ASYMPTOTIC ANALYSIS OF LOGICAL SYSTEMS WITH UNRELIABLE ELEMENTS

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In this paper models of networks with unreliable arcs are investigated. Asymptotic formulas for probabilities of the networks work or failure and the networks lifetime distributions are obtained. Direct calculations of these characteristics in general case [1], [2] demand sufficiently large volumes of arithmetical operations. Main parameters of the asymptotic formulas are minimal way length and minimal section ability to handle. A series of new algorithms and formulas to calculate parameters of asymptotic formulas are developed.

Main characteristics. Define oriented graph Γ with finite number of nodes U and the set W of arcs (u,v). In this graph there is single node u_*, without input arcs and single node u^*, without output arcs, the graph has not arcs (u,u).

Suppose that $n(s)$ is a number of arcs of a subgraph s, $s \subseteq W$. For $S \subseteq \{s : s \subseteq W\}$ put

$$n(S) = \min_{s\in S} n(s),\ D(S) = \sum_{s:n(s)=n(S)} \prod_{(u,v)\in s} c(u,v),$$

$$C(S) = \min_{s\in S} C(s),\ C(s) = \sum_{(u,v)\in s} c(u,v),$$

$$C_1(S) = \min_{s\in S} C_1(s),\ C_1(s) = \max_{(u,v)\in s} c(u,v),$$

$$T_h(S) = \sum_{s:C_1(s)=C_1(S)} \prod_{(u,v)\in s} \exp\left(-h^{-c(u,v)}\right),$$

$c(u,v)$ - is positive and integer function. Designate by $N(S)$, $N_1(S)$, $N_*(S)$ - numbers of $s \in S : C(s) = C(S)$, $C_1(s) = C_1(S)$, $n(s) = n(S)$ correspondingly.

Put $\Re$ the set of all ways R from u_* to u^* without selfintersections. Consider the sets $\mathcal{A} = \{A \subset U, u_* \in A, u^* \notin A\}$, $L = L(A) = \{(u,v) : u \in A, v \notin A\}$ and $\mathcal{L} = \{L(A), A \in \mathcal{A}\}$ - is the set of all sections in the graph Γ.

Graphs with unreliable arcs. For each the graph Γ define arc define the number $\alpha(u,v) = I$(the arc (u,v) works), where $I(G)$ - is an indicator function of the event G. It is not difficult to confirm, that

$$\bigvee_{R\in\mathcal{R}}\ \bigwedge_{(u,v)\in R}\alpha(u,v)=\bigvee_{L\in\mathcal{L}}\ \bigwedge_{(u,v)\in L}\alpha(u,v). \qquad (1)$$

Denote $\alpha(\Gamma)$ the quantity of both sides of the equality (1) which characterizes the graph Γ work.

Suppose that $\alpha(u,v),(u,v)\in W$ are independent random variables, $P(\alpha(u,v)=1)=p_{u,v}(h)$, $q_{u,v}(h)=1-p_{u,v}(h)$, where h - is small parameter: $h\to 0$. Then the following asymptotic formulas are true for $h\to 0$.

1. If $p_{u,v}(h)\sim c(u,v)h$, then $P(\alpha(\Gamma)=1)\sim h^{n(\mathcal{R})}D(\mathcal{R})$.
2. If $p_{u,v}(h)\sim h^{c(u,v)}$, then $P(\alpha(\Gamma)=1)\sim N(\mathcal{R})h^{C(\mathcal{R})}$.
3. If $p_{u,v}(h)\sim \exp\left(-h^{-c(u,v)}\right)$, then $P(\alpha(\Gamma)=1)\sim T_h(\mathcal{R})$.
4. If $q_{u,v}(h)\sim c(u,v)h$, then $P(\alpha(\Gamma)=0)\sim h^{n(\mathcal{L})}D(\mathcal{L})$.
5. If $q_{u,v}(h)\sim h^{c(u,v)}$, then $P(\alpha(\Gamma)=0)\sim N(\mathcal{L})h^{C(\mathcal{L})}$.
6. If $q_{u,v}(h)\sim \exp\left(-h^{-c(u,v)}\right)$, then $P(\alpha(\Gamma)=0)\sim T_h(\mathcal{L})$.

