Kołowrocki Krzysztof
Maritime University, Gdynia, Poland

Reliability modelling of complex systems - Part 1

Keywords

reliability, large system, asymptotic approach, limit reliability function

Abstract

The paper is concerned with the application of limit reliability functions to the reliability evaluation of large systems. Two-state large non-repaired systems composed of independent components are considered. The asymptotic approach to the system reliability investigation and the system limit reliability function are defined. Two-state homogeneous series, parallel and series-parallel systems are defined and their exact reliability functions are determined. The classes of limit reliability functions of these systems are presented. The results of the investigation concerned with domains of attraction for the limit reliability functions of the considered systems and the investigation concerned with the reliability of large hierarchical systems as well are discussed in the paper. The paper contains exemplary applications of the presented facts to the reliability evaluation of large technical systems.

1. Introduction

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and their complicated operating processes. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet large series systems, for instance, in piping transportation of water, gas, oil and various chemical substances. Large systems of these kinds are also used in electrical energy distribution. A city bus transportation system composed of a number of communication lines each serviced by one bus may be a model series system, if we treat it as not failed, when all its lines are able to transport passengers. If the communication lines have at their disposal several buses we may consider it as either a parallel-series system or an "m out of n" system. The simplest example of a parallel system or an "m out of n" system may be an electrical cable composed of a number of wires, which are its basic components, whereas the transmitting electrical network may be either a parallel-series system or an "m out of n"-series system. Large systems of these types are also used in telecommunication, in rope transportation and in transport using belt conveyers and elevators. Rope transportation systems like port elevators and ship-rope elevators used in shipyards during ship docking are model examples of series-parallel and parallel-series systems.

In the case of large systems, the determination of the exact reliability functions of the systems leads us to complicated formulae that are often useless for reliability practitioners. One of the important techniques in this situation is the asymptotic approach to system reliability evaluation. In this approach, instead of the preliminary complex formula for the system reliability function, after assuming that the number of system components tends to infinity and finding the limit reliability of the system, we obtain its simplified form.

The mathematical methods used in the asymptotic approach to the system reliability analysis of large systems are based on limit theorems on order statistics distributions, considered in very wide literature, for instance in [4]-[5], [7], [12]. These theorems have generated the investigation concerned with limit reliability functions of the systems composed of two-state components. The main and fundamental results on this subject that determine the three-element classes

of limit reliability functions for homogeneous series systems and for homogeneous parallel systems have been established by Gniedenko in [6]. These results are also presented, sometimes with different proofs, for instance in subsequent works [1], [8]. The generalizations of these results for homogeneous "*m* out of *n*" systems have been formulated and proved by Smirnow in [13], where the seven-element class of possible limit reliability functions for these systems has been fixed. As it has been done for homogeneous series and parallel systems classes of limit reliability functions have been fixed by Chernoff and Teicher in [2] for homogeneous series-parallel and parallel-series systems. Their results were concerned with so-called "quadratic" systems only. They have fixed limit reliability functions for the homogeneous series-parallel systems with the number of series subsystems equal to the number of components in these subsystems, and for the homogeneous parallel-series systems with the number of parallel subsystems equal to the number of components in these subsystems. Kolowrocki has generalized their results for non-"quadratic" and non-homogeneous series-parallel and parallel-series systems in [8]. These all results may also be found for instance in [9].
The results concerned with the asymptotic approach to system reliability analysis have become the basis for the investigation concerned with domains of attraction ([9], [11]) for the limit reliability functions of the considered systems and the investigation concerned with the reliability of large hierarchical systems as well ([3], [9]). Domains of attraction for limit reliability functions of two-state systems are introduced. They are understood as the conditions that the reliability functions of the particular components of the system have to satisfy in order that the system limit reliability function is one of the limit reliability functions from the previously fixed class for this system. Exemplary theorems concerned with domains of attraction for limit reliability functions of homogeneous series systems are presented here and the application of one of them is illustrated. Hierarchical series-parallel and parallel-series systems of any order are defined, their reliability functions are determined and limit theorems on their reliability functions are applied to reliability evaluation of exemplary hierarchical systems of order two.
All the results so far described have been obtained under the linear normalization of the system lifetimes. The paper contains the results described above and comments on their newest generalizations recently presented in [9].

2. Reliability of two-state systems

We assume that

$$E_i,\ i = 1,2,...,n,\ n \in N,$$

are two-state components of the system having reliability functions

$$R_i(t) = P(T_i > t),\ t \in (-\infty, \infty),$$

where

$$T_i,\ i = 1,2,...,n,$$

are independent random variables representing the lifetimes of components E_i with distribution functions

$$F_i(t) = P(T_i \le t),\ t \in (-\infty, \infty).$$

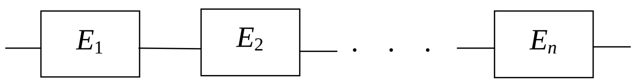
The simplest two-state reliability structures are series and parallel systems. We define these systems first.

Definition 1. We call a two-state system series if its lifetime T is given by

$$T = \min_{1 \le i \le n}\{T_i\}.$$

The scheme of a series system is given in *Figure 1*.

Figure 1. The scheme of a series system



Definition 1 means that the series system is not failed if and only if all its components are not failed, and therefore its reliability function is given by

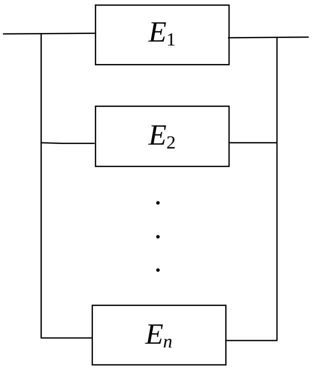
$$\overline{\boldsymbol{R}}_n(t) = \prod_{i=1}^{n} R_i(t),\ t \in (-\infty, \infty). \quad (1)$$

Definition 2. We call a two-state system parallel if its lifetime T is given by

$$T = \max_{1 \le i \le n}\{T_i\}.$$

The scheme of a parallel system is given in *Figure 2*.

