Grabski Franciszek
Naval University, Gdynia, Poland

The random failure rate

Keywords

reliability, random failure rate, semi-Markov process

Abstract

A failure rate of the object is assumed to be a stochastic process with nonnegative, right continuous trajectories. A reliability function is defined as an expectation of a function of a random failure rate process. The properties and examples of the reliability function with the random failure rate are presented in the paper. A semi-Markov process as the random failure rate is considered in this paper.

1. Introduction

Often, the environmental conditions are randomly changeable and they cause a random load of an object. Thus, the failure rate depending on the random load is a random process. The reliability function with semi-Markov failure rate was considered in the following papers Kopociński & Kopocińska [5], [6], Grabski [3], [4].

2. Reliability function with random failure rate

Let $\{\text{л}(t):\ t \geq 0\}$ be a random failure rate of an object. We assume that the stochastic process has the nonnegative, right continuous trajectories. The reliability function is defined as

$$R(t) = E\left[\exp\left(-\int_0^t \text{л}(x)dx\right)\right],\ t \geq 0. \qquad (1)$$

It means that the reliability function is an expectation of the process $\{\mathbf{o}(t): t \geq 0\}$, where

$$\mathbf{o}(t) = \exp\left(-\int_0^t \text{л}(x)dx\right),\ t \geq 0. \qquad (2)$$

Let

$$\overset{E}{R}(t) = \exp\left(-\int_0^t E[\text{л}(x)]dx\right),\ t \geq 0. \qquad (3)$$

From Jensen's inequality we get very important result

$$R(t) = E\left[\exp\left(-\int_0^t \text{л}(x)dx\right)\right]$$
$$\geq \exp\left(-\int_0^t E[\text{л}(x)]dx\right) = \overset{E}{R}(t),\ t \geq 0. \qquad (4)$$

The above mentioned inequality means that the reliability function defined by the stochastic process $\{\text{л}(t):\ t \geq 0\}$ is greater than or equal to the reliability function with the deterministic failure rate, equal to the expectation $\bar{\lambda}(t) = E[\text{л}(t)]$.
It is obvious, that for the stationary stochastic process $\{\text{л}(t):\ t \geq 0\}$, that has a constant mean value $\bar{\lambda}(t) = E[\text{л}(t)] = \lambda$, the reliability function defined by (3) is

$$\overset{E}{R}(t) = \exp\left(-\lambda\int_0^t dx\right) = \exp(-\lambda\, t),\ t \geq 0.$$

(5)

Hence, we come to conclusion: for each stationary random failure rate process, the according reliability function for each $t \geq 0$, has values greater than or equal to the exponential reliability function with parameter λ.

Example 1.
Suppose that, the failure rate of an object is a stochastic process $\{\lambda(t): t \geq 0\}$, given by $\lambda(t) = C\,t,\ t \geq 0$, where C is a nonnegative random variable. Trajectories of the process $\{\xi(t): t \geq 0\}$, are

$$\xi(t) = \exp(-c\frac{t^2}{2}),\ t \geq 0,$$

where c is a value of the random variable C. Assume that the random variable C has the exponential distribution with parameter β :

$$P(C \leq u) = 1 - e^{-\beta u},\ u \geq 0.$$

Then, according to (1), we compute the reliability function

$$R(t) = E\left[\exp\left(-\int_0^t Cx\,dx\right)\right] = \int_0^\infty e^{-u\frac{t^2}{2}} \beta e^{-\beta u}\,du$$

$$= \beta \int_0^\infty e^{-u\left(\frac{t^2}{2}+\beta\right)} du = \frac{2\beta}{t^2 + 2\beta}$$

Figure 1 shows that function.

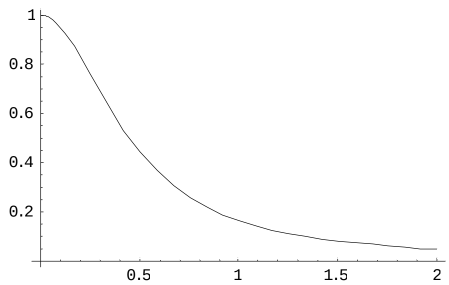

Figure 1. Reliability function $R(t)$

In that case the function (3) is

$$\breve{R}(t) = \exp\left(-\int_0^t E[Cx]dx\right) = \exp\left(-\frac{t^2}{2\beta}\right),\ t \geq 0.$$

Figure 2 shows that function.

