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On asymptotic approach to reliability improvement of multi-state systems with components quantitative and qualitative redundancy: series and parallel systems

Keywords

reliability improvement, limit reliability functions

Abstract

The paper is composed of two parts, in this part after introducing the multi-state and the asymptotic approaches to system reliability evaluation the multi-state homogeneous series and parallel systems with reserve components are defined and their multi-state limit reliability functions are determined. In order to improve of the reliability of these systems the following methods are used: (i) a warm duplication of components, (ii) a cold duplication of components, (iii) a mixed duplication of components, (iv) improving the reliability of components by reducing their failure rate. Next, the effects of the systems' reliability different improvements are compared.

1. Introduction

Most real systems are very complex and it is difficult to analyze and to improve their reliability. Large numbers of components and subsystems and their operating complexity cause that the evaluation of their reliability is complicated. As a rule these are series systems, parallel systems or "*m* out of *n*" systems composed of a large number of components. One of the important techniques for reliability evaluation of large systems is the asymptotic approach. The mathematical methods are based on the limit theorems of order statistics distributions considered in a wide literature. These theorems generated investigations on limit reliability functions for systems with two-state components. Next, more general systems with multi-state components began to be considered. The asymptotic approach is also very useful in reliability improvement of large multi-state systems because of simplifying the calculation.

2. Multi-state and asymptotic approach

In multi-state reliability analysis presented in this paper it is supposed that:

- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the state set $\{0,1,...,z\}$,
- the state indices are ordered, the state 0 is the worst and the state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of the components E_i in the state subset $\{u,u+1,...,z\}$ while they were in the state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
- the system state degrades with time t without repair,
- $e_i(t)$ is a component E_i state at the time t, $t > 0$,
- $s(t)$ is a system state at the moment t, $t > 0$.

Definition 2.1. A vector

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1),..., R_i(t,z)],$$
$$t \in (-\infty,\infty),\ i = 1,2,...,n,$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t),$$
$$t \in (-\infty,\infty),\ u = 0,1,...,z,$$

is the probability that the component E_i is in the state subset $\{u,u+1,...,z\}$ at the time t, $t \in (-\infty,\infty)$ while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Definition 2.2. A vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)],$$
$$t \in (-\infty,\infty),$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t),$$
$$t \in (-\infty,\infty),\ u = 0,1,...,z,$$

is the probability that the system is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in (-\infty,\infty)$ while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a system.

In the asymptotic approach to system reliability analysis we are interested in limit distributions of a standardized random variable

$$(T(u) - b_n(u))/a_n(u),\ u = 1,2,...,z,$$

where $T(u)$ is the lifetime of the system in the state subset $\{u,u+1,...,z\}$ and $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$, are some suitably chosen numbers, called normalizing constants.

Since

$$P((T(u) - b_n(u))/a_n(u) > t)$$

$$= P(T(u) > a_n(u)t + b_n(u))$$

$$= \boldsymbol{R}_n(a_n(u)t + b_n(u), u),\quad u = 1,2,...,z,$$

where

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state reliability function of the system, then we assume the following definition.

Definition 2.3. A vector

$$\Re(t,\cdot) = [1, \Re(t,1), ..., \Re(t,z)],\ t \in (-\infty,\infty),$$

is called the limit multi-state reliability function of the system if there exist normalizing constants $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$ such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n(u)t + b_n(u), u) = \Re(t,u),$$
$$t \in C_{\Re(u)},\ u = 1,2,...,z,$$

where $C_{\Re(u)}$ is the set of continuity points of $\Re(t,u)$.

The knowledge of the system limit reliability function allow us, for sufficiently large n, to apply the following approximate formula

$$R_n(t,\cdot) \cong \Re((t - b_n(u))/a_n(u),\cdot),\ t \in (-\infty,\infty). \qquad (1)$$

3. System reliability improvement

3.1. Reliability improvement of a multi-state series system

Definition 3.1. A multi-state system is called a series system if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \min_{1 \leq i \leq n}\{T_i(u)\},\ u = 1,2,...,z.$$