Applications to lifetime models. Suppose that $\tau(u,v)$ - independent random variables are arcs $(u,v)\in W$ lifetimes. Denote $P(\tau(u,v)>t)=p_{u,v}(h)$ and put the graph Γ lifetime $\tau(\Gamma)=\min\limits_{R\in\mathcal{R}}\ \max\limits_{(u,v)\in R}\tau(u,v)$.

Suppose that $h=h(t)$ s monotonically decreasing and continuous function and $h\to 0$, $t\to\infty$, then asymptotic formulas 1, 2, 3 are true if $P(\alpha(\Gamma)=1)$ is replaced by $P(\tau(\Gamma)>t)$. Suppose that h is monotonically increasing and continuous function and $h\to 0$, $t\to 0$, then the formulas 4, 5, 6 are true if $P(\alpha(\Gamma)=0)$ is replased by $P(\tau(\Gamma)\le t)$.

Calculation of graph characteristics. For $A\in\mathcal{A}$ define $Q(A)=\{v\notin A:\exists u\in A,(u,v)\in W\}$ and construct the sets $A_1=Q(A_0)=\{u_*\}$, $A_{k+1}=A_k \, r \, Q(A_k)$, $k=1,2,\ldots$ Denote $n=n(\mathcal{R})=\min\left(k:u^*\in A_k\right)$.

Designate by $\varphi(u,v)$, $(u,v)\in W$ integer and nonnegative function: $\sum\limits_{(u,v)\in W}\varphi(u,v)=\sum\limits_{(v,u)\in W}\varphi(v,u)$, $\varphi(v,u)\le c(u,v)$, $(u,v)\in W$, and call it a flow. A quantity of the flow is the sum $\sum\limits_{(u_*,v)\in W}\varphi(u_*,v)$.

Denote by $\Gamma_1\ \ \Gamma_2$ he graph constructed from the graphs Γ_1, Γ_2 by a connection of their initial and final nodes, correspondingly, and by $\Gamma_1\to\Gamma_2$ the graph constructed by a connection of the graph Γ_1 final node with the graph Γ_2 initial node. Consider the sets $\mathcal{R}_1$, $\mathcal{L}_1$, $\mathcal{R}_2,\mathcal{L}_2$ for the graphs Γ_1, in the same sense as the sets $\mathcal{R}$, $\mathcal{L}$ for the graph Γ. Suppose that further $u_i\in Q(A_{i-1})$, $i=1,\ldots,n$.

Calculation of $D(R)$: $D(u_1)=1,\ u_1\in A_1,\ \ D(u_{k+1})=\sum_{u_k\in Q(A_{k-1})} D(u_k)c(u_k,u_{k+1}),\ \ 1\le k<n,$ $D(R)=D\left(u^*\right).$

Calculation of $N_*(R)$: $N_*(u_{n-1})=1,\ u_{n-1}\in Q(A_{n-2}),$
$N_*(u_{n-k-1})=\sum_{u_{n-k}\in Q(A_{n-k-1})} N_*(u_{n-k})I((u_{n-k-1},u_{n-k})\in W),\ 1\le k<n-1,\ N_*(R)=N_*(u_*).$

Calculation of $C(R), N(R)$: each arc (u,v) of the graph is devided into arcs with initial lengths (because the function $c(u,v)$ is integer). Then the graph Γ is transformed into the graph Γ_1 with single lengths arcs and applying the $n=n(R),\ N_*(R)$ calculation procedures to the graph Γ_1 obtain $C(R),\ N(R)$ forthe graph Γ.

Calculation of $C(L), n(L)$: using the theorem [3] of coincidence of maximal flow value and minimal section ability to handle $C(L)$ and Ford-Falkerson algorithm define $C(L)$. Then $n(L)$ equals to $C(L)$ for $c(u,v)\equiv 1$.

Suppose that $W=\{(u_k,u_{k+1}),u_i\in Q(A_{i-1}),i=1,...,n\}$ in next five points.

