exp\{-\int_t^{t+x} \beta(u)\, du\} \qquad (5.6)$$

In [4] is proven the validity of the following lemmas:
Lemma 5.1. With $s \geq 2$ from $[\hat{\lambda}^s\, m_R^{(s)}\, /\, (m_R)^{s-1}] \to 0$ it follows that $q \to 0$.
Lemma 5.2. From $\hat{\lambda}\, m_R \to 0$ it follows that $\lambda T_1 \to 0$ and $\lambda T_2 \to 0$, and from $[\hat{\lambda}^s\, m_R^{(s)}\, /\, (m_R)^{s-1}] \to 0$ additionally it follows that $\lambda T_3 \to 0$.
Lemma 5.3. From $[\hat{\lambda}^s\, m_R^{(s)}\, /\, (m_R)^{s-1}] \to 0$ it follows that $q_* \to 0$.

Thus it is proven
Theorem 5.3. For the system of the 1st and the 2nd type with $\hat{\lambda}\, m_R \to 0$ for $i = 1$ and with $[\hat{\lambda}^s\, m_R^{(s)}\, /\, (m_R)^{s-1}] \to 0$ for $i > 1$

$$P\{\tau_i(t) \geq x\} \approx \exp\{-\int_t^{t+x} \beta(u) du\}, \qquad (5.7)$$

where $\beta(u)$ is the rate of the appearance of the failure IM in the system at the moment u.

Let us examine systems of the 3d type. As a rule, the parameters, that describe the behavior of systems of the 3d type are determined with the condition *B(t,z)*. For the simplification of record instead of *B(t,z)* we will write *t*, understanding that instead of *t* in this place must be *B(t,z)*.

Under the conditions of fast restoration the summary failure rate of the elements is limited from the top and is not equal to zero ($\hat{\lambda} < \infty$ и $\underline{\lambda} > 0$). Relying on the general results of Renyi [25] and Yu. Belyaev [24], it is possible to make a heuristic assumption, that estimation, analogous (5.7) are accurate for the systems of the 3d type.

Under the conditions for the fast restoration of those given in theorem 5.3 for the systems of the 3d type with $i \geq 1$

$$P\{\tau_i(t) \geq x\} \approx \exp\{-\int_t^{t+x} \beta(u \mid t) du\}, \qquad (5.8)$$

where the value $\beta(u \mid t) \equiv \beta(u \mid B(t,u))$ is determined with the condition $B(t,u)$ and it depends not only from u but also from *t*. Validity of (5.8) for the systems with the uniform elements and the loaded reserve is shown by A. Solovyev [10].

Let $p_{\vec{b}}(z \mid t) \equiv p_{\vec{b}}(z \mid B(t,z))$ is the probability of that, that $\vec{v}(z) = \vec{b} = (b_1, ..., b_n)$ with condition $B(t,z)$ and $z \in$ IO, $\lambda(z \mid \vec{b}, t) \equiv \lambda(z \mid \vec{b}, B(t,z))$ is the summary failure rate of elements in the system at the moment *z* with the condition, that $\vec{v}(z) = \vec{b}, B(t,z)$ and $z \in$ IO, $q^{(z|\vec{b},t)} \equiv q^{(z|\vec{b},B(t,z))}$ is the probability of the system failure on IM *z* with the condition $B(t,z)$ and $v(\vec{z}) = \vec{b}$. If $z \in$ IO, then according to the formula of the total probability

$$\beta(z\,|\,t) = \sum_{\{\vec{b}:\vec{v}(z)=\vec{b}\}} p_{\vec{b}}(z\,|\,t)\lambda(z\,|\,\vec{b},t)q^{(z|\vec{b},t)}. \qquad (5.9)$$

Through $\beta(z\,|\,t)$ are expressed not only the indices of failure-free operation, but also the indices of maintainability and the readiness factor of system (see sections 6 - 9). Below will be given recommendations regarding the estimation of $\beta(z\,|\,t)$ and $\lambda(z\,|\,\vec{b},t)$ for examined systems. However, estimation of $p_{\vec{b}}(z\,|\,t)$ must be carried out for each concrete structure of system individually (see section 9).

