Briš Radim
VŠB Technical University of Ostrava, Czech Republic

Stochastic ageing models – extensions of the classic renewal theory

Keywords

renewal theory, alternating renewal theory, maintenance processes, ageing, unavailability coefficient

Abstract

Exact knowledge of the reliability characteristics as the time dependent unavailability coefficient for example, under influence of different ageing processes as well as under different failure types is very useful to the practitioners who have to find the optimal maintenance policy for their equipment. In this paper found models and their solutions have potential to face the optimisation task under the conflicting issues of safety and economics. Most of the solved models take into account ageing processes. An increasing tendency lately exists to include aging effects into the risk assessment models to evaluate its contribution. We developed different renewal models taking into account different ageing distributions of failures (Weibull, Erlang, log-normal): models with negligible renewal time, models with periodical preventive maintenance, alternating renewal process with lognormal distribution of failure time, and with two types of failures.

1. Introduction

This paper mainly concentrates on the modelling of various types of renewal processes and on the computation of principal characteristics of these processes – the time dependent coefficient of availability, possibly unavailability. The aim is to generate models, most often found in practice, which describe the processes of ageing, further the occurrence of dormant failures that are eliminated by periodical inspections as well as monitored failures which are detectable immediately after their occurrence.

Renewal theory seems to be a feasible option to quantify time-dependent effects on component unavailability due to ageing, periodical inspections, or repairs [1]. Closed form solutions for the asymptotic the failure rate and unavailability can be obtained using Laplace transform. Obtaining the detailed time behaviour may not be a trivial numerical task.

Basic information from renewal theory brings Appendix [4], [3]. The following chapter 2 is devoted to models with a negligible renewal time in which a main impact is given on flexible models with the Erlang and Weibull distribution. The solution of these models is received from a Laplace and discrete Fourier transformation. In the following chapter 3 we introduce different models with maintenance. Main attention is paid to models with periodical preventive maintenance - basic equations for the model are formulated. The solution of a system of equations is demonstrated for the situations with an exponential and Weibull distribution. In the next chapter the alternating model with an inconsiderable renewal time is solved, this is demonstrated for lognormal distribution of time to failure. The final part involves generally formulated alternating models with the occurrence of two types of independent failures.

Time-dependent unavailability of components under maintenance and ageing processes can exhibit mathematically complex behaviour [5]. The unavailability may be also dependent on maintenance history. First failure distributions may not be continuous functions. Within this paper we can say that renewal theory provides a feasible approach in selected cases to implement and evaluate interventions given by maintenance and aging processes.

A lot of notable asymptotic results on availability analyses are focused on the situation that the components have exponential lifetime distributions. Using so-called phase-type approach, author in [2] shows that the multi-state model also provides a framework for covering other types of distributions, but with limitations - the approach makes use of the fact that a distribution function can be approximated by a mixture of Erlang distributions (with the same scale parameter). Asymptotic analysis of highly available systems has been carried out by a number of researchers. A survey is given by Gertsbakh [7], with emphasis on results related to the convergence of the distribution of the first system failure to the exponential distribution.

If the lifetimes are distributed arbitrarily, then the system can be described by a semi-Markov process or Markov renewal process. Semi-Markov processes and Markov renewal processes are based on a marriage of renewal processes and Markov chains. Pyke [8] gave a careful definition and discussion of Markov renewal processes in detail. In reliability, these processes are one of the most powerful mathematical techniques for analysing maintenance and random models. A detailed analysis of the non-exponential case (non-regenerative case) is however outside of the scope of the introduction part. Further research is needed to present formally proved results for the general case. Presently, the literature covers only some particular cases, what is also the case of this presentation.

2. Models with a negligible renewal period

In some cases we can take into account a renewal period equal to zero. For example the situation when a time to a renewal is substantially smaller than a time to a failure and its implementation would not influence an expected result. This case was intensively studied in [6]. Basic relationships for Poisson process are derived in [4]:

Renewal function and renewal density are given as follows

$$H(t) = EN_t = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda t.$$

Basic definitions from renewal theory see in Appendix. Renewal density is constant

$$h(t) = H'(t) = \lambda.$$

Methodology based on Laplace transforms was dramatically extended in [6] for the case when a time to failure is modelled by the Erlang distribution, which has a probability density function

$$f(t) = \frac{\lambda (\lambda t)^a e^{-\lambda t}}{\Gamma(a)}, \quad t \geq 0,\ \lambda ¤ 0,\ a ¤ 0.$$

