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Some remarks on mean time between failures of repairable systems

Keywords

age replacement, increasing failure rate on average, mean time between failures, mean time between failures or repairs

Abstract

This paper describes a simple technique for approximating the mean time between failures (MTBF) of a system that has periodic maintenance at regular intervals. We propose an approximation of MTBF for vide range of systems than IFRA class of distributions.

1. Introduction

One important research area in reliability engineering is studying various maintenance policies. Maintenance can be classified by two major categories: corrective and preventive. Corrective maintenance is any maintenance that occurs when the system is failed. Some authors refer to corrective maintenance as repair and we will use this approach in this paper. Preventive maintenance is any maintenance that occurs when system is not failed.

A common measure used to describe the reliability characteristics of a repairable system is mean time between failures (MTBF). In most repairable systems, preventive maintenance used to reduce system failure frequency and hence increase the MTBF. It is easy that MTBF is the mean time to a repair service or an age replacement.

The MTBF for a system that has periodic maintenance at a time t can be described by [1], [10]:

$$MTBF = \frac{\int_0^t R(s)\,ds}{F(t)}. \qquad (1)$$

In paper [1], [10] an approximation to MTBF is provided. Moreover in [10] is proved that

$$\frac{tR(t)}{F(t)} \le MTBF \le \frac{t}{F(t)},$$

and for increasing failure rate on average (IFRA) class of distributions in [1] the following relation by proposed

$$-\frac{t}{ln(R(t))} \le MTBF \le \frac{t}{F(t)}.$$

In this paper, we propose an approximation to MTBF for wide class than IFRA. The equality (1) can be a basis to introduction a new class of ageing distributions (see [7], [8]).

Let us assumption the following notation

$$M(t) = \frac{1}{MTBF}.$$

The case when $M(t)$ is monotonic was considered by Barlow and Campo [2], Marshall and Proschan [9], Klefsjö [5] and Knopik [7], [8].

Definition 1. The lifetime T belongs to the class (mean time to failure or replacement) MTFR, if the function $M(t)$ is non-decreasing for $t \in \{ t : F(t) > 0 \}$.

It has been shown in Barlow [2] and Klefsjö [5] that

$$IFR \subset MTFR \subset NBUE,$$

where IFR is increasing failure rate class, NBUE is new better than used in expectation class.

For absolutely continuous in [2] and for any random variable in [8] it has been proved, that

IFRA $\subset$ MTFR,

where IFRA is increasing failure rate on average class of distributions. Preservation of life distribution classes under reliability operations has been studied in [7], [8]. The class MTFR is closed under the operations:

(a) formation of a parallel system for absolutely continuous random variables,

(b) formation of series system with identically distributed and absolutely continuous random variables,

(c) weak convergence of distributions,

(d) convolution.

In this paper we propose new approximation and bounds, applicable for a wide range of systems. It is MTFR class of distributions, such that MTBF is non-increasing. The class MTFR contains distributions with unimodal failure rate function. We analyze special case of distribution with MTBF non-increasing as mixture of an exponential distribution and Rayleigh's distributions. This distribution has unimodal failure rate function. However, it is not easy to obtain distribution from mixtures with unimodal failure rate function [12]. The mixtures of two increasing linear failures rate functions were studied in [4]. In [4] showed that a mixture of two distributions with increasing linear failure rate functions does not give distributions with unimodal failure rate function.

In section 2, we introduce the proposed model of mixture and we estimate their parameters for an example of lifetime data [11]. In section 3 we propose a simple approximation of to MTBF for lifetime distributions with MTBF non-increasing.

2. Mixture of distributions

We consider a mixture of two lifetimes X_1 and X_2 with densities $f_1(t)$, $f_2(t)$, reliability functions $R_1(t)$, $R_2(t)$, failure rate functions $r_1(t)$, $r_2(t)$ and weights p and $q=1-p$, where $0 < p < 1$. The mixed density is then written as

$$f(t) = pf_1(t) + (1-p)f_2(t)$$

and the mixed reliability function is

$$R(t) = pR_1(t) + (1-p)R_2(t).$$

The failure rate function of the mixture can be written as the mixture

$$r(t) = \omega(t)r_1(t) + [1 - \omega(t)]r_2(t),$$

where $\omega(t) = pR_1(t)/R(t)$.