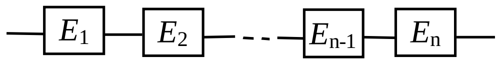

Figure 1. The scheme of a homogeneous series system

Definition 3.2. A multi-state series system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets $\{u,u+1,...,z\}$ have an identical distribution function

$$F_i(t,u) = F(t,u),\ u = 1,2,...,z,\ t \in (-\infty,\infty),\ i = 1,2,...,n,$$

The reliability function of the homogeneous multi-state series system is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1), ..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = [R(t,u)]^n,\ t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Definition 3.3. A multi-state series system is called a system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = \min_{1 \leq i \leq n}\{\max_{1 \leq j \leq 2}\{T_{ij}(u)\}\},\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state series system with a hot reserve of its components is given by

$$\boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,\cdot) = [1, \boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,1), ..., \boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,z)],$$

where

$$\boldsymbol{I\overline{R}}_n^{(1)}(t,u) = [1-(F(t,u))^2]^n,\ t \in (-\infty,\infty). \quad (2)$$

Lemma 3.1. If

(i) $\mathrm{I}\overline{\mathfrak{R}}^{(1)}(t,u) = \exp[-\overline{V}(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}_n^{(1)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state series system whit a hot reserve of its components defined by (2),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}_n}^{(1)}(a_n(u)t + b_n(u)) = \mathrm{I}\overline{\mathfrak{R}}^{(1)}(t,u),\ t \in C_{\overline{\mathfrak{R}}},$$
$$u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u))]^2 = \overline{V}(t,u),\ t \in C_{\overline{V}},$$
$$u = 1,2,...,z.$$

Proposition 3.1. If components of the homogeneous multi-state series system with a hot reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)}$, $b_n(u) = 0$, $u = 1,2,...,z$,

then

$$\mathrm{I}\overline{\mathfrak{R}}^{(1)}(t,u) = 1,\ t < 0,$$

$$\mathrm{I}\overline{\mathfrak{R}}^{(1)}(t,u) = \exp[-t^2],\ t \geq 0,\ u = 1,2,...,z,$$

is its limit reliability function.

The proof of Proposition 3.1 is given in [9].

Corollary 3.1. The reliability function of exponential series system whit a hot reserve of its components is given by

$$\boldsymbol{I\overline{R}_n}^{(1)}(t,u) = 1,\ t < 0,$$

$$\boldsymbol{I\overline{R}}_n^{(1)}(t,u) \cong \exp[-\lambda^2(u)nt^2],\ t \geq 0,\ u = 1,2,...,z. \quad (3)$$

Definition 3.4. A multi-state series system is called a system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u, u+1,...,z\}$ is given by

$$T^{(2)}(u) = \min_{1\leq i\leq n}\{\sum_{j=1}^{2} T_{ij}(u)\},\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components. The reliability function of the homogeneous multi-state series system with cold reserve of its components is given by

$$\boldsymbol{I\overline{R}}_n^{(2)}(t,\cdot) = [1,\ \boldsymbol{I\overline{R}}_n^{(2)}(t,1),...,\ \boldsymbol{I\overline{R}}_n^{(2)}(t,z)],$$

where

$$\boldsymbol{I\overline{R}_n}^{(2)}(t,u) = [1 - F(t,u) * F(t,u)]^n,\ t \in (-\infty,\infty), \quad (4)$$
$$u = 1,2,...,z.$$

Lemma 3.2. If

(i) $\mathrm{I}\overline{\mathfrak{R}}^{(2)}(t,u) = \exp[-\overline{V}(t,u)]$ $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}_n^{(2)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state series system whit a cold reserve of its components defined by (4),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}_n}^{(2)}(a_n(u)t + b_n(u)) = \mathrm{I}\overline{\mathfrak{R}}^{(2)}(t,u),\ t \in C_{\mathrm{I}\overline{\mathfrak{R}}},$$
$$u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= \overline{V}(t,u),\ t \in C_{\overline{V}},\ u = 1,2,...,z.$$

Proposition 3.2. If components of the homogeneous multi-state series system with a cold reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{\sqrt{2}}{\lambda(u)\sqrt{n}}$, $b_n(u) = 0$, $u = 1,2,...,z$,

then

$$\mathrm{I}\overline{\mathfrak{R}}^{(2)}(t,u) = 1,\ t < 0,$$

$$\mathrm{I}\overline{\mathfrak{R}}^{(2)}(t,u) = \exp[-t^2],\ t \geq 0,\ u = 1,2,...,z,$$

is its limit reliability function.