Figure 2. The scheme of a parallel system



Definition 2 means that the parallel system is failed if and only if all its components are failed and therefore its reliability function is given by

$$\boldsymbol{R}_n(t) = 1 - \prod_{i=1}^{n} F_i(t), \; t \in (-\infty, \infty). \quad (2)$$

Another basic, a bit more complex, two-state reliability structure is a series-parallel system. To define it, we assume that

$$E_{ij},\; i = 1,2,...,k_n,\; j = 1,2,...,l_i,\; k_n,\; l_1,\; l_2,...,\; l_{k_n} \in N,$$

are two-state components of the system having reliability functions

$$R_{ij}(t) = P(T_{ij} > t),\; t \in (-\infty, \infty),$$

where

$$T_{ij},\; i = 1,2,...,k_n,\; j = 1,2,...,l_i,$$

are independent random variables representing the lifetimes of components E_{ij} with distribution functions

$$F_{ij}(t) = P(T_{ij} \le t),\; t \in (-\infty, \infty).$$

Definition 3. We call a two-state system series-parallel if its lifetime T is given by

$$T = \max_{1 \le i \le k_n}\{\min_{1 \le j \le l_i}\{T_{ij}\}\}.$$

By joining the formulae (1) and (2) for the reliability functions of two-state series and parallel systems it is easy to conclude that the reliability function of the two-state series-parallel system is given by

$$\boldsymbol{R}_{k_n, l_1, l_2, ..., l_{k_n}}(t) = 1 - \prod_{i=1}^{k_n}[1 - \prod_{j=1}^{l_i} R_{ij}(t)],\; t \in (-\infty, \infty), \quad (3)$$

where k_n is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Definition 4. We call a two-state series-parallel system regular if

$$l_1 = l_2 = . . . = l_{k_n} = l_n,\; l_n \in N,$$

i.e. if the numbers of components in its series subsystems are equal.

The scheme of a regular series-parallel system is given in *Figure 3*.

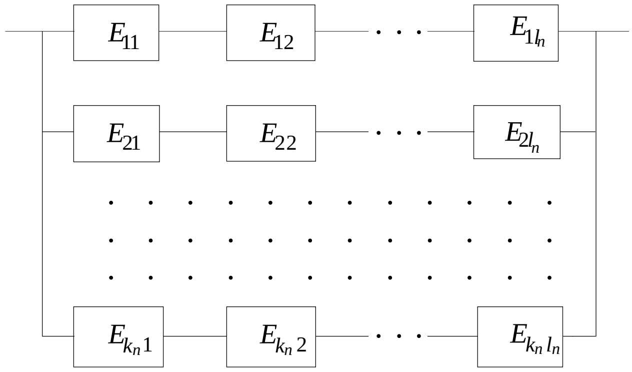


Figure 3. The scheme of a regular series-parallel system

Definition 5. We call a two-state system homogeneous if its component lifetimes have an identical distribution function $F(t)$, i.e. if its components have the same reliability function

$$R(t) = 1 - F(t),\; t \in (-\infty, \infty).$$

The above definition and equations (1)-(3) result in the simplified formulae for the reliability functions of the homogeneous systems stated in the following corollary.

Corollary 1. The reliability function of the homogeneous two-state system is given by

- for a series system

$$\overline{\boldsymbol{R}}_n(t) = [R(t)]^n,\; t \in (-\infty, \infty), \quad (4)$$

- for a parallel system

$$\boldsymbol{R}_n(t) = 1 - [F(t)]^n,\; t \in (-\infty, \infty), \quad (5)$$

- for a regular series-parallel system

$$\boldsymbol{R}_{k_n, l_n}(t) = 1 - [1 - [R(t)]^{l_n}]^{k_n},\; t \in (-\infty, \infty). \quad (6)$$

3. Asymptotic approach to system reliability

The asymptotic approach to the reliability of two-state systems depends on the investigation of limit distributions of a standardized random variable

$$(T - b_n)/a_n,$$

where T is the lifetime of a system and $a_n > 0$ and $b_n \in (-\infty, \infty)$ are suitably chosen numbers called normalizing constants.
Since

$$P((T - b_n)/a_n > t) = P(T > a_n t + b_n) = \boldsymbol{R}_n(a_n t + b_n),$$

where $\boldsymbol{R}_n(t)$ is a reliability function of a system composed of n components, then the following definition becomes natural.

Definition 6. We call a reliability function $\mathfrak{R}(t)$ the limit reliability function of a system having a reliability function $\boldsymbol{R}_n(t)$ if there exist normalizing constants $a_n > 0$, $b_n \in (-\infty, \infty)$ such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}},$$

where $C_{\mathfrak{R}}$ is the set of continuity points of $\mathfrak{R}(t)$.

Thus, if the asymptotic reliability function $\mathfrak{R}(t)$ of a system is known, then for sufficiently large n, the approximate formula

$$\boldsymbol{R}_n(t) \cong \mathfrak{R}((t - b_n)/a_n),\ t \in (-\infty, \infty). \quad (7)$$

may be used instead of the system exact reliability function $\boldsymbol{R}_n$(t).

