SOME BAYESIAN QUEUEING AND RELIABILITY MODELS[2]

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Abstract

The Bayesian approach for certain tasks of queueing systems theory and reliability theory is introduced. The method provides the randomization of system characteristics with regard of a priori distributions of input parameters. This approach could be used, for instance, for calculating average values and for construction of confidential intervals applicable for performance and reliability characteristics of large groups of systems or devices.

1. Introduction and main assumptions

Theory of queueing systems is a well-developed mathematical discipline. Based on it a substantial number of positive R&D results have been generated. The results obtained in studying queueing systems and networks proved to be of significant profundity and importance from mathematical and practical points of view. In fact queueing systems and networks are able to model a broad class of real systems, info-telecommunication systems and networks being in the first place. In order to reflect real processes in a more adequate way, the present development of queueing theory is being carried out mostly with a focus on studying more complex service disciplines, input flows and service time distributions with more and more complicated probabilistic characteristics.

One of the directions of generalization of problem formulations is the complication of probabilistic structure of one or more queueing systems input parameters. Instead of considering traditional input flows, the researchers study Cox flows, self-similar flows, Markovian and semi-Markovian flows, etc. Similar generalizations are made regarding service times distributions. To some extent, these generalizations can be interpreted as the randomization result of these or those parameters of more "simple" flows and service times distributions. Thus, Cox process is obtained as a result of special randomization of Poisson flow intensity, etc.

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All these generalized modern formulations assume that stochastic method of randomization “affects” the parameters of a system precisely during its functioning, meaning that we primarily know the kind of the system we are “dealing with”, even when the system is rather complicated and then we study characteristics of this particular “primarily fixed” system. However, in real life often the system under study is specified in some sense vaguely, or inaccurately. For example, even when we deal with the simplest systems of M|G|1 type, we may not know a priori the input flow parameter λ and the service parameters μ and σ^2. Such situations can occur studying the whole class of queueing systems when the only known characteristics are the input flow types the service distribution and the service discipline, but at the same time the concrete parameters of these flows and distributions, generally speaking, vary for different queueing systems of a given class. A researcher does not know a priori the queueing system belonging to the given class he is dealing with. For example, such situation can take place testing a series of uniformed commutation or transmission devices manufactured by the same company. Spread in some of their performances can be explained by natural technological deviations during manufacturing process. In this particular case, since the unknown characteristics are the “initial” parameters of the flows and service times, a natural thing could be the use of a randomized approach according to whch the values λ, μ and σ^2 become the elements of a probabilistic space, but in general, one can speak about probabilistic space with uniformed queueing systems being its elements. In this situation it is quite natural that the calculated characteristics of such randomized queueing system are randomization of similar characteristics of “usual” queueing system of similar type taking into account a priori distribution of queueing system input parameters.

So, in the same example concerning a M|G|1 queueing system there arise the tasks of “common” characteristics randomization of such systems with regard for a priori input parameters distributions. In other words, we can make assumption about exponential, uniform or any other distribution of one or several values λ, μ or σ^2 (that become random variables under such approach), about their dependence or independence, etc. The obtained results could be used, for instance, to calculate “in general” average values and to construct confidential intervals applicable for these or those characteristics of the queueing system class under consideration. Naturally, such approach queueing models development can be called Bayesian and it was formulated for the first time in [1].

The Bayesian approach can be used also in problems of reliability estimation. As it is known (see [2]), the availability factor of the restorable device in a stationary mode can be calculated using the formula

$$k = \frac{\lambda^{-1}}{\lambda^{-1} + \mu^{-1}} = \frac{\mu}{\lambda + \mu},$$

where λ^{-1} is the average operating time between failures, and μ^{-1} is the average restoration time. If we accept the hypothesis stated above that the device under consideration is randomly selected from some set of similar devices whose average reliability characteristics vary, then according to the reasonings presented above, values λ and μ could be considered as random. Hence, under these assumption the availability factor *k* is random, too, and its distribution depends on distributions of values λ, μ. The obtained results in this field could be used, for instance, for calculating “in general” average values and for the construction of confidential intervals for reliability characteristics of the overall set of investigated devices.

