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The model of non-renewal reliability systems with dependent time lengths of components

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Abstract
The models of the non-renewal reliability systems with dependent times to failure of components are presented. The dependence arises from some common environmental stresses and shocks. It is assumed that the failure occurs only because of two independent sources common for two neighbour components. The reliability function of series and parallel systems with components depending on common sources are computed. The reliability functions of the systems with dependent and independent life lengths of components are compared.

1. Introduction

The problem of determining the reliability function of the system with dependent components is important but difficult to solve. Many papers are devoted to it i.e. [1], [2], [3], [4]. In the Barlow & Proshan book [1975], there is defined, based on the reliability theory, multivariate exponential distribution as a distribution of a random vector, the coordinates of which are dependent random variables defining life lengths of the components. Their dependence arises from some environmental common sources of shocks. Using that idea we are going to present some examples of systems with dependent components, giving up the assumption that the joint survival probability is exponential and accepting the assumption that the failure occurs only because of two independent sources common for two neighbour components.

Assume that due to reliability there are n ordered components

$$E = (e_1, e_2, ..., e_n).$$

Assume also that $n+1$ independent sources of shocks are present in the environment

$$Z = (z_1, z_2, ..., z_n, z_{n+1})$$

and each component e_i can be destroyed only because of shocks from two sources z_i and z_{i+1}.

Let U_i be non-negative random variable defining the time to failure of the component caused by the shock from the source z_i. Thus the life length of the object depends on the random vector

$$U = (U_1, U_2, ..., U_n, U_{n+1}). \qquad (1)$$

Admit that the coordinates of the vector are independent random variables with distributions defined as follows

$$G_i(u_i) = P(U_i \le u_i), \quad i = 1, 2, ..., n+1. \qquad (2)$$

The life length of the component e_i is a random variable satisfy

$$T_i = \min(U_i, U_{i+1}), \quad i = 1, 2, ..., n. \qquad (3)$$

Notice that two neighbour components in the sequence $(e_1, e_2, ..., e_n)$ have one common source of shock – depend on the same random variable. The random variables T_1, $T_2, \ldots, T_n$ are *dependent*. Their joint distribution is expressed by means of the multivariate reliability function and it can be easily determined:

$$R(t_1, t_2, ..., t_n) = P(T_1 > t_1, T_2 > t_2, ..., T_n > t_n)$$

$$= P(\min(U_1, U_2) > t_1, \min(U_2, U_3) > t_2,$$

$$..., \min(U_n, U_{n+1}) > t_n)$$

$$= P(U_1 > t_1, U_2 > \max(t_1, t_2),$$

$$..., U_n > \max(t_{n-1}, t_n), U_{n+1} > t_n)$$

$$= P(U_1 > t_1) P(U_2 > \max(t_1, t_2))$$

$$... P(U_n > \max(t_{n-1}, t_n)) P(U_{n+1} > t_n) .$$

Thus

$$R(t_1, t_2, ..., t_n) = \overline{G}_1(t_1)\, \overline{G}_2(\max(t_1, t_2)) ... \overline{G}_n(\max(t_{n-1}, t_n)\, \overline{G}_{n+1}(t_n) \quad (4)$$

where

$$\overline{G}_i(u_i) = P(U_i > u_i) = 1 - G_i(u_i) = P(T \le u_i),$$
$$i = 1, 2, ..., n+1 .$$

The reliability functions of the components can be obtained as marginal distributions computing the limit of the function (4), when

$$t_1 \to 0^+, ..., t_{i-1} \to 0^+, \quad t_{i+1} \to 0^+, ..., t_n \to 0^+ \quad .$$

$$R_i(t_i) = P(T_i > t_i) = \overline{G}_i(t_i)\, \overline{G}_{i+1}(t_i), \quad i = 1,2,..., n. \quad (5)$$

The bivariate reliability functions can be determined by computing the limit of (4), when

$$t_1 \to 0^+, ..., t_{i-1} \to 0^+, \quad t_{i+1} \to 0^+, ..., t_{j-1} \to 0^+,$$
$$t_{j+1} \to 0^+, ..., t_n \to 0^+.$$

$$R_{ij}(t_i, t_j) = P(T_i > t_i, T_j > t_j)$$

$$= \overline{G}_i(t_i)\, \overline{G}_{i+1}(t_i)\, \overline{G}_j(t_j)\, \overline{G}_{j+1}(t_j), \quad (6)$$
$$i + 1 < j, \quad i, j = 1,2,..., n-1,$$