Let $q_M^{(z|\vec{b},t)} \equiv q_M^{(z|\vec{b},B(t,z))}$ is the probability of the system failure on IM *z* only on the minimum paths for $\vec{v}(z)=\vec{b}$ of length $l=l(\vec{b})$ with the condition $B(t,z)$ and $\vec{v}(z)=\vec{b}$.

For systems of the 1st type by A. Solovyev [13] and for the systems of the 2nd type by author [5] it is shown, that with $d=d_1 \in D$ and fast restoration in the estimation $q^{(z|\vec{b},t)}$ it is possible to take into account only the minimum paths.

Let $\tau_i^{"}(t)$ is the recovery time of system after the *i* - th system failure with the condition, that the *i* - th failure of system occurred on IM *t*.

Let us examine the general model of system of the 1st and the 2nd type, for which $\lambda(t)dt$ is the probability of appearance of IM *t*; $q^{(t)}$ and $q_x^{(t)}$ are the probabilities of failure and *x*-failure of system on IM *t*; $q_M^{(t)}$ and $q_{xM}^{(t)}$ are the probabilities of failure and *x*-failure of system on IM *t* taking into account only minimum paths; $\beta_{xM}(t)=\lambda(t)q_{xM}^{(t)}$, $\beta_M(t)=\beta_{xM}(t)|_{x=0}=\lambda(t)q_M^{(t)}$ are the rates of *x*-failure and failure of system on IM *t* taking into account only the minimum paths of failure.

With the work of system under the conditions of fast restoration the probability of the failure of system converges to the probability of the failure of system along the minimum monotonic paths [2], that is next estimations are carried out:

$$\beta(t) \approx \beta_M(t) \qquad (5.10)$$

and

$$\beta_x(t) \approx \beta_{xM}(t) \qquad (5.11)$$

According to [1] it is true

<u>Theorem 5.4</u>. If $\varphi_1 = [\hat{\lambda}^s\, m_R^{(s)} / \underline{\lambda}^{s-1}[m_{R*}(\eta)]^{s-1}] \to 0$, than evenly on $i \geq 1$

$$P\{\tau_i^{"}(t) \geq x\} \approx \beta_{xM}(t)/\beta_M(t). \qquad (5.12)$$

Let $T_R^{(t)}$ is the mean recovery time of the system with the condition that it failed on IM *t*.

Corollary 5.2. With $\varphi_1 \to 0$

$$T_R^{(t)} \approx \int_0^\infty \beta_{xM}(t)dx / \beta_M(t)\,. \qquad (5.13)$$

6. Estimation of the indices of reliability for systems of the first and the third type, working in the steady-state operation

Only under the conditions when system works in steady0-state operation (this regime excludes systems of the 2nd type) there are obtained simple engineering formulas for evaluating the reliability of a system.

In steady-state operation mode $q^{(z|\vec{b},t)} = q^{(\vec{b})}$, $q_M^{(z|\vec{b},t)} = q_M^{(\vec{b})}$, $\lambda(z\,|\,\vec{b},t) = \lambda(\vec{b})$ and $\beta(z\,|\,t) = \beta$. Let us estimate β.

Let from $\vec{v}(z) = \vec{b} \in E_+$, $z \in$ IO, there are possible the minimum paths $l = l(\vec{b})$, leading into the certain state $\vec{b}^j \in E_-$. States $\vec{b}^j$ are characterized by the set l of number of failed elements belonging to the set $J = J_+ \cup J_-$, where J_+ and J_- are accordingly the sets of the numbers of those elements, which in the state $\vec{b}$ were located in the loaded and unloaded regime. Let the set of minimum paths leading from $\vec{b}$ into $\vec{b}^j$ is Π^j. Let path $\pi \in \Pi^j$ and the order of the failures of elements on this path is $\{i_1,...,i_l\}$. If $i_k \in J_-$, then on the minimum path π i_k-th element was switched on in the loaded regime after the failure of $i_{m(k)}$- th element, $m(k) < k$, $k \in \overline{2,l}$. Let $0 = x_1 < ... < x_l$ are the moments of the failures of elements on the path π, counted from the beginning of IM. Let $A(d,k,\pi,u)$ is the probability that on IM to the moment $(x_l + u)$ will not end not one restoration with the condition that the system failure occurred on this IM along the path $\pi \in \Pi^j$ with discipline $d \in D$ and k RU.