2. The Bayesian approach to queuing systems.

In order to explain the essence of the task formulation we present the following example. Let us consider a situation when an observer deals with rather large series of queueing systems M|M|1|0 that differ only in service distribution parameter. In particular, these can be certain devices, commutators, routers or any other servicing tools. It is known in advance that their functioning can be modelled by a system belonging to the above-described type., i.e these systems have identical service discipline, types of input flow and of service times distribution.

This example assumes that the input flow characteristics are also identical for all the systems of a given series; only numerical characteristics of **service** are different (i.e. the parameters of exponential distribution).

Dispersion in characteristics of service is due to technological (design) reasons and the main aspect of the problem statement is the fact that the researcher does not know what the real value of service parameter of the system belonging to a given series under study that was selected by him at random. The only thing that he knows is “a priori” distribution of this parameter (since the series is supposed to be large, one can consider stochastic phenomena in relation with that series and introduce probabilistic distributions). The researcher is interested in finding out service characteristics for a series as a whole (or characteristics of the system “selected at random”). Obviously, along with traditional factors of stochasticity that occur in queueing systems (stochasticity of input flow and service processes), there appears one more factor of stochasticity related **to randomized selection of the system under study**.

Let us assume that the service parameter μ of the systems under study can take only two values: μ_1 and μ_2 with probability p_1 and p_2, respectively. In “physical terms” it means that among the system series under study (routers, machine tools, etc.) only two “varieties” of servicing devices occur. Devices belonging to the first variety provide the service with parameters μ_1, while devices of the second variety provide the service with parameter μ_2. Then the loading factor of the system “selected at random” becomes the random variable that takes the values λ/μ_1 with probability p_1 and λ/μ_2 with probability p_2. The steady-state probability of blocking the “selected” system due to the interference of the random factor of selecting a concrete system becomes “random” itself and takes the values $\lambda/(\lambda+\mu_1)$ with probability p_1 (it is the probability that a system belonging to the first variety has “fallen into the researcher’s hands”) and $\lambda/(\lambda+\mu_2)$ with probability p_2 (meaning that a system of the second variety “has fallen into the researcher’s hands”). It is natural that the “averaged” blocking probability of such “Bayesian” queueing system is equal to $p_1\lambda/(\lambda+\mu_1)+p_2\lambda/(\lambda+\mu_2)$.

As we can see, there is no need to use the methods of queueing theory for studying the Bayesian queueing systems. Bayesian system is “randomization” of a certain “ordinary” queueing system, meaning that the Bayesian queueing system characteristics can be calculated by means of randomizing subsequent averaging (by a priori distribution of the parameter or parameters) of the “ordinary” queueing system characteristics that have been calculated earlier by using the methods of queueing theory. In other words, the mathematical part of the job comes to this particular randomization and averaging. At the same time, it is an expedient from both technological and conceptual points of view to accomplish randomization of stationary characteristics of “ordinary” queueing systems and obtain the steady-state characteristics of Bayesian queueing systems.

We would like to point out one more substantial model that can be described mathematically with the help of Bayesian queueing system. Let's assume that a researcher considers not a series of systems with quantitative parameters that change with the time. For example, there exists a servicing device, one of its elements being replaced by another one at the moments that we do not know beforehand, then being replaced by the third one, etc. Such a system can be the frontier post at the airport, where an officer on duty is relieved from time to time at the moments not known by the observers (passengers). The only things an observer knows are the probability that he will have "come upon" a certain concrete frontier officer and an average time of passport checking by each frontier.

Under such approach the system structure and service discipline do not change with the time while only quantitative parameter of distribution of service changes (e.g. intensity). The input flow parameter can change in a similar way. There is no information about the moment when changes occur. The researcher is aware only of **distribution** of the values of "changeable", random parameters he "comes across" while examining the system at a "random" moment of time.

Since it is assumed that the researcher does not have any information about the moments of the system "reorganization", and even about distribution of these moments, it is impossible to describe transient processes within such kind of a system. Therefore, it is possible to carry out analysis (and subsequent randomization) of only **steady-state** distributions of the queueing system under analysis. In order to give meaning to this problem statement, it is necessary to make an assumption that the system changes quite "rarely" so that at each interval of constancy of the parameters, the queueing system "had time" to reach steady-state condition. Of course, the results of such analysis will be rough because steady-state condition, strictly speaking, cannot be reached in real life.