$$R_{ij}(t_i, t_j) = P(T_i > t_i, T_j > t_j)$$

$$= \overline{G}_i(t_i)\, \overline{G}_{i+1}(\max(t_i, t_{i+1}))\overline{G}_{i+2}(t_{i+1}), \quad (7)$$
$$i + 1 = j, \quad i, j = 1,2,..., n-1,$$

it could be proved that

$$P(T_1 > t_1 \mid T_2 > t_2, ..., T_n > t_n)$$

$$= P(T_1 > t_1 \mid T_2 > t_2) \quad (8)$$

and generally

$$P(T_i > t_1 \mid T_{i+1} > t_{i+1}, ..., T_n > t_n)$$

$$= P(T_i > t_i \mid T_{i+1} > t_{i+1}), \; i = 1,2,..., n-1. \quad (9)$$

That property asserts that the life length of e_i depends only on the life length of the next component e_{i+1}, does not depend on the life lengths of the rest of the components. That is a certain kind of Markov property.

2. Reliability of the object with the series structure

If the object has a series reliability structure then its life length T is the random variable defined by the formula

$$T = \min(T_1, T_2, ..., T_n) . \quad (10)$$

Using (4) we can determine the reliability function:

$$\mathbf{R}(t) = P(T > t) = P(T_1 > t, T_2 > t, ..., T_n > t)$$

$$= R(t, t, ..., t) \quad (11)$$

$$= \overline{G}_1(t)\, \overline{G}_2(t) ... \overline{G}_n(t)\, \overline{G}_{n+1}(t).$$

Let us compare the function with the reliability function of a series system in which the life lengths of the components $T_1, T_2, \ldots, T_n$ are independent and their reliability functions are defined by (5). Let $\tilde{\mathbf{R}}(t), t \ge 0$ be a reliability function of that system. It satisfies

$$\tilde{\mathbf{R}}(t) = P(T > t) = P(T_1 > t, T_2 > t, \ldots, T_n > t)$$

$$= R_1(t) R_2(t) \ldots R_n(t)$$

$$=\overline{G}_1(t)\overline{G}_2(t)\overline{G}_2(t)\overline{G}_3(t)\ldots\overline{G}_n(t)\overline{G}_n(t)\overline{G}_{n+1}(t)$$

$$=\overline{G}_2(t)\overline{G}_3(t)\ldots\overline{G}_n(t)\mathbf{R}(t) \qquad (12)$$

Thus, for $t \geq 0$

$$\tilde{\mathbf{R}}(t) \leq \mathbf{R}(t)$$

holds.

The inequality means that the reliability of a series system with dependent (in the considering sense) life lengths of components is greater than (or equal) to the reliability of that system with independent life lengths of components and the same distributions as the marginals of $T_1, T_2, \ldots, T_n$.

Accepting the assumption about independence of the life lengths of the components even though the random variables describing the life lengths are dependent, we make an obvious mistake but that error is „safe" because the real series system has a greater reliability. That estimation is very conservative.

Example 1.

Assume that a non-negative random variable *Ui*, describing time to failure of the component caused by the shock from source z_i has a Weibull distribution with parameters

$$\alpha_i, \lambda_i, \; i = 1,2,\ldots,n+1$$

for $u_i > 0$

$$\overline{G}_i(u_i) = P(U_i > u_i) = e^{-\lambda_i u_i^{\alpha_i}}, \quad i = 1, 2,\ldots, n+1.$$

The reliability function of a series system with dependent components satisfies

$$\mathbf{R}(t) = P(T > t) = \overline{G}_1(t)\,\overline{G}_2(t)\ldots\overline{G}_n(t)\,\overline{G}_{n+1}(t)$$

$$= e^{-(\lambda_1 t^{a_1} + \ldots + \lambda_{n+1} t^{a_{n+1}})}$$

For $n = 3$ and

$$\alpha_1 = 1.2 \quad, \lambda_1 = 0.1,\; \alpha_2 = 2, \quad \lambda_2 = 0.2,$$