3.1. Reliability of large two-state series systems

The investigations of limit reliability functions of homogeneous two-state series systems are based on the following auxiliary theorem.

Lemma 1. If

(i) $\overline{\mathfrak{R}}(t) = \exp[-\overline{V}(t)]$ is a non-degenerate reliability function,

(ii) $\overline{\boldsymbol{R}}_n(t)$ is the reliability function of a homogeneous two-state series system defined by (4),

(iii) $a_n > 0,\ \ b_n \in (-\infty, \infty)$,

then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_n(a_n t + b_n) = \overline{\mathfrak{R}}(t) \text{ for } t \in C_{\overline{\mathfrak{R}}}$$

if and only if

$$\lim_{n\to\infty} nF(a_n t + b_n) = \overline{V}(t) \text{ for } t \in C_{\overline{V}}$$

Proof. The proof may be found in [1], [6], [8].

Lemma 1 is an essential tool in finding limit reliability functions of two-state series systems. It also is the basis for fixing the class of all possible limit reliability functions of these systems. This class is determined by the following theorem.

Theorem 1. The only non-degenerate limit reliability functions of the homogeneous two-state series system are:

$$\overline{\mathfrak{R}}_1(t) = \exp[-(-t)^{-\alpha}] \text{ for } t < 0,$$

$$\overline{\mathfrak{R}}_1(t) = 0 \text{ for } t \ge 0,\ \alpha > 0;$$

$$\overline{\mathfrak{R}}_2(t) = 1 \text{ for } t < 0,$$

$$\overline{\mathfrak{R}}_2(t) = \exp[-t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0;$$

$$\overline{\mathfrak{R}}_3(t) = \exp[-\exp[t]] \text{ for } t \in (-\infty, \infty).$$

Proof. The proof may be found in [1], [6], [8].

3.2. Reliability of large two-state parallel systems

The class of limit reliability functions for homogeneous two-state parallel systems may be determined on the basis of the following auxiliary theorem.

Lemma 2. If

(i) $\mathfrak{R}(t) = 1 - \exp[-V(t)]$ is a non-degenerate reliability function,

(ii) $\boldsymbol{R}_n(t)$ is the reliability function of a homogeneous two-state parallel system defined by (5),

(iii) $a_n > 0,\ \ b_n \in (-\infty, \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}},$$

if and only if

$$\lim_{n\to\infty} nR(a_n t + b_n) = V(t) \text{ for } t \in C_V.$$

Proof. The proof may be found in [1], [6], [8].

By applying *Lemma 2* it is possible to fix the class of limit reliability functions for homogeneous two-state parallel systems. However, it is easier to obtain this result using the duality property of parallel and series systems expressed in the relationship

$$\boldsymbol{R}_n(t) = 1 - \overline{\boldsymbol{R}}_n(-t) \text{ for } t \in (-\infty, \infty),$$

that results in the following lemma, [1], [6], [8]-[9].

Lemma 3. If $\overline{\mathfrak{R}}(t)$ is the limit reliability function of a homogeneous two-state series system with reliability functions of particular components $\overline{R}(t)$, then

$$\mathfrak{R}(t) = 1 - \overline{\mathfrak{R}}(-t) \text{ for } t \in C_{\overline{\mathfrak{R}}}$$

is the limit reliability function of a homogeneous two-state parallel system with reliability functions of particular components

$$R(t) = 1 - \overline{R}(-t) \text{ for } t \in C_{\overline{R}}.$$

At the same time, if $(a_{n,} b_n)$ is a pair of normalizing constants in the first case, then $(a_{n,} - b_n)$ is such a pair in the second case.
The application of *Lemma 3* and *Theorem 1* yields the following result.

Theorem 2. The only non-degenerate limit reliability functions of the homogeneous parallel system are:

$$\mathfrak{R}_1(t) = 1 \text{ for } t \le 0,$$

$$\mathfrak{R}_1(t) = 1 - \exp[-t^{-\alpha}] \text{ for } t > 0, \alpha > 0;$$

$$\mathfrak{R}_2(t) = 1 - \exp[-(-t)^{\alpha}] \text{ for } t < 0,$$

$$\mathfrak{R}_2(t) = 0 \text{ for } t \ge 0, \alpha > 0;$$

$$\mathfrak{R}_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty,\infty).$$

Proof. The proof may be found in [1], [6], [8].

3.3. Reliability evaluation of large two-state series-parallel systems

The proofs of the theorems on limit reliability functions for homogeneous regular series-parallel systems and methods of finding such functions for individual systems are based on the following essential lemmas.
Lemma 4. If
(i) $k_n \to \infty$,
(ii) $\mathfrak{R}(t) = 1 - \exp[-V(t)]$ is a non-degenerate reliability function,
(iii) $\boldsymbol{R}_{k_n,l_n}(t)$ is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),
(iv) $a_n > 0$, $b_n \in (-\infty,\infty)$,
then

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n,l_n}(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}}$$

if and only if

$$\lim_{n\to\infty} k_n[R(a_n t + b_n)]^{l_n} = V(t) \text{ for } t \in C_V.$$

Proof. The proof may be found in [8].

Lemma 5. If
(i) $k_n \to k$, $k > 0$, $l_n \to \infty$,
(ii) $\mathfrak{R}(t)$ is a non-degenerate reliability function,
(iii) $\boldsymbol{R}_{k_n,l_n}(t)$ is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),
(iv) $a_n > 0$, $b_n \in (-\infty,\infty)$,
then

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n,l_n}(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}},$$

if and only if

$$\lim_{n\to\infty} [R(a_n t + b_n)]^{l_n} = \mathfrak{R}_0(t) \text{ for } t \in C_{\mathfrak{R}_0},$$

where $\mathfrak{R}_0(t)$ is a non-degenerate reliability function and moreover

$$\mathfrak{R}(t) = 1 - [1 - \mathfrak{R}_0(t)]^k \text{ for } t \in (-\infty,\infty).$$

Proof. The proof may be found in [8].