The proof of Proposition 3.2 is given in [9].

Corollary 3.2. The reliability function of exponential series system whit a cold reserve of its components is given by

$$\boldsymbol{I\overline{R}_n}^{(2)}(t,u) = 1,\ t < 0,$$

$$\boldsymbol{I\overline{R}}_{\mathrm{n}}^{(2)}(t,u) \cong \exp[-\lambda^2(u)nt^2/2],\ t \ge 0, \qquad (5)$$
u =1,2,...,z.

Definition 3.5. A multi-state series system is called a system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u) = \min\{\min_{1\le i\le s_1 n}\{\max_{1\le j\le 2}\{T_{ij}(u)\}\},\ \min_{s_1 n+1\le i\le n}\{\sum_{j=1}^{2} T_{ij}(u)\}\},$$
u = 1,2,...,z,

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1 + s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state series system with a mixed reserve of its components is given by

$$\boldsymbol{I\overline{R}}_n^{(3)}(t,\cdot) = [1, \boldsymbol{I\overline{R}}_n^{(3)}(t,1),..., \boldsymbol{I\overline{R}}_n^{(3)}(t,z)\ ],$$

where

$$\boldsymbol{I\overline{R}}_n^{(3)}\ (t,u) = [1-(F(t,u))^2]^{s_1 n}[1-F(t,u)*F(t,u)]^{s_2 n}, \qquad (6)$$
$t \in (-\infty,\infty)$, u = 1,2,...,z.

Lemma 3.3. If

(i) $\mathrm{I}\overline{\mathfrak{R}}^{(3)}(t,u) = \exp[-\overline{V}(t,u)]$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}_n^{(3)}(t,u)$, $t \in (-\infty,\infty)$, u = 1,2,...,z, is the reliability function of non-degenerate multi-state series system whit a mixed reserve of its components defined by (6),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, u = 1,2,...,z,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}_n}^{(3)}(a_n(u)t+b_n(u)) = \mathrm{I}\overline{\mathfrak{R}}^{(3)}(t,u),\ t \in C_{\mathrm{I}\overline{\mathfrak{R}}},$$
u = 1,2,...,z,

if and only if

$$\lim_{n\to\infty} 2ns_1[F(a_n(u)t+b_n(u))] + ns_2[F(a_n(u)t+b_n(u)) * F(a_n(u)t+b_n(u))] = \overline{V}(t,u),$$
$t \in C_{\overline{V}}$, u = 1,2,...,z.

Proposition 3.3. If components of the homogeneous multi-state series system with a mixed reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)\sqrt{n}}$, $b_n(u) = 0$, u = 1,2,...,z,

then

$$\mathrm{I}\overline{\mathfrak{R}}(t,u) = 1 \text{ for } t < 0,$$

$$\mathrm{I}\overline{\mathfrak{R}}(t,u) = \exp[-(2s_1+s_2)\,t^2/2],\ t \ge 0,\ u = 1,2,\ldots,z,$$

is its limit reliability function.

Corollary 3.3. The reliability function of exponential series system whit mixed reserve of its components is given by

$$\boldsymbol{I\overline{R}_n}^{(3)}(t,u) = 1,\ \mathrm{t} < 0,$$

$$\boldsymbol{I\overline{R}_n}^{(3)}(t,u) \cong \exp[-\lambda^2(u)n(2s_1+s_2)t^2/2],\ t \ge 0, \qquad (7)$$
u = 1,2,...,z.

Proposition 3.4. If components of the homogeneous multi-state series system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u) \in <0,1>$, u = 1,2,...,z, i.e.

$$\tilde{R}(t,u) = 1,\ t < 0,$$

$$\tilde{R}(t,u) = \exp[-\lambda(u)\rho(u)t\ ],\ t \ge 0,\ \lambda(u) > 0,$$
$u = 1,2,\ldots,z$,

and $a_n(u) = \dfrac{1}{\lambda(u)\rho(u)n}$, $b_n(u) = 0$, u = 1,2,...,z,

then

$$\mathrm{I}\overline{\mathfrak{R}}^{(4)}(t,u) = 1,\ t < 0,$$

$$\mathrm{I}\overline{\mathfrak{R}}^{(4)}(t,u) = \exp[-t],\ t \ge 0,\ u = 1,2,...,z,$$

is its limit reliability function.

