

A NEW GENERALIZATION OF RAYLEIGH DISTRIBUTION

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Abstract. In this paper we propose a new generalized Rayleigh distribution different from that introduced in "Aplikace Matematiky" (1976) [23]: the construction makes use of the so-called "conservability approach" (see "Kybernetika", 1989 [26]), namely if X is a positive continuous random variable with a finite mean-value $E(X)$, then a new density is set $-f_1(x) = x \cdot f(x) / E(x)$ where $f(x)$ is the p.d.f. of X . The new generalized Rayleigh variate is obtained using as $f(x)$ a generalized form of the exponential distribution introduced by Isaic-Maniu and the present author in 1996 (see [12]).

Key words: GRV-generalized Rayleigh variate, GE-generalized exponential, GDE-generating differential equation, conservability, p.d.f.-probability density function, pseudo-Weibull variable.

1. A little bit of statistical history

Some decades ago, I did publish in the Prague journal "Aplikace Matematiky" (see Vodă, 1976 [23, 24]) two papers regarding a generalized variant of the Rayleigh density function:

$$(1) \quad X : f(x; \theta, k) = \frac{2\theta^{k+1}}{\Gamma(k+1)} x^{2k+1} \exp(-\theta x^2), \quad x \geq 0, \quad \theta > 0, \quad k \geq 0,$$

where $\Gamma(u)$ is the well-known Gamma function:

$$(2) \quad \Gamma(u) = \int_0^{\infty} t^{u-1} e^{-t} dt.$$

The form (1) includes – apart from the classical Rayleigh density (for $k = 0$) some others, such as Maxwell (for $k = 1/2$) and Chi (χ) one (for $k = \frac{1}{2}a - 1$, $a \in \mathbf{N}$, $a > 2$ and $\theta = 1/2b^2$, $b > 0$).

Also, if we quit the positivity request for k and take $k = -1/2$ and $\theta = 1/2\sigma^2$, $\sigma > 0$ we shall obtain the "half-normal" density.

This form (1) has found its place in the well-known book of Johnson-Kotz-Balakrishnan (vol. 1, 1994, page 479 [14]).

There exist a lot of density functions which may be considered as generalizations for the Rayleigh one. For instance, in a monograph published in Kiev in 1987 by E. S. Pereverzev (see [17]) I detected two forms, namely:

$$(3) \quad F(x; k, a) = 1 - \exp(-x^{2k} / 2a^2) \quad x \geq 0, \quad k, a > 0,$$

this distribution becoming for $k = 1/2$ the exponential one and for $k = 1$, the Rayleigh one:

$$(4) \quad F(x) = 1 - \exp\left[-\frac{a(x-x_1)^n}{(x_2-x_1)^m}\right], \quad x \in [x_1, x_2] \subset \mathbf{R}$$

with $a > 0$ and $n, m \in \mathbf{N}$, which generalizes Rayleigh ($m = 0, x_1 = 0, n = 2$ and $x \in [0, \infty)$).

The best known generalization is probably that of Walloddi Weibull (1887-1979):

$$(5) \quad X : f(x; \theta, k) = \theta k x^{k-1} \exp(-\theta x^k), \quad x \geq 0, \quad k, \theta > 0$$

which gives Rayleigh for $k = 2$.

Blischke and Murthy (2000, [5]) consider that this distribution was a response to the overused exponential model – which has a constant hazard rate $h(x) = \theta$ for $k = 1$ – inapplicable to strength of materials or cutting-tool durability studies, for instance.

Weibull distribution had a tremendous career amongst the practitioners. Weibull himself collected in 1977 in a technical report of Förvarets Teletekniska Laboratorium (in Stockholm, Sweden) a number of 1019 references (articles) and 36 titles of books in which his model is mentioned (all these titles are only in English). Facing the enthusiasm related to this distribution, A. C. Giorski (1968 [9]) draws the attention of what he calls “Weibull euphoria” arguing that the model is very useful but it is not “universal” (one year later, Ravenis (see [18]) just proclaimed Weibull’s model as a “*potentially universal p.d.f. for scientists and engineers*”...