Suppose that a failure rate process $\{\lambda(t): t \geq 0\}$ is a linear function of a random load process $\{u(t): t \geq 0\}$:

$$\lambda(t) = \varepsilon\, u(t).$$

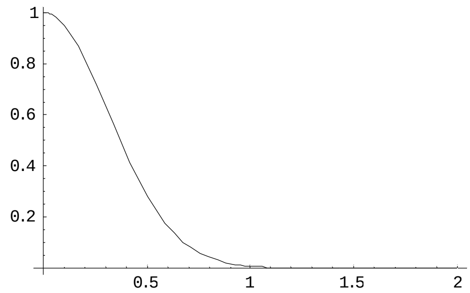

Figure 2. Reliability function $\breve{R}(t)$

Assume that the process $\{u(t): t \geq 0\}$ has an ergodic mean, i.e.

$$\lim_{T\to\infty} \tfrac{1}{T} \int_0^T u(x)dx = E[u(t)] = \bar{u}\,.$$

Then, [2], [3]

$$\lim_{\varepsilon\to 0} R(\tfrac{t}{\varepsilon}) = \exp[-\bar{u}t]\,.$$

It means, that for small ε

$$R(x) \approx \exp[-\varepsilon\,\bar{u}x]\,.$$

3. Semi-Markov process as a random failure rate

The semi-Markov process as a failure rate and the reliability function with that failure rate was introduced by Kopociński & Kopocińska [5]. Some extensions and developments of the results from [3] were obtained by Grabski [3], [4].

3.1. Semi-Markov processes with a discrete state space

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy, W.L. Smith, and L.Takacs in 1954-55. The essential developments of semi-Markov processes theory were achieved by Cinlar [1], Koroluk & Turbin [8], Limnios & Oprisan [7], Silvestrov [9]. We will apply only semi-Markov processes with a finite or countable state space. The semi-Markov processes are connected to the Markov renewal processes.

Let S be a discrete (finite or countable) state space and let $R_+ = [0,\infty)$, $N_0 = \{0,1,2,...\}$. Suppose, that $\xi_n, \vartheta_n,\ n = 0,1,2,...$ are the random variables defined on a joint probabilistic space (Ω, $\mathcal{F}$, P) with values on S and R_+ respectively. A two-dimensional random sequence $\{(\xi_n, \vartheta_n),\ n = 0,1,2,...\}$ is called a Markov renewal chain if for all $i_0,....,i_{n-1}, i \in S,\ t_0,...,t_n \in R_+,\ n \in N_0$.

The equalities

1. $P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i, \vartheta_n = t_n, ..., \xi_0 = i_0, \vartheta_0 = t_0\}$

$= P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i\} = Q_{ij}(t)$ (6)

2. $P\{\xi_0 = i_o, \vartheta_0 = 0\} = P\{\xi_0 = i_0\} = p_{i_o}$ (7)

hold.

It follows from the above definition that a Markov renewal chain is a homogeneous two-dimensional Markov chain such that the transition probabilities do not depend on the second component. It is easy to notice that a random sequence $\{\xi_n : n = 0,1,2,...\}$ is a homogeneous one-dimensional Markov chain with the transition probabilities

$$p_{ij} = P\{\xi_{n+!} = j \mid \xi_n = i\} = \lim_{t\to\infty} Q_{ij}(t). \quad (8)$$

A matrix

$$Q(t) = \lfloor Q_{ij}(t):\ i, j \in S \rfloor$$

Is called a Markov renewal kernel. The Markov renewal kernel and the initial distribution $p = [p_i : i \in S]$ define the Markov renewal chain. That chain allows us to construct a semi-Markov process.