$$\alpha_3 = 2.2,\; \lambda_3 = 0.1, \quad \alpha_4 = 3,\; \lambda_4 = 0.2$$

we get

$$\mathbf{R}(t) = P(T > t) = e^{-(0.1t^{1.2} + 0.2t^2 + 0.1t^{2.2} + 0.2t^3)}$$

The graph of the function is presented in *Figure 1.*

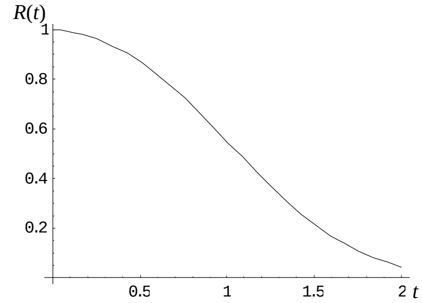

Figure 1. The graph of the series reliability function with dependent components

The reliability function $\tilde{\mathbf{R}}(t), t \geq 0$ of the series system with independent life lengths of components, the same marginals satisfies

$$\tilde{\mathbf{R}}(t) = P(T > t) = e^{-(0.1t^{1.2} + 0.4t^2 + 0.2t^{2.2} + 0.2t^3)}.$$

3. Reliability of the object of the parallel structure

The life length of the object of a parallel structure is a random variable defined by

$$T = \max(T_1, T_2,\ldots,T_n). \qquad (13)$$

Let us compute the reliability function of the object:

$$\mathbf{R}(t) = P(T > t) = 1 - P(T \leq t)$$

$$= 1 - P(T_1 \leq t, T_2 \leq t,\ldots, T_n \leq t) \qquad (14)$$

$$= P(\{T_1 > t\} \cup \{T_2 > t\} \cup \ldots \cup \{T_n > t\}).$$

Using the formula of probability of a sum of events we obtain

$$\mathbf{R}(t) = P(T > t) = \sum_{i=1}^{n} P(T_i > t) - \sum_{\substack{i,j=1 \\ i<j}}^{n} P(T_i > t, T_j > t)$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} P(T_i > t, T_j > t, T_k > t) - \ldots$$

$$+ (-1)^{n+1} P(T_1 > t, T_2 > t,\ldots, T_n > t).$$

Hence and from (4), (6), (7) we get

$$\mathbf{R}(t)=\sum_{i=1}^{n}\overline{G}_i(t)\,\overline{G}_{i+1}(t)-\sum_{\substack{i,j=1\\ i+1<j}}^{n}\overline{G}_i(t)\,\overline{G}_{i+1}(t)\,\overline{G}_j(t)\,\overline{G}_{j+1}(t)$$

$$-\sum_{i=1}^{n-1}\overline{G}_i(t)\,\overline{G}_{i+1}(t)\overline{G}_{i+2}(t) \qquad (15)$$

$$\ldots+(-1)^{n+1}\overline{G}_1(t)\,\overline{G}_2(t)\ldots\overline{G}_n(t)\,\overline{G}_{n+1}(t).$$

Particularly, for $n=3$ we have,

$$\begin{aligned}\mathbf{R}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)-\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\ &+\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\ &=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\ &+\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)-\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t).\end{aligned} \qquad (16)$$

If $T_1, T_2, \ldots, T_n$ are independent then

$$\begin{aligned}\breve{\mathbf{R}}(t)&=1-P(T_1\le t, T_2\le t,\ldots,T_n\le t)\\ &=1-P(T_1\le t)P(T_2\le t)\ldots P(T_n\le t)\\ &=1-[1-R_1(t)][1-R_2(t)]\ldots[1-R_n(t)]\\ &=1-[1-\overline{G}_1(t)\,\overline{G}_2(t)][1-\overline{G}_2(t)\,\overline{G}_3(t)]\\ &\ldots[1-\overline{G}_n(t)\,\overline{G}_{n+1}(t)].\end{aligned}$$

For $n = 3$

$$\breve{\mathbf{R}}(t)=1-[1-\overline{G}_1(t)\,\overline{G}_2(t)][1-\overline{G}_2(t)\,\overline{G}_3(t)]$$

$$\cdot[1-\overline{G}_3(t)\,\overline{G}_4(t)].$$

After multiplication we get

$$\begin{aligned}\breve{\mathbf{R}}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t)\\ &+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t).\end{aligned}$$

Notice that

$$\begin{aligned}\breve{\mathbf{R}}(t)-\mathbf{R}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)\\ &-\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t)+\\ &+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t)\\ &=\overline{G}_2(t)\,\overline{G}_3(t)\,[\overline{G}_1(t)+\overline{G}_4(t)\\ &-\overline{G}_1(t)\,\overline{G}_2(t)-\overline{G}_1(t)\,\overline{G}_4(t)\\ &-\overline{G}_3(t)\,\overline{G}_4(t)+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)].\end{aligned}$$

