LIMIT RARING PROCESSES AND RAREFACTION OF TWO INTERACTED RENEWAL PROCESSES

This research was supported (in part) by the Ministry
of Education and Science of Ukraine, project No 01.07/103

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Keywords:raring processes, mixing coefficient, generating function, renewall processes.

Abstract: This paper deals with study of the sufficient condition of approximation of raring process with mixing by renewal process. We consider use the proved results to practice problem too.

study

The limit theorems for raring processes were obtained by many .authors with use the different technics [1-8] . In the article [1] it was constructed the first model of Bernoullis' rarefaction of renewal process and it was obtained the elegant proof of approximation of such processes by Poisson process. The problem of necessary and sufficient conditions of such approximation was solved in the articles [3, 5]. The general procedures of construction a raring processes from initial processes were considered in works [2, 4, 6, 7, 8, 9]. The authors of articles [2,7, 9, 13] obtained new results for concrete apllied models with help the proved theorems of raring processes.

This article is to some degree a continuation of [9]. In section 1 it is proved the limit theorem. This proof is self-depended. It does not apply results of other offers. In section 2 it is considered the application of obtained results to concrete models.

If we have a strictly increasing almost sure sequence of positive random values $\{\tau_i, i \geq 0\}$, $\tau_{i+1} > \tau_i$, $i \geq 0$ then we can define random flow of points-events on the time axes. The moment appearance of i-th event coincides with time τ_i. Any underflow this flow is named raring flow. Thus i- th event in raring flow has number $\beta(i)$ in initial flow (it is clear that $\beta(i) \geq i$). At the beginning we shall study the sequence $\beta(i)$, $i \geq 0$ and then we shall construct this sequence for concrete model of raring process.

1. Limit theorem.

Let us consider the sequence of discrete random values

$$\xi(t),\ t \in \{0,1,2,\dots\},\quad \xi(t) \in \{1,2,\dots\}.$$

We are going to investigate distribution the following sequence

$$\beta(1) = \xi(0),\quad \beta(m+1) = \beta(m) + \xi(\beta(m)),\quad m \geq 1.$$

For this purpose, we introduce the following objects

$$v(t) = \max\{m \geq 1 : \beta(m) \leq t\},\qquad \alpha(k) = \sup_{x \geq 0} \sup_{A \in F_{\leq x},\, B \in F_{\geq x+k}} |P(B/A) - P(B)|,$$

here $F_{\leq x} = \sigma(\xi(s), s \leq x)$, $F_{\geq x} = \sigma(\xi(s), s \geq x)$.

Statement.*The following inequality holds for any* $x > 0$

$$P(\beta(m) < x) = \max_{t \leq x} P(\xi(t) < \frac{x}{m})([x]+1).$$

Proof. We have by definition of $\beta(m)$

$$\{\beta(m) < x\} \subseteq \bigcup_{i=1}^{[x]+1} \left\{\xi(i) < \frac{x}{m}\right\}$$

from latter one proof follows.

Now we will proof the limit theorem for random values $\beta(m)$ in case when process $\xi(t)$ depends on parameter n. The dependence on n means , in this case, that sequence processes $\xi_n(t)$ must convergence to infinity (in some sense) at fixed t under $n \to \infty$. Such situation occurs in practice problem very often. The parameter n is index for all values which are defined by $\xi_n(t)$. For example, the values $v(t)$ transform to $v_n(t)$.

Let $\underset{n\to\infty}{\Rightarrow}$ denotes weak convergence of random values or distribution functions. Let $N(t)$ is equal to number of renewals on the interval $[0,T]$of renewal process $\left\{\sum_{k=1}^{i} \eta_k,\ i \geq 1\right\}$. This process has the following property $P(\eta_1 \leq x) = R_1(x)$, $P(\eta_i \leq x) = R_2(x)$, $i \geq 2$. Here $R_1(\cdot)$, $R_2(\cdot)$ are a distribution function.