Let

$$\tau_0 = \vartheta_0 = 0, \tau_n = \vartheta_1 + ... + \vartheta_n, \tau_\infty = \sup\{\tau_n : n \in N_0\}$$

A stochastic process $\{X(t) : t \ge 0\}$ given by the following relation

$$X(t) = \xi_n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}) \quad (9)$$

is called a semi-Markov process on S generated by the Markov renewal chain related to the kernel $Q(t), t \ge 0$ and the initial distribution p.

Since the trajectory of the semi-Markov process keeps the constant values on the half-intervals $[\tau_n, \tau_{n+1})$ and it is a right-continuous function, from equality $X(\tau_n) = \xi_n$, it follows that the sequence $\{X(\tau_n) : n = 0,1,2,...\}$ is a Markov chain with the transition probabilities matrix

$$P = [p_{ij} : i, j \in S]. \quad (10)$$

The sequence $\{X(\tau_n) : n = 0,1,2,...\}$ is called an embedded Markov chain in a semi-Markov process $\{X(t) : t \ge 0\}$.

The function

$$F_{ij}(t) = P\{\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i, X(\tau_{n+1}) = j\}$$

$$= \frac{Q_{ij}(t)}{p_{ij}} \quad (11)$$

is a cumulative probability distribution of a holding time of a state i, if the next state will be j. From (11) we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \quad (12)$$

The function

$$G_i(t) = P\{\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i\} = \sum_{j\in S} Q_{ij}(t) \quad (13)$$

is a cumulative probability distribution of an occupation time of the state i.

A stochastic process $\{N(t) : t \ge 0\}$ defined by

$$N(t) = n \ \text{ for } \ t \in [\tau_n, \tau_{n+1}) \quad (14)$$

is called a counting process of the semi-Markov process $\{X(t) : t \ge 0\}$.

The semi-Markov process $\{X(t) : t \ge 0\}$ is said to be regular if for all $t \ge 0$

$$P\{N(t) < \infty\} = 1. \quad (15)$$

It means that the process $\{X(t) : t \ge 0\}$ has the finite number of state changes on a finite period.

Every Markov process $\{X(t) : t \ge 0\}$ with the discrete space S and the right-continuous trajectories keeping constant values on the half-intervals, with the generating matrix of the transition rates $\mathrm{A} = [\alpha_{ij} : i, j \in S]$, $0 < -\alpha_{ii} = \alpha_i < \infty$ is the semi-Markov process with the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in S],$$

where

$$Q_{ij}(t) = p_{ij}(1 - e^{-\alpha_i t}), t \ge 0,$$

$$p_{ij} = \frac{\alpha_{ij}}{\alpha_i} \ \text{for } i \ne j$$

and

$p_{ii} = 0$.

3.2. Semi-Markov failure rate

Suppose that the random failure rate $\{\lambda(t): t \geq 0\}$ is the semi-Markov process with the discrete state space $S = \{\lambda_j : j \in J\}$, $J = \{0,1,...,m\}$ or $J = \{0,1,2,...\}$, $0 \leq \lambda_0 < \lambda_1 < ...$ with the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in J]$$

and the initial distribution $p = [p_i : i \in J]$.
We define a conditional reliability function as

$$R_i(t) = E\left[\exp\left(-\int_0^t \lambda(u)du\right) \middle| \lambda(0) = \lambda_i\right], \; t \geq 0, \; i \in J. \quad (16)$$

In [3] it is proved, that for the regular semi-Markov process $\{\lambda(t): t \geq 0\}$ the conditional reliability functions $R_i(t)$, $t \geq 0$, $i \in J$ defined by (16), satisfy the system of equations

$$R_i(t) = e^{-\lambda_i t}[1 - G_i(t)] + \sum_j \int_0^t e^{-\lambda_i x} R_j(t-x) dQ_{ij}(x), \quad (17)$$

$i \in J$.