Moreover, from [3], we have under some mild conditions, that

$$\lim_{t \to \infty} r(t) = \lim_{t \to \infty} min\{r_1(t), r_2(t)\}.$$

In the following propositions, we give some properties for the mixture failure rate function.

Proposition 1. For the first derivative of $\omega(t)$, we have

$$\omega'(t) = \omega(t)[1 - \omega(t)][r_2(t) - r_1(t)].$$

Proposition 2. For the first derivative of $r(t)$, we have

$$r'(t) = [1 - \omega(t)](-\omega(t)(r_2(t) - r_1(t))^2 + r'_2(t)$$
$$+ \omega(t)r'_1(t).$$

Proposition 3. If X_1 is exponentially distributed with parameter λ, then

$$r'(t) = [1 - \omega(t)](-\omega(t)(r_2(t) - \lambda)^2 + r'_2(t)).$$

We suppose that $r_2(t) = \alpha t$, where $\alpha > 0$. Consequently, the reliability function of X_2 is the reliability function of Rayleigh's distribution of the form

$$R_2(t) = exp\{-\frac{\alpha}{2}t^2\} \quad \text{for } t \geq 0.$$

Proposition 4. If $p\lambda^2 \leq \alpha$, then r (t) is unimodal.

Proof. By *Proposition 1*, we conclude that $\omega(t)$ is decreasing for $t \in (0, t_1)$, where $t_1 = \lambda/\alpha$ and is increasing for $t \in (t_1, \infty)$. Hence, if $t < t_1$, then $\omega(t)(r_2(t) - \lambda)^2$ is decreasing from $p\lambda^2 < \alpha$ to 0, and if $t > t_1$ then $\omega(t)(r_2(t) - \lambda)^2$ is increasing from 0 to ∞. Thus the equation $\omega(t)(r_2(t) - \lambda)^2 = \lambda$ has only one solution t_2 and r (t) is increasing for $t < t_2$ and decreasing for $t > t_2$.

Proposition 5. If $p\lambda^2 > \alpha$, then there exist t_3, and t_4, $t_3 < t_1 < t_4$, such that r (t) decreases in $(0, t_3)$, increases in (t_3, t_4) and decreases in (t_4, ∞).

Proof. Let $h(t) = \alpha - \omega(t)(r_2(t) - \lambda)^2$. It is easy to find that $h(0) = \alpha - p\lambda^2 < 0$, $h(t_1) = \alpha$, $h(\infty) = -\infty$.
The function $h(t)$ is increasing from $h(0) < 0$ to $h(t_1) = \alpha > 0$, and is decreasing from $h(t_1) = \alpha > 0$ to $h(\infty) = -\infty$.
Thus, there exist t_3 and t_4, $0 < t_3 < t_1 < t_4$ such that r (t) decreases on $(0, t_3)$, increases in (t_3, t_4) and decreases in (t_4, ∞). This completes the proof.

Example 1. In this example we consider a real life time data from the [11]. We estimate the parameters p, α, λ of the model with reliability function

$$R(t) = p \, exp(-\lambda t) + (1-p) \, exp(-0.5\alpha t^2) \text{ for } x \geq 0.$$

By maximizing the logarithm of likelihood function for grouped data, we calculate $p = 0.643316$, $\alpha = 0.001284$, $\lambda = 0.0288$. For these values of parameters, we prove Pearson's test of fit and compute $\chi^2 = 0.68$. By *Proposition 4*, we conclude that $r(t)$ is unimodal.