The types of limit reliability functions of a series-parallel system depend on the system shape [7], i.e. on the relationships between the number k_n of its series subsystems linked in parallel and the number l_n of components in its series subsystems. The results based on *Lemma 4* and *Lemma 5* may be formulated in the form of the following theorem.

Theorem 3. The only non-degenerate limit reliability functions of the homogeneous regular two-state series-parallel system are:
Case 1. $k_n = n$, $|l_n - c\log n| >> s$, $s > 0$, $c > 0$.

$$\mathfrak{R}_1(t) = 1 \text{ for } t \le 0,$$

$\Re_1(t) = 1 - \exp[-t^{-\alpha}]$ for $t > 0$, $\alpha > 0$;

$\Re_2(t) = 1 - \exp[-(-t)^{\alpha}]$ for $t < 0$,

$\Re_2(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_3(t) = 1 - \exp[-\exp[-t]]$ for $t \in (-\infty,\infty)$;

Case 2. $k_n = n$, $l_n - c \log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$.

$\Re_4(t) = 1$ for $t < 0$,

$\Re_4(t) = 1 - \exp[-\exp[-t^{\alpha} - s/c]]$ for $t \geq 0$, $\alpha > 0$;

$\Re_5(t) = 1 - \exp[-\exp[(-t)^{\alpha} - s/c]]$ for $t < 0$,

$\Re_5(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_6(t) = 1 - \exp[-\exp[\beta(-t)^{\alpha} - s/c]]$ for $t < 0$,

$\Re_6(t) = 1 - \exp[-\exp[-t^{\alpha} - s/c]]$ for $t \geq 0$,
$\alpha > 0$, $\beta > 0$;

$\Re_7(t) = 1$ for $t < t_1$,

$\Re_7(t) = 1 - \exp[-\exp[-s/c]]$ for $t_1 \leq t < t_2$,

$\Re_7(t) = 0$ for $t \geq t_2$, $t_1 < t_2$;

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$.

$\Re_8(t) = 1 - [1 - \exp[-(-t)^{-\alpha}]]^k$ for $t < 0$,

$\Re_8(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_9(t) = 1$ for $t < 0$,

$\Re_9(t) = 1 - [1 - \exp[-t^{\alpha}]]^k$ for $t \geq 0$, $\alpha > 0$;

$\Re_{10}(t) = 1 - [1 - \exp[-\exp t]]^k$ for $t \in (-\infty,\infty)$.

Proof. The proof may be found in [8].

Using the duality property of parallel-series and series-parallel systems similar to this given in *Lemma 3* for parallel and series systems it is possible to prove that the only limit reliability functions of the homogeneous regular two-state parallel-series system are

$$\overline{\Re}_i(t) = 1 - \Re_i(-t) \text{ for } t \in C_{\Re_i},\ i = 1,2,...,10.$$

Applying *Lemma 2*, it is possible to prove the following fact ([9]).

Corollary 2. If components of the homogeneous two-state parallel system have Weibull reliability functions

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \geq 0,\ \alpha > 0,\ \beta > 0$$

and

$$a_n = b_n/(\alpha \log n),\ b_n = (\log n/\beta)^{1/\alpha},$$

then

$$\Re_3(t) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),$$

is its limit reliability function.

Example 1 (*a steel rope, durability*). Let us consider a steel rope composed of 36 strands used in ship rope elevator and assume that it is not failed if at least one of its strands is not broken. Under this assumption we may consider the rope as a homogeneous parallel system composed of $n = 36$ basic components. Further, assuming that the strands have Weibull reliability functions with parameters

$$\alpha = 2,\ \beta = (7.07)^{-6},$$

by (5), the rope's exact reliability function takes the form

$$\boldsymbol{R}_{36}(t) = 1 - [1 - \exp[-(7.07)^{-6} t^2]]^{36} \text{ for } t \geq 0.$$

Thus, according to *Corollary 2*, assuming

$$a_n = (7.07)^3/(2\sqrt{\log 36}),\ \ b_n = (7.07)^3\sqrt{\log 36}$$

and applying (7), we arrive at the approximate formula for the rope reliability function of the form

$$\boldsymbol{R}_{36}(t) \cong \Re_3((t - b_n)/a_n)$$

$$= 1 - \exp[-\exp[-0.01071t + 7.167]]$$

for $t \in (-\infty,\infty)$.

The mean value of the rope lifetime T and its standard deviation, in months, calculated on the basis of the above approximate result and according to the formulae

$$E[T] = Ca_n + b_n,\ \ \sigma = \pi a_n/\sqrt{6},$$

where $C \cong 0.5772$ is Euler's constant, respectively are:

$$E[T] \cong 723,\ \sigma \cong 120.$$

The values of the exact and approximate reliability functions of the rope are presented in *Table 1* and graphically in *Figure 4*. The differences between them are not large, which means that the mistakes in replacing the exact rope reliability function by its approximate form are practically not significant.