After the backward transformation a renewal density is equal to

$$h(t) = \frac{\lambda}{a} + \sum_{k=1}^{a-1} \frac{\lambda + s_k}{a} e^{s_k t},\ t ¤ 0,$$

in the expression there is $s_k \in C$,

which is k[th] nonzero root of the equation

$$(s + \lambda)^a = \lambda^a \quad s \in C,$$

For example for a = 4 nonzero roots are equal to

$$s_1 = \lambda(e^{\frac{i\pi}{2}} - 1) = (-1 + i)\lambda,$$

$$s_2 = \lambda(e^{i\pi} - 1) = -2\lambda,$$

$$s_3 = \lambda(e^{\frac{i3\pi}{2}} - 1) = (-1 - i)\lambda,$$

and a renewal density

$$h(t) = \frac{\lambda}{4} + \sum_{k=1}^{3} \frac{\lambda + s_k}{4} e^{s_k t}$$

$$= \frac{\lambda}{4}[1 - e^{-2\lambda t} - e^{-\lambda t} \sin(\lambda t)].$$

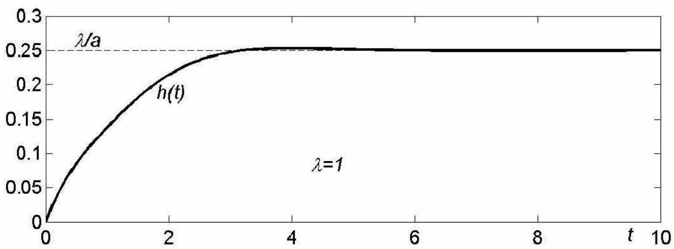

Figure 1. Renewal density for Erlang distribution

Another calculation method was applied for Weibull distribution of time to failure, which has a probability density

$$f(t) = \alpha\lambda(\lambda t)^{\alpha - 1} e^{-(\lambda t)^{\alpha}}, \quad t \geq 0,$$

$\alpha > 0$ is a parameter of the shape, $\lambda > 0$ is a parameter of a scale.

A probability density $f_n(t)$, $n = 2,3,\ldots$, or a probability density of time to n[th] failure can be calculated as a convolution of the function ($f_{n-1} * f$). We can express it numerically e.g. with the help of discrete Fourier transformation [6].

By a numerical integration we can determine a vector of a distribution function $\mathbf{F_n}$.

A formula (1) for a calculation of the renewal function

$$H(t) = \sum_{n=0}^{\infty} F_n(t) \qquad (1)$$

is necessary to substitute by a finite sum of the first K computed terms at the numerical calculation. It can be conducted because these terms converge quickly to a zero at a definite interval [0, T]. Equally, $S_n = X_1 + X_2 +\ldots+ X_n$ has an asymptotic normal distribution $N(n\mu, n\sigma^2)$ where μ and σ^2 are definite expected value and a dispersion of X_i.

Considering that Weibull distribution of time to failure for $\alpha > 1$ has an increasing failure rate

$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}$$

a distribution function $F_n(t)$ can be upper estimated by the function

$$F_n(t) \le 1 - \sum_{i=0}^{n-1} \frac{(\frac{t}{\mu})^i}{i!} e^{-\mu t} = G_n(t),\ \ t \circledR n\mu,$$

where μ is an expected value of a time to failure [3].

$$\mu = \frac{\Gamma(1+\frac{1}{\alpha})}{\lambda}.$$

We can estimate in this way an error of a finite sum

$$H(t) = \sum_{n=0}^{K} F_n(t),$$

because a remainder is limited

$$\sum_{n=K+1}^{\infty} F_{n(t)} \le \sum_{n=K+1}^{\infty} G_n(t),\ \ t \circledR n\mu.$$

Exact relationship for the reminder is derived in [6]. The behaviour of the renewal function estimated for different number of members in the above finite sum we can see in *Figure 2.*

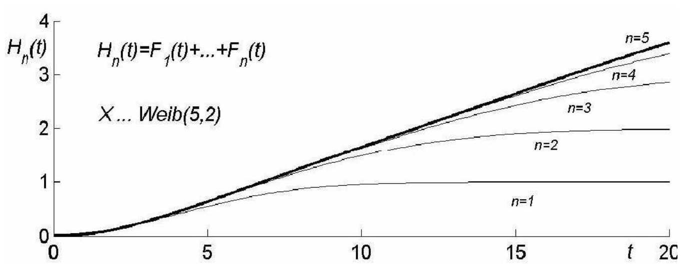

Figure 2. Renewal function for Weibull distribution

3. Models with maintenance

In many situations, failure of a unit during actual operation is costly or dangerous. If the unit is characterized by a failure rate that increases with age, it may be wise to replace it before it has aged too greatly. In this section we shall concentrate on the operating characteristics of some commonly employed replacement policies.