Corollary 3.4. The reliability function of exponential series system whit improved reliability functions of its components is given by

$\bar{\boldsymbol{IR}}_{\boldsymbol{n}}^{(4)}(t,u) = 1$, t < 0,

$\bar{\boldsymbol{IR}}_{\boldsymbol{n}}^{(4)}(t,u) = \exp[-\lambda(u) n \rho(u) t]$, $t \geq 0$, $u = 1,2,...,z$. (8)

3.2. Reliability improvement of a multi-state parallel system

Definition 3.6. A multi-state system is called a parallel system if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \max_{1 \leq i \leq n}\{T_i(u)\}, \; u = 1,2,...,z.$$

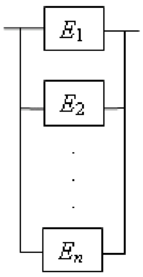

Figure 2. The scheme of a homogeneous parallel system

Definition 3.7. A multi-state parallel system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets $\{u,u+1,...,z\}$ have an identical distribution function

$$F_i(t,u) = F(t,u), \; u = 1,2,...,z, \; t \in (-\infty,\infty), \; i = 1,2,...,n.$$

The reliability function of the homogeneous multi-state parallel system is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1 - [F(t,u)]^n, \; t \in (-\infty,\infty), \; u = 1,2,...,z.$$

Definition 3.8. A multi-state parallel system is called a system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = \max_{1 \leq i \leq n}\{\max_{1 \leq j \leq 2}\{T_{ij}(u)\}\}, \; u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state parallel system with a hot reserve of its components is given by

$$\boldsymbol{IR}_n^{(1)}(t,\cdot) = [1, \boldsymbol{IR}_n^{(1)}(t,1),..., \boldsymbol{IR}_n^{(1)}(t,z)],$$

where

$$\boldsymbol{IR}_n^{(1)}(t,u) = 1 - [(F(t,u))^2]^n, \; t \in (-\infty,\infty), \quad (9)$$
$u = 1,2,...,z.$

Lemma 3.4. If

(i) $\mathfrak{I}\mathfrak{R}^{(1)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(1)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state parallel system whit a hot reserve of its components defined by (9),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(1)}(a_n(u)t + b_n(u)) = \mathfrak{I}\mathfrak{R}(t,u), \; t \in C_{\mathfrak{I}\mathfrak{R}},$$
$u = 1,2,...,z,$

if and only if

$$\lim_{n\to\infty} 2n[R(a_n(u)t + b_n(u))] = V(t,u), \; t \in C_V,$$
$u = 1,2,...,z.$

Proposition 3.5. If components of the homogeneous multi-state parallel system with a hot reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)}$, $b_n(u) = \dfrac{\log 2n}{\lambda(u)}$, $u = 1,2,...,z$,

then

$$\boldsymbol{I\mathfrak{R}}^{(1)}(t) = 1 - \exp[-\exp[-t]], \; t \in (-\infty,\infty), \; u = 1,2,...,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{t + \log 2n}{\lambda(u)} \to \infty \text{ as } n \to \infty$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Therefore

$$V(t,u) = \lim_{n\to\infty} 2nR(a_n(u)t + b_n(u))$$

$$= \lim_{n\to\infty} 2n \exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} 2n \exp[-t - \log 2n]$$

$$= \exp[-t], t \in (-\infty, \infty),$$

which by Lemma 3.4 completes the proof.

Corollary 3.5. The reliability function of exponential parallel system whit a hot reserve of its components is given by

$$\boldsymbol{IR}_n^{(1)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)t + \log 2n], \quad (10)$$
$$t \in (-\infty, \infty), \; u = 1,2,...,z.$$

Definition 3.9. A multi-state parallel system is called a system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(2)}(u) = \max_{1 \le i \le n}\{\sum_{j=1}^{2} T_{ij}(u)\}, \; u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state parallel system with a cold reserve of its components is given by

$$\boldsymbol{IR}_n^{(2)}(t,\cdot) = [1, \boldsymbol{IR}_n^{(2)}(t,1),..., \boldsymbol{IR}_n^{(2)}(t,z)],$$

where

$$\boldsymbol{IR}_n^{(2)}(t,u) = 1 - [F(t,u) * F(t,u)]^n, \quad (11)$$
$$t \in (-\infty,\infty), \; u = 1,2,...,z.$$