Theorem 1. *If sequence of numbers* $c_n \to \infty$ *exists such that the following conditions hold :*

1) $\lim\limits_{n\to\infty} P\left(\xi_n(0)c_n^{-1} \le x\right) = R_1(x);$

2) $\lim\limits_{n\to\infty} \sup\limits_{a\le\delta\le t} \mid P\left(\xi_n([c_n\delta])c_n^{-1} \le y\right) - R_2(y) \mid = 0;$

δ *-- any positive number,* $t<\infty$*, functions* $R_i(y)$ *are continuous distribution functions for* $y>0$*;*

3) $\lim\limits_{n\to\infty} \alpha_n(c_n)c_n = 0,$

then $v_n(c_n t) \underset{n\to\infty}{\Rightarrow} N(t)$ *for every fixed* t.

Proof. We denote by $\beta_k(m),\ m\ge 1$ the sequence which is defined by the sequence $\beta(m)$ under condition $\xi(0)=k$. That is $P(\beta_k(m)=s)=P(\beta_k(m)=s/\xi(0)=k)$.

Further $v(k,t)=\max\{m\ge 1:\beta_k(m)<t\}$.

We define the following sequence of random values $\nu_k(m)$:

$$\nu_k(0)\equiv 0,\ \nu_k(1)=\xi(k),\quad \nu_k(m+1)=\nu_k(m)+\xi(\nu_k(m)),\ m\ge 1.$$

Further let $V_k(t)=\max\{m\ge 1:\nu_k(m)\le t\}$.

Now we introduce the sequence of random integer numbers $\beta_{l,k}(m),\ m\ge 1$, which have the following distribution function

$$P\left(\beta_{l,k}(m)=s\right)=P\left(\beta(m)=s/\xi(0)=l,\xi(l)=k\right)=P\left\{\nu_{l+k}(m)=s-l-k/\ \xi(0)=l,\xi(l)=k\right\}.$$

We will denote by $v_{l,k}(t)=\max\{m\ge 2:\beta_{l,k}(m)\le t\}$.

By the definition of $v(t)$ and $v(l,t)$ we have stochastic equalities (right and left parts have the same distribution function)

$$v(t)=\sum_{l=1}^{[t]} I(\xi(0)=l)\left(v(l,t)+1\right),\quad v(l,t)=\sum_{k=1}^{[t-l]} I(\xi(l)=k/\xi(0)=l)\left(v_{l,k}(t)+1\right),$$

here the function $I(\cdot)$ is indicator function of sets.

Applying indicator identity

$$s^{I(\cdot)x}=1+I(\cdot)\left(s^x-1\right),$$

we get

$$M\, s^{\nu(t)} = 1 + \sum_{l=1}^{[t]} M\, I\left(\xi(0)=l\right)\left(s^{\nu(l,t)+1} - 1\right),$$

$$M\, s^{\nu(l,t)} = 1 + \sum_{k=1}^{[t-l]} M\, I\left(\xi(l)=k/\xi(0)=l\right)\left(s^{\nu_{l,k}(t)+1} - 1\right),$$

here $s \in (0,1)$.

If the $\xi(t)$ depends on the parameter n, then latter equalities have the following forms.

Put

$$M\, s^{\nu_n(c_n t)} = g_n(c_n t, s), \qquad M\, s^{V_{n,k}(c_n t)} = f_{n,k}(c_n t, s).$$

Further

$$g_n(c_n t,\, s) = 1 - P\left(\xi_n(0) \le t\right) + s \sum_{l=1}^{[t]} M\, I\left(\xi_n(0)=l\right) s^{\nu_n(l,\, c_n t)} ,$$

$$M\, s^{\nu_n(l,\, c_n t)} = 1 - P\left(\xi_n(l) \le c_n t/\xi_n(0)=l\right) + s \sum_{k=1}^{[c_n t - l]} M I\left(\xi_n(l)=k/\xi_n(0)=l\right) s^{\nu_{n,l,k}(c_n t)} . \quad (1)$$

We will divide the sums in the right parts equalities (1) into two sums:

$$\sum_{1}^{[c_n \delta]} + \sum_{[c_n \delta]+1}^{[c_n t]} \quad (2)$$

The first sum we can make less than given number. This follows from the conditions 1,2 and continuous of functions $R_i(\cdot)$ in zero.