A more general model (which includes the Weibull one) is that called generalized Gamma:

$$(6) \quad X : f(x; b, k, p) = \frac{k}{b\Gamma(p)} \cdot \left(\frac{x}{b}\right)^{pk-1} \exp\left\{-\left(\frac{x}{b}\right)^k\right\},$$

where $x \geq 0$, $b, p, k > 0$, proposed by Stacy in 1962 [21] which for $p = 1$ becomes Weibull one, with (b^k) as a scale parameter and (k) as shape parameter. It is interesting to notice that a similar form of (6) has been used in 1925 by an Italian economist, Luigi Amoroso in “Annali di Matematica Pura ed Applicata” in a long paper (nr. 421, pp. 123-159) entitled “Ricerche intorno alla curva dei redditi” (Researches on the curve of incomes). Since it was published in a journal not very widespread, Amoroso’s work remained unknown for a long time (until late sixties) when Henrick J. Malik discovered it and seized the priority regarding this generalized Gamma (see Bârsan-Pipu et al. 1999, [4]). Amoroso-Stacy’s distribution has been intensively studied by two Polish engineers: K. Ciechanowicz (1972, [6]) and Szymon Firkowicz (1969, [8]). Urban Hjorth (1980, [15]) proposed another generalization of Rayleigh distribution as follows:

$$(7) \quad X : F(x; \theta; \beta, \delta) = 1 - (1 + \beta x)^{-\theta/\beta} \cdot \exp\{-\delta x^2 / 2\}$$

where $x \geq 0$, $\theta \geq 0$, $\beta, \delta > 0$. If we take $\theta = 0$, one obtains classical Rayleigh form. It is interesting to notice that exponential distribution is obtained if $\delta = 0$ and $\beta \rightarrow 0$. In the next paragraph we shall examine some generalized forms of the exponential p.d.f., choosing one of them as working material.

2. Generalized exponentials

Various forms to generalize exponential distribution have been proposed. We shall mention two of them:

Dobó's form (1976, [7]):

$$(8) \quad F(x; \lambda, a, \theta) = 1 - \left[\frac{\lambda + \theta}{\lambda + \theta e^{ax}} \right]^{1/a\theta}$$

where $x \geq 0$, $a > 0$, $\lambda, \theta \geq 0$, which for $\lambda = 0$ gives the classical $F(x; \theta) = 1 - \exp(-x/\theta)$.

Khan's form (1987, [16] or [4, pp. 44-46])

$$(9) \quad f(x; r, \theta) = \frac{e^{-x/\theta} \cdot x^{rx}}{\theta^{1+rx} \cdot \Gamma(2+rx)}$$

with $x \geq 0$, $r \geq 0$ but $r\theta < 1$, which for $r = 0$ gives $f(x; \theta) = \theta^{-1} \exp(-x/\theta)$.

Consider now the following differential equation:

$$(10) \quad \frac{d\varphi}{dx} = a(x) \cdot \varphi^\alpha + b(x) \cdot \varphi^\beta$$

where φ is a positive real function, $x \in [a, b] \subseteq \mathbf{R}$ where for a reliability context one may choose $[a, b] \equiv [0, +\infty)$.

We shall call (10) as a generating differential equation (GDE) since for various choices of $a(x)$, $b(x)$ – real continuous functions on \mathbf{R} and α, β two real numbers one could obtain a wide range of densities (φ).

Now, in (10) let us take $\alpha = -1$, $b(x) = 0$ and β - an arbitrary real number. We have:

$$(11) \quad \frac{d\varphi}{dx} = a(x) \cdot \frac{1}{\varphi} \quad \text{or} \quad \frac{1}{2} \varphi^2 = \int a(x) dx.$$

If $x \geq 0$, and $a(x) = -A \cdot \frac{k}{\lambda} x^{k-1} \exp\left(-\frac{x^k}{\lambda}\right)$ where A is a norming factor, we obtain:

$$(12) \quad \frac{1}{2}\varphi^2 = A \cdot \exp\left(-\frac{x^k}{\lambda}\right) \text{ or } \varphi = \sqrt{2A} \exp\left(-\frac{x^k}{2\lambda}\right).$$

If we denote $2\lambda = \theta > 0$ and take A as $k^2 / 2\theta^{2/k} \cdot \Gamma^2(1/k)$ where $k > 0$. We get $\varphi(x)$ as a density function, namely:

$$(13) \quad \varphi(x; \theta, k) = \frac{k}{\theta^{1/k} \Gamma(1/k)} \cdot \left(-\frac{x^k}{\theta}\right)$$

which for $k=1$ gives the classical exponential p.d.f.