3. Simple models of " Bayesian" queueing systems

Below two more simplest models of "Bayesian" queueing systems are presented in order to provide further elucidation of specific character of the problems that emerge under such an approach and of the obtained obtained results.

Uniform distribution of λ and μ : loading factor

Let us consider an arbitrary queueing system with input flow intensity λ and service intensity μ. The loading of such system is equal to $\rho = \lambda / \mu$. As it is generally known, the availability of steady-state mode of the system under consideration depends on the value ρ which apperas in many formulae that describe characteristics of different queueing systems. Hence, the study of the value ρ should be considered within the frames Bayesian theory of queueing systems.

The variety of possible and interesting distributions of variables λ and μ for their joint applications is rather wide. We consider one of the simplest but at the same time very common in practice cases when the values λ and μ are independent and uniformly distributed on some certain pre-determined segments. Such model is good for describing situations when some legitimate interval of values have been assigned for both values λ and μ (or for any of them), but the real value λ or/and μ can vary within such limits.

Assume that the random variable λ has a uniform distribution on the segment $[a_\lambda, b_\lambda]$, the random variable μ has a uniform distribution on $[a_\mu, b_\mu]$, with $0 \le a_\lambda \le b_\lambda$, $0 \le a_\mu \le b_\mu$.

In this case, the cumulative function of the random variable $\rho = \lambda / \mu$ distribution can be written down as follows:

$$P\{\rho < x\} = \iint_{\lambda/\mu < x} \frac{1}{b_\lambda - a_\lambda} \frac{1}{b_\mu - a_\mu} d\lambda d\mu$$

Subsequent calculations depend essentially on relation between the values a_λ / a_μ and b_λ / b_μ. Let us suppose for the sake of definiteness that $a_\lambda / a_\mu \le b_\lambda / b_\mu$. Then:

provided $x \le a_\lambda / b_\mu$

$$P\{\rho < x\} = 0,$$

provided $a_\lambda / b_\mu \le x \le a_\lambda / a_\mu$

$$P\{\rho < x\} = K \frac{(b_\mu x - a_\lambda)^2}{2x},$$

provided $a_\lambda / a_\mu \le x \le b_\lambda / b_\mu$

$$P\{\rho < x\} = K\left(\frac{a_\mu + b_\mu}{2} x - a_\lambda\right)(b_\mu - a_\mu),$$

provided $b_\lambda / b_\mu \le x \le b_\lambda / a_\mu$

$$P\{\rho < x\} = 1 - K \frac{(b_\lambda - a_\mu x)^2}{2x},$$

provided $x \ge b_\lambda / a_\mu$

$$P\{\rho < x\} = 1,$$

when

$$K = \frac{1}{(b_\mu - a_\mu)(b_\lambda - a_\lambda)}.$$

Let us derive the density of random variable ρ:

provided $x \le a_\lambda / b_\mu$

$$f_\rho(x) = 0,$$

provided $a_\lambda / b_\mu \le x \le a_\lambda / a_\mu$

$$f_\rho(x) = K\left(\frac{b_\mu^2}{2} - \frac{a_\lambda^2}{2x^2}\right),$$

provided $a_\lambda / a_\mu \le x \le b_\lambda / b_\mu$

$$f_\rho(x) = K\left(\frac{b_\mu^2 - a_\mu^2}{2}\right),$$

provided $b_\lambda / b_\mu \le x \le b_\lambda / a_\mu$

$$f_\rho(x) = K\left(\frac{b_\lambda^2}{2x^2} - \frac{a_\mu^2}{2}\right),$$

provided $x \ge b_\lambda / a_\mu$

$$f_\rho(x) = 0.$$

Through accomplished elementary calculations, we derive the average value and the second moment of random variable ρ, that are respectively equal to:

$$\mathbf{E}\,\rho = \frac{b_\lambda + a_\lambda}{2(b_\mu - a_\mu)} \ln \frac{b_\mu}{a_\mu},$$

$$\mathbf{E}\,\rho^2 = \frac{a_\lambda^2 + a_\lambda b_\lambda + b_\lambda^2}{3 a_\mu b_\mu}.$$

It is evident that if $b_\lambda - a_\lambda \to 0$ and $b_\mu - a_\mu \to 0$, i.e. contracting the range of the random variable λ to some fixed point λ_0, and the range of the random variable μ to some fixed point μ_0, the value $\mathbf{E}\,\rho$, as it should be, tends to λ_0 / μ_0, and the value $\mathrm{E}\,\rho^2$ tends to λ_0^2 / μ_0^2.