The second sum consists of the expectations of two random factors. These factors are bounded by one and measured with respect to σ -algebras $F_{\le x}$, $F_{\ge x + c_n \delta}$ respectively. The latter one enables us to change every summand of second part of (2) by factor of expectations of the given random values with error less than $2\alpha_m(c_n \delta)$ (look for example (20.29)[10]):

We have the following estimates under $l \ge c_n\, \delta$

$$|\, MI\left(\xi_n(0)=l\right) s^{\nu_n(l,\, c_n t)} - MI\left(\xi_n(0)=l\right)\; M\, s^{\nu_n(l,\, c_n t)} \,| \le 2\alpha_n(c_n \delta),$$

$$M\, s^{\nu_n(l,c_n t)} = \sum_{d=0}^{[c_n t]-l} s^d \left(P\left(\nu_{n,l}(d) \le c_n t - l, \nu_{n,l}(d+1) > c_n t - l \,/\, \xi_n(0)=l\right) \pm\right.$$

$$\left.\pm P\left(\nu_{n,l}(d) \le c_n t - l, \nu_{n,l}(d+1) > c_n t - l\right)\right) = M\, s^{V_{n,l}(c_n t - l)} + \pi_{n},$$

here $|\pi_n| \le K \alpha_n(c_n \delta), \quad K < \infty$.

$$\left|P\left(\xi_n(l) \le c_n t \,/\, \xi_n(0)=l\right) - P\left(\xi_n(l) \le c_n t\right)\right| \le \alpha_n(c_n \delta).$$

Further we have estimates in case $k \ge c_n \delta$:

$$|MI\left(\xi_n(l)=k \,/\, \xi_n(0)=l\right) s^{\nu_{n,l,k}(c_n t)} - MI\left(\xi_n(l)=k \,/\, \xi_n(0)=l\right) M\, s^{\nu_{n,l,k}(c_n t)}| \le 2\, \alpha_n(c_n \delta);$$

$$|M\, s^{\nu_{n,l,k}(c_n t)} - M\, s^{V_{n,l+k}(c_n t - l - k)}| \le K_1 \alpha_n(c_n \delta), \quad K_1 < \infty.$$

Now we can rewrite of (1) in the following form

$$g_n(c_n t, s) = 1 - P\left(\xi_n(0) \le c_n t\right) + a_{n,1}(\delta) + b_{n,1} + s \sum_{l=[c_n \delta]+1}^{[c_n t]} P\left(\xi_n(0)=l\right) f_{n,l}(c_n t - l, s),$$

$$f_{n,l}(c_n t - l) =$$

$$1 - P\left(\xi_n(l) \le c_n t\right) + a_{n,2}(\delta) + b_{n,2} + s \sum_{k=[c_n \delta]+1}^{[c_n t]} P\left(\xi_n(l)=k\right) f_{n,l+k}(c_n t - l - k, s), \quad l \ge [c_n \delta],$$

here

$$|b_{n,i}| \le k_i \alpha_n(c_n \delta), k_i < \infty, i = 1,2; \quad a_{n,1}(\delta) \le P\left(0 < \xi_n(0) \le c_n \delta\right),$$

$$a_{n,2}(\delta) \le \sup_{q \ge [c_n \delta]} P\left(0 < \xi_n(q) \le c_n \delta\right).$$