A commonly considered replacement policy is the policy based on age (age replacement). Such a policy is in force if a unit is always replaced at the time of failure or τ_c hours after its installation, whichever occurs first; τ_c is a constant unless otherwise specified. If τ_c is a random variable, we shall refer to the policy as a random age replacement policy. Under a policy of block replacement the unit is replaced at times $k\tau_c$ (k = 1,2,…), and at failure. This replacement policy derives its name from the commonly employed practice of replacing a block or group of units in a system at prescribed times $k\tau_c$(k = 1,2,…) independent of the failure history of the system.

3.1. Replacement based on age

A unit is replaced τ_c hours after its installation or at failure, whichever occurs first; τ_c is considered constant. Let $R(t)$ denote the probability that an item does not fail in service before time t. Then

$$R(t) = R(\tau_c)^n R(t - n\tau_c),$$

$$n \in N \cup \{0\}:\ n\tau_c \le t < (n+1)\tau_c.$$

The distribution function of a time to failure X is

$$F(t) = 1 - R(t) = 1 - R(\tau_c)^n R(t - n\tau_c),$$

$$t \ge 0,\ n \in N \cup \{0\}:\ n\tau_c \le t < (n+1)\tau_c.$$

Expected time to failure E(X) is

$$E(x) = \int_0^{\infty} R(x)dx$$

$$= \sum_{n=0}^{\infty} \int_{n\tau_c}^{(n+1)\tau_c} R(\tau_c)^n R(t - n\tau_c)dt$$

$$= \sum_{n=0}^{\infty} R(\tau_c)^n \int_0^{\tau_c} R(t)dt = \frac{1}{F(\tau_c)} \int_0^{\tau_c} R(t)dt,$$

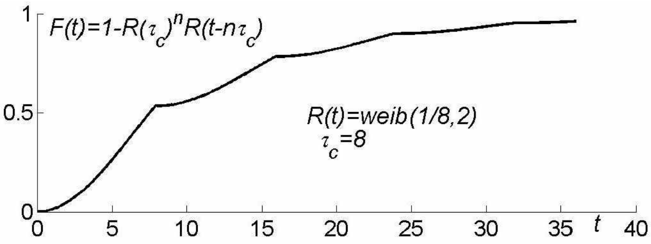

Figure 3. The Weibull distribution function for a unit with the replacement based on age.

When the time to failure is exponentially distributed $X \sim \exp(1/\mu)$, then we have

$$F(t) = 1 - R(\tau_c)^n R(t - n\tau_c)$$

$$= 1 - e^{-n\mu\tau_c} e^{-m\mu(t - n\tau_c)} = 1 - e^{-\mu t},$$

which means that distribution function is independent on replacements. Other words, the unit does not age.

3.2. Block replacement policy

Under a policy of block replacement all components of a given type are replaced simultaneously at times $k\tau_c$ (k = 1,2,…) independent of the failure history of the system. If X_i is time of *i-th* failure of a unit which has distribution function F_i and probability density f_i and $R_i(t)=1- F_i(t)$, then

$$R_1(t) = R(\tau_c)^n R(t - n\tau_c),$$

$$n \in N \cup \{0\}: n\tau_c \le t < (n+1)\tau_c,$$

$$f_1(t) = R(\tau_c)^n \frac{d}{dt}[1 - R(t - n\tau_c)].$$

Distribution of time to *i-th* failure for $i > 1$ we can derive on the basis of conditional probability:

$$x > 0,\ y > 0,\ \ k = [x/\tau_c],\ l = [(x+y)/\tau_c],\ \ l > k,$$

$$\Pr(X_i > y \setminus X_{i-1} = x)$$

$$= R((k+1)\tau_c - x) R(\tau_c)^{(n-(k+1))} R(x + y - n\tau_c).$$

Then

$$\Pr(X_i > y) = \int_0^\infty R((k+1)\tau_c - x) R(\tau_c)^{(l-(k+1)}$$

$$\cdot R(x + y - n\tau_c) f_{i-1}(x)dx$$

$$= \sum_{k=0}^{\infty} \int_{k\tau_c}^{(k+1)\tau_c} R((k+1)\tau_c - x) R(\tau_c)^{l-(k+1)}$$

$$\cdot R(x + y - l\tau_c) f_{i-1}(x)dx$$

$$z = x - k\tau_c,\ n = [(z+y)/\tau_c]$$

$$= \sum_{k=0}^{\infty} R_{i-1}(\tau_c)^k \int_0^{\tau_c} R(\tau_c - z) R(\tau_c)^{n-1}$$

$$\cdot R(z + y - n\tau_c) f_{i-1}(z)dz$$

$$= \frac{R(\tau_c)^{n-1}}{1 - R_{i-1}(\tau_c)} \int_0^{\tau_c} R(\tau_c - z) R(z + y - n\tau_c) f_{i-1}(z)dz.$$