Using the Laplace transform we obtain the system of linear equations

$$\tilde{R}_i(s) = \frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) + \sum_j \tilde{R}_j(s)\tilde{q}_{ij}(s+\lambda_i), \; i \in J \quad (18)$$

where

$$\tilde{R}_i(s) = \int_0^\infty e^{-st} R_i(t)dt,$$

$$\tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t)dt,$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t).$$

In matrix notation we have

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]\tilde{\mathbf{R}}(s) = \tilde{\mathbf{H}}(s), \quad (19)$$

where

$$\tilde{\mathbf{R}}(s) = \left[\tilde{R}_i(s) : i \in J\right]^T,$$

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] = \left[\delta_{ij} - \tilde{q}_{ij}(s+\lambda_i) : i, j \in J\right],$$

$$\tilde{\mathbf{H}}(s) = \left[\frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) : i \in J\right].$$

The conditional mean times to failure we obtain from the formula

$$\mu_i = \lim_{p \to 0^+} \tilde{R}_i(p), \; p \in (0,\infty), \; i \in J \quad (20)$$

The unconditional mean time to failure has a form

$$\mu = \sum_{i \in J} P(\lambda(0) = \lambda_i)\, \mu_i. \quad (21)$$

3.3. 3-state random walk process as a failure rate

Assume that the failure rate is a semi-Markov process $\{\lambda(t): t \geq 0\}$ with the state space $S = \{\lambda_0, \lambda_1, \lambda_2\}$ and the kernel

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & G_0(t) & 0 \\ aG_1(t) & 0 & (1-a)G_1(t) \\ 0 & G_2(t) & 0 \end{bmatrix},$$

where $G_0(t), G_1(t), G_2(t)$ are the cumulative probability distribution functions with nonnegative support. Suppose that at least one of the functions is absolutely continuous with respect to the Lebesgue measure. Let $p = [p_0, p_1, p_2]$ be an initial probability distribution of the process. That stochastic process is called the 3-state random walk process. In that case the matrices from the equation (19) are

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] =$$

$$= \begin{bmatrix} 1 & -\tilde{g}_0(s+\lambda_0) & 0 \\ -a\tilde{g}_1(s+\lambda_1) & 1 & -(1-a)\tilde{g}_1(s+\lambda_1) \\ 0 & -\tilde{g}_2(s+\lambda_2) & 1 \end{bmatrix}, \quad (22)$$

where

$$\tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t), i = 0,1,2.$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \\ \tilde{R}_2(s) \end{bmatrix},$$

$$\tilde{\mathbf{H}}(s) = \begin{bmatrix} \frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) \\ \frac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1) \\ \frac{1}{s+\lambda_2} - \tilde{G}_2(s+\lambda_2) \end{bmatrix}. \qquad (23)$$

The Laplace transform of unconditional reliability function is

$$\tilde{R}(s) = p_0\tilde{R}_0(s) + p_1\tilde{R}_1(s) + p_1\tilde{R}_1(s)$$

Example 2.
Assume that

$$p_0 = 1, \quad p_1 = 0, \quad p_2 = 0$$

and

$$G_0(t) = 1-(1+\alpha\, t)e^{-\alpha\, t},$$

$$G_1(t) = 1-e^{-\beta\, t},$$

$$G_2(t) = 1-(1+\gamma\, t)e^{-\gamma\, t},\ t \geq 0.$$

The corresponding Laplace transforms are

$$\tilde{G}_0(s) = \frac{\alpha^2}{s(s+\alpha)^2},$$

$$\tilde{G}_1(s) = \frac{\beta}{s(s+\beta)},$$

$$\tilde{G}_2(s) = \frac{\gamma^2}{s(s+\gamma)^2},$$

$$\tilde{g}_0(s) = \frac{\alpha^2}{(s+\alpha)^2},$$

$$\tilde{g}_1(s) = \frac{\beta}{s+\beta},$$

$$\tilde{g}_2(s) = \frac{\gamma^2}{(s+\gamma)^2}.$$

Let

$$p = [1, 0, 0], \quad a = 0.4$$

and

$$\alpha = 0.4,\ \beta = 0.04, \gamma = 0.02,\ \lambda_0 = 0,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2\ .$$