It is interesting to draw the attention on the so-called "generalized error distribution" studied by T. Taguchi (1978, [22]) who states that it was introduced by a Russian mathematician M. T. Subbotin in 1923 ("Matematicheskyy Sbornik", vol.31, pp. 296-301). Its density is:

$$(14) \quad f(x; p) = \frac{1}{2p^{\frac{1}{p}-1} \Gamma(1/p)} \cdot \exp\left\{-\frac{|x|^p}{p}\right\}$$

where $x \in \mathbf{R}$ and $p > 0$. This form resembles to (13) but it does not include the exponential, since for $p=1$ one obtains Laplace density function. Our form (13) has two parameters (scale and shape ones) and x is restricted to $[0, +\infty)$.

Now, if we compute the theoretical mean-value for (13) we shall obtain easily:

$$(15) \quad E(X) = \theta^{1/k} \cdot \frac{\Gamma(2/k)}{\Gamma(1/k)}$$

(see for other details [12]).

3. The new GRV (Generalized Rayleigh Variate)

Our variable will be obtained in a more general framework which was presented in [26]. It is known that in the reliability theory some classes of time-to-failure distributions are obtained using a so-called "generator" – which is also a p.d.f. One of the problem of interest is for instance the following: if we have an IFR (increasing failure rate) distribution function (d.f.) $F(x)$, and if we construct a new d.f. such as

$$(16) \quad F_1(x) = \mu \cdot \int_0^{\infty} R(u) du, \text{ where } R(x) = 1 - F(x), \quad \mu = \int_0^{\infty} x \cdot dF(x)$$

does this $F_1(x)$ preserve the IFR property? In accordance with Barlow and Proschan (1975, [3]), the answer is yes.

Now, if we define a p.d.f. as below:

$$(17) \quad f_1(x) = \frac{x}{E(x)} \cdot f(x) \quad \text{and} \quad E(x) = \int_0^{\infty} x \cdot f(x) dx < +\infty$$

where $f(x)$ is known- that is it has a well - defined form and specified parameters- which means it belongs to a certain class – say Weibull – then, the new $f_1(x)$ it is also a Weibull type density? In this example, the answer is no (see [26], page 211). The new p.d.f. was nicknamed as a “pseudo-Weibull distribution”.

The property to preserve the initial class of belongness has been called conservativeness. Therefore, in this approach, Weibull (and also the classical exponential) are not conservative.

Now, if we take as $f(x)$ the form (13) – that is our generalized exponential, we obtain:

$$(18) \quad f_1(x) = \frac{x}{E(x)} \cdot f(x) = \frac{kx}{\theta^{2/k} \Gamma(2/k)} \cdot \exp\left(-\frac{x^k}{\theta}\right)$$

with $x \geq 0$, $k, \theta > 0$, which for $k=2$ provides the usual Rayleigh p.d.f.:
 $f_1(x) = (2/\theta) \cdot x \cdot \exp(-x^2/\theta)$.

Hence, this new generalization of Rayleigh p.d.f. is not conservative as regards the transformation given by (17).

One interesting thing is the following: if $k=1$, then we obtain – curiously – the pseudo-Weibull p.d.f., $PW(x; \theta, 1)$, that is $f_1(x) = \theta^{-2} \cdot x \cdot \exp(-x/\theta)$, since general $PW(x; \theta, k)$ is :

$$(19) \quad f_1(x; \theta, k) = kx^k \left[\theta^{1+1/k} \cdot \Gamma(1+1/k) \right]^{-1} \cdot \exp(-x^k/\theta).$$

The p.d.f. $PW(x; \theta, 1)$ has been studied in [26]. In our case (18) we have, for instance (for $m \in \mathbf{N}$) :

$$(20) \quad E(X^m) = \frac{k}{\theta^{2/k} \Gamma(2/k)} \int_0^{\infty} x^{m+1} \cdot e^{-x^k/\theta} dx = \theta^{m/k} \cdot \frac{\Gamma\left(\frac{m+2}{k}\right)}{\Gamma(2/k)}.$$

