NARROW PLACES IN LOGICAL SYSTEMS WITH ANRELIABLE ELEMENTS

Gurami Tsitsiashvili

●

e-mail: guram@iam.dvo.ru
640041, Vladivostok, Radio str. 7, IAM, FEB RAS

In this paper models of logical systems with anreliable elements are considered [1], [2]. Definitions of narrow places in these systems are made, algorithms of narrow places constructions are built. An asymptotic analysis of work probability (or failure probability) of logical systems in appropriate asymptotic conditions for a work probability (or a failure probability) of their elements is made. All main definitions and algorithms are based on an idea of a recursive construction of logical models with anreliable elements.

Preliminaries

Suppose that Z is a set which consists of $|Z|$ logical variables z. Define the recursive class $\mathcal{G}$ of logical expressions of variables $z \in Z$:

$$z \in Z \Rightarrow z \in \mathcal{G},\ A_1 \in \mathcal{G},\ A_2 \in \mathcal{G} \Rightarrow (A_1 \wedge A_2) \in \mathcal{G},\ (A_1 \vee A_2) \in \mathcal{G}. \qquad (1)$$

Denote $2^Z = \{Z_i,\ i \in I = \{1,\dots,2^Z\}\}$ the family of all subsets of the set Z. Define the disjunctive normal form of the logical expression $A \in \mathcal{G}$: for $z \in Z$, $A_1 \in \mathcal{G}$, $A_2 \in \mathcal{G}$, $I_1, I_2 \subseteq I$

$$\begin{gathered} D(z) = z,\ D(A_1) = \bigvee_{i \in I_1}\left(\bigwedge_{z \in Z_i} z\right),\ D(A_2) = \bigvee_{i \in I_2}\left(\bigwedge_{z \in Z_i} z\right) \Rightarrow \\ D(A_1 \vee A_2) = \bigvee_{i \in I_1 \cup I_2}\left(\bigwedge_{z \in Z_i} z\right),\ D(A_1 \wedge A_2) = \bigvee_{i \in I_1,\, j \in I_2}\left(\bigwedge_{z \in Z_i \cup Z_j} z\right) \end{gathered} \qquad (2)$$

Analogously define the conjunctive normal form $K(A)$, $A \in \mathcal{G}$: for $z \in Z$, $A_1 \in \mathcal{G}$, $A_2 \in \mathcal{G}$, $I_1, I_2 \subseteq I$

$$\begin{gathered} K(z) = z,\ K(A_1) = \bigwedge_{i \in I_1}\left(\bigvee_{z \in Z_i} z\right),\ K(A_2) = \bigwedge_{i \in I_2}\left(\bigvee_{z \in Z_i} z\right) \Rightarrow \\ K(A_1 \vee A_2) = \bigwedge_{i \in I_1,\, j \in I_2}\left(\bigvee_{z \in Z_i \cup Z_j} z\right),\ K(A_1 \wedge A_2) = \bigwedge_{i \in I_1 \cup I_2}\left(\bigvee_{z \in Z_i} z\right) \end{gathered} \qquad (3)$$

For the families of sets $\mathcal{X} = \{X\} \subseteq 2^Z$, $\mathcal{Y} = \{Y\} \subseteq 2^Z$ put

$\mathcal{X} \otimes \mathcal{Y} = \{ X \cup Y : X \in \mathcal{X}, Y \in \mathcal{Y} \}$, $Z(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$, $N(\mathcal{X}) = \min(|X| : X \in \mathcal{X})$.

If $Z(\mathcal{X}) \cap Z(\mathcal{Y}) = \emptyset$, then

$$N(\mathcal{X} \otimes \mathcal{Y}) = N(\mathcal{X}) + N(\mathcal{Y}),\ N(\mathcal{X} \cup \mathcal{Y}) = \min(N(\mathcal{X}), N(\mathcal{Y})). \qquad (4)$$

Suppose that $p_z = P(z=1)$, $q_z = P(z=0)$, $p_z + q_z = 1$, and the random variables $z \in Z$ are independent. The logical function A with random $z \in Z$ call the logical system $\mathcal{A}$.

Lowreliable elements

Suppose that $\exists\ d > 0$ so that for $\forall z \in Z\ \exists$ the natural number $c(z)$:

$$p_z = p_z(h) \sim \exp\left(-h^{-dc(z)}\right),\ h \to 0. \qquad (5)$$

Denote by $\tau(z)$ variables which equal to lifetimes of logical elements $z \in Z$, and $P(\tau(z) > t) = p_z(h)$. If $h = h(t)$ is monotonically decreasing and continuous function and $h \to 0$, $t \to \infty$, then the formula (5) may be transformed to the form

$$P(\tau(z) > t) \sim \exp\left(-h(t)^{-dc(z)}\right),\ t \to \infty,$$

characteristic of Weibull asymptotic used in lifetime models of systems which consist of lowreliable elements [3], [4].