Let

$$\Lambda^j = \prod_{k \in J_-} c_k \prod_{i \in J_{+-}} 1/m_i\,. \qquad (6.1)$$

Then the following statements are true.

Theorem 6.1. In the steady-state operation of the system with the FR condition (4.2)

$$\lambda(\vec{b}) q_M^{\vec{b}} \approx \sum_{\vec{b}^j \in E_-} \Lambda^j \sum_{\pi \in \Pi^j} \int_0^\infty \int\limits_{0<x_2<...<x_l} \!\!\!...\!\int A(d,k,\pi,u)dx_2...dx_l dH(u)\,. \qquad (6.2)$$

Theorem 6.2. With the FR condition (4.2)

$$q^{(\vec{b})} \approx q_M^{(\vec{b})} \qquad (6.3)$$

Let us note, that theorem 6.2, proven for the steady-state operation of the system, has more general nature and can be disseminated also to the non-stationary regime of system work.

Let us refine (6.2) for restoration disciplines, introduced above in section 2. Let all RU are of the same type, each of them can restore any failed element and $G_{ij}(x) = G_i(x)$. Let the order of the failures of elements on the minimum path π is $\{i_{1,},...,i_l\}$. Then [1] it is true

<u>Corollary 6.1</u>. If $G_{ij}(x) = G_i(x)$, $i = \overline{1,n}$, then with the fast restoration condition (4.2)

1) with discipline d_1 and $k < l$

$$\lambda(\vec{b})q^{(\vec{b})} \approx \sum_{\vec{b}^j \in E_-} \Lambda^j \sum_{\pi \in \Pi^j} \int_0^\infty \int_{0<y_1<...<y_k} \!\!\!\!...\int \frac{y_1^{l-k-1}}{(l-k-1)!} *$$

$$*\overline{G}_{i_k}(y_1 + u)\overline{G}_{i_{k-1}}(y_2 + u)...\overline{G}_{i_1}(y_k + u)dy_1...dy_k dH(u), \qquad (6.4)$$

and with $k \geq l$

$$\lambda(\vec{b})q^{(\vec{b})} \approx \sum_{\vec{b}^j \in E_-} \Lambda^j \sum_{\pi \in \Pi^j} \int_0^\infty \int_{0<y_1<...<y_k} \!\!\!\!...\int \overline{G}_{i_l}(u)\overline{G}_{i_{l-1}}(y_1 + u)...$$

$$...\overline{G}_{i_1}(y_{l-1} + u)dy_1...dy_{l-1}dH(u); \qquad (6.5)$$

2) with discipline d_2 and $k < l$

$$\lambda(\vec{b})q^{(\vec{b})} \approx \sum_{\vec{b}^j \in E_-} \Lambda^j \sum_{\pi \in \Pi^j} \int_0^\infty \int_{0<x_2<...<x_l} \!\!\!\!...\int \overline{G}_{i_1}(x_{k+1})\overline{G}_{i_2}(x_{k+2} - x_2)...$$

$$...\overline{G}_{i_{l-k}}(x_k - x_{l-k})\overline{G}_{i_{l-k+1}}(x_l - x_{l-r+1} + u) *$$

$$*\overline{G}_{i_{l-1}}(x_l - x_{l-1} + u)\overline{G}_{i_l}(u)dx_2...dx_l dH(u), \qquad (6.6)$$

and with $k \geq l$ the estimation (6.5) is accurate;

3) with discipline d_3 and $k < l$

$$\lambda(\bar{b})q^{(\bar{b})} \approx \sum_{\bar{b}^j \in E_-} \frac{(l-1)!\Lambda^j}{k!k^{l-k-1}} *$$

$$* \sum_{\pi \in \Pi^j} \int_0^\infty \int_{0<y_1<...<y_{l-1}} \!\!\!\!...\int \overline{G}_{i_l}\left(\frac{ku}{l}\right)\overline{G}_{i_{l-1}}\left(y_1 + \frac{ku}{l}\right)...$$

$$*\overline{G}_{i_1}\left(y_{l-1} + \frac{ku}{l}\right)dy_1...dy_{l-1}dH(u), \qquad (6.7)$$

and with $k \geq l$ the estimation (6.5) is carried out;

4) with discipline d_4 and $k \geq 1$ the estimation (6.5) is carried out.

