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Applications of semi-Markov processes in reliability

Keywords

semi-Markov processes, reliability, random failure rate, cold standby system with repair

Abstract

The basic definitions and theorems from the semi-Markov processes theory are discussed in the paper. The semi-Markov processes theory allows us to construct the models of the reliability systems evolution within the time frame. Applications of semi-Markov processes in reliability are considered. Semi-Markov model of the cold standby system with repair, semi-Markov process as the reliability model of the operation with perturbations and semi-Markov process as a failure rate are presented in the paper.

1. Introduction

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy, W.L. Smith, and L.Takacs in 1954-55. The essential developments of semi-Markov processes theory were proposed by Cinlar [3], Koroluk & Turbin [13], Limnios & Oprisan [14]. We would apply only semi-Markov processes with a finite or countable state space. The semi-Markov processes are connected to the Markov renewal processes.
The semi-Markov processes theory allows us to construct many models of the reliability systems evolution through the time frame.

2. Definition of semi-Markov processes with a discrete state space

Let *S* be a discrete (finite or countable) state space and let $R_+ = [0,\infty)$, $N_0 = \{0,1,2,...\}$. Suppose, that ξ_n, ϑ_n, $n = 0,1,2,...$ are the random variables defined on a joint probabilistic space (Ω, F, *P*) with values on *S* and R_+ respectively. A two-dimensional random sequence $\{(\xi_n, \vartheta_n), n = 0,1,2,...\}$ is called a Markov renewal chain if for all
$i_0,....,i_{n-1}, i \in S, t_0,...,t_n \in R_+, \quad n \in N_0$:

$$1.\ P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \,|\, \xi_n = i, \vartheta_n = t_n,...,\xi_0 = i_0, \vartheta_0 = t_0\}$$

$$= P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \,|\, \xi_n = i\} = Q_{ij}(t), \qquad (1)$$

$$2.\ P\{\xi_0 = i_o, \vartheta_0 = 0\} = P\{\xi_0 = i_0\} = p_{i_o} \qquad (2)$$

hold.
From the above definition it follows that a Markov renewal chain is a homogeneous two-dimensional Markov chain such that the transition probabilities do not depend on the second component. It is easy to notice that a random sequence $\{\xi_n : n = 0,1,2,...\}$ is a homogeneous one-dimensional Markov chain with the transition probabilities

$$p_{ij} = P\{\xi_{n+!} = j \,|\, \xi_n = i\} = \lim_{t\to\infty} Q_{ij}(t). \qquad (3)$$

The matrix

$$\mathbf{Q}(t) = \left\lfloor Q_{ij}(t): \quad i, j \in S \right\rfloor, \qquad (4)$$

Is called a Markov renewal kernel. Both Markov renewal kernel and the initial distribution define the Markov renewal chain. This fact allows us to construct a semi-Markov process.
Let

$$\tau_0 = \vartheta_0 = 0,$$

$$\tau_n = \vartheta_1 + ... + \vartheta_n, \tau_\infty = \sup\{\tau_n : n \in N_0\}$$

A stochastic process $\{X(t) : t \ge 0\}$ given by the following relation

$$X(t)=\xi_n \quad \text{for} \quad t\in[\tau_n,\tau_{n+1}) \tag{5}$$

is called a semi-Markov process on S generated by the Markov renewal chain related to the kernel $Q(t)$, $t\geq 0$ and the initial distribution *p*.
Since the trajectory of the semi-Markov process keeps the constant values on the half-intervals $[\tau_n,\tau_{n+1})$ and it is a right-continuous function, from equality $X(\tau_n)=\xi_n$, it follows that the sequence $\{X(\tau_n): n=0,1,2,...\}$ is a Markov chain with the transition probabilities matrix

$$\mathbf{P}=[p_{ij}: i,j\in S] \tag{6}$$

The sequence $\{X(\tau_n): n=0,1,2,...\}$ is called an embedded Markov chain in a semi-Markov process $\{X(t): t\geq 0\}$.
The function

$$F_{ij}(t)=P\{\tau_{n+1}-\tau_n\leq t \mid X(\tau_n)=i, X(\tau_{n+1})=j\}$$

$$=\frac{Q_{ij}(t)}{p_{ij}} \tag{7}$$

is a cumulative probability distribution of a random variable T_{ij} that is called holding time of a state i, if the next state will be j. From (11) we have

$$Q_{ij}(t)=p_{ij}F_{ij}(t). \tag{8}$$

The function

$$G_i(t)=P\{\tau_{n+1}-\tau_n\leq t \mid X(\tau_n)=i\}=\sum_{j\in S}Q_{ij}(t) \tag{9}$$

is a cumulative probability distribution of a random variable T_i that is called waiting time of the state i. The waiting time T_i means the time being spent in state i *when* we do not know the successor state.
A stochastic process $\{N(t): t\geq 0\}$ defined by

$$N(t)=n \quad for \quad t\in[\tau_n,\tau_{n+1}) \tag{10}$$

is called a counting process of the semi-Markov process $\{X(t): t\geq 0\}$.
The semi-Markov process $\{X(t): t\geq 0\}$ is said to be regular if for all $t\geq 0$

$$P\{N(t)<\infty\}=1$$

It means that the process $\{X(t): t\geq 0\}$ *has* the finite number of state changes on a finite period.
Every Markov process $\{X(t): t\geq 0\}$ with the discrete space S and the right-continuous trajectories keeping constant values on the half-intervals, with the generating matrix of the transition rates $A=[\alpha_{ij}: \ i,j\in S]$, $0<-\alpha_{ii}=\alpha_i<\infty$ is the semi-Markov process with the kernel

$$\mathbf{Q}(t)=[Q_{ij}(t): i,j\in S],$$

Where

$$Q_{ij}(t)=p_{ij}(1-e^{-\alpha_{ij}t}),\ t\geq 0,$$

$$p_{ij}=\frac{\alpha_{ij}}{\alpha_i} \text{ for } i\neq j \text{ and } p_{ii}=0$$

In the reliability models the parameters and characteristics of a semi-Markov process are interpreted as the reliability characteristics and parameters of the system.

3. Transition probabilities of a semi-Markov process

The transition probabilities of the semi-Markov process are introduced as follows:

$$P_{ij}(t)=P\{X(t)=j \mid X(0)=i\},\ i,j\in S. \tag{11}$$

Applying the Markov property of the semi-Markov process at the jump moments, as a result, we obtain Markov renewal equation for the transitions probabilities, [4], [12]

$$P_{ij}(t)=\delta_{ij}[1-G_i(t)]+\sum_{k\in S}\int_0^t P_{kj}(t-x)dQ_{ik}(x), \tag{12}$$
$$i,j\in S.$$

Using Laplace-Stieltjes transformation we obtain the system of linear equation

$$\tilde{p}_{ij}(s)=\delta_{ij}[1-\tilde{g}_i(s)]+\sum_{k\in S}\tilde{q}_{ik}(s)\tilde{p}_{kj}(s), \tag{13}$$
$$i,j\in S$$

where the transforms

$$\tilde{p}_{ij}(s)=\int_0^\infty e^{-st}dP_{ij}(t)$$

are unknown while the transforms

$$\tilde{q}_{ik}(s) = \int_0^\infty e^{-st} dQ_{ik}(t), \quad \tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t)$$

are given.
Passing to matrices we obtain the following equation

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{g}}(s)] + \tilde{\mathbf{q}}(s)\tilde{\mathbf{p}}(s), \tag{14}$$

where

$$\tilde{\mathbf{p}}(s) = [\tilde{p}_{ij}(s) : i, j \in S], \quad \tilde{\mathbf{q}}(s) = [\tilde{q}_{ij}(s) : i, j \in S],$$

$$\tilde{\mathbf{g}}(s) = [\delta_{ij}(1 - \tilde{g}_i(s)) : i, j \in S].$$

In many cases the transitions probabilities $P_{ij}(t)$ and the states probabilities

$$P_j(t) = P\{X(t) = j\}, \quad j \in S, \tag{15}$$

approach constant values for large t

$$P_{ij} = \lim_{t\to\infty} P_{ij}(t), \quad P_j = \lim_{t\to\infty} P_j(t) \ . \tag{16}$$