Table 1. The values of the exact and approximate reliability functions of the steel rope

t	$\boldsymbol{R}_{36}(t)$	$\mathfrak{R}_3\left(\frac{t-b_n}{a_n}\right)$	$\Delta = \boldsymbol{R}_{36} - \mathfrak{R}_3$
0	1.000	1.000	0.000
400	1.000	1.000	0.000
500	0.995	0.988	–0.003
550	0.965	0.972	–0.007
600	0.874	0.877	–0.003
650	0.712	0.707	0.005
700	0.513	0.513	0.000
750	0.330	0.344	–0.014
800	0.193	0.218	–0.025
900	0.053	0.081	**–0.028**
1000	0.012	0.029	–0.017
1100	0.002	0.010	–0.008
1200	0.000	0.003	–0.003

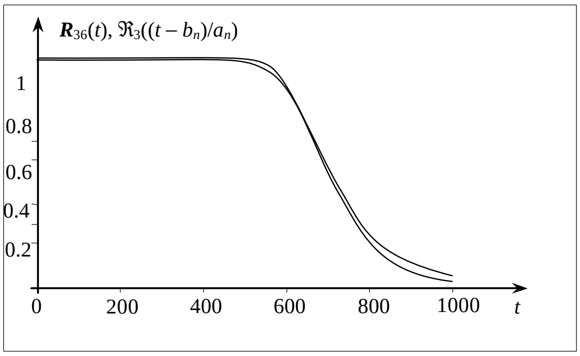


Figure 4. The graphs of the exact and approximate reliability functions of the steel rope

4. Domains of attraction for system limit reliability functions

The problem of domains of attraction for the limit reliability functions of two-state systems solved completely in [11] we will illustrate partly for two-state series homogeneous systems only. From *Theorem 1* it follows that the class of limit reliability functions for a homogeneous series system is composed of three functions, $\overline{\mathfrak{R}}_i(t)$, $i = 1,2,3$. Now we will determine domains of attraction $D_{\overline{\mathfrak{R}}_i}$ for these fixed functions, i.e. we will determine the conditions which the reliability functions $R(t)$ of the particular components of the homogeneous series system have to satisfy in order that the system limit reliability function is one of the reliability functions $\overline{\mathfrak{R}}_i(t)$, i = 1,2,3.

Proposition 1. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\mathfrak{R}}_1}$$

if and only if

$$\lim_{r \to -\infty} \frac{1 - R(r)}{1 - R(rt)} = t^{\text{б}} \quad \text{for } t > 0.$$

Proposition 2. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\mathfrak{R}}_2}$$

if and only if

(i) $\exists\ y \in (-\infty, \infty)$ $R(y) = 1$ and $R(y + \varepsilon) < 1$ for $\varepsilon > 0$,

(ii) $$\lim_{r \to 0^+} \frac{1 - R(rt + y)}{1 - R(r + y)} = t^{\text{б}} \quad \text{for } t > 0.$$

Proposition 3. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\mathfrak{R}}_3}$$

if and only if

$$\lim_{n \to \infty} n[1 - R(a_n t + b_n)] = e^t \quad \text{for } t \in (-\infty, \infty)$$

with

$$b_n = \inf\{t : R(t+0) \le 1 - \frac{1}{n} \le R(t-0)\},$$

$$a_n = \inf\{t : R(t(1+0) + b_n)$$

$$\le 1 - \frac{e}{n} \le R(t(1-0) + b_n)\}.$$

Example 2. If components of the homogeneous series system have reliability functions

$$R(t) = \begin{cases} 1, & t < 0 \\ 1 - t, & 0 \le t < 1 \\ 0, & t \ge 1, \end{cases}$$

then

$$R(t) \in D_{\overline{\mathfrak{R}}_2}.$$

The results of the analysis on domains of attraction for limit reliability functions of two-state systems may automatically be transmitted to multi-state systems. To do this, it is sufficient to apply theorems about two-state systems such as the ones presented here to each vector co-ordinate of the multi-state reliability functions ([9], [14]).

5. Reliability of large hierarchical systems

Prior to defining the hierarchical systems of any order we once again consider a series-parallel system like a system presented in *Figure 3*. This system here is called a series-parallel system of order 1.
It is made up of components

$$E_{i_1 j_1},\ i_1 = 1,2,\dots,k_n,\ \ j_1 = 1,2,\dots,l_{i_1},$$

with the lifetimes respectively

$$T_{i_1 j_1},\ i_1 = 1,2,\dots,k_n,\ \ j_1 = 1,2,\dots,l_{i_1}.$$

Its lifetime is given by

$$T = \max_{1 \le i_1 \le k_n}\{\min_{1 \le j_1 \le l_{i_1}}\{T_{i_1 j_1}\}\}. \tag{8}$$

Now we assume that each component

$$E_{i_1 j_1},\ i_1 = 1,2,\dots,k_n,\ \ j_1 = 1,2,\dots,l_{i_1},$$

of the series-parallel system of order 1 is a subsystem composed of components

$$E_{i_1 j_1 i_2 j_2},\ i_2 = 1,2,\dots,k_n^{(i_1 j_1)},\ \ j_2 = 1,2,\dots,l_{i_2}^{(i_1 j_1)},$$

and has a series-parallel structure.
This means that each subsystem lifetime $T_{i_1 j_1}$ is given by

$$T_{i_1 j_1} = \max_{1 \le i_2 \le k_n^{(i_1 j_1)}}\{\min_{1 \le j_2 \le l_{i_2}^{(i_1 j_1)}}\{T_{i_1 j_1 i_2 j_2}\}\}, \tag{9}$$

$$i_1 = 1,2,\dots,k_n,\ j_1 = 1,2,\dots,l_{i_1},$$

where

$$T_{i_1 j_1 i_2 j_2},\ i_2 = 1,2,\dots,k_n^{(i_1 j_1)},\ \ j_2 = 1,2,\dots,l_{i_2}^{(i_1 j_1)},$$