Moreover, we note that the dependence of the average value of ρ on distribution λ is reduced to dependence on the mathematical expectation λ. At the same time, dependence of $\mathbf{E}\,\rho$ on parameters of distribution μ has a more complex look.

In the case $a_\lambda / a_\mu \ge b_\lambda / b_\mu$, the formulae for calculating the cummulative and density functions of the random variable ρ are similar. The mathematical expectation and the second moment of the random variable ρ in this particular case coincide with the values that have been calculated previously.

Based on the obtained results, it would be easy to calculate other necessary characteristics of value ρ.

It is worthwhile to observe that the examined model allows to study an important situation when $\lambda < \mu$ has the probability 1. In this case $\rho < 1$, which is the condition of ergodicity of the systems having one servicing device. By virtue of postulated independence of random values λ and μ, and the condition for $\lambda < \mu$ is satisfied only if the condition $0 \le a_\lambda \le b_\lambda \le a_\mu \le b_\mu$ holds.

Exponential λ and μ distribution: loading factor, probability of losses in the system M|M|1|0 and avalaibility factor

Let us consider another probabilistic model for the values λ and μ. In a situation when there is no a priori information about their mean values, it we can consider as a "first approximation" a model where λ and μ are exponentially distributed with known averages, *1/l* and *1/m* respecticely). Assumption about λ and μ has been retained.

So, the cummulative function of the random variable λ distribution is equal to $1-\exp(-lu)$ and the cummulative function of the random variable μ distribution is equal to $1-\exp(-mu)$. As we did in the previous section, le us first of all consider $\rho = \lambda / \mu$. Obviously, for $x \geq 0$ we get

$$P\{\rho < x\} = P\{\lambda < \mu x\} = \int_0^\infty P\{\lambda < xy\} dP\{\mu < y\} = \int_0^\infty [1-\exp(-lxy)] m \exp(-my) dy =$$

$$= \frac{lx}{m+lx}$$

Hence, it follows in particular that the random variable ρ in this case does not have any moments of the first and higher orders, as distinct from the situation described in the previous section. However, some other characteristics of Bayesian queueing systems, depending on random variable $\rho = \lambda / \mu$, can have finite moments. Let us consider, for example, the queueing system of M|M|1|0 type. The probability that a signal has been received by the system will not be lost in a steady-state mode is equal to $\pi = 1/(1+\rho)$ according to Erlangian formulae. As for the Bayesian problem statement, this probability becomes "random" by itself. Let us consider the distribution of the random variable π under the conditions of the model under study.

Provided $0 \leq y \leq 1$

$$P\{\pi < y\} = P\{\rho > (1-y)/y\} = \frac{my}{my + l(1-y)}$$

Correspondingly, the random variable π density is equal to $\frac{ml}{[my+l(1-y)]^2}$, while the averaged probability that the call is not lost looks as follows

$$\mathbf{E}\pi = \int_0^1 \frac{mly}{[my+l(1-y)]^2} dy = \frac{ml}{(m-l)^2}\left(\ln\frac{m}{l} + \frac{l}{m} - 1\right).$$

It would be easy to calculate also the second moment of the random variable π as well as its other characteristics. Let us note hat for *m=l*

$\mathbf{E}\pi$ = *1/2*.

The **v**alue

$$\pi = 1/(1+\rho) = \frac{\mu}{\lambda + \mu}$$

is equal to value of the avalaibility factor *k* (see above). Hence, the distribution of the random avalaibility factor in case of exponentially distributed λ and μ is presented above as the distribution of random value π .

4. Conclusions

The results presented in this article, related to Bayesian approach for queueing systems' and reliability problems, are very preliminary, or "trial" ones. It is obvious that further advancement will require consideration of such a priori distributions of the values λ , μ and other traditional queueing systems as well restorable devices input parameters that can be of practical interest. The distributions of the variables that characterize the functioning of different system types can be calculated after they have been randomized taking into account of the given a priori distributions.

Reference

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