To formulate the appropriate theorem, we have to introduce a random variable

$$\Delta_j = \min\{n \in N : X(\tau_n) = j\}, \tag{17}$$

That denotes the time of first arrival at state *j*. A number

$$f_{ij} = P\{\Delta_j < \infty \mid X(\tau_n) = i\} \tag{18}$$

is the probability that the chain that leaves state *i* will sooner or later achieve the state *j*.
As a conclusion of theorems presented by Korolyuk and Turbin [13], we have obtained following theorem

Theorem 1.
Let $\{X(t) : t \geq 0\}$ be a semi-Markov process with a discrete state space S and continuous kernel $\mathbf{Q}(t) = [Q_{ij}(t): \ i, j \in S]$. *If* the embedded Markov chain $\{X(\tau_n) : n = 0,1,2,...\}$, contains one positive recurrent class *C*, such that for each state $i \in S, \ j \in C, \quad f_{ij} = 1$ and $0 < E(T_i) < \infty, i \in S$, then

$$P_{ij} = \lim_{t\to\infty} P_{ij}(t) = P_j = \lim_{t\to\infty} P_j(t) = \frac{\pi_j E(T_j)}{\sum_{i\in S} \pi_j E(T_j)} \tag{19}$$

where $\pi = [\pi_j, \quad j \in S]$ is the unique stationary distribution of the embedded Markov chain that satisfies system of equations.

$$\sum_{i\in S} \pi_i p_{ij} = \pi_j, \ j \in S, \ \sum_{i\in S} \pi_i = 1. \tag{20}$$

4. First passage time from the state *i* to the states subset *A*.

The random variable

$$\Theta_A = \tau_{\Delta_A},$$

where

$$\Delta_A = \min\{n \in N : X(\tau_n) \in A\},$$

denotes the time of first arrival of semi-Markov process, at the set of states *A*.
The function

$$\Phi_{iA}(t) = P\{\Theta_A \leq t \mid X(0) = i\}. \tag{21}$$

is the cumulative distribution of the random variable Θ_{iA} that denotes the first passage time from the state *i* to the states subset *A*.

Theorem 2. [4], [13]
For the regular semi-Markov processes such that,

$$f_{iA} = P\{\Delta_A < \infty \mid X(0) = i\} = 1, \quad i \in A', \tag{22}$$

the distributions $\Phi_{iA}(t), \quad i \in A'$ are proper and they are the unique solutions of the system of equations

$$\Phi_{iA}(t) = \sum_{j\in A} Q_{ij}(t) + \sum_{k\in S} \int_0^t \Phi_{kA}(t - x) dQ_{ik}(x),$$
$$i \in A'$$

Applying Laplace-Stieltjes transformation we obtain the system of linear equations

$$\tilde{\phi}_{iA}(s) = \sum_{j\in A} \tilde{q}_{ij}(s) + \sum_{k\in A'} \tilde{\phi}_{kA}(s)\tilde{q}_{ik}(s), \ i \in A' \tag{23}$$

with unknown transforms

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t).$$

Generating matrix form we get equation

$$\left(I-\tilde{\mathbf{q}}_{A'}(s)\right)\tilde{\mathbf{u}}_{A'}(s)=\tilde{\mathbf{b}}(s), \tag{24}$$

where

$$\mathbf{I}=[\delta_{ij}: i, j\in A'],\ \tilde{\mathbf{q}}_{A'}(s)=[\tilde{q}_{ij}(s): i, j\in A]$$

are the square matrices and

$$\tilde{\mathbf{u}}_{A'}(s)=\left[\tilde{u}_{iA}(s): i\in A'\right]^T,$$

$$\tilde{\mathbf{b}}(s)=\left[\sum_{j\in A}\tilde{q}_{ij}(s): i\in A'\right]^T$$

are the one-column matrices of transforms. The formal solution of the equation is

$$\tilde{\mathbf{u}}_{A'}(s)=\left(I-\tilde{\mathbf{q}}_{A'}(s)\right)^{-1}\tilde{\mathbf{b}}(s).$$

To solve this equation we use any computer programs, for example MATHEMATICA. Obtaining the inverse Laplace transform is much more complicated.
It is essentially simpler to find the expected values and the second moment of the random variables $\Theta_{iA}, i\in A'$. If the second moments of the waiting times $T_i, i\in A'$ are positive and commonly bounded, and $f_{iA}=1, \quad i\in A'$, then the expected values of the random variables $\Theta_{iA}, i\in A'$ are the unique solution of equation

$$\left(I-\mathbf{P}_{A'}\right)\overline{\mathbf{\Pi}}_{A'}=\overline{\mathbf{T}}_{A'}, \tag{25}$$

where

$$\mathbf{I}=\left[\delta_{ij}: i, j\in A'\right],\ \mathbf{P}_{A'}=\left[p_{ij}: i, j\in A'\right],$$

$$\overline{\mathbf{\Pi}}_{A'}=\left[E(\Theta_{iA}):\ i\in A'\right]^T,\ \overline{\mathbf{T}}_{A'}=\left[E(T_i): i\in A'\right]^T$$

and the second moments of the time to failure are the unique solution of equation

$$\left(\mathbf{I}-\mathbf{P}_{A'}\right)\overline{\mathbf{\Pi}}^2_{A'}=\mathbf{B}_{A'} \tag{26}$$

where

$$\mathbf{I}=\left[\delta_{ij}: i, j\in A'\right],\ \mathbf{P}_{A'}=\left[p_{ij}: i, j\in A'\right],$$

$$\overline{\mathbf{\Pi}}^2_{A'}=\left[E(\Theta^2{}_{iA}):\ i\in A'\right]^T,\ \mathbf{B}_{A'}=\left[b_i: i\in A'\right]^T,$$

$$b_i=E(T_i)+2\sum_{k\in A'}p_{ik}E(T_{ik})E(\Theta_{kA}).$$

5. Semi-Markov model of the cold standby system with repair

The problem is well known in reliability theory (Barlow & Proschan [1]). The model presented here is some modification of the model that was considered by Brodi & Pogosian [2].

5.1. Description and assumptions

A system consists of one operating component, an identical stand-by component and a switch, (*Figure1*).

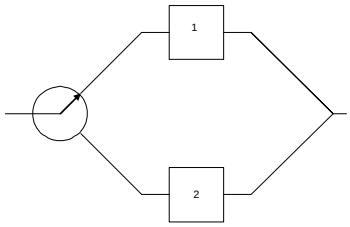

Figure 1. Diagram of the system

When the operating component fails, the spare is put in motion by the switch immediately. The failed unit (component) is repaired. There is a single repair facility. The repairs fully restore the components i.e. the components repairs means their renewals. The system fails when the operating component fails and component that was sooner failed in not repaired yet or when the operating units fail and the switch fails. We assume that the time to failure of the operating components are represented by the independent copies of a non-negative random variable ς with distribution given by a probability density function (pdf) $f(x), x\geq 0$. We suppose that the lengths of the repair periods of the components are represented by the identical copies of the non-negative random variables γ with cumulative distribution function (CDF) $G(x)=P(\gamma\leq x)$. Let U be a random variable having binary distribution

$$b(k)=P(U=k)=a^k(1-a)^{1-k}, k=0,1,\ 0<a<1,$$

where $U=0$, when a switch is failed at the moment of the operating component failure, and $U=1$, when the switch work at that moment. We suppose that the whole failed system is replaced by the new identical system. The replacing time is a non-negative random variable η with CDF $H(x)=P(\eta\leq x)$.