Further we introduce a sequence of independence random values with the same distribution function $\{\eta_k(n, \delta), \; k \ge 1\}$ under fixed δ. The distribution function is defined by the following equality

$$P\left(\eta_1(n, \delta) \le x\right) = P\left(\xi_n(c_n \delta) \le x\right).$$

We will denote

$$S_m(n,\delta)=\sum_{k=1}^{m}\eta_k(n,\delta),\qquad D_{n,\delta}(t)=\sup_{m\geq 1}\{m: S_m(n,\delta)\leq t\}.$$

$$M\, s^{D_{n,\delta}(t)}=\sum_{d=0}^{\infty}s^d\, P\big(D_{n,\delta}(t)=d\big)=:F_{n,\delta}(t,s)$$

We will estimate of difference of $f_{n,l}(c_n t-l,s),\, l\geq c_n\delta$ and $F_{n,\delta}(c_n t-l,s)$.
The definition leads to

$$f_{n,l}(c_n t-l,s)=\sum_{d=0}^{[c_n t-l]}s^d\, P\big(\nu_{n,l}(d)\leq c_n t-l, \nu_{n,l}(d+1)>c_n t-l\big).$$

Further we get for $d=0$ by condition 2

$$P\big(\xi_n(l)\geq c_n t-l\big)\pm P\big(\xi_n(c_n\delta)\geq c_n t-l\big)=\theta_n+P\big(\xi_n(c_n\delta)\geq c_n t-l\big).$$

Here and after the designation θ_n means that we have some sequence of numbers such that it convergences to zero under $n\to\infty$ and the following condition holds

$$|\theta_n|\leq 2\sup_{y\leq t}\sup_{\delta\leq\Delta\leq t}|P\big(\xi_n([c_n\Delta])c_n^{-1}<y\big)-R_2(y)|\,.$$

We have for $d=1$: $P\big(\nu_{n,l}(1)\leq c_n t-l, \nu_{n,l}(2)>c_n t-l\big)=$

$$=\sum_{k=1}^{[c_n t-l]}P\big(\xi_n(l)=k,\ \xi_n(k+l)>c_n t-l-k\big)=$$

$$=a_{n,\delta}+r_{n,1,\delta}+\sum_{k=[c_n\delta]}^{[c_n t-l]}P\big(\xi_n(l)=k\big)P\big(\xi_n(k+l)>c_n t-l-k\big)=$$

$$=a_{n,\delta}+r_{n,1,\delta}+\theta_n+\sum_{k=[c_n\delta]}^{[c_n t-l]}P\big(\xi_n(l)=k\big)P\big(\eta_2(n,\delta)>c_n t-l-k\big)=$$

$$=a_{n,\delta}+r_{n,1,\delta}+\theta_n+P\big(D_{n,\delta}(c_n t-l)=1\big)-$$

$$-\sum_{k=[c_n\delta]}^{[c_n t-l]}\sum_{s=[c_n\delta]}^{k}\Big(P\big(\xi_n(l)=s\big)-P\big(\eta_1(n,\delta)=s\big)\Big)\,P\big(\eta_2(n,\delta)=c_n t-l-k\big)= \qquad (3)$$

$$=P\big(D_{n,\delta}(c_n t-l)=1\big)+\pi_{n,1}.$$

Here

$$|\pi_{n,1}|\leq 2\big(a_{n,\delta}+\alpha_n(c_n\delta)+\theta_n\big),\quad a_{n,\delta}=\max_{i=1,2}\big(a_{n,i}\big)\quad |r_{1,n,\delta}|=2\,\alpha_n(c_n\delta)\,.$$