In the steady-state operation section of work with the restoration discipline $d \in D$ and k RU $\beta(t) = \beta(d,k)$, $\beta_M(t) = \beta_M(d,k)$, $\beta_{xM}(t) = \beta_{xM}(d,k)$, $\tau_i^{"}(t) = \tau^{"}(d,k)$, $T_R(d,k) = M\tau^{"}(d,k)$.

Let $K_A(d,k)$ is the availability function of system in steady-state operation. Since $(1/\beta_M(d,k) - T_R(d,k))$ is the estimation of the mean failure-free operating time of the system in steady-state operation, then it takes place [2]

Corollary 6.2. In the steady-state operation with $\varphi_1 \to 0$ and restoration discipline $d \in D$ for first and third type systems the estimations are carried out:

$$\beta(d,k) \approx \beta_M(d,k), \tag{6.8}$$

$$P\{\tau^{"}(d,k) \geq x\} \approx \beta_{xM}(d,k)/\beta_M(d,k), \tag{6.9}$$

$$T_R(d,k) \approx \int_0^\infty \beta_{xM}(d,k)dx/\beta_M(d,k), \tag{6.10}$$

$$K_A(d,k) \approx 1 - \int_0^\infty \beta_{xM}(d,k)dx/\beta_M(d,k). \tag{6.11}$$

Theorem 6.1 and estimation (6.8) make following sense. The failure rate of system with fast restoration can be determined only taking into account the minimum paths of failure. In this case by duration of IM in comparison with the time of the failure-free operation of elements is possible to disregard. Therefore during estimation of the failure rate of the system DD of the time of failure-free operation in zero for the elements, which on IO were located in the unloaded regime, is taken equal to $c_i = f_i(0)$, and elements, located in the steady-state section in the loaded regime, is taken equal to their density in zero $1/m_i$.

Let us discuss the accuracy of estimations (6.9) - (6.11). The replacement in them of the probability of the system failure to the probability of the system failure only along the monotonic paths leads to the errors, which for the denominator and the numerator have the same sign. Therefore estimations of DF of $\tau^{"}(d,k)$ and $T_R(d,k)$ may be sufficiently precise. Actually, for a number of systems with the loaded and unloaded reserve with not limited ($k \geq n$) and completely limited (k = 1) restoration the approximate estimate $T_R(d,k)$, obtained according to formula (6.10), coincide with the precise estimations $T_R(d,k)$ (in such cases when the precise estimations of $T_R(d,k)$ can be determined).

Corollary 6.3. For the systems in question with $s \geq 2$ and $[\hat{\lambda} m_R^s/(m_R)^{s-1}] \to 0$ or s=1, $\hat{\lambda} m_R \to 0$ and $q \to 0$

$$P\{\tau_i(t) \geq x\} \approx \exp\{-\beta_M x\}, i \geq 1, \tag{6.12}$$

only in steady-state operation. However, in the non-stationary section of the work of system error from the replacement of DF $\tau_i(t)$ to the exponential may be essential.

Only in the systems with exponential DF of the time of the failure-free operation of elements, in which or elements are uniform or there is provided only loaded reserve, the non-stationary regime of the work of system is absent. For such systems the use of (6.8) is justifiable in any section of the work of system, if in the initial state all elements of system were operational.

But already the presence of the unloaded reserve with the different-type elements in the systems with the exponential time of the failure-free operation of elements leads to the appearance of a non-stationary section of work. Systems of the second type always lack the steady-state operation. For all such systems the application of estimations (6.8) in the non-stationary section (for example, in the initial section of the work of system) is incorrect.

7. Estimation of the indices of reliability for systems second type

In the engineering practice such tasks of evaluating the reliability of systems sometimes appear, for which the reliability of elements, the regime of maintenance, the condition of environment, requirements on the reliability and other parameters of system are changed in the time in a no random or random way.