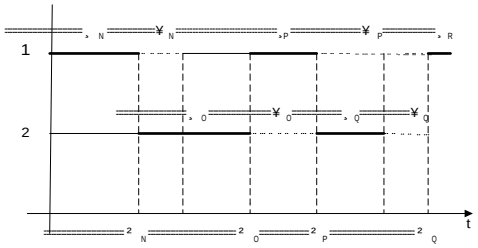

Figure 2. Reliability evolution of the standby system

Moreover we assume that the all random variables, mentioned above are independent.

5.2. Construction of the semi-Markov model

To describe reliability evolution of the system, we have to define the states and the renewal kernel. We introduce the following states:
0 - the system is failed

1 - the failed component is repaired, spare is operated

2 - both operating component and spare are "up".

Let $0=\tau_0^*,\tau_1^*,\tau_2^*,...$ denote the instants of the states changes, and $\{Y(t):t\geq 0\}$ be a random process with the state space $S=\{0,1,2\}$, which keeps constant values on the half-intervals $[\tau_n^*,\tau_{n+1}^*)$, 0,1,... and is right-continuous. The realization of this process is shown in *Figure 1*. This process is not semi-Markov, because the condition (1) of definition (2) is not satisfied for all instants of the state changes of the process.
Let us construct a new random process a following way. Let $0=\tau_0$ and $\tau_1,\tau_2,...$ denote the instants of the system components failures or the instants of whole system renewal. The random process $\{X(t):t\geq 0\}$ defined by equation

$$X(0)=0,\ X(t)=Y(\tau_n) \text{ for } t\in[\tau_n,\tau_{n+1})$$

is the semi-Markov process.
To have semi-Markov process as a model we must define its initial distribution and all elements of its kernel

$$\mathbf{Q}(t)=\begin{bmatrix} 0 & 0 & Q_{02}(t) \\ Q_{10}(t) & Q_{11}(t) & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 \end{bmatrix}$$

For $t\geq 0$ we obtain

$$Q_{02}(t)=P(\eta\leq t)=H(t),$$

$$Q_{10}(t)=P(\varsigma\leq t,\gamma>\zeta)$$

$$+P(U=0,\varsigma\leq t,\gamma<\zeta)$$

$$=\int_0^t[1-G(x)]dF(x)+(1-a)\int_0^t G(x)dF(x)$$

$$=F(t)-a\int_0^t G(x)dF(x),$$

$$Q_{11}(t)=P(U=1,\varsigma\leq t,\gamma<\zeta)=a\int_0^t G(x)dF(x),$$

$$Q_{20}(t)=P(U=0,\varsigma\leq t)=(1-a)F(t),$$

$$Q_{21}(t)=P(U=1,\varsigma\leq t)=aF(t).$$

We assume that, the initial state is 2. It means that an initial distribution is

$$\mathbf{p}(0)=[0\quad 0\quad 1].$$

Hence, the semi-Markov model is constructed.

5.3. The reliability characteristics

The random variable Θ_{iA}, that denotes the first passage time from the state i to the states subset A, for $i=2$ and $A=\{0\}$ in our model, represents the time to failure of the system. The function

$$R(t)=P(\Theta_{20}>t)=1-\Phi_{20}(t),\quad t\geq 0 \tag{27}$$

is the reliability function of the considered cold standby system with repair.
System of linear equation (23) for the Laplace-Stieltjes transforms of the functions

$$\Phi_{i0}(t),\ t\geq 0,\ i=1,2,$$

in this case is

$$\tilde{\phi}_{10}(s)=\tilde{q}_{10}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{11}(s)$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{21}(s)$$

The solution is

$$\tilde{\phi}_{10}(s)=\frac{\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)},$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\frac{\tilde{q}_{21}(s)\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}. \tag{28}$$

Hence, we obtain the Laplace transform of the reliability function

$$\tilde{R}(s) = \frac{1-\tilde{\phi}_{20}(s)}{s}. \tag{29}$$

The transition probabilities matrix of the embedded Markov chain in the semi-Markov process $\{X(t): t \geq 0\}$ is

$$\mathrm{P} = \begin{bmatrix} 0 & 0 & 1 \\ p_{10} & p_{11} & 0 \\ p_{20} & p_{21} & 0 \end{bmatrix} \tag{30}$$

Where

$$p_{10} = 1 - p_{11}$$

$$p_{11} = P(U = 1, \gamma < \zeta) = a\int_0^\infty G(x)dF(x)\,,$$

$$p_{20} = 1 - a, \quad p_{21} = P(U = 1) = a.$$

Using formula (9) we obtain the CDF of the waiting times of $T_i, i = 0,1,2.$

$$G_0(t) = H(t),\ G_1(t) = F(t),\ G_2(t) = F(t)\,.$$

Hence

$$E(T_0) = E(\eta),\ E(T_1) = E(\varsigma),\ E(T_3) = E(\varsigma)\,.$$

The equation (25) in this case has form

$$\begin{bmatrix} 1-p_{11} & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} E(\Theta_{10}) \\ E(\Theta_{20}) \end{bmatrix} = \begin{bmatrix} E(\varsigma) \\ E(\varsigma) \end{bmatrix}$$

The solution is

$$E(\Theta_{10}) = \frac{E(\varsigma)}{1-p_{11}},$$

$$E(\Theta_{20}) = E(\varsigma) + \frac{a\,E(\varsigma)}{1-p_{11}}\,. \tag{31}$$

We will apply theorem 1 to calculate the limit probability distribution of the state. Now, the system of linear equation (20) is

$$\pi_1 p_{10} + \pi_2 p_{20} = \pi_0,$$

$$\pi_1 p_{12} + \pi_2 p_{21} = \pi_1,$$

$$\pi_0 = \pi_2,$$

$$\pi_0 + \pi_1 + \pi_2 = 1.$$

Since, the stationary distribution of the embedded Markov chain is

$$\pi_0 = \frac{p_{11}}{2p_{11} + p_{21}},$$

$$\pi_1 = \frac{p_{21}}{2p_{11} + p_{21}},$$

$$\pi_2 = \frac{p_{11}}{2p_{11} + p_{21}}.$$

Using formula (19) we obtain the limit distribution of semi-Markov process

$$P_0 = \frac{p_{11}E(\eta)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)} \tag{32}$$

$$P_1 = \frac{p_{21}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}$$

$$P_2 = \frac{p_{11}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}$$

5.4. Conclusion

The expectation $E(\Theta_{20})$ denoting the mean time to failure is

$$E(\Theta_{20}) = E(\varsigma) + \frac{a\,E(\varsigma)}{1-p_{11}},$$

where

$$p_{11} = a\int_0^\infty G(x)dF(x)\,.$$

Let us notice, that the cold standby determines increase the meantime to failure $1 + \frac{a}{1-p_{11}}$ times.

The limiting availability coefficient of the system is

$$A = P_1 + P_2 = \frac{p_{21}E(\varsigma) + p_{11}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}\,.$$

6. Semi-Markov process as the reliability model of the operation with perturbation

Semi-Markov process as the reliability model of multi-stage operation was considered by F. Grabski in [8] and [10]. Many operations consist of some elementary tasks, which are realized in turn. Duration of the each task realization is assumed to be positive random variable. Each elementary operation may be perturbed or failed. The perturbations increase the time of operation and the probability of failure as well.