are the lifetimes of the subsystem components $E_{i_1 j_1 i_2 j_2}$.
The system defined this way is called a hierarchical series-parallel system of order 2. Its lifetime, from (8) and (9), is given by the formula

$$T = \max_{1 \le i_1 \le k_n}\{\min_{1 \le j_1 \le l_{i_1}}[\max_{1 \le i_2 \le k_n^{(i_1 j_1)}}(\min_{1 \le j_2 \le l_{i_2}^{(i_1 j_1)}} T_{i_1 j_1 i_2 j_2})]\},$$

where k_n is the number of series systems linked in parallel and composed of series-parallel subsystems $E_{i_1 j_1}$, l_{i_1} are the numbers of series-parallel subsystems $E_{i_1 j_1}$ in these series systems, $k_n^{(i_1 j_1)}$ are the numbers of series systems in the series-parallel subsystems $E_{i_1 j_1}$ linked in parallel, and $l_{i_2}^{(i_1 j_1)}$ are the numbers of components in these series systems of the series-parallel subsystems $E_{i_1 j_1}$.
In an analogous way it is possible to define two-state parallel-series systems of order 2.
Generally, in order to define hierarchical series-parallel and parallel-series systems of any order r, $r \ge 1$, we assume that

$$E_{i_1 j_1 \dots i_r j_r},$$

where

$$i_1 = 1,2,\dots,k_n,\ j_1 = 1,2,\dots,l_{i_1},\ i_2 = 1,2,\dots,k_n^{(i_1 j_1)},$$
$$j_2 = 1,2,\dots,l_{i_2}^{(i_1 j_1)},\ \dots,\ \ i_r = 1,2,\dots,k_n^{(i_1 j_1 \dots i_{r-1} j_{r-1})},$$
$$j_r = 1,2,\dots,l_{i_r}^{(i_1 j_1 \dots i_{r-1} j_{r-1})}$$

and

$$k_n,\ l_{i_1},\ k_n^{(i_1 j_1)},\ l_{i_2}^{(i_1 j_1)},\ \dots,\ \ k_n^{(i_1 j_1 \dots i_{r-1} j_{r-1})},$$
$$l_{i_r}^{(i_1 j_1 \dots i_{r-1} j_{r-1})} \in N,$$

are two-state components having reliability functions

$$R_{i_1 j_1 \dots i_r j_r}(t) = P(T_{i_1 j_1 \dots i_r j_r} > t),\ t \in (-\infty, \infty),$$

and random variables

$$T_{i_1 j_1 \dots i_r j_r},$$

where

$$i_1 = 1,2,\dots,k_n,\ j_1 = 1,2,\dots,l_{i_1},\ i_2 = 1,2,\dots,k_n^{(i_1 j_1)},$$

$j_2 = 1,2,\ldots,l_{i_2}^{(i_1 j_1)}$, ..., $i_r = 1,2,\ldots,k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}$,
$j_r = 1,2,\ldots,l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}$,

are independent random variables with distribution functions

$$F_{i_1 j_1 \ldots i_r j_r}(t) = P(T_{i_1 j_1 \ldots i_r j_r} \le t),\ t \in (-\infty,\infty),$$

representing the lifetimes of the components $E_{i_1 j_1 \ldots i_r j_r}$.

Definition 7. A two-state system is called a series-parallel system of order *r* if its lifetime *T* is given by

$$T = \max_{1 \le i_1 \le k_n}\{ \min_{1 \le j_1 \le l_{i_1}} \{ \max_{1 \le i_2 \le k_n^{(i_1 j_1)}} \{ \min_{1 \le j_2 \le l_{i_2}^{(i_1 j_1)}}$$

$$\ldots \max_{1 \le i_r \le k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}} (\min_{1 \le j_r \le l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}} T_{i_1 j_1 \ldots i_r j_r})] \ldots \}\}\},$$

where k_n, $k_n^{(i_1 j_1)}$, ..., $k_n^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of suitable series systems of the system composed of series-parallel subsystems and linked in parallel, l_{i_1}, $l_{i_2}^{(i_1 j_1)}$, ..., $l_{i_{r-1}}^{(i_1 j_1 i_2 j_2 \ldots i_{r-2} j_{r-2})}$ are the numbers of suitable series-parallel subsystems in these series systems, and $l_{i_r}^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of components in the series systems of the series-parallel subsystems.

Definition 8. A two-state series-parallel system of order *r* is called homogeneous if its component lifetimes $T_{i_1 j_1 \ldots i_r j_r}$ have an identical distribution function

$$F(t) = P(T_{i_1 j_1 \ldots i_r j_r} \le t),\ t \in (-\infty, \infty),$$

where

$i_1 = 1,2,\ldots,k_n$, $j_1 = 1,2,\ldots,l_{i_1}$, $i_2 = 1,2,\ldots,k_n^{(i_1 j_1)}$,
$j_2 = 1,2,\ldots,l_{i_2}^{(i_1 j_1)}$, ..., $i_r = 1,2,\ldots,k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}$,
$j_r = 1,2,\ldots,l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}$,

i.e. if its components $E_{i_1 j_1 \ldots i_r j_r}$ have the same reliability function
$R(t) = 1 - F(t)$, $t \in (-\infty,\infty)$.

Definition 9. A two-state series-parallel system of order *r* is called regular if

$$l_{i_1} = l_{i_2}^{(i_1 j_1)} = \ldots = l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = l_n$$

and

$$k_n^{(i_1 j_1)} = \ldots = k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = k_n$$

where k_n is the number of series systems in the series-parallel subsystems and l_n are the numbers of series-parallel subsystems or respectively the numbers of components in these series systems.