We used the Abel's transformation ([12], Chapter XI , Sec.383), for sum of pair factors of (3).
Similar considerations apply to the case $d=2$. Thus applying (3) we get

$$P\big(\nu_{n,l}(2)\leq c_n t-l, \nu_{n,l}(3)>c_n t-l\big)=a_{n,\delta}+r_{n,2}+$$

$$+\sum_{k=[c_n\delta]}^{[c_n t-l]} P\big(\xi_n(l)=k\big)\ P\big(\nu_{n,l+k}(1)\le c_n t-l-k,\nu_{n,l+k}(2)>c_n t-l\ -k\big)=$$

$$= a_{n,\delta}+r_{n,2,\delta}+\theta_n+\pi_{n,1,\delta}+P\big(D_{n,\delta}(c_n t-l)=2\big)-$$

$$-\sum_{k=[c_n\delta]}^{[c_n t-l]}\sum_{s=[c_n\delta]}^{k}\big(P(\xi_n(l)=s)-P(\eta_1(n,\delta)=s)\big)\big(P\big(D_{n,\delta}(c_n t-l-k-1)=1\big)-P\big(D_{n,\delta}(c_n t-l-k)=1\big)\big)=$$

$$=P\big(D_{n,\delta}(c_n t-l)=2\big)+\pi_{n,2}.$$

For the latter one we used the Abel's transformation too and the following equality which is checked easy.

$$P\big(D_{n,\delta}(c_n t-l-k-1)=1\big)-P\big(D_{n,\delta}(c_n t-l-k)=1\big)=$$

$$=P\big(S_2(n,\delta)=[c_n t]-l-k\big)+P\big(\eta_1(n,\delta)=[c_n t]-l-k\big).$$

The implicit introduced sequences have obvious sense and the following estimates take place $|r_{n,2,\delta}|\le 2\,\alpha_n(c_n\delta)$, $|\pi_{n,2}|\le 4\big(a_{n,\delta}+\alpha_n(c_n\delta)+\theta_n\big)$. It is no difficult to show with help induction that we have for $d=p$ the following formulas

$$P\big(\nu_{n,l}(p)\le c_n t-l,\nu_{n,l}(p+1)>c_n t-l\ \big)\ =$$

$$=P\big(D_{n,\delta}(c_n t-l)=p\big)+\pi_{n,p},\quad |\pi_{n,p}|\le 2\,p\big(a_{n,\delta}+\alpha_n(c_n\delta)+\theta_n\big).$$

Thus we obtained next representations for fixed $s\in(0,1)$

$$f_{n,l}(c_n t-l,s)=\ F_{n,\delta}(c_n t-l,s)+L_{n,\delta},\quad |L_{n,\delta}|\le L\big(a_{n,\delta}+\alpha_n(c_n\delta)+o_n(1)\big),\quad L<\infty.$$

$$g_n(c_n t,\ s)=1-P\big(\xi_n(0)\le c_n t\big)+K_{n,\delta}+s\sum_{l=[c_n\delta]+1}^{[c_n t]}P\big(\xi_n(0)=l\big)F_{n,\delta}(c_n t-l,s),$$

$$F_{n,\delta}(c_n t-l,s)=1-P\big(\xi_n(c_n\delta)\le c_n t-l\big)+Z_{n,\delta}+$$

$$+s\sum_{l=[c_n\delta]+1}^{[c_n t]}P\big(\xi_n(c_n\delta)=k\big)F_{n,\delta}(c_n t-l-k,s),\quad l\ge[c_n\delta].$$

Here the constructions of $K_{n,\delta}$ and $Z_{n,\delta}$ lead to the following relations

$$\lim_{n\to\infty}K_{n,\delta}=l_1 K_\delta;\quad \lim_{n\to\infty}Z_{n,\delta}=l_2 Z_\delta;\quad \max(l_1,l_2)\le\infty,$$

and $|K_\delta| \le 2\left(R_1(\delta) - R_1(0)\right)$, $|Z_\delta| \le 2\left(R_2(\delta) - R_2(0)\right)$.