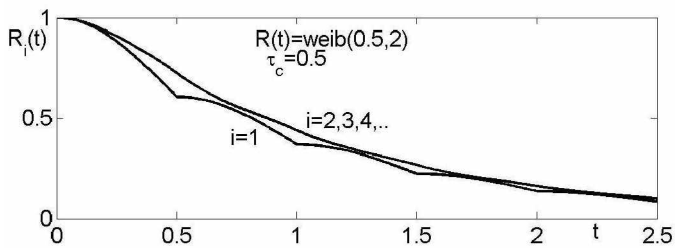

Figure 4. The Weibull distribution function for a unit with block replacement

In *Figure 4*, we can see time dependencies for $R_i(t)$ for Weibull distribution of time to failure.

3.3. Periodical preventive maintenance

May a device goes through a periodical maintenance after a time interval of the operation τ_c, whose intention is a detection of possible dormant flaws and their possible elimination. The period of device maintenance is τ_d and after this period the device starts operating again. *F(t)* is here a time distribution to a failure *X*. Then in the interval $[0, \tau_c + \tau_d)$ there is a probability that the device appears in the not operating state equal to

$$P(t) = F(t),\ \ t\ ®\ \tau_c$$

$$= 1,\ t \ge \tau_c.$$

The state of a failure is considered then both a time to the maintenance after a possible failure and the time when the device is under maintenance. The probability *P(t)* (also a coefficient of unavailability) for $t \in [0, \infty)$ is generated by following system of equations:

$$P(t) = P_n(t),$$
$$n \in N \cup \{0\}: n(\tau_c + \tau_d)\ ®\ t \le (n+1)(\tau_c + \tau_d),$$

…

$$P_i(t) = P_{i-1}(t) - P_{i-1}(i\tau_c)[1 - P_0(t - i(\tau_c + \tau_d))]$$

$$i = 1, 2, ..., n-1,$$

…

$$P_0(t) = F(t),\ \ t \text{ ¤ } 0.$$

Here $P_i(t)$ stands for a probability that a device exists in the failure state provided that before it had gone through i inspections. A term $P_{i-1}(t)$ - $P_{i-1}(i\tau_c)$ represents a probability that a device was all right at the previous inspection and it failed in the interval $(i\tau_c, t)$, $P_{i-1}(i\tau_c)\ P_0(t - i(\tau_c + \tau_d))$ is a probability that it had failed in the previous inspection and since then it failed again.

3.3.1. Exponential distribution of time to failure

May

$$F(t) = 1 - e^{-\lambda t}.$$

For the time $t \in [0, \infty)$ is then a probability $P(t)$ equal to

$$P(t) = F(t - n(\tau_c + \tau_d)),$$

$$t \in [n(\tau_c + \tau_d), n(\tau_c + \tau_d) + \tau_c),$$

$$P(t) = 1,\ t \in [n(\tau_c + \tau_d) + \tau_c, (n+1)(\tau_c + \tau_d)),$$

If τ_d = 0, the expression for $P(t)$ can be further simplified into the form

$$P(t) = F(t - n\tau_c),$$

$$n \in N \cup \{0\}:\ n(\tau_c + \tau_d) \le t \text{ ® } (n+1)(\tau_c + \tau_d).$$

3.3.2. Weibull distribution of time to failure

Let the intensity of failures of the given distribution of time to failure is not constant, it is then a function of time past since the last renewal. In this case it is necessary for the given t and n, related with it which sets a number of done inspections to solve above mentioned system of n equations and the solution of the given system is not eliminated anyhow as it is at an exponential distribution.

In the *Figures 5* and *Figure* 6 a slightly marked curve draws points of the local extremes in the case of shortening a time to the first inspection.

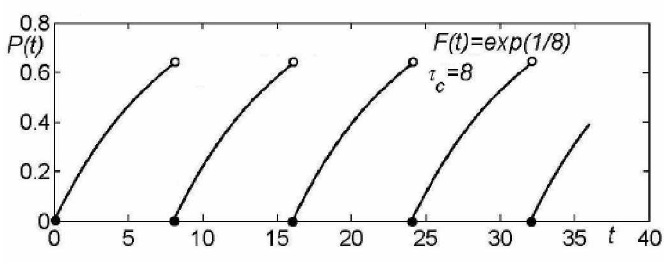

Figure 5. Coefficient of unavailability for exponential distribution

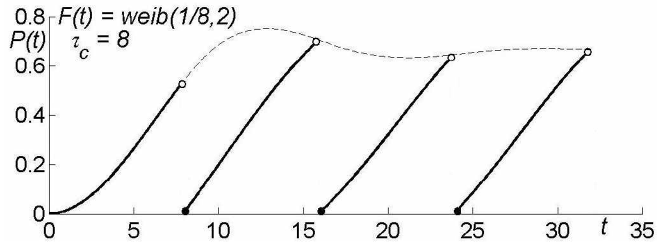

Figure 6. Coefficient of unavailability for Weibull distribution.