6.1. Description and assumptions

Suppose, that the operation consists of *n* stages which following in turn. We assume that duration of an *i*-th stage, ($i = 1, \ldots, n$) is a nonnegative random variable $\xi_i; i = 1, \ldots, n$ with a cumulative probability distribution

$$F_i(t) = P(\xi_i \le t) = \int_0^t f_i(x)dx, \quad i = 1, \ldots, n,$$

where $f_i(x)$ denotes its probability density function in an extended sense.
Time to failure of the operation on the *i*-th stage (component) is the nonnegative random variable η_i, $i = 1, \ldots, n$ with exponential distribution

$$P(\eta_i \le t) = 1 - e^{-\lambda_i t}; \quad i = 1, \ldots, n.$$

The operation on each step may be perturbed. We assume that no more then one event causing perturbation on each stage of the operation may occur. Time to event causing of an operation perturbation on *i*-th stage is a nonnegative random variable $\zeta_i; i = 1, \ldots, n$ with exponential distribution

$$P(\zeta_i \le t) = 1 - e^{-\alpha_i t}; \quad i = 1, \ldots, n.$$

The perturbation degreases the probability of the operation fail. We suppose that time to failure of the perturbed operation on the i-th stage is the nonnegative random variable $\nu_i, i = 1,2\ldots,n$ that has the exponential distribution with a parameter $\beta_i > \lambda_i$

$$P(\nu_i \le t) = 1 - e^{-\beta_i t}; \quad i = 1, \ldots, n.$$

We assume that the operation is cyclical.
We assume that random variables $\xi_i, \eta_i, \nu_i, \zeta_i, \quad i = 1,\ldots,n$ are mutually independent.

6.2. Semi-Markov model

To construct reliability model of operation, we have to start from definition of the process states.
Let e_{ij}, *i=1,...,n, j=0,1* denotes *j-th* reliability state on *i-th* step of the operation where, *j=0* denotes perturbation and *j=1* denotes success
e_{2n+1} - failure (un-success) of the operation
e_{11} - an initial state.
For convenience we numerate the states

$$e_{i1} \leftrightarrow i, \quad i = 1,\ldots,n$$

$$e_{i0} \leftrightarrow i+n, \quad i = 1,\ldots,n$$

$$e_{2n+1} \leftrightarrow 2n+1,$$

Under the above assumptions, stochastic process describing of the overall operation in reliability aspect, is a semi-Markov process $\{X(t): \ t \ge 0\}$ with a space of states $S = \{1, 2, \ldots, 2n, 2n+1\}$ and flow graph shown in *Figure 3.*

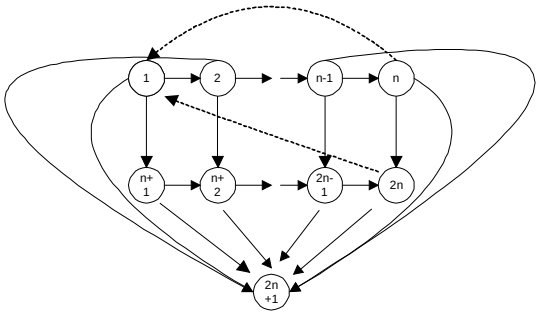

Figure 3. Transition graph for n-stage cyclic operation

To obtain a semi-Markov model we have to define all nonnegative elements of semi- Markov kernel

$$\mathbf{Q}(t) = \lfloor Q_{ij}(t) : i, j \in S \rfloor$$

$$Q_{ij}(t) = P\{X(\tau_{n+1}) = j, \tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i\}$$

First, we define transition probabilities from the state *i to* the state *j* for time not greater than t for *i=1,…,n-1.*

$$Q_{i\,i+1}(t) = P(\xi_i \le t, \eta_i > \xi_i, \zeta_i > \xi_i)$$
$$= \iiint_D \alpha_i e^{-\alpha_i y} \lambda_i e^{-\lambda_i z} f_i(x)dx\,dy\,dz$$

where

$$D = \{(x, y, z): \quad x \ge 0, \ y \ge 0, \ z \ge 0,$$
$$x \le t, \quad z > x, \quad x > y\}$$

Since, we have

$$Q_{i\,i+1}(t)=\int_0^t f_i(x)dx\int_0^x \alpha_i e^{-\alpha_i y}dy\int_x^\infty \lambda_i e^{-\lambda_i z}dz$$

$$=\int_0^t e^{-(\lambda_i+\alpha_i)x}\,f_i(x)\,dx\ .$$

For $i=n+1,...,2n-1$ we obtain

$$Q_{i\,i+n}(t)=P(\zeta_i\le t,\eta_i>\zeta_i,\zeta_i>\xi_i)$$

$$=\int_0^t \alpha_i e^{-(\lambda_i+\alpha_i)u}[1-F_i(u)]du\ ,\ i=1,...,n-1.$$

For $i=1,...,n$ we get

$$Q_{i\,2n+1}(t)=P(\eta_i\le t,\eta_i<\zeta_i,\eta_i<\xi_i)$$

$$=\int_0^t \lambda_i e^{-(\lambda_i+\alpha_i)u}[1-F_i(u)]du\ ,$$

$i=1,...,n-1.$

If on *i*-th stage a perturbation has happened the transition probability to next state for time less then or equal to t is

$$Q_{n+i\,n+i+1}(t)=P(\xi_i-\zeta_i\le t,\nu_i>\xi_i-\zeta_i\,|\,\xi_i>\zeta_i)$$

$$=\frac{\iiint_D \alpha_i e^{-\alpha_i y}\beta_i e^{-\beta_i z}\,f_i(x)dx\,dy\,dz}{\iint_E \alpha_i e^{-\alpha_i y}\,f_i(x)dx\,dy},$$

$i=1,...,n-1,$

where

$$D=\{(x,y,z):\quad x\ge 0,\ y\ge 0,\ z\ge 0,$$

$$0\le x-y\le t,\quad z>x-y,\quad x>y\}$$

$$E=\{(x,y):\quad x\ge 0,\ y\ge 0,\ x>y\}.$$

To find the triple integral over the region *D*, we apply change of coordinates:

$$u=x-y,\quad v=y,\quad w=z.$$

Hence

$$x=u+v,\quad y=v,\quad z=w.$$

This mapping assigns to points from set

$$\Delta=\{(u,v,w):0\le u\le t,\ v\ge 0,\ w>u\}$$

the points from plane region *D*. The Jacobian of this mapping is

$$J(u,v,w)=1.$$

Since, we get

$$\iiint_D \alpha_i e^{-\alpha_i y}\beta_i e^{-\beta_i z}\,f_i(x)dx\,dy\,dz$$

$$=\iiint_\Delta \alpha_i e^{-\alpha_i v}\beta_i e^{-\beta_i w}\,f_i(u+v)J(u,v,w)du\,dv\,dw$$

$$=\int_0^\infty \alpha_i e^{-\alpha_i v}dv\int_u^\infty \beta_i e^{-\beta_i w}dw\int_0^t f_i(u+v)du$$

$$=\int_0^t e^{-\beta_i u}du\int_0^\infty \alpha_i e^{-\alpha_i v}\,f_i(u+v)dv\ .$$

Let us notice that

$$\iint_E \alpha_i e^{-\alpha_i y}\,f_i(x)dx\,dy=\int_0^\infty \alpha_i e^{-\alpha_i y}\int_y^\infty f_i(x)dx$$

$$=\int_0^\infty \alpha_i e^{-\alpha_i y}[1-F_i(y)]dy$$

$$=1-\int_0^\infty \alpha_i e^{-\alpha_i y}F_i(y)dy.$$

Finally, we obtain

$$Q_{n+i\,n+i+1}(t)=\frac{\int_0^t e^{-\beta_i u}[\int_0^\infty \alpha_i e^{-\alpha_i v}\,f_i(u+v)dv]du}{\int_0^\infty \alpha_i e^{-\alpha_i y}[1-F_i(y)]dy},\ .$$

$i=1,...,n-1.$

In the same way we get

$$Q_{n+i\,2n+1}(t) = P(\nu_i \le t, \nu_i < \xi_i - \zeta_i \mid \xi_i > \zeta_i)$$

$$= \frac{\iiint_D \alpha_i e^{-\alpha_i y} \beta_i e^{-\beta_i z} f_i(x)\, dx\, dy\, dz}{\iint_E \alpha_i e^{-\alpha_i y} f_i(x) dx\, dy},$$