Using mathematical induction it is possible to prove that the reliability function of the homogeneous and regular two-state hierarchical series-parallel system of order *r* is given by

$$\boldsymbol{R}_{k,k_n,l_n}(t) = 1 - [1 - [\boldsymbol{R}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \ \text{for}\ k = 2,3,\ldots,r$$

and

$$\mathrm{R}_{1,k_n,l_n}(t) = 1 - [1 - [R(t)]^{l_n}]^{k_n},\ t \in (-\infty,\infty),$$

where k_n and l_n are defined in *Definition 9.*

Corollary 3. If components of the homogeneous and regular two-state hierarchical series-parallel system of order *r* have an exponential reliability function

$$R(t) = \exp[-\lambda t] \ \text{for}\ t \ge 0,\ \lambda > 0,$$

then its reliability function is given by

$$\mathrm{R}_{k,k_n,l_n}(t) = 1 - [1 - [\mathrm{R}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \ \text{for}\ t \ge 0$$

for $k = 2,3,\ldots,r$ and

$$\boldsymbol{R}_{1,k_n,l_n}(t) = 1 - [1 - \exp[-\lambda l_n t]]^{k_n} \ \text{for}\ t \ge 0.$$

Theorem 4. If

(i) $\Re(t) = 1 - \exp[-V(t)]$, $t \in (-\infty,\infty)$, is a non-degenerate reliability function,

(ii) $\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0$ for $r \ge 1$,

(iii) $\lim_{n\to\infty} k_n^{l_n^{r-1}+\ldots+1} [R(a_n t + b_n)]^{l_n^r} = V(t)$ for $t \in C_V$, $r \ge 1$, $t \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{R}_{r,k_n,l_n}(a_n t + b_n) = \Re(t) \ \text{for}\ t \in C_{\Re},\ r \ge 1,$$
$t \in (-\infty,\infty)$.

Proposition 4. If components of the homogeneous and regular two-state hierarchical series-parallel system of order r have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \ge 0, \ \lambda > 0,$$

$$\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0 \quad \text{for } r \ge 1,$$

and

$$a_n = \frac{1}{\lambda l_n^r}, \ b_n = \frac{1}{\lambda}(\frac{1}{l_n} + \frac{1}{l_n^2} + \ldots + \frac{1}{l_n^r}) \log k_n,$$

then

$$\mathfrak{R}_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty), \qquad (10)$$

is its limit reliability function.

Example 3. A hierarchical regular series-parallel homogeneous system of order $r = 2$ is such that $k_n = 200,$ $l_n = 3.$ The system components have identical exponential reliability functions with the failure rate $\lambda = 0.01.$

Under these assumptions its exact reliability function, according to *Corollary 3*, is given by

$$R_{2,200,3}(t) = 1 - [1 - [\,1 - [1 - \exp[-0.01 \cdot 3t]]^{200}\,]^3]^{200}$$

for $t \ge 0.$

Next applying *Proposition 4* with normalising constants

$$a_n = \frac{1}{0.01 \cdot 9} = 11.1,$$

$$b_n = \frac{1}{0.01}(\frac{1}{3} + \frac{1}{9}) \log 200 = 235.5,$$

we conclude that the system limit reliability function is given by

$$\mathfrak{R}_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty),$$

and from (7), the following approximate formula is valid

$$\boldsymbol{R}_{2,200,3}(t) \cong \mathfrak{R}_3(0.09t - 21.2)$$

$$= 1 - \exp[-\exp[-0.09t + 21.2]] \text{ for } t \in (-\infty, \infty).$$

Definition 10. A two-state system is called a parallel-series system of order r if its lifetime T is given by

$$T = \min_{1 \le i_1 \le k_n} \{ \max_{1 \le j_1 \le l_{i_1}} \{ \min_{1 \le i_2 \le k_n^{(i_1 j_1)}} \{ \max_{1 \le j_2 \le l_{i_2}^{(i_1 j_1)}}$$

$$\ldots[\min_{1 \le i_r \le k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}} (\max_{1 \le j_r \le l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}} T_{i_1 j_1 \ldots i_r j_r})]\ldots\}\}\},$$

where k_n, $k_n^{(i_1 j_1)}$, …, $k_n^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of suitable parallel systems of the system composed of parallel-series subsystems and linked in series, l_{i_1}, $l_{i_2}^{(i_1 j_1)}$, …, $l_{i_{r-1}}^{(i_1 j_1 i_2 j_2 \ldots i_{r-2} j_{r-2})}$ are the numbers of suitable parallel-series subsystems in these parallel systems, and $l_{i_r}^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of components in the parallel systems of the parallel-series subsystems.

Definition 11. A two-state parallel-series system of order r is called homogeneous if its component lifetimes $T_{i_1 j_1 \ldots i_r j_r}$ have an identical distribution function

$$F(t) = P(T_{i_1 j_1 \ldots i_r j_r} \le t),$$

where

$$i_1 = 1,2,\ldots,k_n, \ j_1 = 1,2,\ldots,l_{i_1}, \ i_2 = 1,2,\ldots,k_n^{(i_1 j_1)},$$
$$j_2 = 1,2,\ldots,l_{i_2}^{(i_1 j_1)}, \ldots,$$
$$i_r = 1,2,\ldots,k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}, \ j_r = 1,2,\ldots,l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})},$$

i.e. if its components $E_{i_1 j_1 \ldots i_r j_r}$ have the same reliability function

$$R(t) = 1 - F(t), \ t \in (-\infty, \infty).$$

Definition 12. A two-state parallel-series system of order r is called regular if

$$l_{i_1} = l_{i_2}^{(i_1 j_1)} = \ldots = l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = l_n$$

and

$$k_n^{(i_1 j_1)} = \ldots = k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = k_n$$

where k_n is the number of parallel systems in the parallel-series subsystems and l_n are the numbers of parallel-series subsystems or, respectively, the numbers of components in these parallel systems.