Lemma 3.5. If

(i) $\mathrm{I}\Re^{(2)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(2)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state parallel system whit a cold reserve of its components defined by (11),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(2)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(2)}(t,u),$$
$$t \in C_{\Re}, \; u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$
$$= V(t,u), \; t \in C_V, \; u = 1,2,...,z.$$

Proposition 3.6. If components of the homogeneous multi-state parallel system with a cold reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)}, \; \dfrac{\exp[\lambda b_n(u)]}{\lambda(u) b_n(u)} = n, \; u = 1,2,...,z,$

then

$$\boldsymbol{I}\Re^{(2)}(t,u) = 1 - \exp[-\exp[-t]], \; t \in (-\infty,\infty),$$
$$u = 1,2,...,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have

$a_n(u)t + b_n(u) \to \infty$ as $n \to \infty$, $t \in (-\infty, \infty)$, $u = 1,2,...,z$.

Therefore

$$V(t,u)$$
$$= \lim_{n\to\infty} n[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$
$$= \lim_{n\to\infty} n[(1 + \lambda(u)(a_n(u)t + b_n(u))) \cdot$$
$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$
$$= \lim_{n\to\infty} n[(1 + t + \lambda(u)b_n(u)) \exp[-(t + \lambda(u)b_n(u))]$$
$$= \lim_{n\to\infty} n[\frac{1+t}{\exp[\lambda(u)b_n(u)]} + \frac{\lambda(u)b_n(u)}{\exp[\lambda(u)b_n(u)]}] \exp[-t]$$
$$= \exp[-t] \text{ for } t \in (-\infty, \infty),$$

which by Lemma 3.5 completes the proof.

Corollary 3.6. The reliability function of exponential parallel system whit a cold reserve of its components is given by

$$\boldsymbol{IR}_n^{(2)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)t + \lambda(u)b_n(u)], \quad (12)$$
$$t \in (-\infty, \infty), \; u = 1,2,...,z.$$

Definition 3.10. A multi-state parallel system is called a system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u) = \max\{ \max_{1\le i\le s_1 n}\{\max_{1\le j\le 2}\{T_{ij}(u)\}\},$$
$$\max_{s_1 n+1\le i\le n}\{\sum_{j=1}^{2} T_{ij}(u)\}\},$$
$u = 1,2,...,z,$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1 + s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state parallel system with mixed reserve of its components is given by

$$\boldsymbol{IR}_n^{(3)}(t,\cdot) = [1, \boldsymbol{IR}_n^{(3)}(t,1),..., \boldsymbol{IR}_n^{(3)}(t,z)],$$

where

$$\boldsymbol{IR}_n^{(3)}(t,u) = 1-[(F(t,u))^2]^{s_1 n}[F(t,u)*F(t,u)]^{s_2 n}, \quad (13)$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Lemma 3.6. If

(i) $I\Re^{(3)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(3)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state parallel system with a mixed reserve of its components defined by (13),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(3)}(a_n(u)t+b_n(u)) = I\Re^{(3)}(t,u), t \in C_{\Re},$$
$u = 1,...,z,$

if and only if

$$\lim_{n\to\infty} 2ns_1[R(a_n(u)t+b_n(u))]$$
$$+ns_2[1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]$$
$$= \overline{V}(t,u),\ t \in C_V,\ u = 1,2,...,z.$$

Proposition 3.7. If components of the homogeneous multi-state parallel system with a mixed reserve of its components have multi-state exponential reliability functions and

$$a_n(u) = \frac{1}{\lambda(u)b_n(u)-1},\quad \frac{\exp[\lambda(u)b_n(u)]}{\lambda(u)b_n(u)} = s_2 n,$$

then

$$\boldsymbol{I}\Re^{(3)}(t,u) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),\ u = 1,\dots,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have
$a_n(u)t + b_n(u) \to \infty$ as $n \to \infty$, $t \in (-\infty,\infty)$,

and

$$\frac{1}{\lambda(u)b_n(u)-1} \to 0 \text{ as } n\to\infty,\ t \in (-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[2s_1 \exp[-\lambda(u)(a_n(u)t+b_n(u))]$$
$$+s_2[(1+\lambda(u)(a_n(u)t+b_n(u)))$$
$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]]$$
$$= \lim_{n\to\infty} n\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$
$$s_2\lambda(u)(a_n(u)t+b_n(u))$$
$$[1+\frac{s_2+2s_1}{s_2\lambda(u)(a_n(u)t+b_n(u))}]$$
$$= \lim_{n\to\infty} \exp[-\lambda(u)a_n(u)t]$$
$$\exp[-\lambda(u)b_n(u) + \log ns_2\lambda(u)b_n(u)$$
$$[1++o(1)]][1+o(1)] = \exp[-t],$$
$t \in (-\infty,\infty)$, $u = 1,2,\dots,z$,

which by Lemma 3.6 completes the proof.