Define $C(A) = \min_{i \in I} \max_{z \in Z_i} c(z)$ by known $D(A) = \bigvee_{i \in I_1} \left(\bigwedge_{z \in Z_i} z \right)$. Then from (2) obtain

$$C(A_1 \wedge A_2) = \max(C(A_1), C(A_2)),\ C(A_1 \vee A_2) = \min(C(A_1), C(A_2)). \qquad (6)$$

Correspond to the logical function A the families of sets $\mathcal{S}(A) \subseteq 2^Z$, $\mathcal{T}(A) \subseteq 2^Z$ by recursive formulas: $\mathcal{S}(z) = \{z\}, \mathcal{T}(z) = \{z\}$,

$$\mathcal{S}(A_1 \wedge A_2) = \begin{cases} \mathcal{S}(A_1), C(A_1) > C(A_2), \\ \mathcal{S}(A_2), C(A_1) < C(A_2), \\ \mathcal{S}(A_1) \otimes \mathcal{S}(A_2), C(A_1) = C(A_2), \end{cases} \quad \mathcal{T}(A_1 \wedge A_2) = \begin{cases} \mathcal{T}(A_1), C(A_1) > C(A_2), \\ \mathcal{T}(A_2), C(A_1) < C(A_2), \\ \mathcal{T}(A_1) \cup \mathcal{T}(A_2), C(A_1) = C(A_2), \end{cases}$$

$$\mathcal{S}(A_1 \vee A_2) = \begin{cases} \mathcal{S}(A_1), C(A_1) < C(A_2), \\ \mathcal{S}(A_2), C(A_1) > C(A_2), \\ \mathcal{S}(A_1) \cup \mathcal{S}(A_2), C(A_1) = C(A_2), \end{cases} \quad \mathcal{T}(A_1 \vee A_2) = \begin{cases} \mathcal{T}(A_1), C(A_1) < C(A_2), \\ \mathcal{T}(A_2), C(A_1) > C(A_2), \\ \mathcal{T}(A_1) \otimes \mathcal{T}(A_2), C(A_1) = C(A_2). \end{cases}$$

Put $I' = \left\{ i \in I : \max_{z \in Z_i} c(z) = C(A) \right\}$, then from the formulas (2), (6) obtain

$$\mathcal{S}(A) = \left\{ \left\{ z \in Z_i : c(z) = C(A) \right\} : i \in I' \right\}. \quad (7)$$

The formulas (1), (2), (6), (7) lead to the statements.

Theorem 1. *In conditions (5) the formula* $-\ln P(A=1) \sim N(\mathcal{S}(A)) h^{-C(A)}$, $h \to 0$, *is true.*

Theorem 2. *In conditions (5) the following statements take place:*

1. for each $S \in \mathcal{S}(A)$ *the following formula is true*

$$\left(c(z) \to c(z) - \varepsilon,\, z \in S \right) \Rightarrow \left(C(A) \to C(A) - \varepsilon \right),\, 0 < \varepsilon < 1; \quad (8)$$

2. if a set $S \subseteq Z$ *and satisfies (8), then* $\exists\, S_* \in \mathcal{S}(A)$: $S_* \subseteq S$;

3. for each $T \in \mathcal{T}(A)$ *the following formula is true*

$$\left(c(z) \to c(z) + \varepsilon,\, z \in T \right) \Rightarrow \left(C(A) \to C(A) + \varepsilon \right),\, 0 < \varepsilon < 1; \quad (9)$$

4. if $T \subseteq Z$ $S \subseteq Z$ *and satisfies the formula (9), then* $\exists\; T_* \in \mathcal{T}(A)$: $T_* \subseteq T$.

The theorem 2 allows to call sets from the families $\mathcal{S}(A), \mathcal{T}(A)$ by narrow places in the logical system $\mathcal{A}$ with lowreliable elements. For any $S \in \mathcal{S}(A)$(for any $T \in \mathcal{T}(A)$) an increase of elements $z \in S$ (a decrease of elements $z \in T$) reliabilities leads to an increase (to a decrease) of the logical system $\mathcal{A}$ reliability. The formulas (6) and the theorem 2 allow to calculate recursively the numbers $C(A)$, $N(\mathcal{S}(A))$ and to define the families $\mathcal{S}(A), \mathcal{T}(A)$ and their subfamilies

$$\mathcal{S}'(A) = \left\{ S \in \mathcal{S}(A) : |S| = N(\mathcal{S}(A)) \right\},\; \mathcal{T}'(A) = \left\{ T \in \mathcal{T}(A) : |T| = N(\mathcal{T}(A)) \right\}.$$

Highreliable elements

Suppose that $\exists\; d > 0$ so that for $\forall\;\; z \in Z\;\; \exists$ the natural number $c(z)$:

$$q_z = q_z(h) \sim \exp\left(-h^{-dc(z)} \right),\; h \to 0. \quad (10)$$

Denote $P(\tau(z) \le t) = q_z(h)$. If h is monotonically increasing and continuous function and $h \to 0$, $t \to 0$, then the formula (10) may be transformed to the form

$$P(\tau(z) \le t) \sim \exp\left(-h(t)^{-dc(z)} \right),\; t \to 0\,,$$

characteristic of Weibull asymptotic used in lifetime models of systems which consist of highreliable elements.