Let us examine the following sufficiently common model of system of second type. Model consists of *n* of elements, which are included in the system according to the assigned timetable. Each element can be only in operational or failed state. Operational element depending on the states of system is included in the loaded or unloaded regime (simplicity for we do not examine the reduced regime). DF of the time of failure-free operation of i-th element, included in moment t into the loaded regime is equal to $\bar{F}_i(x,t) = \exp\{-\int_t^{t+x} \lambda_i(u)du\}$.

He failed element is sent into the maintenance crew, being of *k* RU, where it is restored in accordance with discipline $d \in D$. Restoration is complete. After restoration element, that is included in the system, is returned to the system.

If the number of elements in the system, their type, the number of repair units, the distribution functions, the criteria of the failure of system and so on are changed sufficiently slowly and during the time of one IM these parameters can be considered as constants, then this model is included in the model, described in the sections 2 – 5. Therefore [1] it is accurate

Theorem 7.1. For systems of the second type with restoration discipline $d \in D$, $\hat{\lambda} m_R \to 0$ for $i = 1$ and with $[\hat{\lambda} m_R^s/(m_R)^{s-1}] \to 0$ for $i > 1$

$$P\{\tau_i(t,d) \ge x\} \approx \exp\{\int_t^{t+x} \beta(u,d)du\} \qquad (7.1)$$

Let we examine systems of second type under the conditions of fast restoration with $d \in D$, and $\lambda(t)$ is the summary failure rate of the elements of system at the moment t (or the intensity of appearance IM), $q^{(t)}(d)$, $q_M^{(t)}(d)$ and $q_{xM}^{(t)}(d)$ are correspondingly the probability of the system failure, failure and x- failure taking into account only the minimum paths of the system failure on IM, that begun in the interval $(t, t + dt)$. Then it is carried out

Corollary 7.1. For systems of the second type with $[\hat{\lambda}^{s}\, m_R^{(s)} / \underline{\lambda}^{s-1}[m_{R*}(\eta]^{s-1}] \to 0$

$$P\{\tau''(t,d) \geq x\} \approx q_{xM}^{(t)}(d) / q_M^{(t)}(d), \tag{7.2}$$

and

$$T_R^{(t)}(d) \approx \int_0^\infty q_{xM}^{(t)}(d) dx / q_M^{(t)}(d). \tag{7.3}$$

8. Estimation of the reliability indices of the complex systems

Let the system consists of N subsystems connected in series in the sense of reliability.
A system malfunction occurs when at least one of the subsystems malfunctions. The failure of system occurs when its malfunction lasts for a time not shorter than $\eta, P\{\eta < x\} = H(x)$.

Let us name the scheme *p* out of *m* a system, containing *m* of elements, malfunction of which begins when the number of failed elements in it is not less than *p*, $p \leq m$, and its failure begins when the malfunction of scheme lasts for a time not shorter than $\eta, P\{\eta < x\} = H(x)$.

Let in the *i*-th subsystem it is possible to separate M_i *p* out of *m* schemes that are connected in series in the reliability sense and the *j*-th of them is the scheme p^{ij} out of m^{ij} (*ij*-th scheme). Some elements of subsystem can participate in the operation of more than one scheme *p* out of *m*. The malfunction of *i*-th subsystem occurs when at least one of these M_i schemes malfunctions.

Let $\beta^{ij}(d, m)$, $\beta_x^{ij}(d, m)$, $\beta_M^{ij}(d,m)$, $\beta_{xM}^{ij}(d,m)$, $\tau''_{ij}(d, m)$ and $T_R^{ij}(d, m)$ denote values described above, but relating to *ij*-th scheme, and $\beta(d,m)$ ($\beta_x(d,m)$) is the failure (x- failures) rate of complex system in the steady-state mode of its operation. Let the summation with respect to "*ij*" indicates summation with respect to $j = \overline{1, M_i}$ and $i = \overline{1, N}$.