$i = 1,..., n$

where

$$D = \{(x, y, z): \quad x \ge 0, y \ge 0,$$

$$0 \le z \le t, \quad z < x - y, \quad x > y\}$$

$$E = \{(x, y): \quad x \ge 0, y \ge 0, \ x > y\}.$$

Hence

$$Q_{n+i\,2n+1}(t) = \frac{\beta_i \int_0^t e^{-\beta_i w} dw \int_w^\infty \alpha_i e^{-\alpha_i v} dv \int_0^\infty f_i(u+v) du}{\int_0^\infty \alpha_i e^{-\alpha_i y}[1 - F_i(y)] dy}$$

$i = 1,..., n.$

Similar way we obtain

$$Q_{n1}(t) = P(\xi_n \le t, \eta_n > \xi_n\ \zeta_n > \xi_n)$$

$$= \int_0^t e^{-(\lambda_n + \alpha_n)u} f_n(u)\, du$$

$$Q_{2n1}(t) = P(\xi_n - \zeta_n \le t, \nu_n > \xi_n - \zeta_n \mid \xi_n > \zeta_n)$$

$$= \frac{\int_0^t e^{-\beta_n u} du \int_0^\infty \alpha_n e^{-\alpha_i v} f_n(u+v) dv}{\int_0^\infty \alpha_n e^{-\alpha_n x}[1 - F_n(x)]\, dx}.$$

Therefore the semi-Markov reliability model of operation has been constructed.

6.3. Two-stage cyclical operation

We will investigate particular case of that model, assuming $n = 2$. A transition matrix for the semi-Markov model of the *2*-stage cyclic operation in reliability aspect takes the following form

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) & 0 & Q_{15}(t) \\ Q_{21}(t) & 0 & 0 & Q_{24}(t) & Q_{25}(t) \\ 0 & 0 & 0 & Q_{34}(t) & Q_{35}(t) \\ Q_{41}(t) & 0 & 0 & 0 & Q_{45}(t) \\ 0 & 0 & 0 & 0 & Q_{55}(t) \end{bmatrix},$$

where

$$Q_{12}(t) = \int_0^t e^{-(\lambda_1 + \alpha_1)u} dF_1(u),$$

$$Q_{13}(t) = \int_0^t \alpha_1 e^{-(\lambda_{1i} + \alpha_1)u}[1 - F_1(u)] du,$$

$$Q_{15}(t) = \int_0^t \lambda_1 e^{-(\lambda_{1i} + \alpha_1)u}[1 - F_1(u)] du,$$

$$Q_{21}(t) = \int_0^t e^{-(\lambda_2 + \alpha_2)u} dF_2(u),$$

$$Q_{24}(t) = \int_0^t \alpha_2 e^{-(\lambda_2 + \alpha_2)u}[1 - F_2(u)] du,$$

$$Q_{25}(t) = \int_0^t \lambda_2 e^{-(\lambda_2 + \alpha_2)u}[1 - F_2(u)] du,$$

$$Q_{34}(t) = \frac{\int_0^t e^{-\beta_1 u}[\int_0^\infty \alpha_1 e^{-\alpha_1 v} f_1(u+v) dv] du}{\int_0^\infty \alpha_1 e^{-\alpha_i y}[1 - F_1(y)] dy},$$

$$Q_{35}(t) = \frac{\beta_1 \int_0^t e^{-\beta_i w} dw \int_w^\infty \alpha_1 e^{-\alpha_1 v} dv \int_0^\infty f_1(u+v) du}{\int_0^\infty \alpha_1 e^{-\alpha_1 x}[1 - F_1(x)] dx},$$

$$Q_{41}(t) = \frac{\int_0^t e^{-\beta_2 u} du \int_0^\infty \alpha_2 e^{-\alpha_2 v} f_2(u+v)\ ] dv}{\int_0^\infty \alpha_2 e^{-\alpha_2 x}[1 - F_2(x)] dx},$$

$$Q_{45}(t) = \frac{\beta_2 \int_0^t e^{-\beta_2 w} dw \int_w^\infty \alpha_2 e^{-\alpha_2 v} dv \int_0^\infty f_2(u+v) du}{\int_0^\infty \alpha_2 e^{-\alpha_2 x}[1 - F_2(x)] dx},$$

$$Q_{55}(t) = U(t).$$

We will consider 3

That model allows us to obtain some reliability characteristics of the operation. The random variable Θ_{15} denoting the first passage time from state 1 to state 5 in our model, means time to failure of the operation. The Laplace-Stieltjes transform for the cumulative distribution function of that random variable we will obtain from a matrix equation (25).

In this case we have $A' = \{1,2,3,4\}$, $A = \{5\}$ and

$$\tilde{\mathbf{u}}_{A'}(s) = \begin{bmatrix} \tilde{\phi}_{15}(s) \\ \tilde{\phi}_{25}(s) \\ \tilde{\phi}_{35}(s) \\ \tilde{\phi}_{45}(s) \end{bmatrix}, \quad \tilde{\mathbf{b}}(s) = \begin{bmatrix} \tilde{q}_{15}(s) \\ \tilde{q}_{25}(s) \\ \tilde{q}_{35}(s) \\ \tilde{q}_{45}(s) \end{bmatrix},$$

$$\tilde{\mathbf{q}}_{A'}(s) = \begin{bmatrix} 0 & \tilde{q}_{12}(s) & \tilde{q}_{13}(s) & 0 \\ \tilde{q}_{21}(s) & 0 & 0 & \tilde{q}_{24}(s) \\ 0 & 0 & 0 & \tilde{q}_{34}(s) \\ \tilde{q}_{41}(s) & 0 & 0 & 0 \end{bmatrix}.$$

From the solution of equation (24) we obtain Laplace-Stieltjes transform of the cumulative distribution function of the random variable Θ_{15} denoting time to failure of the operation

$$\tilde{\phi}_{15}(s) = \frac{\tilde{a}(s)}{\tilde{b}(s)}, \tag{33}$$

$$\tilde{a}(s) = \tilde{q}_{15}(s) + \tilde{q}_{12}(s)\tilde{q}_{25}(s) + \tilde{q}_{13}(s)\tilde{q}_{35}(s)$$

$$+ \tilde{q}_{12}(s)\tilde{q}_{24}(s)\tilde{q}_{45}(s) + \tilde{q}_{13}(s)\tilde{q}_{34}(s)\tilde{q}_{45}(s)$$

$$\tilde{b}(s) = 1 - \tilde{q}_{12}(s)\tilde{q}_{21}(s) - \tilde{q}_{12}(s)\tilde{q}_{24}(s)\tilde{q}_{41}(s)$$

$$- \tilde{q}_{13}(s)\tilde{q}_{34}(s)\tilde{q}_{41}(s).$$

The Laplace transform of the reliability function is given by the formula

$$\tilde{R}(s) = \frac{1 - \tilde{\phi}_{15}(s)}{s}. \tag{34}$$

6.4. Examples

Example 1
We suppose that

$$F_i(t) = 1 - e^{-\kappa_i t}, \; t \geq 0, \; i = 1,2.$$

Then

$$Q_{12}(t) = \frac{\kappa_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{13}(t) = \frac{\alpha_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{15}(t) = \frac{\lambda_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{21}(t) = \frac{\kappa_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{24}(t) = \frac{\alpha_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{25}(t) = \frac{\lambda_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{34}(t) = \frac{\kappa_1}{\beta_1 + \kappa_1}\left(1 - e^{-(\beta_1+\kappa_1)t}\right),$$

$$Q_{35}(t) = \frac{\beta_1}{\beta_1 + \kappa_1}\left(1 - e^{-(\beta_1+\kappa_1)t}\right),$$

$$Q_{41}(t) = \frac{\kappa_2}{\beta_2 + \kappa_2}\left(1 - e^{-(\beta_2+\kappa_2)t}\right),$$

$$Q_{45}(t) = \frac{\beta_2}{\beta_2 + \kappa_2}\left(1 - e^{-(\beta_2+\kappa_2)t}\right),$$