If $m=1$ and $m=2$ we have the first two non-central moments:

$$(21) \quad E(x) = \theta^{1/k} \cdot \frac{\Gamma(3/k)}{\Gamma(2/k)} \quad \text{and} \quad E(x^2) = \theta^{2/k} \cdot \frac{\Gamma(4/k)}{\Gamma(2/k)}$$

which give the variance of the variable:

$$(22) \quad \text{Var}(x) = \theta^{2/k} \left[\frac{\Gamma(4/k)}{\Gamma(2/k)} - \frac{\Gamma^2(3/k)}{\Gamma^2(2/k)} \right].$$

We shall prove now the following:

Lemma. If X is a GRV with k -known, then the variable $Y = X^k$ is a Gamma type random variate with parameters θ and $2/k$.

Proof. We write the distribution function of Y , namely:

$$(23) \quad F(y) = \text{Pr ob} \{ X^k < y \} = \text{Pr ob} \{ X < y^{1/k} \} = \int_0^{y^{1/k}} f(x; \theta, k) dx$$

where $f(x; \theta, k)$ is the p.d.f. of X . We have hence:

$$(24) \quad F(y) = \frac{k}{\theta^{2/k} \Gamma(2/k)} \int_0^{y^{1/k}} x \cdot \exp(-x^k / \theta) dx.$$

Taking into account of the general formula:

$$(25) \quad \frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(u) du \right) = b'(x) \cdot f[b(x)] - a'(x) f[a(x)]$$

from (24) we obtain:

$$(26) \quad F'(y) = f(y) = \frac{y^{\frac{2}{k}-1}}{\theta^{2/k} \cdot \Gamma(2/k)} \exp(-y/\theta), \quad y \geq 0, \theta, k > 0$$

which is just the p.d.f. of a Gamma random variable (one may denote $(2/k=a)$, to have the usual form).

If k is known, then the estimation of θ is easily to find by maximum likelihood method. Indeed, if we have a sample $\{x_1, x_2, \dots, x_n\}$ on X , the likelihood function is:

$$(27) \quad L = \frac{K^n \cdot \prod_1^n x_i}{\theta^{2n/k} \cdot \Gamma^n(2/k)} \cdot \exp \left[-\frac{1}{\theta} \sum_1^n x_i^k \right]$$

and taking the logarithms and the derivative with respect to θ , we find:

$$(28) \quad \ln L = n \ln k + \sum_1^n \ln x_i - \frac{2n}{k} \cdot \ln \theta - n \ln \Gamma(2/k) - \frac{1}{\theta} \sum_1^n x_i^k$$

$$(29) \quad \frac{\partial \ln L}{\partial \hat{\theta}} = -\frac{2n}{k} \cdot \frac{1}{\hat{\theta}} + \frac{1}{\hat{\theta}^2} \cdot \sum_1^n x_i^k = 0$$

which provides the solution;

$$(30) \quad \hat{\theta}_{ML} = \left(\frac{K}{2n} \right) \cdot \sum_1^n x_i^k.$$

The distribution of $\hat{\theta}_{ML}$ is now almost obvious: since x_i^k has a Gamma distribution, the sum of i.i.d. (independent and identically distributed) Gamma variables is also Gamma (the stability property of Gamma distribution – see [3]).

If both parameters are unknown then we shall obtain the system:

$$(31) \quad \begin{cases} \frac{\partial \ln L}{\partial \hat{k}} = \frac{n}{\hat{k}} + 2n \cdot \frac{1}{\hat{k}^2} \cdot \ln \hat{\theta} - n \cdot \frac{\Gamma'(2/k)}{\Gamma(2/k)} - \frac{1}{\hat{\theta}} \sum_1^n x_i^{\hat{k}} \ln x_i = 0 \\ \frac{\partial \ln L}{\partial \hat{\theta}} = -\frac{2n}{\hat{k}} + \frac{1}{\hat{\theta}^2} \cdot \sum_1^n x_i^{\hat{k}} = 0. \end{cases}$$