4. Alternating renewal models

Alternating models are those where two of the significantly diverse states appear, between which a model converts from one to another. A faulty device is the example of the alternating model where a time to a repair is compared with a time to failure and it cannot be neglected.

In the case that both a time to failure and a time to a repair follows an exponential distribution, general solution for a calculation of a coefficient of availability can be found in [4].

4.1. Lognormal distribution of a time to failure

If a distribution to a failure X_f has a lognormal distribution, then a probability density is in the form

$$f_f(t) = \frac{1}{\sigma t}\varphi\left(\frac{\ln t - \lambda}{\delta}\right),\ t \text{ ¤ } 0$$

where $\varphi(x)$ is standard normal density.

In this case a numerical calculation is offered again for the computation of the coefficient of availability.

We can compute a probability density of a sum of random quantities $X_{f\ and}\ X_r$ (X_r is an exponential time to a repair) from a discrete Fourier transformation [6], equally as a convolution in the equation

$$K(t) = R_f(t) + \int_0^t h(x) R_f(t-x)\,dx.$$

The calculation of a renewal density is substituted by a finite sum

$$h(t) = \sum_{n=1}^{N} f_n(t).$$

An example: In the following example a calculation for parameter values $\sigma=1/4$, $\lambda=8\sigma$, $\tau=1/2$, is done. In the *Figure 7* there is a renewal density. The asymptotic value is marked by dots, which is in this case equal to

$$\lim_{t\to\infty} h(t)=\frac{1}{EX_f+EX_r}=\frac{1}{e^{-\lambda+\frac{1}{2}\sigma^2}+\tau}\cong 0.123$$

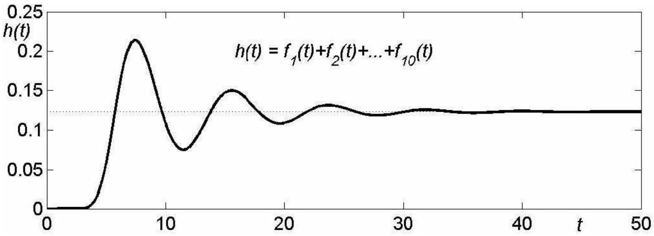

Figure 7. A renewal density for lognormal distribution

Figure 8 shows a procedure of the coefficient of availability *K(t)*, the asymptotic value is marked by dots again and it is given by the following formula:

$$K=\lim_{t\to\infty} K(t)=\frac{1}{EX_f+EX_r}\cong 0.938$$

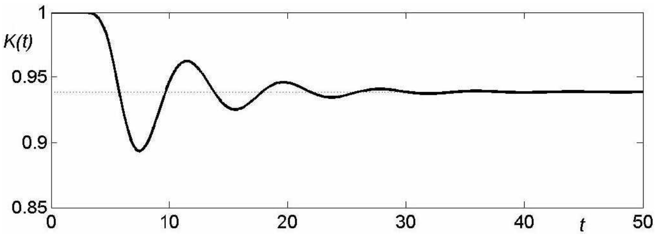

Figure 8. Coefficient of availability for a lognormal distribution

5. Alternating renewal models with two types of failures

The following part presents models, which consider an appearance of two different independent failures. These failures can be described by an equal distribution with different parameters or by different distributions.

5.1. Common repair

A device composed of two serial elements can be an example whereas a failure of one of them causes a failure of the whole device. A time to a renewal is common for both the failures and begins immediately after one of them. It is described by an exponential distribution with a mean value $1/\tau$.

A failure occurrence in the renewal time is not taken into account, after the renewal both the parts are considered to be new.

May X_{f1} and X_{f2} are independent random values describing time of failures with probability densities $f_{f1}(t)$ and $f_{f2}(t)$, further a time to a repair is X_r with a density $f_r(t)$. A probability that no failure occurs in the interval [0,t) is equal to

$$R_f(t)=P(X_{f1}\geq t \wedge X_{f2}\geq t)$$

$$=[1-F_{f1}(t)][1-F_{f2}(t)]=R_{f_1}(t)R_{f2}(t).$$

and is a reliability function of the time to failure X_f of the whole device. Then X_f *has* a probability density

$$f_f(t)=-\frac{d}{dt}R_f(t).$$

With the knowledge $f_f(t)$ we can calculate the functions describing this alternating process.