Laplace-Stieltjes transform of these functions are:

$$\tilde{q}_{12}(s) = \frac{\kappa_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{13}(s) = \frac{\alpha_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{15}(s) = \frac{\lambda_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{21}(s) = \frac{\kappa_2}{s + \lambda_2 + \alpha_2 + \kappa_2},$$

$$\tilde{q}_{24}(s) = \frac{\alpha_2}{s + \lambda_2 + \alpha_2 + \kappa_2},$$

$$\tilde{q}_{25}(s)=\frac{\lambda_2}{s+\lambda_2+\alpha_2+\kappa_2},$$

$$\tilde{q}_{34}(s)=\frac{\kappa_1}{s+\beta_1+\kappa_1},$$

$$\tilde{q}_{35}(s)=\frac{\beta_1}{s+\beta_1+\kappa_1},$$

$$\tilde{q}_{41}(s)=\frac{\kappa_2}{s+\beta_2+\kappa_2},$$

$$\tilde{q}_{45}(s)=\frac{\beta_2}{s+\beta_2+\kappa_2}.$$

For $\kappa_1=0{,}1$; $\kappa_2=0{,}12$; $\lambda_1=0{,}002$; $\lambda_2=0{,}001$; $\alpha_1=0{,}02$; $\alpha_2=0{,}04$; $\beta_1=0{,}01$; $\beta_1=0{,}01$, applying (33) and (34), with help of MATHEMATICA computer program, we obtain the density function and the reliability function as inverse Laplace transforms .
The density function is given by the formula

$$\phi_{15}(t)$$

$$=0.00712201\,e^{-0.213357\,t}-0.00872613\,e^{-0.180053\,t}$$

$$-0.000144434\,e^{-0.125901\,t}+0.00374856\,e^{-0.00368869\,t}$$

This function is shown in *Figure 4.*
The reliability function is

$$R(t)=3.79823\times10^{-16}+0.0333808\,e^{-0.213357\,t}$$

$$-0.0484641\,e^{-0.180053\,t}-0.0011472\,e^{-0.125901\,t}$$

$$+1.01623\,e^{-0.00368869\,t}$$

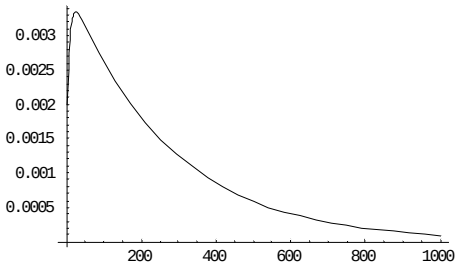

Figure 4. The density function the time to failure of 2-stage cyclic operation

This function is shown on *Figure 5*.

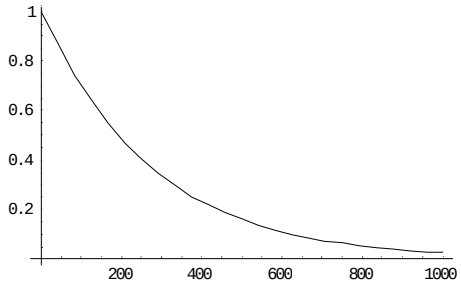

Figure 5. The reliability function of 2-stage cyclic operation

Mean time to failure we can find solving the matrix equation

$$\left(I-P_{A'}\right)\overline{\Theta}_{A'}=\overline{T}_{A'} \tag{35}$$

where

$$\mathbf{P}_{A'}=\begin{bmatrix}0 & p_{12} & p_{13} & 0\\ p_{21} & 0 & 0 & p_{24}\\ 0 & 0 & 0 & p_{34}\\ p_{41} & 0 & 0 & 0\end{bmatrix},$$

$$\overline{\mathbf{T}}=\begin{bmatrix}E(T_1)\\ E(T_2)\\ E(T_3)\\ E(T_4)\end{bmatrix},\qquad \overline{\Theta}_{A'}=\begin{bmatrix}E(\Theta_{15})\\ E(\Theta_{25})\\ E(\Theta_{35})\\ E(\Theta_{45})\end{bmatrix}.$$

From this equation we obtain the mean time to failure

$$E(\Theta_{15})=275{,}378$$

Example 2
Now we assume that

$$F_i\ (t)=\begin{cases}0 & \text{dla}\quad t\le L_i\\ 1 & \text{dla}\quad t>L_i\end{cases},\ i=1{,}2\ .$$

It means that the duration of the stages are determined and they are equal $\xi_i=L_i$ for $i=1{,}2$.
In this case the elements of $\boldsymbol{Q}(t)$ are:

$$Q_{12}(t)=\begin{cases}0 & \text{for}\quad t\le L_1\\ e^{-(\lambda_1+\alpha_1)L_1} & \text{for}\quad t>L_1\end{cases}$$

$$Q_{13}(t)=\begin{cases}\dfrac{\alpha_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)t}\right) & \text{for} \quad t\le L_1\\ \dfrac{\alpha_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)L_1}\right) & \text{for} \quad t>L_1\end{cases}$$

$$Q_{15}(t)=\begin{cases}\dfrac{\lambda_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)t}\right) & \text{for} \quad t\le L_1\\ \dfrac{\lambda_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)L_1}\right) & \text{for} \quad t>L_1\end{cases}$$

$$Q_{21}(t)=\begin{cases}0 & \text{for} \quad t\le L_2\\ e^{-(\lambda_2+\alpha_2)L_2} & \text{for} \quad t>L_2\end{cases}$$

$$Q_{24}(t)=\begin{cases}\dfrac{\alpha_2}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)t}\right) & \text{for} \quad t\le L_2\\ \dfrac{\alpha_1}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)L_2}\right) & \text{for} \quad t>L_2\end{cases}$$

$$Q_{25}(t)=\begin{cases}\dfrac{\lambda_2}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)t}\right) & \text{for} \quad t\le L_2\\ \dfrac{\lambda_1}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)L_2}\right) & \text{for} \quad t>L_2\end{cases}$$

$$Q_{34}(t)=\begin{cases}\dfrac{\alpha_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)(1-e^{-a_1L_1})}\left(1-e^{-(\beta_1-\alpha_1)t}\right), & 0\le t\le L_1\\ \dfrac{\alpha_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)(1-e^{-a_1L_1})}\left(1-e^{-(\beta_1-\alpha_1)L_1}\right), & t>L\end{cases},$$

$$Q_{35}(t)=\begin{cases}\dfrac{1}{(1-e^{-a_1L_1})}\left[\left(1-e^{-\beta_1 t}\right)-\dfrac{\beta_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)}\left(1-e^{-(\beta_1-\alpha_1)t}\right)\right], & 0\le t\le L_1\\ \dfrac{1}{(1-e^{-a_1L_1})}\left[\left(1-e^{-\beta_1L_1}\right)-\dfrac{\beta_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)}\left(1-e^{-(\beta_1-\alpha_1)L_1}\right)\right], & t>L_1\end{cases}$$

$$Q_{41}(t)=\begin{cases}\dfrac{\alpha_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)(1-e^{-a_2L_2})}\left(1-e^{-(\beta_2-\alpha_2)t}\right), & 0\le t\le L_2\\ \dfrac{\alpha_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)(1-e^{-a_2L_2})}\left(1-e^{-(\beta_2-\alpha_2)L_2}\right), & t>L_2\end{cases}$$

$$Q_{45}(t)=\begin{cases}\dfrac{1}{(1-e^{-a_2L_2})}\left[\left(1-e^{-\beta_2 t}\right)-\dfrac{\beta_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)}\left(1-e^{-(\beta_2-\alpha_2)t}\right)\right], & 0\le t\le L_2\\ \dfrac{1}{(1-e^{-a_2L_2})}\left[\left(1-e^{-\beta_2L_1}\right)-\dfrac{\beta_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)}\left(1-e^{-(\beta_2-\alpha_2)L_1}\right)\right], & t>L_2\end{cases}$$