Combining construction of $F_{n,\delta}(c_n t, s)$, condition 2 and continuity of convolution we conclude that the following limit exists

$$\lim_{\delta \to 0} \lim_{n \to \infty} F_{n,\delta}(c_n t, s) = F(t, s)$$

as this limit is unique solution the following equation

$$F(t, s) = 1 + R_2(t) + s R_2(\cdot) * F(t, s), \qquad (4)$$

here symbol $*$ denotes of convolution of two functions.

The sequence of generating functions $g_n(c_n t, s)$ has limit too

$$\lim_{\delta \to 0} \lim_{n \to \infty} g_n(c_n t, s) = g(t, s)$$

This limit is solution of the following equation

$$g(t, s) = 1 + R_1(t) + s R_2(\cdot) * F(t, s).$$

The latter one and (4) lead to proof of theorem.

Remark 1. We consider the extension of theorem 1. It consists in definition more weakly the mixing coefficient than $\alpha(\cdot)$.

Suppose that sequence c_n, $n \ge 1$ from theorem1 is defined. Now we take any sequence r_n, $n \ge 1$, which satisfies the following condition $r_n \to \infty$, $r_n = o(c_n)$ under $n \to \infty$.

Further we construct truncated process:

$$\overline{\xi_n}(t) = \begin{cases} \xi_n(t), & \xi_n(t) \le c_n - r_n, \\ c_n - r_n, & \xi_n(t) \ge c_n - r_n. \end{cases}$$

and construct the σ - algebra $F_{\le x}(r_n) = \sigma\left(\overline{\xi_n}(t), t \le x\right)$ too.

Now we define new mixing coefficient

$$\alpha_n(r_n, c_n) = \sup_{x \ge 0} \quad \sup\left\{ |P(B/A) - P(B)| : A \in F_{\le x}(r_n),\ B \in F_{\ge x + c_n} \right\}.$$

Thus this coefficient is constructed only on those events from $F_{\le x}$ on which the process $\xi_n(t)$ is less of value $c_n - r_n$ under $t \le x$. Such coefficient is useful in those cases when time dependence of researched events is controlled by values of process $\xi_n(t)$. For example, if the event $\{\xi_n(x)=k\}$ restricts of investigation of such events by interval $[0, x+k]$.

Now we divide second sum of (2) in this way:

$$\sum_{[c_n\delta]+1}^{[c_n t]} = \sum_{[c_n\delta]+1}^{[(c_n - r_n)t]} + \sum_{[(c_n - r_n)t]+1}^{[c_n t]} . \qquad (5)$$

We can do second sum from (5) less any given value due to continuity of $R_i(\cdot)$.

Further we apply the transformation from the proof of theorem 1 to first sum with use coefficient $\alpha_n(r_n, c_n)$.

Thus we can replace the condition 3 of theorem 1 the following condition

3' it exists such sequence $r_n, n\ge 1$: $r_n \to \infty$, $r_n = o(c_n)$ *under* $n\to\infty$ *that*

$$c_n \alpha_n(r_n, c_n) \underset{n\to\infty}{\to} 0 .$$

Remark 2.*If the sequence* $\{\tau_i, i\ge 0\}$ *be such that*

$$\lim_{i\to\infty} i^{-1}\tau_i = \mu^{-1}, \quad a.s., \ \mu = const.,$$

then we get the following convergence under conditions theorem 1

$$P\left(\tau_{\beta_n(i)} \le x c_n\right) \underset{n\to\infty}{\Rightarrow} R_1 * R_2^{*(i-1)}(x\mu) .$$

It follows from the known theorems of transfer (look, for example [11]).

2. Interaction of two renewal processes.

The model of raring process which is considered below is result interaction two renewal processes. This model was offered in [13] as the mathematical model of practice problem.

Let us denote by Z and H two renewal processes : $Z=\{\varsigma_i, i\ge 1\}$, $H=\{\eta_i, i\ge 1\}$.