If the time to failure has an exponential distribution with mean values $1/\lambda$ and $1/\mu$, then

$$f_f(t)=-\frac{d}{dt}e^{-(\lambda+\mu)t}=(\lambda+\mu)e^{-(\lambda+\mu)t}.$$

Then $f_f(t)$ has an exponential distribution with a mean value $1/(\lambda+\mu)$ and the coefficient of availability is equal

$$K(t)=\frac{\tau}{\lambda+\mu+\tau}+\frac{\lambda+\mu}{\lambda+\mu+\tau}e^{-(\lambda+\mu+\tau)t},\ t\ \text{¤}\ 0.$$

If the analytical procedure is uneasy or impossible, a numerical calculation can be used. For a renewal density computation is desirable instead of the equation

$$h(t)=f_f(t)+\int_0^\infty h(x)f_f(t-x)dx$$

use a renewal equation for a renewal density

$$h(t)=\sum_{n=1}^{\infty} f_n(t)$$

and conduct a sum of the only definite number of elements with a fault stated above. $f_n(t)$ is a probability density of time to n^{th} failure. Then for the calculation of convolutions is used for example a quick discrete Fourier's transformation.

In the *Figure 9* there is a graph of a coefficient of availability in the case that a time to a failure X_{f1} and

X_{f2} have Weibull distribution. Expected value to the failure EX_f is equal to

$$EX_f = \sqrt{\frac{EX_{f1}^2 EX_{f2}^2}{EX_{f1}^2 + EX_{f2}^2}} = \sqrt{\frac{2^2 6^2}{2^2 + 6^2}} = \sqrt{3.6}$$

and that is why an asymptotic coefficient of availability is equal to

$$K = \frac{EX_f}{EX_f + EX_r} = 0.79$$

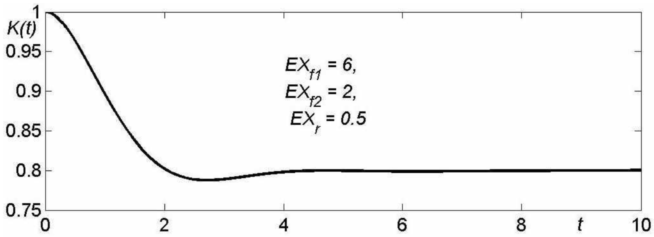

Figure 9. Coefficient of availability for Weibull distribution

5.2. Two independent parts

Supposing the device consists of two independent parts. The behaviour of each one is described by its alternating model with a given time to a failure and a time to a repair. Maintenance proceeds for both differently and independently. Equally, the failure of one of them can appear regardless of the state of the other part, even in the state of a failure.
Let us consider the whole device to be in the state of a failure when at least one of the parts is in the state of a failure. $K_a(t)$ is a coefficient of availability of the first part and $K_b t)$ is a coefficient of availability of the second one. For the whole device $K(t)$ is equal to

$$K(t) = K_a(t) K_b(t).$$

May dormant faults occur in the first part, with Weibull's distribution and with an expected value $EX = 2$ and a parameter of the form $\alpha = 2$ which are eliminated by periodical inspections with a period $\tau_c = 2$ (See Models with periodical preventive maintenance) and the second part is equal as in the previous model. The course of the coefficient of availability as the product of already computed partial ones is designed in the *Figure 10.*

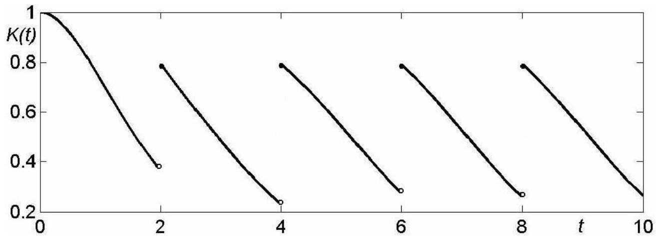

Figure 10. Coefficient of availability for independent parts

6. Conclusion

In this paper a few types of renewal processes, which differentiate in a renewal course and a type of probability distribution of a time to failure, were described. These processes were mathematically modelled by the means of a renewal theory and these models were subsequently solved.
In the cases, when the solving of integral equations was not analytically feasible, numerical computations were successfully applied. It was known from the theory that the cases with the exponential probability distribution are analytically easy to solve.
With the gained results and gathered experience it would be possible to continue in modelling and solving more complex mathematical models which would precisely describe real problems. For example by the involvement of certain relations which would specify the emergence, or a possible renewal of individual types of failures which in reality do not have to be independent on each other. Equally, it would be practically efficient to continue towards the calculation of optimal maintenance strategies with the set costs connected with failures, exchanges and inspections of individual components of the system and determination of the expected number of these events at a given time interval.