Since the matrices (22) and (23) are

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] =$$

$$= \begin{bmatrix} 1 & -\frac{0.0025}{(0.05+s)^2} & 0 \\ -0.4\frac{0.04}{0.14+s} & 1 & -0.6\frac{0.04}{0.14+s} \\ 0 & -\frac{0.0004}{(0.22+s)^2} & 1 \end{bmatrix},$$

$$\tilde{\mathbf{H}}(s) = \begin{bmatrix} \frac{1}{s} - \frac{0.0025}{s(0.05+s)^2} \\ \frac{1}{s+0.1} - \frac{0.04}{(s+0.1)(0.14+s)} \\ \frac{1}{s+\lambda_2} - \frac{0.0004}{(s+0.2)(0.22+s)^2} \end{bmatrix}.$$

From solution of equation (19), in this case, we obtain

$$\tilde{R}(s) = \tilde{R}_0(s) = \frac{\tilde{a}(s)}{\tilde{b}(s)}$$

where

$$\tilde{a}(s) = (0.01623 + 0.23349s + s^2)$$

$$\cdot (0.05002 + 0.44655s + s^2)$$

$$\tilde{b}(s) = (0.03083 + s)(0.07486 + s)(0.13292 + s)$$

$$\cdot (0.04882 + 0.44138s + s^2)$$

Using the MATHEMATICA computer program we obtain the reliability function as the inverse Laplace transform

$$R(t) = 0.51646e^{-0.13292\,t} + 0.23349e^{-0.07486t}$$

$$+ 2.28565e^{-0.13292\,t}$$

$$- 2\cdot 0.01539e^{-0.22069\,t}\cos(0.01075t)$$

$$- 2\cdot 0.01343e^{-0.22069\,t}\cos(0.01075t)\,.$$

Figure 3 shows this reliability function.

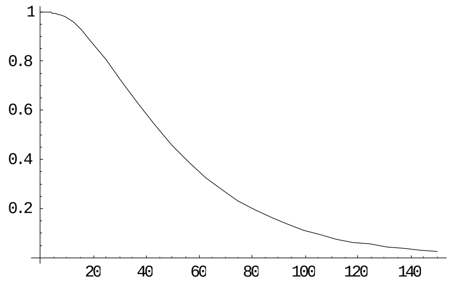

Figure 3. The reliability function from example 2

The corresponding density function is shown in *Figure 4.*

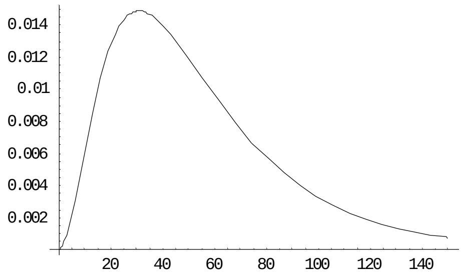

Figure 4. The density function from example 2

3.4. The Poisson process as a failure rate

Suppose that the random failure rate $\{\lambda(t): t \geq 0\}$ is the Poisson process with parameter $\lambda > 0$. Of course, the Poisson process is the Markov process with the counting state space $S = \{0,1,2,...\}$. That process can be treated as the semi-Markov process defined on by the initial distribution $p = [1,0,0,...]$ and the kernel

$$Q(t) = \begin{bmatrix} 0 & G_0(t) & 0 & 0 & ... \\ 0 & 0 & G_1(t) & 0 & ... \\ 0 & 0 & 0 & G_2(t) & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \end{bmatrix},$$

where

$$G_i(t) = 1 - e^{-\lambda t}, t \geq 0, \; i = 0,1,2,...$$

The Poisson process is of course a Markov process too.