Corollary 3.7. The reliability function of parallel system whit a mixed reserve of its components is given by

$$\boldsymbol{IR}_n^{(3)}(t,u)$$
$$\cong 1-\exp[-\exp[-\frac{\lambda(u)b_n(u)-1}{b_n(u)}t + (\lambda(u)b_n(u)-1)],$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z.$ (14)

Proposition 3.8. If components of the homogeneous multi-state parallel system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u) \in$ <0,1>,
$u = 1,2,...,z$, and

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)},\ b_n(u) = \frac{\log n}{\lambda(u)\rho(u)},\ u = 1,2,...,z,$$

then

$$I\Re^{(4)}(t,u) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),$$
$$u = 1,2,...,z,$$

is its limit reliability function.

Corollary 3.8. The reliability function of exponential parallel system whit improved reliability functions of its components is given by

$$\boldsymbol{IR}_n^{(4)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)\rho(u)t + \log n],\quad (15)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z.$$

4. Comparison of reliability improvement effects

The comparisons of the limit reliability functions of the systems with different kinds of reserve and such systems with improved components allow us to find the value of the components decreasing failure rate factor $\rho(u)$, which warrants an equivalent effect of the system reliability improvement.

4.1 Series system

The comparisons of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equations

$$\boldsymbol{I}\overline{\Re}^{(4)}\left((t - b_n(u))/a_n(u)\right)$$

$$= I\overline{\Re}^{(k)}\left((t - b_n(u))/a_n(u)\right),\ u = 1,2,...,z,\quad (16)$$
$$k = 1,2,3.$$

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state series system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (16) are respectively given by

$$k = 1\quad \rho(u) = \rho(t,u) = \lambda(u)t,\quad u = 1,2,...,z,$$

$$k = 2\quad \rho(u) = \rho(t,u) = \frac{\lambda(u)t}{2},\quad u = 1,2,...,z,$$

$$k = 3\quad \rho(u) = \rho(t,u) = \frac{2s_1 + s_2}{2}\lambda(u)t,\ u = 1,2,...,z.$$

4.2. Parallel system

The comparisons of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equation

$$\boldsymbol{I}\Re^{(4)}\left((t - b_n(u))/a_n(u)\right)$$

$$= \boldsymbol{I}\Re^{(k)}\left((t - b_n(u))/a_n(u)\right),\ u = 1,2,...,z,\quad (17)$$
$$k = 1,2,3.$$

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state parallel system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (17) are respectively given by

$$k = 1\quad \rho(u) = \rho(t,u) = 1 - \frac{\log 2}{\lambda(u)t},\quad u = 1,2,...,z,$$

$$k = 2\quad \rho(u) = \rho(t,u) = 1 - \frac{\lambda(u)b_n(u) - \log n}{\lambda(u)t},$$
$$u = 1,2,...,z,$$

$$k = 3$$
$$\rho(u) = \rho(t,u) = \frac{\lambda(u)b_n(u) - 1}{\lambda(u)}$$

$$- \frac{\lambda(u)b_n(u) - 1 - \log n}{\lambda(u)t},$$
$$u = 1,2,...,z.$$

5. Conclusion

Proposed in the paper application of the limit multi-state reliability functions for reliability of large systems evaluation and improvement simplifies calculations. The methods may be useful not only in the technical objects operation processes but also in their new processes designing, especially in their optimization. The case of series, parallel and ”*m* out of *n*” (in part 2) systems composed of components having exponential reliability functions with double reserve of their components is considered only. It seems to be possible to extend the results to the systems having other much complicated reliability structures and components with different from the exponential reliability function. Further, it seems to be reasonable to elaborate a computer programs supporting calculations and accelerating decision making, addressed to reliability practitioners.

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