Then [1] there is carried out
Theorem 8.1. For the considered complex systems in steady-state mode of their operation with the FR condition (4.2)

$$\beta(d,m) \approx \sum_{ij} \beta^{ij}(d, m)$$
$$\approx \sum_{ij} \beta_M^{ij}(d,m), \tag{8.1}$$

$$\beta_x(d,m) \approx \sum_{ij} \beta_x^{ij}(d,\, m)$$
$$\approx \sum_{ij} \beta_{xM}^{ij}(d,m)\,. \qquad (8.2)$$

Theorem 8.1 allows one, under conditions of fast restoration, to evaluate the failure (x- failures) rate of system as the sum of the failure (x- failures) rates of its series-connected in the sense of the reliability the *ij*-th schemes of form *p* out of *m*, calculated under the assumption, that these schemes operate autonomously. In this case the failure rate (x- failures) of the schemes p^{ij} out of m^{ij} can be determined only taking into account the minimum ways p^{ij} (see (8.1) and (8.2)).

Let $\alpha_{ij} = \beta^{ij}(d,\, k)\, /\, \beta(d,\, k)$ is the probability of that, that the failure of the system is caused by the failure of the *ij*-th scheme. Then it is carried out

<u>Corollary 8.1</u>. Under the FR conditions (4.2) for the complex system operating in the steady-state mode

$$P\{\tau''(d,\, m) \geq x\} \approx \sum_{ij} \alpha_{ij} P\{\tau''_{ij}(d,\, m) \geq x\}, \qquad (8.3)$$

$$T_r(d,\, m) \simeq \sum_{ij} \alpha_{ij}\, T_r^{ij}(d,\, m)$$
$$\approx \sum_{ij} \int_0^\infty \beta_x^{ij}(d,\, m)\, dx\, /\, \beta(d,\, m), \qquad (8.4)$$

$$K_A(d,\, m) \approx 1 - \sum_{ij} \beta^{ij}(d,\, m)\; T_R^{ij}\;(d,\, m)$$
$$\approx 1 - \sum_{ij} \int_0^\infty \beta_x^{ij}(d,\, m)\, dx. \qquad (8.5)$$

Corollary 8.1 makes it possible under the conditions of fast restoration to estimate the indices of maintainability and the availability function of complex system as the weighted sum of the corresponding indices of the reliability of the *ij*-th schemes p^{ij} out of m^{ij} calculated under the assumption of their autonomous operation.

9. Example. System with the unloaded reserve

It is examined a system with the unloaded standby reserve and with elements of different types that consists of a single basic element and $(n-1)$ standby elements with one RU [3]. The $G_i(x)$ is the DF of the recovery time of *i*-th element, $i \in \overline{1,n}$. The restoration discipline is FIFO $d_1 \in D$. With the failure of the basic element its place occupies the reserve element with the smallest number from the elements, stayed in the reserve the greatest time. At *t*=0, the first element became the basic.

The malfunction of system begins when the basic element fails and there is no standby element capable to replace it (that is all of *n* elements failed). The system failure occurs when its malfunction lasts for a time not shorter than $\eta, P\{\eta < x\} = H(x)$. Let $m_{iR}^{(j)}(\eta) = \int_0^\infty \int_0^\infty x^j d_x G_i(x+u) dH(u)$.

For this system the state of the elements of the system $\vec{v}(z)$ under the condition $B(t,z)$ and $z \in \mathrm{IO}$ is uniquely determined by the index *i* of the basic element at the instant *z*. Under the fast restoration in the steady-state operating conditions of the system

$$p_i(z \mid t) = m_i / \sum_{j \in \overline{1,n}} m_j \,,$$

$$\lambda(z \mid i,t) = 1/m_i \,,$$

$$q^{(z|i,t)} \approx \prod_{j \neq i,\, j=\overline{1,n}} c_j m_{iR}^{(n-1)}(\eta)/(n-1)!$$

So we obtained theorem 9.1.

Theorem 9.1. For the system *n* out of *n* with the unloaded standby reserve and the elements of different types, with a single repair unit and restoration discipline $d = d_1$ under fast restoration in steady-state operating conditions

$$P\{\tau_j \geq x\} \approx \exp\{-\beta x\},\, j \geq 1\,,$$

where

$$\beta \approx \sum_{i=\overline{1,n}} \prod_{j \neq i} c_j m_{iR}^{(n-1)}(\eta)/[(n-1)! \sum_{l=\overline{1,n}} m_l]\,.$$

In conclusion the author expresses the deepest gratitude to Prof. V. Rykov for the appeared interest and his notes.

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