Acknowledgements

This research is supported by The Ministry of Education, Youth and Sports of the Czech Republic. Project CEZ MSM6198910007.

References

[1] Aldemir, T. (2004). Characterization of time dependent effects due to ageing. *Proc. of the Kick off meeting, JRC Network on Incorporating Ageing Effects into PSA Applications*, European Commission, DG JRC – Institute for Energy, Nuclear Safety Unit, Probabilistic Risk & Availability Assessment Sector. Edited by M. Patrik and C. Kirchsteiger.

[2] Aven, T. & Jensen, U. (1999). *Stochastic models in reliability*. Springer-Verlag , ISBN 0-387-98633-2.

[3] Barlow, R. & Proschan, F. (1965). *Mathematical Theory of Reliability*. Wiley.

[4] Bartlett, M.S. (1970). *Renewal Theory*. Meten & Co. Ltd. Science paperbacks, ISBN 412 20570 X.
[5] Briš, R., Châtelet, E. & Yalaoui, F. (2003). *New method to minimize the preventive maintenance cost of series–parallel systems*. In Reliability Engineering & System Safety, ELSEVIER, Vol. 82, Issue 3, p.247-255.
[6] Briš, R. & Čepin, M. (2006). Stochastic ageing models under two types of failures. *Proc. of the 31st ESReDA SEMINAR on Ageing,* Smolenice Castle, Slovakia.
[7] Gertsbakh, I.B. Asymptotic methods in reliability: A review. *Adv. Appl. Prob.* 16, 147-175.
[8] Pyke, R. Markov renewal processes: Definitions and preliminary properties. *Ann Math Statist* 32, 1231-1242.

Appendix

Renewal Process

Renewal process serves for example to model mathematically a device behaviour which is maintained in such a way that it stays running as most effectively and longest as possible. May a file of components or the whole device with a time to a failure X (non-negative absolutely continuous random variable) with a dispersion given by a probability density $f(t)$ exists and may a symbol t denotes for clearness a time. The first component is put into operation at time $t = 0$. Further, X_1 is a period when the first component comes to the failure and at the same time it is substituted by a new identical component from a given file. It means that a renewal period (in this case a change period) is negligible, or equal to zero. This second component breaks down after the period X_2 since it started to operate. At the time $X_1 + X_2$ the second component is renewed by the exchange for the third one and the process continues further in such a way. The r-th renewal will happen at the time $S_r = X_1 + X_2 + \ldots + X_r$.
If $X_1, X_2 \ldots$ are independent non-negative equally distributed random variables with a finite expected value and dispersion,

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, n \in N,$$

then a random process $\{S_n\}_{n=0}^{\infty}$ is called a renewal process in a renewal theory. Sometimes an order of stated random variables $\{X_n\}^{\infty}_{n=0}$ is denoted in this way. In the case when a time distribution until the failure is exponential, we speak about Poisson process. A function $F_n(t)$ indicates a distribution function of a random variable S_n. There are a few other random variables connected with the renewal process, which describe its behaviour (at time). Let we call N_t *a number of renewals in the interval* [0, t] for a firm t ≥ 0, it means

$$N_t = \max\{n : S_n \text{®} t\}$$

From this we also get that $S_{Nt} \leq t < S_{Nt+1.}$ Regarding the fact that the interval [0, t] contains n failures (as well as renewals) only if n^{th} failure happens at the latest at the time t

$$P\{N_t \geq n\} = P\{S_n \text{®} t\} = F_n(t)$$

and the probability that at the time t there are n renewals in the given renewal process can be described in the following way

$$P\{N_t = n\} = P\{S_n \text{®} t \wedge S_{n+1} \geq t\}$$

$$= F_n(t)[1 - F_{n+1}(t)] = F_n(t) - F_{n+1}(t)$$

Provided that $X_1, X_2 \ldots$ are independent non-negative equally distributed random variables and $P_r(X_1 = 0) < 1$, then a random variable N_t has finite moments of all the series (Stein`s theorem).
And if N_t , $t \geq 0$ gives a number of renewals in the interval [0, t], then a function

$$H(t) = EN_t, \quad t \geq 0$$

is called *a renewal function.* As it is apparent it gives an expected number of renewals in the interval [0, t]. The expected number of renewals in the interval [t_1, t_2], $0 < t_1 < t_2$ can be quantified from $H(t_1) - H(t_2)$, because a number of renewals in this interval is $N_{t2} - N_{t1}$.
A renewal function can be also expressed from distributional functions $F_n(t)$ of random variables S_n

$$H(t) = \sum_{n=0}^{\infty} nP(N_t = n)$$

$$= \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)] = \sum_{n=1}^{\infty} F_n(t) .$$

A renewal equation is important for the renewal function computation $H(t)$. It provides a mutual unique relation between distributional function of a time to a renewal and a renewal function: if a distributional function of a time to the renewal $F(t)$ is continuous, then a renewal function $H(t)$ is convenient with an integral equation

$$H(t) = F(t) + \int_0^t H(t-u) F(u) du.$$

This equation can be easily derived from the previous equation with help of its integral transformation (e.g. Laplace).