Applying equation (19), Grabski [3] proved the following theorem:

If the random failure rate $\{\lambda(t): t \geq 0\}$ is the Poisson process with parameter $\lambda > 0$, than the reliability function defined by (16) takes form

$$R(t) = \exp\{-\lambda\,[t - 1 + \exp(-t)]\}, t \geq 0.$$

The corresponding density function is given by the formula

$$f(t) = \lambda \exp\{-\lambda\,[t - 1 + \exp(-t)]\}[1 - \exp(-t)], t \geq 0.$$

Those functions with parameter $\lambda = 0.2$ are shown in *Figure 5* and *Figure 6.*

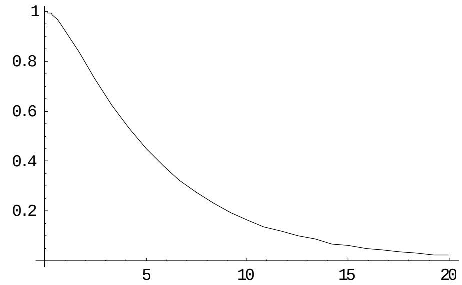

Figure 5. The reliability function for the Poisson process

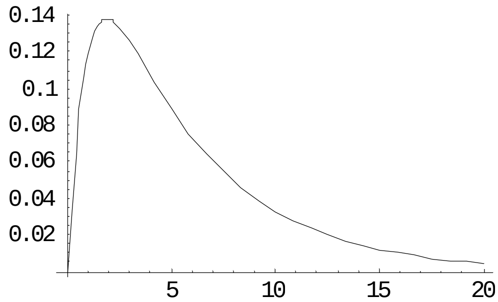

Figure 6. The density function for the Poisson process

3.5. The Furry-Yule process as a failure rate

The Furry-Yule is the semi-Markov process on the counting state space $S = \{0,1,2,...\}$ with the initial distribution $p = [1,0,0,...]$ and the kernel similar to the Poison process

$$Q(t) = \begin{bmatrix} 0 & G_0(t) & 0 & 0 & ... \\ 0 & 0 & G_1(t) & 0 & ... \\ 0 & 0 & 0 & G_2(t) & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \end{bmatrix},$$

where

$$G_i(t) = 1 - e^{-\lambda(i+1)t}, t \geq 0, \; i = 0,1,2,...$$

The Furry-Yule process is also the Markov process.

Assume that the random failure rate $\{\lambda(t): t \geq 0\}$ is the Furry-Yule process with parameter $\lambda > 0$. The following theorem is proved by Grab ski [4]:

If the random failure rate $\{\lambda(t): t \geq 0\}$ is the Furry-Yule process with parameter $\lambda > 0$, then the reliability function defined by (1) is given by

$$R(t) = \frac{(\lambda + 1)\exp(-\lambda\, t)}{1 + \lambda\, \exp[-(\lambda + 1)t]},\ t \geq 0.$$

The corresponding density function is

$$f(t) = \frac{\lambda(\lambda + 1)\exp[1 - (\lambda + 1)\,t]}{\{1 + \lambda\, \exp[-(\lambda + 1)t]\}^2},\ t \geq 0.$$

Those functions with parameter $\lambda = 0.2$ are shown in *Figure 7* and *Figure 8*.

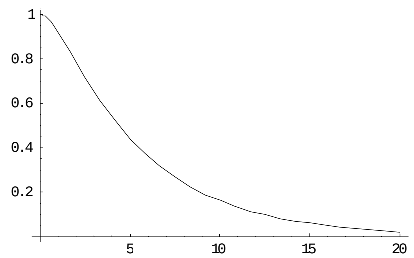

Figure 7. The reliability function for the Furry-Yule process

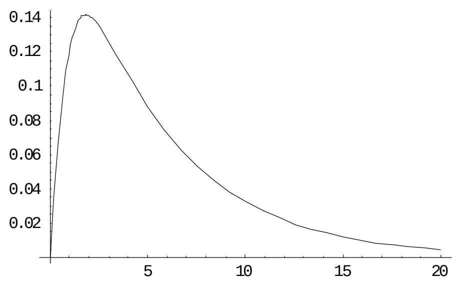

Figure 8. The density function for the Furry-Yule process

4. Conclusion

Frequently, because of the randomly changeable environmental conditions and tasks, the assumption that a failure rate of an object is a random process seems to be proper and natural. We obtain the new interesting classes of reliability functions for the different stochastic failure rate processes.

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