As one can see, the so-called DiGamma function is involved (that is, the derivative of Gamma function $\Gamma(u)$), and consequently, some numerical methods are needed. Using some formulas from Ryzhyk-Gradsteyn tables (1965, [19]), namely

$$(32) \quad \frac{d \ln \Gamma(u)}{du} = \psi(u) = \ln + \int_0^\infty e^{-ut} \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) dt$$

one may approximate $\psi(u+1)$ as:

$$(33) \quad \psi(u+1) \approx -0.577215664 + 1.644934067 \cdot k - 1.202056904 \cdot k^2.$$

As regards $\ln \theta$, if $0 < \theta < 1$, we could use the well-known Taylor series, and if $\theta > 1$, we may approximate $\ln \theta$ with the upper bound $(\theta-1)/\sqrt{\theta}$ since we have the elementary inequality:

$$(34) \quad \frac{\ln x}{x-1} \leq \frac{1}{\sqrt{x}} \quad \text{if } x \in (0, +\infty) \setminus \{1\}.$$

Let us remark that in (33), the first constant is just Euler's

$$C = \lim_{n \rightarrow \infty} (1 + 1/2 + 1/3 + \dots + 1/n - \ln n) \approx 0.57721.$$

4. Some comments

a). One can detect in the literature some more general p.d.f. (s) than the generalized Gamma – as for instance the so-called generalized Gompertz-Verhulst one (see Ajuha and Nash 1967 [1], Ajuha, 1969 [2] or Vodă, 1980 [25]). Such p.d.f. (s) contain two or more special functions, which are difficult to be analytically treated.

b). The generating differential equation (10) may provide not only p.d.f. (s) *but also distribution functions*. As, for instance, if we take $a(x) = \theta$, $b(x) = -\theta$, $\theta > 0$, $\alpha = 1$ and $\beta = 2$, we obtain:

$$(35) \quad \frac{d\varphi}{dx} = \theta \cdot \varphi(1-\varphi) \quad \text{or} \quad \frac{d\varphi}{\varphi(1-\varphi)} = \theta dx$$

where from by straightforward integration, one gets:

$$\varphi(x; \theta) = \frac{1}{1 + e^{-\theta x}}, \quad x \in \mathbf{R}, \quad \theta > 0$$

which is just a reduced Verhulst distribution function, proposed by Pierre François

Verhulst (1804-1849) in 1846 (see Iosifescu et al., 1985 [10]).

c). An interesting form of a GRV can be obtained from the reliability function proposed

in 1998 by Silvia Spătaru and Angela Galupa (see [20]). Their construction was the following: let $Y \geq 0$ be a random variable and let $G(x) = \text{Prob}\{Y \leq x\}$, $x \geq 0$ its distribution function. Starting from a generalized form of the power distribution (see [11]) the subsequent reliability function is advanced:

$$(36) \quad R(x) = [1 - G(x)]^\lambda \cdot e^{-\theta x}, \quad x \geq 0, \quad \theta \geq 0$$

which provides a hazard rate as :

$$(37) \quad h(x) = \lambda \cdot \frac{g(x)}{1 - G(x)} + \theta, \quad g(x) = G'(x)$$

having the following interpretation : if $\lambda = 1$, the first element in the right-hand-side of $h(x)$ represents the hazard rate of Y component, and θ is the failure rate of a second element having an exponential distribution. If we take now $G(x)$ as a classical Rayleigh namely $G(x) = 1 - \exp(-ax^2)$, then the below distribution is obtained:

$$(38) \quad F(x) = 1 - R(x) = 1 - \exp(-\lambda ax^2 + \theta x)$$

with $x \geq 0$, $a > 0$, $\lambda, \theta \geq 0$, (if $\lambda = 1$, $\theta = 0$, we have $F(x) = 1 - \exp[-ax^2]$).

d) Professor I. C. Bacivarof (Bucharest) has drawn my attention on a paper of D. Kundu and M. Z. Raqab (2004, "Generalized Rayleigh distribution: different methods of estimation", posted on

Internet) in which these authors study the distribution $F(x; \alpha, \lambda) = [1 - e^{-(\lambda x)^2}]^\alpha$ $x \geq 0$, $\alpha, \lambda > 0$ which for $\alpha = 1$ becomes the classical Rayleigh.

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