The Laplace'a-Stieltjes transform of these functions are:

$$\tilde{q}_{12}(s)=e^{-(\lambda_1+\alpha_1+s)L_1}$$

$$\tilde{q}_{13}(s)=\frac{\alpha_1}{s+\lambda_1+\alpha_1}-\frac{\alpha_1 e^{-(\lambda_1+\alpha_1+s)L_1}}{s+\lambda_1+\alpha_1}$$

$$\tilde{q}_{15}(s)=\frac{\lambda_1}{s+\lambda_1+\alpha_1}-\frac{\lambda_1 e^{-(\lambda_1+\alpha_1+s)L_1}}{s+\lambda_1+\alpha_1}$$

$$\tilde{q}_{21}(s)=e^{-(\lambda_2+\alpha_2+s)L_2},$$

$$\tilde{q}_{24}(s)=\frac{\alpha_2}{s+\lambda_2+\alpha_2}-\frac{\alpha_2 e^{-(\lambda_2+\alpha_2+s)L_2}}{s+\lambda_2+\alpha_2}$$

$$\tilde{q}_{25}(s)=\frac{\lambda_2}{s+\lambda_2+\alpha_2}-\frac{\lambda_2 e^{-(\lambda_2+\alpha_2+s)L_{21}}}{s+\lambda_2+\alpha_2}$$

$$\tilde{q}_{34}(s)=\frac{\alpha_1 e^{-\alpha_1L_1}}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1-\alpha_1+s)L_1}}{s+\beta_1-\alpha_1}\right)$$

$$\tilde{q}_{35}(s)=\frac{\beta_1}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1+s)L_1}}{s+\beta_1}-\frac{e^{-\alpha_1L_1}(1-e^{-(\beta_1-\alpha_1+s)L_1})}{s+\beta_1-\alpha_1}\right)$$

$$\tilde{q}_{41}(s)=\frac{\alpha_2 e^{-\alpha_2L_2}}{1-e^{-\alpha_2L_2}}\left(\frac{1-e^{-(\beta_2-\alpha_2+s)L_2}}{s+\beta_2-\alpha_2}\right)$$

$$\tilde{q}_{45}(s)=\frac{\beta_2}{1-e^{-\alpha_2L_2}}\left(\frac{1-e^{-(\beta_2+s)L_2}}{s+\beta_2}-\frac{e^{-\alpha_2L_2}(1-e^{-(\beta-\alpha_2+s)L_2})}{s+\beta_2-\alpha_2}\right)$$

The mean time to failure we can find solving the matrix equation (35), where

$$p_{12}=e^{-(\lambda_1+\alpha_1)L_1}$$

$$p_{13}=\frac{\alpha_1(1-e^{-(\lambda_1+\alpha_1)L_1})}{\lambda_1+\alpha_1} \qquad p_{15}=\frac{\lambda_1(1-e^{-(\lambda_1+\alpha_1)L_1})}{\lambda_1+\alpha_1}$$

$$p_{21}=e^{-(\lambda_2+\alpha_2)L_2}$$

$$p_{24}=\frac{\alpha_2(1-e^{-(\lambda_2+\alpha_2)L_2})}{\lambda_2+\alpha_2}$$

$$p_{25}=\frac{\lambda_2(1-e^{-(\lambda_2+\alpha_2)L_{21}})}{\lambda_2+\alpha_2}$$

$$p_{34}=\frac{\alpha_1 e^{-\alpha_1L_1}}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1-\alpha_1)L_1}}{\beta_1-\alpha_1}\right)$$

$$p_{35} = \frac{\beta_1}{1-e^{-\alpha_1 L_1}}\left(\frac{1-e^{-\beta_1 L_1}}{\beta_1}\right.$$

$$\left. - \frac{e^{-\alpha_1 L_1}(1-e^{-(\beta_1-\alpha_1)L_1}}{\beta_1-\alpha_1}\right)$$

$$p_{41} = \frac{\alpha_2 e^{-\alpha_2 L_2}}{1-e^{-\alpha_2 L_2}}\left(\frac{1-e^{-(\beta_2-\alpha_2)L_2}}{\beta_2-\alpha_2}\right)$$

$$p_{45} = \frac{\beta_2}{1-e^{-\alpha_2 L_2}}$$

$$\left(\frac{1-e^{-\beta_2 L_2}}{\beta_2} - \frac{e^{-\alpha_2 L_2}(1-e^{-(\beta-\alpha_2)L_2})}{\beta_2-\alpha_2}\right)$$

$$p_{51} = 1$$

$$E(T_1) = \frac{1}{\alpha_1+\lambda_1} - \frac{e^{-(\alpha_1+\lambda_1)L_1}}{\alpha_1+\lambda_1}$$

$$E(T_2) = \frac{1}{\alpha_2+\lambda_2} - \frac{e^{-(\alpha_2+\lambda_2)L_2}}{\alpha_2+\lambda_2}$$

$$E(T_3) = -\frac{d(\tilde{q}_{34}(s)+\tilde{q}_{35}(s))}{ds}\Big|_{s=0}$$

$$E(T_4) = -\frac{d(\tilde{q}_{41}(s)+\tilde{q}_{45}(s))}{ds}\Big|_{s=0}$$

For the same parameters $\kappa_1 = 0{,}1$; $\kappa_2 = 0{,}12$; $\lambda_1 = 0{,}002$; $\lambda_2 = 0{,}001$; $\alpha_1 = 0{,}02$; $\alpha_2 = 0{,}04$; $\beta_1 = 0{,}01$; $\beta_1 = 0{,}01$, and $L_1 = \frac{1}{\kappa_1}$ i $L_2 = \frac{1}{\kappa_1}$, the mean time to failure of the operation is

$$E(\Theta_{15}) = 366.284 .$$

In previous case the mean time to failure is

$$E(\Theta_{15}) = 275{,}378 .$$

6.5. Conclusion

It means that for the determined duration of the stages mean time to failure of the operation is essentially greater than for exponentially distributed duration of the stages with the same expectations.

To assess reliability of the many stage operation we can apply a semi-Markov process. Construction of the semi-Markov model consist in defining a kernel of that process. A way of building the kernel for the semi-Markov model of the many stage operation is presented in this paper. From Semi-Markov model we can obtain many interesting parameters and characteristics for analysing reliability of the operation.
From presented examples we get conclusion that for the determined duration of the stages, mean time to failure of the operation is essentially greater than for exponentially distributed duration of the stages with the same expectations.

7. Semi-Markov process as a failure rate

The reliability function with semi-Markov failure rate was considered by Kopociński & Kopocińska [11], Kopocińska [12] and by Grabski [4], [6], [9]. Suppose that the failure rate $\{\lambda(t): t \geq 0\}$ is the semi-Markov process with the discrete state space $S = \{\lambda_j : j \in J\}$, $J = \{0,1,...,m\}$ or $J = \{0,1,2,...\}$, $0 \leq \lambda_0 < \lambda_1 < ...$ with the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in J]$$

and the initial distribution $p = [p_i : i \in J]$.
We define a conditional reliability function as

$$R_i(t) = E\left[\exp\left(-\int_0^t л(u)du\right)\middle|\lambda(0) = \lambda_i\right], \quad t \geq 0, \quad i \in J$$

In [6] it is proved, that for the regular semi-Markov process $\{\lambda(t): t \geq 0\}$ the conditional reliability functions $R_i(t)$, $t \geq 0$, $i \in J$ defined by (17), satisfy the system of equations

$$R_i(t) = e^{-\lambda_i t}[1-G_i(t)] + \sum_j \int_0^t e^{-\lambda_i x} R_j(t-x)dQ_j(x), i \in J$$