Calculation of $C_1(R)$: $C_1(u_1)=0,\ u_1\in A_1,\ \ C_1(u_{k+1})=\min_{u_k\in Q(A_{k-1})}\max(C_1(u_k),c(u_k,u_{k+1})),$ $1\le k<n,\ C_1(R)=C_1\left(u^*\right).$

Calculation of $N_1(R)$: $N_1(u_1)=1,\ u_1\in A_1,\ \ N_1(u_{k+1})=\sum_{u_k:C_1(u_{k+1})=\max(C_1(u_k),c(u_k,u_{k+1}))} N_1(u_k),$ $1\le k<n,\ N_1(R)=N_1\left(u^*\right).$

Calculation of $C_1(L)$: as the formula (1) leads to $C_1(L)=\max_{R\in R}\min_{(u,v)\in R} c(u,v)$, then $C_1(L)$ is defined by $C_1(u_{k+1})=\max_{u_k\in Q(A_{k-1})}\min(C_1(u_k),c(u_k,u_{k+1})),\ 1\le k<n,\ C_1(u_1)=\infty,$ $u_1\in A_1, C_1(L)=C_1\left(u^*\right).$

Direct formulas of $C(L),\ N(L)$: if $c(u_k,u_{k+1})\equiv c_k,\ \ 1\le k<n-1,\ c\left(u_{n-1},u^*\right)=c_n,\ \ N_k$ - is a number of nodes in $Q(A_{k-1}),\ 1\le k<n-1,\ N_n=1$, then $C(L)=\min_{1\le k<n} c_kN_kN_{k+1}$, and $N(L)$ - is a number of elements in the set $\{k: M=c_kN_kN_{k+1}, 1\le k<n\}$.

Weak elements in the graph Γ. Suppose that for any pairs of arcs $(u_1,v_1),(u_2,v_2)\in W$, such that $(u_1,v_1)\ne(u_2,v_2)$, the inequality $c(u_1,v_1)\ne c(u_2,v_2)$ is true. Then there is single arc $(u(S),v(S))\in s: C_1(s)=c(u(S),v(S))$, and $-\ln T_h(S)\sim h^{-c(u(S),v(S))},\ h\to 0$.

Call this arc $(u(S),v(S))$ a weak element of (Γ,S).

3′. If $p_{u,v}(h)\sim\exp\left(-h^{-c(u,v)}\right),\ h\to 0$, then $-\ln P(\alpha(\Gamma)=1)\sim -h^{-c(u(R),v(R))}$.

6′. If $q_{u,v}(h)\sim\exp\left(-h^{-c(u,v)}\right),\ h\to 0$, then $-\ln P(\alpha(\Gamma)=0)\sim -h^{-c(u(L),v(L))}$.

In conditions of the statement $\mathbf{3'}$ or the statement $\mathbf{6'}$ a definition of a weak element $(u(S),v(S))$ is made by *the procedure for* $C_1(S)$ with $S=R$ or with $S=L$, correspondingly. A definition of the weak element and related asymptotic formula may be spread from a network onto arbitrary logic function represented in a disjunctive or in a conjunctive normal form.

Recursive formulas for the graph $\Gamma_1 \quad \Gamma_2$:

$$C(R)=\min\left(C(R_1),C(R_2)\right), \tag{2}$$

$$N(R)=\begin{cases} N(R_1), C(R_1)<C(R_2), \\ N(R_2), C(R_2)>C(R_1), \\ N(R_1)+N(R_2), C(R_1)=C(R_2), \end{cases} \tag{3}$$

$$C(L)=C(L_1)+C(L_2), \tag{4}$$

$$N(L)=N(L_1)N(L_2), \tag{5}$$

$$C_1(L)=\max\left(C(L_1),C(L_2)\right), \tag{6}$$

$C_1(R)$, $n(R)$ are defined analogously (2), $N_1(R), n(L)$ are defined analogously (3), (4), correspondingly.

Recursive formulas for the graph $\Gamma_1 \to \Gamma_2$. $C(R)$, $n(R)$ are defined analogously (4), $N(R)$, $N_1(R)$ are defined analogously (5), $C_1(R)$ are defined analogously (6), $C(L), C_1(L), n(L)$ are defined analogously (2), $N(L), N_1(L)$ are defined analogously (3).

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