ISSN 1932-2321

RELIABILITY:

Theory & Applications

Vol.2 No.3-4,
December, 2007

Special Issue # 1 on SSARS 2007

Invited Editors E. Zio and K. Kołowrocki

Electronic Journal
Of International Group On Reliability

San Diego

ISSN 1932-2321

http://www.gnedenko-forum.org/Journal/index.htm

Journal of International Group on Reliability

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Guest Editorial

The papers collected in these 2 issues are works presented at the 1st Summer Safety and Reliability Seminars, SSARS 2007, held in Gdansk/Sopot, Poland, from 22nd of July 2007 until 29th of July 2007.

The Seminar was attended by 46 participants from 14 countries (Canada, Czech Republic, Germany, Greece, Italy, Lithuania, Netherlands, Poland, Portugal, Slovakia, South Africa, South Korea, Tunisia, United States).

The motivation behind this annual event series is to provide a forum for discussing, advancing and developing methods for the safety and reliability analysis of the complex systems and processes, which form the backbone of our modern societies. The subjects of the Seminars are chosen each year by a Program Board of selected experts in an effort to dynamically represent the methodological advancements developed to meet the newly arising challenges in the field of safety and reliability analysis.

This year the emphasis was addressed to the following subjects:

- Natural Hazards Analysis and Environment Protection Modeling;
- Reliability and Safety Data Collection and Analysis;
- System Safety and Reliability Modeling, Dependence, Dynamic Reliability;
- Risk Assessment and Management;
- Maintenance Modeling and Optimization.

Both 1-2 hours lectures on advanced methods and technical presentations of 20-30 minutes on applications of such methods were offered during the plenary sessions and the seminar sessions, respectively.

The Advisory, Editorial and Organizing Boards have carried out the preliminary evaluation of the 52 contributions selected for this year Seminars and sent out recommendations to the authors for improving their work.

The extended abstracts of all lectures and technical papers were collected in the SSARS Proceedings, which constitute an up-to-date reference textbook for the participants to the Seminars and all the researchers in the field.

The 43 papers and lectures presented at SSARS 2007 are presented in the two special issues of the Journal, grouped into a System Safety, Reliability and Maintenance Modeling Section and a Natural Hazards and Risk Assessment and Management Section.

Guest Editors

Krzysztof Kołowrocki, Enrico Zio

SSARS 2007 Chairmen

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Section 1: System Safety, Reliability and Maintenance Modeling

List of Papers & Lectures

Blokus-Roszkowska Agnieszka
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Analysis of component failures dependency influence on system lifetime

Keywords

component failure dependency, reliability characteristics, limit reliability function, transportation system

Abstract

In the paper the results of the reliability investigation of multi-state homogeneous parallel-series systems with independent and dependent components are presented. The multi-state reliability functions of such systems and other reliability characteristics in both cases are determined under the assumption that their components have exponential reliability function. Moreover the asymptotic approach to the reliability evaluation of these systems is also presented and the classes of limit reliability functions for the considered systems in both cases are fixed. Finally, the presented theoretical results are applied to the reliability evaluation of the shipyard rope transportation system. The comparison of the multi-state exact and limit reliability functions of the considered transportation system under the assumption that its components are independent and under the assumption that its components have failure dependency is performed and illustrated graphically.

1. Introduction

The paper is concentrated on the reliability analysis of large multi-state parallel-series systems with dependent and independent failures of components. A parallel-series system with dependent components is considered as a system of a number of parallel subsystems linked in series, each of them composed of components with dependent failures. The system is failed if all components in at least one of its subsystem are failed. In the reliability analysis of their parallel subsystems, it seems natural to assume that the failures of one or several of their components may cause the reliability characteristics of their un-failed components worsening. In such systems the increased load caused by one or several of its components' failures may cause the increase of the failure rates of remaining un-failed components. The rules of load sharing between remaining not failed components that are the rules of their failure rates increase may be different. In the paper it is assumed that the load is distributed equally among all un-failed components of considered parallel systems and subsystems. This means that the failure rates of these components are changing in an analogical way that is the failure rates are increasing with the same level.

Some results on limit reliability functions of two-state parallel-series systems with equal load sharing among components of parallel subsystems in a case when the number of components in these subsystems is large were obtained by Smith [6]-[7]. Other results on limit reliability functions of two-state parallel-series systems in a case of the parallel-series system structure's shape when the number of parallel subsystems is large were fixed by Harlow and Smith [8]. Some partial results on limit reliability functions in the first of these two cases for multi-state parallel-series systems can be found in my recent publications as well [1]-[2].