We define stochastic characteristics of Z, H:

$$\tau_i = \sum_{l=1}^{i}\eta_l, \quad \vartheta_i = \sum_{l=1}^{i}\varsigma_l, \quad i=1,2,\ldots ; \quad N_1(t)=\sup\{n:\tau_n < t\}, \quad N_2(t)=\sup\{n:\vartheta < t\},$$

$$\gamma_1^+(t)=\tau_{N_1(t)+1} - t, \quad \gamma_2^+(t)=\vartheta_{N_2(t)+1} - t, \quad t>0 .$$

The moments of time τ_i, ϑ_i, $i \geq 1$ are named renewal points processes H and Z respectively.

If we have a renewal points of the process Z in interval $(\tau_{n-1}, \tau_n]$ then we will say that the renewal point τ_n is marked by process Z. The process H marks a renewal points of process Z analogy.

Let us denote by $T_0'' = 0, T_1'', \ldots$ renewal points of H which were marked by Z and $T_1', T_2', \ldots$ renewal points of Z which were marked by H. It is clear that the following inequalities $T_0'' = 0 < T_1' \leq T_1'' \leq T_2' \leq \ldots$ take place. It is shown in [4] that sequence random values

$$V_n = T_n' - T_{n-1}'', \quad U_n = T_n'' - T_n', \quad n = 1, 2, \ldots,$$

be Markov's chain. This chain is defined by transition probabilities

$$P(V_1 < x) = P(\varsigma_1 < x), \; P(U_n < x / V_n = y), \quad P(V_{n+1} < x / U_n = y), \quad n = 1, 2, \ldots .$$

It is easy to see that for investigation V_n, U_n it is necessary simultaneously to observe two raring processes:

$$T'' = \{T_0'' = 0, T_1'', T_2'' - T_1'', \ldots\} \quad T' = \{T_1', T_2' - T_1', \ldots\}.$$

We will investigate these raring processes separately. We will use that the processes T'', T' are raring processes respect to processes H, Z respectively.

We take, for example, T''. The T'', as underflow of H, defines the following indicators

$$\chi(i) = \begin{cases} 1, & \text{if } i-th \text{ renewal point of } H \text{ belongs to } T'', \\ 0, & \text{otherwise.} \end{cases}$$

$$\xi(l) = \inf\{j \geq 1 : \chi(l + j) = 1\}, \quad l \geq 0.$$

Thus $\beta(i) = \beta(i-1) + \xi(\beta(i-1))$, $i \geq 1$ be number of the i-th event from H which belongs to T''. The moment $\tau_{\beta(i)}$ is moment of appearance this event. We shall suppose that processes H and Z depend on a parameter $n, n \to \infty$ such that $H_n = \{\eta_{n,i}, i \geq 1\}$, $Z_n = \{\varsigma_{n,i}, i \geq 1\}$. Now the characteristics these processes have forms: $\tau_{n,i}$, $\vartheta_{n,i}$, $i = 1, 2, \ldots$, $\gamma^{+}_{n,k}(t)$, $k = 1, 2$.

Theorem 2. *If the following conditions:*

1) there are a positive numbers $c_n \to \infty$ and distribution function $G(x), x \geq 0$ guaranteeing the following limit

$$\lim_{n\to\infty}\sup_{t\geq 0}\ |\,P\left(\gamma^{+}_{n,2}(t)<\tau_{n,[nx]}\right)-G(x)\,|=0,$$

here x *is continuous point of* $G(x)$*;*

2) $\lim\limits_{n\to\infty} c_n^{-1}\tau_{n,c_n}=\mu,\quad \mu=const.$.

hold then $P\left(\tau_{\beta_n(k)}<xc_n\right)\underset{n\to\infty}{\to}G^{*k}\left(\dfrac{x}{\mu}\right)$.