Applying the Laplace transformation we obtain the system of linear equations

$$\tilde{R}_i(s) = \frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) + \sum_j \tilde{R}_j(s)\tilde{q}_{ij}(s+\lambda_i), i \in J,$$

where

$$\tilde{R}_i(s) = \int_0^\infty e^{-st} R_i(t)dt, \ \tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t)dt,$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t) .$$

In matrix notation we have

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]\tilde{\mathbf{R}}(s) = \tilde{\mathbf{H}}(s),$$

where

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]$$

$$= \begin{bmatrix} 1-\tilde{q}_{00}(s+\lambda_0) & -\tilde{q}_{01}(s+\lambda_0) & -\tilde{q}_{02}(s+\lambda_0) & \mathrm{i} \\ -\tilde{q}_{10}(s+\lambda_1) & 1-\tilde{q}_{11}(s+\lambda_1) & -\tilde{q}_{12}(s+\lambda_1) & \mathrm{i} \\ -\tilde{q}_{20}(s+\lambda_2) & -\tilde{q}_{21}(s+\lambda_2) & 1-\tilde{q}_{22}(s+\lambda_2) & \mathrm{i} \\ \mathrm{j} & \mathrm{j} & \mathrm{j} & \mathrm{I} \end{bmatrix}$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \\ \tilde{R}_2(s) \\ \mathrm{j} \end{bmatrix} \qquad \tilde{\mathbf{H}}(s) = \begin{bmatrix} \dfrac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) \\ \dfrac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1) \\ \dfrac{1}{s+\lambda_2} - \tilde{G}_1(s+\lambda_2) \\ \mathrm{j} \end{bmatrix}$$

The conditional mean times to failure we obtain from the formula

$$\mu_i = \lim_{p\to 0^+} \tilde{R}_i(p),\ p \in (0,\infty), \quad i \in J \tag{21}$$

The unconditional mean time to failure has a form

$$\mu = \sum_{i=1}^{N} P(\lambda(0) = \lambda_i)\, \mu_i .$$

7.1. Alternating random process as a failure rate

Assume that the failure rate is a semi-Markov process with the state space $S = \{\lambda_0, \lambda_1\}$ and the kernel

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & G_0(t) \\ G_1(t) & 0 \end{bmatrix},$$

where $G_0(t), G_1(t),$ are the cumulative probability distribution functions with nonnegative support. Suppose that at least one of the functions is absolutely continuous with respect to the Lebesgue measure. Let $p = [p_0,\ p_1]$ be an initial probability distribution of the process. That stochastic process is called the *alternating random process*. In that case the matrices from the equation (20) are

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] = \begin{bmatrix} 1 & -\tilde{g}_0(s+\lambda_0) \\ -\tilde{g}_1(s+\lambda_1) & 1 \end{bmatrix},$$

where

$$\tilde{g}_0(s) = \tilde{q}_{01}(s) = \int_0^\infty e^{-st} dG_0(t) ,$$

$$\tilde{g}_1(s) = \tilde{q}_{10}(s) = \int_0^\infty e^{-st} dG_1(t) ,$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \end{bmatrix},$$

$$\tilde{\mathbf{H}}(s) = \begin{bmatrix} \frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) \\ \frac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1) \end{bmatrix}.$$

The solution of (20) takes the form

$$\tilde{R}_0(s)$$

$$= \frac{\frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) + \tilde{g}_0(s+\lambda_0)\left[\frac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)},$$

$$\tilde{R}_1(s)$$

$$= \frac{\frac{1}{s+\lambda_0} - \tilde{G}_1(s+\lambda_1) + \tilde{g}_1(s+\lambda_1)\left[\frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)}.$$

The Laplace transform of the unconditional reliability function is

$$\tilde{R}(s) = p_0 \tilde{R}_0(s) + p_1 \tilde{R}_1(s).$$

Example 3.
Assume that

$$G_0(t) = \int_0^t g_0(x)dx ,\quad G_1(t) = \int_0^t g_1(x)dx ,$$

where

$$g_0(x) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} x^{\alpha_0 - 1} e^{-\beta_0 x}, \quad x \ge 0 ,$$

$$g_1(t) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_2)} x^{\alpha_1 - 1} e^{-\beta_1 x}, \quad x \ge 0.$$

Suppose that an initial state is λ_0. Hence the initial distribution is $\mathbf{p}(0)=[1 \quad 0]$ and the Laplace transform of the unconditional reliability function is $\tilde{R}(s)=\tilde{R}_0(s)$. Now the equation (20) takes the form of

$$\begin{bmatrix} 1 & -\dfrac{\beta_0^{\alpha_0}}{(s+\beta_0+\lambda_0)^{\alpha}} \\ -\dfrac{\beta_1^{\alpha_1}}{(s+\beta_1+\lambda_1)^{\alpha_1}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s+\lambda_0}-\dfrac{\beta_1^{\alpha_0}}{(s+\lambda_0)(s+\beta_0+\lambda_0)^{\alpha_0}} \\ \dfrac{1}{s+\lambda_1}-\dfrac{\beta_1^{\alpha_1}}{(s+\lambda_1)(s+\beta_1+\lambda_1)^{\alpha_1}} \end{bmatrix}$$

For

$$\alpha_0=2, \alpha_1=3, \beta_0=0.2, \beta_1=0.5, \lambda_0=0, \lambda_1=0.2,$$

we have

$$\tilde{R}_0(s)=\frac{\dfrac{1}{s}-\dfrac{0.04}{s(s+0.2)^2}+\dfrac{0.04}{(s+0.2)^2}\left[\dfrac{1}{s+0.2}-\dfrac{0.125}{(s+0.2)(s+0.7)^3}\right]}{1-\dfrac{0.04}{(s+0.2)^2}\dfrac{0.125}{(s+0.7)^3}}$$

Using the MATHEMATICA computer program we obtain the reliability function as the inverse Laplace transform.

$$R(t)=1.33023\exp(-0.0614293\,t)$$

$$+\exp(-0.02\,t)(1.34007\cdot 10^{-14}+9.9198\cdot 10^{-15}\,t)$$

$$-2\exp(-0.843935\,t)[0.0189459\cos(0.171789\,t)$$

$$+0.00695828\sin(0.171789\,t)]$$

$$-2\exp(-0.37535\,t)[0.146168\cos(0.224699\,t)$$

$$+0.128174\sin(0.224699\,t)]$$

Figure 6 shows the reliability function.

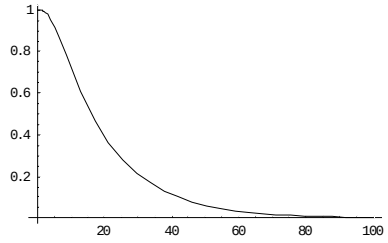

Figure 6. The reliability function from example 2

The corresponding density function

$$f(t)=-R'(t)$$

is shown in *Figure 7*

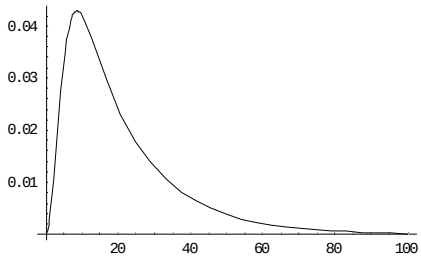

Figure 7. The density function from example 2

8. Conclusion

The semi-Markov processes theory is convenient for description of the reliability systems evolution through the time. The probabilistic characteristics of semi-Markov processes are interpreted as the reliability coefficients of the systems. If A represents the subset of failing states and i is an initial state, the random variable Θ_{iA} designating the first passage time from the state i to the states subset A, denotes the time to failure of the system. Theorems of semi-Markov processes theory allows us to find the reliability characteristic, like the distribution of the time to failure, the reliability function, the mean time to failure, the availability coefficient of the system and many others. We should remember that semi-Markov process might be applied as a model of the real system reliability evolution, only if the basic properties of the semi-Markov process definition are satisfied by the real system.

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