3. Bounds and Approximation

In this section we cover some of the well known bounds and approximations to the MTBF.
By the inequality

$$\int_0^t R(s)ds \leq min\{t, ET\},$$

where ET is the mean value of T, we obtain the upper bound for MTBF:

$$MTBF_U = \frac{1}{F(t)} min\{t, ET\}.$$

In [1] for MTBF proposed is the following average approximation

$$MTBF_A = \frac{t}{2} \frac{1 + R(t)}{F(t)}.$$

Proposition 6. If $f(t)$ is unimodal, then these exist t_1 such that $MTBF_A$ is a lower bound of MTBF for $t \in <0, t_1)$ and it is an upper bound of MTBF for $t \in (t_1, \infty)$

Proof. We consider the difference

$$g(t) = \frac{\int_0^t R(s)ds}{F(t)} - \frac{t}{2} \frac{R(t)+1}{F(t)}$$

and

$$g_1(t) = \int_0^t R(t)dt - \frac{1}{2}t(R(t)+1).$$

It is easy to find that $g_1(0) = 0$, $g_1(+\infty) = -\infty$. The first derivative of $g'_1(t)$ is

$$g'_1(t) = \frac{1}{2}[tf(t) - F(t)].$$

If $f(t)$ is decreasing then $g'_1(t) < 0$ and $MTBF_A$ is a lower bound for MTBF.

If $f(t)$ is unimodal then exists t_m and t_1 such that $f'(t_m) = 0$, $t_m < t_1$ and $g(t) \geq 0$ for $t \in (0, t_1)$, $g(t) \leq 0$ for $t \in (t_1, \infty)$.

Proposition 7. If $T \in MTFR$, then

$$MTBF \geq \frac{1}{r(t)} \quad \text{for } t > 0.$$

Proof. By Definition 1, if $M(t)$ is non-decreasing, then we have

$$[M(t)]' = \frac{f(t)\int_0^t R(s)ds - F(t)R(t)}{(\int_0^t R(s)ds)^2} \geq 0$$

and

$$MTBF \geq MTBF_L = \frac{1}{r(t)} \quad \text{for } t \in \{t: r(t) > 0\}.$$

Proposition 8. If the lifetime T has unimodal failure rate function $r(t)$, then $T \in MTFR$ if and only if $r(\infty) ET - 1 \geq 0$.

Proof. Let

$$h(t) = r(t)\int_0^t R(s)ds - F(t).$$

It is easy to show that $h(0) = 0$ and $h(\infty) = r(\infty) ET - 1$.
The first derivative of $h(t)$ is

$$h'(t) = r'(t)\int_0^t R(s)ds.$$

If $r(t)$ is increasing, then $h(t)$ is increasing and if $r(t)$ is decreasing, then $h(t)$ is decreasing. This completes the proof.

Example 2. Consider the system with failure rate function proposed in Example 1. The exact and approximate results for MTBF are shown in Table 1 for varying $R(t)$ with the corresponding t. The results show that the average approximation $MTBF_A$ is greater than MTBF. For this data, we compute $ET = 34.81$ and $\lambda ET - 1 \geq 0$ and by Proposition 8 we obtain that $T \in MTFR$.

Table 1. The values of the exact and approximate MTBF of lifetime data

R(t)	t	MTBF	MTBF$_U$	MTBF$_L$	MTBF$_A$
0.99999	0.00054	53.97	53.97	53.97	53.97
0.9999	0.0054	53.97	53.97	53.97	53.97
0.999	0.05398	53.95	53.98	53.93	53.95
0,99	0.54031	53.76	54.03	53.55	53.76
0.9	5.43971	51.67	54.40	49.22	51.68
0.8	10.9261	49.15	54.63	44.05	49.17
0.7	16.4696	46.62	54.90	39.17	46.66
0.6	22.1619	44.21	55.40	34.90	44.32
0.5	29.1751	41.98	56.35	34.81	42.26
0.4	34.8007	39.94	58.00	34.81	40.60
0.3	42.5854	38.11	49.73	34.81	39.54
0.2	52.8174	36.50	43.51	34.81	39.61
0.1	70.2655	35.23	38.68	34.81	42.94

4. Conclusion

In this paper we show that, from a practical point view, the unimodal failure rate model can be obtained from a mixture of two common IFR models. This model is flexibility. Practical relevance and applicability have been demonstrated using well known data. In this paper a simple approximation of the MTBF of systems subjected to periodic maintenance has been proposed as well.

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