Proof. We will check all conditions of Theorem 1 for process $\xi_n(l)$. We calculate the probability $P(\xi\ (l)=m)$:

$$P(\xi\ (l)=1)=P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}\right).$$

$$P(\xi\ (l)=2)=P\left(\gamma^{+}_{n,2}(\tau_l)\geq\eta_{l+1},\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}+\eta_{l+2}\right)=$$

$$=P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}+\eta_{l+2}\right)-P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}\right).$$

$$\dots$$

$$P(\xi\ (l)=k)=P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}+\dots+\eta_{l+k}\right)-P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}+\dots+\eta_{l+k-1}\right).$$

Thus

$$P(\xi\ (l)\leq m)=\sum_{k=1}^{m}P(\xi\ (l)=k)\ =\ P\left(\gamma^{+}_{n,2}(\tau_l)<\eta_{l+1}+\dots+\eta_{l+m}\right).\qquad(6)$$

The latter one and condition 1 lead to the following convergence

$$P(\xi_n(l)<c_nx)=\int_0^{\infty}P\left(\gamma^{+}_{n,2}(t)<\tau_{n,[c_nx]}\right)P\left(\tau_{n,l}\in dt\right)\underset{n\to\infty}{\to}G(x),\quad l=0,1,2,\dots\ .$$

here $x>0$ is continuous point of $G(x)$.

We have the following equality when it is considered that (6) holds

$$P\left(\xi\ (l+r)\leq s/\xi\ (l)\leq m\right)-P\left(\xi\ (l+r)\leq s\right)=0,\quad \text{if}\quad m<r\,.$$

Now we have for any sequences of numbers r_n such that $r_n\to\infty,\ r_n<c_n,\ n\geq 0$ $\alpha_n(r_n,c_n)=0,\quad n\geq 0$.

Thus all conditions of theorem 1 hold respect to process $\xi_n(t)$. Now the statement of theorem 2 becomes apparent if it is remembered the theorem of transfer.

Example. Now we consider example of definition of sequence c_n and limits' function $G(x)$ from
Theorem 2. We shall suppose that process Z is Poisson process with parameter λ_n such that $\lambda_n \to 0$ under $n \to \infty$. The process H don't depends on parameter n. The renewal interval of H has finite expectation $\mu = M\eta_1 < \infty$.

All these suppositions led to formula

$$P\left(\gamma^{+}_{n,2}(\tau_l) < \eta_{l+1} + \dots + \eta_{l+m}\right) = \int_0^\infty \lambda_n e^{-\lambda_n y} P(\tau_m > y)\, dy =: G_n(m).$$

If we put $m = [c_n x]$ and make change of variables $\lambda_n y = z$ then we get

$$G_n([c_n x]) = \int_0^\infty e^{-\lambda_n y} P\left(\lambda_n \tau_{[c_n x]} > z\right) dz.$$

Put $c_n =: \lambda_n^{-1}$. The indicator of set A will be denoted by $I(A)$. The following convergences are based on strong law of large numbers.

$$G_n([c_n x]) = \int_0^\infty e^{-z} P\left(x \frac{\tau_{[\lambda_n^{-1} x]}}{[\lambda_n^{-1} x]} > z \right) dz \underset{n\to\infty}{\longrightarrow} \int_0^\infty e^{-z} I(x\mu > z)\, dz = 1 - e^{-\mu x}.$$

and

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \mu \quad a.s.$$

Thus we checked all conditions of Theorem 2. The function $G(x)$ (from condition 1 of theorem 2) be limit for the functions $G_n([x\lambda_n^{-1}])$. In this example the moment of appearance k-th event in flow T_n'' has the following limit distribution function $P\left(\tau_{\beta_n(k)} < x\lambda_n^{-1}\right) \underset{n\to\infty}{\longrightarrow} \left(1 - \exp(\cdot)\right)^{*k}(x), \quad x \geq 0$.

It is clear that similar example we may consider for process T'. In this case the process H must be Poisson with "rare" events and the process Z must be a simple renewal process with bounded expectation of time between neighboring renewal point.

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