2. Reliability of multi-state systems

Taking into account the importance of the safety effectiveness of large systems it seems reasonable to consider the multi-state approach in their reliability analysis. The assumption that the systems are composed of multi-state components with reliability state degrading in time without repair gives the possibility for more precise analysis of their reliability and safety effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the required level of effectiveness of this system exploitation. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-

state reliability function that is a basic characteristic of the multi-state system.

One of basic multi-state reliability structures with components degrading in time is parallel-series system. In the multi-state reliability analysis to define parallel-series systems with degrading components we assume that:

- E_{ij}, $i = 1,2,\ldots,k_n$, $j = 1,2,\ldots,l_i$, $k_n, l_1, l_2, \ldots, l_{k_n}$, $n \in N$, are components of a system,
- all components and a system under consideration have the state set $\{0,1,...,z\}$, $z \geq 1$,
- the reliability states are ordered, the state 0 is the worst and the state z is the best,
- the component and the system reliability states degrade with time t without repair,
- $T_{ij}(u), i = 1,2,\ldots,k_n$, $j = 1,2,\ldots,l_n$, $k_n, l_1, l_2, \ldots, l_{k_n}$, $n \in N$, are independent random variables representing the lifetimes of components Eij in the state subset $\{u, u+1,..., z\}$, while they were in the state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u, u+1,..., z\}$, while it was in the reliability state z at the moment $t = 0$,
- $e_{ij}(t)$ are components E_{ij} states at the moment t, $t \in <0, \infty)$,
- $s(t)$ is the system reliability state at the moment t, $t \in <0, \infty)$.

Under above assumptions we introduce the following definition of multi-state reliability function of a component.

Definition 1. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1), \ldots, R_{ij}(t,z)], \ t \in (-\infty, \infty),$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t), \quad (1)$$

for $t \in <0, \infty)$, $u = 0,1,...,z$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, is the probability that the component E_{ij} is in the reliability state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

It is clear from *Definition 1*, that $R_{ij}(t,0) = 1$.

Definition 2. A vector

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot) = [\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,0), \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,1), \ldots, \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z)]$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \quad (2)$$

for $t \in <0, \infty)$, $u = 0,1,...,z$, is the probability that the system is in the reliability state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a system.

If

$$p(t,\cdot) = [p(t,0), p(t,1), \ldots, p(t,z)] \text{ for } t \in <0, \infty),$$

where

$$p(t,u) = P(s(t) = u \mid s(0) = z)$$

for $t \in <0, \infty)$, $u = 0,1,\ldots,z$, is the probability that the system is in the state u at the moment t, $t \in <0, \infty)$, while it was in the state z at the moment $t = 0$, then

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,0) = 1, \ \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z) = p(t,z), \ t \in <0, \infty), \quad (3)$$

and

$$p(t,u) = \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u) - \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u+1) \quad (4)$$

for $t \in <0, \infty)$, $u = 0,1,\ldots,z-1$.

Moreover, if

$$\overline{\boldsymbol{R}}^{(m)}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u) = 1 \text{ for } t \leq 0, \ u = 1,\ldots,z,$$

then the mean lifetime of the system in the state subset $\{u, u+1,\ldots,z\}$ is

$$M(u) = \int_0^\infty \overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)dt, \ u = 1,\ldots,z, \quad (5)$$

and the standard deviation of the system sojourn time in the state subset $\{u, u+1,\ldots,z\}$ is

$$\sigma(u) = \sqrt{N(u) - [M(u)]^2}, \ u = 1,\ldots,z, \quad (6)$$

where

$$N(u) = 2\int_0^\infty t\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)dt, \ u = 1,\ldots,z. \quad (7)$$

Besides

$$\overline{M}(u) = \int_0^\infty p(t,u)dt, \quad u = 1, \ldots, z, \tag{8}$$

is the mean lifetime of the system in the state u, while the integrals (5), (6) and (7) are convergent. Then, according to (3)-(5) and (7), we get the following relationship

$$\overline{M}(u) = M(u) - M(u+1), \quad u = 1,2, \ldots, z-1,$$

$$\overline{M}(z) = M(z). \tag{9}$$

Definition 3. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t),$$

that the system at the moment t, $t \in <0, \infty)$, is in the subset of states worse than the critical state r, $r \in \{1,2, \ldots, z\}$, while it was In the state z at the moment $t = 0$ is called a risk function of the multi-state system.