Redefine $C(A)=\min_{i\in I}\max_{z\in Z_i} c(z)$ by known $K(A)=\bigwedge_{i\in I_1}\left(\bigvee_{z\in Z_i} z\right)$. From the formula (3) obtain

$$C(A_1\wedge A_2)=\min\left(C(A_1),C(A_2)\right),\ C(A_1\vee A_2)=\max\left(C(A_1),C(A_2)\right). \qquad (11)$$

Redefine $I'=\left\{i\in I:\min_{z\in Z_i} c(z)=C(A)\right\}$, then the formulas (3), (11) lead to the formula (7) for highreliable elements also. The formulas (3), (7), (11) lead to the statements.

Theorem 3. *In conditions (10) the formula* $-\ln P(A=0)\sim N\left(\mathcal{S}(A)\right)h^{-C(A)}$, $h\to 0$, *is true.*

Theorem 4. *In conditions (10) the following statements take place:*

1. for any $S\in\mathcal{S}(A)$ *the following formula is true*

$$\left(c(z)\to c(z)+\varepsilon,\ z\in S\right)\Rightarrow\left(C(A)\to C(A)+\varepsilon\right),\ 0<\varepsilon<1; \qquad (12)$$

2. if a set $S\subseteq Z$ *and satisfies (12), then* $\exists S_*\in\mathcal{S}(A): S_*\subseteq S$ *;*

3. for any $T\in\mathcal{T}(A)$ *the following formula is true*

$$\left(c(z)\to c(z)-\varepsilon,\ z\in T\right)\Rightarrow\left(C(A)\to C(A)-\varepsilon\right),\ 0<\varepsilon<1; \qquad (13)$$

4. if a set $T\subseteq Z$ *and satisfies (13), then* $\exists T_*\in\mathcal{T}(A): T_*\subseteq T$ *.*

The theorem 4 allows to call sets from the families $\mathcal{S}(A),\mathcal{T}(A)$ by narrow places in the logical system $\mathcal{A}$ with highreliable elements. For any $S\in\mathcal{S}(A)$ ($T\in\mathcal{T}(A)$) an increase of elements $z\in S$ (a decrease of elements $z\in T$) reliabilities leads to an increase (to a decrease) of the logical system $\mathcal{A}$ reliability. The formulas (12) and the theorem 4 allow to calculate recursively the numbers $C(A),N\left(\mathcal{S}(A)\right)$ and to define the families $\mathcal{S}(A),\mathcal{T}(A)$ and their subfamilies $\mathcal{S}'(A),\mathcal{T}'(A)$.

Remark 1. *Denote* $\mathcal{X}_1=\{Z_i,i\in I_1\},\mathcal{X}_2=\{Z_i,i\in I_2\}$, *suppose that* $Z(\mathcal{X}_1)\cap Z(\mathcal{X}_2)=\varnothing$. *Then the formula (4) allow to simplify significantly calculations of* $N\left(\mathcal{S}(A_1)\otimes\mathcal{S}(A_2)\right)$, $N\left(\mathcal{S}(A_1)\cup\mathcal{S}(A_2)\right)$, *which are necessary to find recursively* $N\left(\mathcal{S}(A)\right)$ *for the asymptotic formulas of the theorems 1, 3.*

References

[1] Riabinin I.A. Logic-probability calculus as method of reliability and safety investigation in complex systems with complicated structure// Automatics and remote control, 2003, № 7, p. 178-186 (in Russian).

[2] Solojentsev E.D. Specifics of logical-probability risk theory with groups of antithetical events// Automatics and remote control, 2003, № 7, p. 187-203 (in Russian).

[3] Rocchi P. Boltzmann-like Entropy in Reliability Theory// Entropy, 2002, vol. 4, p. 142-150.

[4] Rocchi P. The Notion of Reversibility and Irreversibility at the Base of the Reliability Theory// Proccedings of the International Symposium on Stochastic Models in Reliability, Safety, Security and Logistics. Sami Shamoon College of Engineering. Beer Sheva, February 15-17, 2005, p. 287-291.