Applying mathematical induction it is possible to prove that the reliability function of the homogeneous and regular two-state hierarchical parallel-series system of order r is given by

$$\overline{\boldsymbol{R}}_{k,k_n,l_n}(t) = [1-[1-\overline{\boldsymbol{R}}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \text{ for } k=2,3,...,r$$

and

$$\overline{\boldsymbol{R}}_{1,k_n,l_n}(t) = [1-[F(t)]^{l_n}]^{k_n},\ t \in (-\infty,\infty),$$

where k_n and l_n are defined in *Definition 12*.

Corollary 4. If components of the homogeneous and regular two-state hierarchical parallel-series system of order r have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0,\ \lambda > 0,$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,k_n,l_n}(t) = [1-[1-\overline{\boldsymbol{R}}_{k-1,k_n,l_n}(t)\ ]^{l_n}]^{k_n} \text{ for } k=2,3,...,r$$

and

$$\overline{\boldsymbol{R}}_{1,k_n,l_n}(t) = [1-[1-\exp[-\lambda t]]^{l_n}]^{k_n} \text{ for } t \geq 0.$$

Theorem 5. If

(i) $\overline{\Re}(t) = \exp[-\overline{V}(t)],\ t \in (-\infty,\infty),$ is a non-degenerate reliability function,

(ii) $\lim\limits_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0$ for $r \geq 1$,

(iii) $\lim\limits_{n\to\infty} k_n^{l_n^{r-1}+...+1} [F(a_n t + b_n)]^{l_n^r} = \overline{V}(t)$ for $t \in C_V$, $r \geq 1,\ t \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_{r,k_n,l_n}(a_n t + b_n) = \overline{\Re}(t) \text{ for } t \in C_{\overline{\Re}},\ r \geq 1,\ t \in (-\infty,\infty).$$

Proposition 5. If components of the homogeneous and regular two-state hierarchical parallel-series system of order r have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0,\ \lambda > 0,$$

$$\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0 \text{ for } r \geq 1,\ \lim_{n\to\infty} l_n = l,\ l \in N,$$

and

$$a_n = \frac{1}{\lambda} \frac{1}{k_n^{\frac{1}{l_n}+...+\frac{1}{l_n^r}}},\ b_n = 0,$$

then

$$\overline{\Re}_2(t) = \exp[-t^{l^r}] \text{ for } t \geq 0 \quad (11)$$

is its limit reliability function.

Example 3. We consider a hierarchical regular parallel-series homogeneous system of order $r = 2$ such that $k_n = 200$, $l_n = 3$, whose components have identical exponential reliability functions with the failure rate $\lambda = 0.01$.
Its exact reliability function, according to *Corollary 4*, is given by

$$\overline{\boldsymbol{R}}_{2,200,3}(t) = [1-[1-[\ 1-[1-\exp[-0.01t]]^3\ ]^{200}\ ]^3]^{200}$$

for $t \geq 0$.

Next applying *Proposition 5* with normalising constants

$$a_n = \frac{1}{0.01} \cdot \frac{1}{200^{1/3+1/9}} = 9.4912,\ b_n = 0,$$

we conclude that

$$\overline{\Re}_2(t) = \exp[-t^9] \text{ for } t \geq 0$$

is the system limit reliability function, and from (7), the following approximate formula is valid

$$\overline{\boldsymbol{R}}_{2,200,3}(t) \cong \overline{\Re}_2(0.1054t) = \exp[-(0.1054t)^9]$$

for $t \geq 0$.

6. Conclusion

Generalizations of the results on limit reliability functions of two-state homogeneous systems for these and other systems in case they are non-homogeneous, are mostly given in [8] and [9]. These results allow us to evaluate reliability characteristics of homogeneous and non-homogeneous series-parallel and parallel-series systems with regular reliability structures, i.e. systems composed of subsystems having the same numbers of components. However, this fact does not restrict the completeness of the performed analysis, since by conventional joining of a suitable number of components which do not fail, in series sub-systems of the non-regular series-parallel systems, leads us to the regular non-homogeneous series-parallel systems.

Similarly, conventional joining of a suitable number of failed components in parallel subsystems of the non-regular parallel-series systems we get the regular non-homogeneous parallel-series systems. Thus the problem has been analyzed exhaustively.

The results concerned with the asymptotic approach to system reliability analysis, in a natural way, have led to investigation of the speed of convergence of the system reliability function sequences to their limit reliability functions ([9]). These results have also initiated the investigations of limit reliability functions of "*m* out of *n*"-series, series-"*m* out of *n*" systems, the investigations on the problems of the system reliability improvement and on the reliability of systems with varying in time their structures and their components reliability described briefly in [9] and presented in Part 2 ([10]) of this paper.

More general and practically important complex systems composed of multi-state and degrading in time components are considered in wide literature, for instance in [14]. An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is described in [9] for large multi-state systems with degrading components. The most important results regarding generalizations of the results on limit reliability functions of two-state systems dependent on transferring them to series, parallel, "*m* out of *n*", series-parallel and parallel-series multi-state systems with degrading components are given in [9]. Some practical applications of the asymptotic approach to the reliability evaluation of various technical systems are contained in [9] as well.

The proposed method offers enough simplified formulae to allow significant simplifying of large systems' reliability evaluating and optimizing calculations.

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