An asymptotic behaviour of a renewal process is substantial. An *asymptotic* behaviour of a renewal process is discussed in an *Elementary theorem about a renewal:* if a time distribution to a renewal has a finite expected value μ, then

$$\lim_{t\to\infty}\frac{H(t)}{t}=\frac{1}{\mu}.$$

It is *a Blackwell theorem, which* testifies about a limited behaviour of an expected number of renewals at a finite interval $(t, t+\Delta t]$: if a time to a renewal has a non-lattice distribution with a definite positive expected value μ, then $\forall h$ ¤ 0 is

$$\lim_{t\to 0}\left[H(t+h)-H(t)\right]=\frac{h}{\mu}.$$

If a derivation of a renewal function exists (i.e. $X_1, X_2...$ are absolutely continuous random variables), then for the arbitrary time $\boldsymbol{t > 0}$ a function $h(t)$ that is defined by a relation

$$h(t)=\lim_{\Delta t\to 0+}\frac{H(t)-H(t+\Delta t)}{\Delta t}=H'(t)$$

is a *renewal density*. Then with a help of a probability density $f_n(t) = F'_n(t)$ we have

$$h(t)=\sum_{n=1}^{\infty}f_n(t).$$

A renewal density most often appears in the following integral equation

$$h(t)=f(t)+\int_0^t h(t-u)f(u)du,$$

so called a *renewal equation for a renewal density*. Here $f(t)$ is a probability density of a absolutely continuous non-negative time to the renewal X.

We can describe the equation approximately by words in such a way that for $\Delta t\to 0$ renewal probability $h(t)\Delta t$ in the interval $(t, t + \Delta t]$ is equal to a probability sum $f(t)\ \Delta t$ *that* in the interval $(t, t + \Delta t]$ the first renewal happens and the sum of probabilities for $\forall u\in(0,t)$ that the renewal happens at the time $t - u$ followed by a time to the failure of the length u.

Alternating Renewal Process

Provided that there are two kinds of components with various independent time to a failure X,Y, respectively adequate distributional functions $F(t)$, $G(t)$ (densities $f(t)$, $g(t)$), at the time $t = 0$ the component of the first type is activated and every time at the time of failure is substituted by the component of the opposite type, resulting process is named *Alternating renewal process.*

We can simulate a renewal process with a definite time to a renewal with such a model. At the time $t = 0$ the component begins to work to the moment of failure X_1. The final time to the renewal Y_1 follows. At the moment $X_1 + Y_1$ the renewal ends and a new (or repaired) component is activated with a time to a failure X_2. X_1,X_2…resp. Y_1,Y_2…are independent non-negative random variables with a distr. function $F(t)$ resp. $G(t)$.The n^{th} failure happens at the moment

$$S_n = X_1+Y_1+...+X_{n-1}+Y_{n-1}+X_n,$$

for n^{th} renewal we have

$$T_n = X_1+Y_1+...+X_{n-1}+Y_{n-1}+X_n+Y_n.$$

A random process $\{S_1, T_1, S_2, T_2.....\}$ is then an alternating renewal process. A coefficient of availability $K(t)$ (or also $A(t)$ - availability) is a basic characteristic of a renewal process with a finite time to a renewal. It determines a probability that at the time t the component will work. It is consequently equal to a sum of probabilities that $X_1 > t$, it means that the first component has a time to a failure greater than t, and that the renewal happens in the interval $(u, u + \Delta u]$, $\Delta u \to 0$, $0 < u < t$ and a renewed component will have a time to a failure greater than $t - u$.

Written by an integral equation:

$$K(t)=1-F(t)+\int_0^t h(x)\left[1-F(t-x)\right]dx$$
$$=R(t)+\int_0^t h(x)R(t-x)dx,$$

$h(x)$ is a renewal process density of a renewal $\{T_n\}^{\infty}_{n=0}$, $F(t)$ is a distribution function of the time to a failure, resp. $1 – F(t) = R(t)$ is reliability function.

In particular, *an asymptotic coefficient of availability* of the alternating renewal process is important practical reliability characteristics,

$$K=\lim_{t\to\infty}K(t).$$

It describes behaviour of the alternating renewal process in the situation when the system is stabilized in “a distant time moment t”, i.e. a stationary case, when the influence of the beginning configuration subsides.