Let A_i, i=1, 2, 3, 4 be independent events with probabilities defined by

$$P(A_i)=\overline{G}_i(t),\ \ i=1,2,3,4.$$

The expression

$$\overline{G}_1(t)+\overline{G}_4(t)-\overline{G}_1(t)\,\overline{G}_2(t)-\overline{G}_1(t)\,\overline{G}_4(t)$$

$$-\overline{G}_3(t)\,\overline{G}_4(t)+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)$$

can be rewritten as

$$\begin{aligned}&P(A_1)+P(A_4)-P(A_1)P(A_2)\\ &-P(A_1)P(A_4)-P(A_3)P(A_4)\\ &+P(A_1)P(A_2)P(A_3)P(A_4)\\ &=P(A_1\cup A_4)-P((A_1\cap A_2)\cup P(A_3\cap A_4)).\end{aligned}$$

As

$$A_1 \cup A_4 \supset (A_1 \cap A_2) \cup (A_3 \cap A_4),$$

So

$$P(A_1 \cup A_4) \geq P((A_1 \cap A_2) \cup (A_3 \cap A_4)).$$

Thus

$$\breve{\mathbf{R}}(t) \geq \mathbf{R}(t).$$

The inequality can be proved for any n by the induction. It assures that the reliability of the parallel system with the independent components is greater than (or equal) to the reliability of that system with dependent components.

Computing the reliability of the real systems we often assume that the components life lengths are independent even though the random variables describing the life lengths are dependent. That example shows that such assumption leads towards careless conclusions. The real parallel system may have significantly lower reliability. Moreover, we come to the similar conclusions if we take under consideration more general assumption about the association of the random variables $T_1, T_2, \ldots, T_n$ [1].

Example 2.

Assume as previously that a non-negative random variable U_i, describing time to failure of the component caused by the source z_i has a Weibull distribution with parameters

$$\alpha_i, \lambda_i, \; i = 1,2,\ldots,n+1.$$

Let $n = 3$. Then for $u_i > 0$

$$\overline{G}_i(u_i) = P(U_i > u_i) = e^{-\lambda_i u_i^{\alpha_i}}, \quad i = 1,2,3,4.$$

As previously

$$\alpha_1 = 1.2 \quad, \lambda_1 = 0.1, \; \alpha_2 = 2, \quad \lambda_2 = 0.2,$$

$$\alpha_3 = 2.2, \quad \lambda_3 = 0.1, \quad \alpha_4 = 3, \quad \lambda_4 = 0.2.$$

Using (16) we obtain the reliability function of the parallel system with dependent components. For $t > 0$ it satisfies

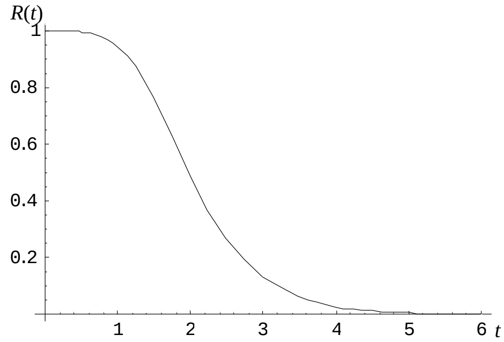

Figure 2. The graph of the reliability function of the parallel system with dependent components

$$\mathbf{R}(t) = \overline{G}_1(t)\,\overline{G}_2(t) + \overline{G}_2(t)\,\overline{G}_3(t) + \overline{G}_3(t)\,\overline{G}_4(t)$$

$$+ \overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t) - \overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)$$

$$= e^{-(0.1t^{1.2}+0.2t^2)} + e^{-(0.2t^2+0.1t^{2.2})}$$

$$+ e^{-(0.1t^{2.2}+0.2t^3)} - e^{-(0.1t^{1.2}+0.2t^2+0.1t^{2.2})}$$

$$- e^{-(0.2t^2+0.1t^{2.2}+0.2t^3)}.$$

Figure 2. Presents its graph.

4. Conclusion

The reliability of a series system with dependent (in the considered sense) life lengths of components is greater than (or equal) to the reliability of that system with independent life lengths of components.

Assuming the independence of the life lengths of the components even though the random variables describing the life lengths are dependent, we make a mistake but that error is „safe" because the real series system has a greater reliability. The estimation of the reliability function is very conservative.

The reliability of the parallel system with the independent components is greater than (or equal) to the reliability of that system with dependent components.

Computing the reliability of the real systems we often assume that the components life lengths are independent even though the random variables describing the life lengths are dependent. The examples presented here show that such assumption leads towards careless conclusions. The real parallel system may have significantly lower reliability.

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