Under this definition, from (1), we have

$$\boldsymbol{r}(t) = 1 - P(s(t) \ge r \mid s(0) = z) = 1 - \overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t, r), \tag{10}$$

for $t \in (-\infty, \infty)$ and moreover, if τ is the moment when the risk exceeds a permitted level δ, $\delta \in <0,1>$, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{11}$$

where $\boldsymbol{r}^{-1}(t)$, if exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

2.1. Multi-state parallel-series systems

Definition 4. A multi-state system is called parallel-series if its lifetime $T(u)$ in the state subset $\{u, u+1, \ldots, z\}$ is given by

$$T(u) = \min_{1 \le i \le k_n} \{\max_{1 \le j \le l_i} \{T_{ij}(u)\}\}, \quad u = 1,2, \ldots, z.$$

Definition 5. A multi-state parallel-series system is called homogeneous if its component lifetimes $T_{ij}(u)$ in the state subset have an identical distribution function

$$F(t,u) = P(T_{ij}(u) \le t), \quad u = 1,2, \ldots, z, \quad i = 1,2, \ldots, k_n, \quad j = 1,2, \ldots, l_i,$$

i.e. if its components E_{ij} have the same reliability function

$$R(t,u) = 1 - F(t,u), \quad u = 1,2, \ldots, z.$$

Definition 6. A multi-state parallel-series system is called regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n, \quad l_n \in N,$$

i.e. if the numbers of components in its parallel subsystems are equal.

2.1.1. Parallel-series systems with independent components

The results presented below can be found in [4].

Proposition 1. If in a homogeneous regular multi-state parallel-series system

(i) components failure independently,

(ii) components have reliability functions $R(t)$,

then the multi-state system reliability function is given by the formula

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t, \cdot) = [1, \overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t,1), \ldots, \overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t, z)],$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t,u) = [1 - [1 - R(t)]^{l_n}]^{k_n} \text{ for } t \in (-\infty, \infty), \quad u = 1, \ldots, z,$$

where k_n is the number of its parallel subsystems linked in series and l_n is the number of components in each subsystem.

In the case when components of a system have exponential reliability functions i.e.

$$R(t, \cdot) = [1, R(t,1), \ldots, R(t, z)], \tag{12}$$

where

$$R(t,u) = 1 \text{ for } t < 0,$$

$$R(t,u) = \exp[-\lambda(u)t] \text{ for } t \ge 0, \quad \lambda(u) > 0, \tag{13}$$

then the multi-state system reliability function takes form

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t, \cdot) = [1, \overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t,1), \ldots, \overline{\boldsymbol{R}}_{\boldsymbol{k}_n, \boldsymbol{l}_n}(t, z)], \tag{14}$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)=1 \text{ for } t<0,$$

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)=[1-[1-\exp[-\lambda(u)t]]^{l_n}]^{k_n} \qquad (15)$$

for $t\geq 0,\ u=1,\mathrm{h},z.$

2.1.2. Parallel-series systems with dependent component failures

A multi-state parallel-series system is in the reliability state subset $\{u,u+1,...,z\}$ if all of its parallel subsystems are in this state subset. Taking into account this fact, a multi-state parallel-series system with dependent components is considered as a system of linked independently in series multi-state parallel subsystems composed of components with failure dependency. In each of these subsystems we assume the following model of failure dependency. After getting out of the reliability state subset $\{u,u+1,...,z\}$ $v, v=0,1,2,\mathrm{h},l_i-1,$ of components in a subsystem the increased load is shared equally among others so as their load increase with the same scale. Then their reliability is getting worse so that the mean values of the *i*-th, $i=1,2,\mathrm{h},k_n,$ subsystem component lifetimes $T_{ij}'(u)$ in the state subset $\{u,u+1,...,z\}$ are of the form

$$E[T_{ij}'(u)]=E[T_{ij}(u)]-\frac{v}{l_i}E[T_{ij}(u)]=\frac{l_i-v}{l_i}E[T_{ij}(u)],$$
$$j=1,2,\mathrm{h},l_i,\ v=0,1,2,\mathrm{h},l_i-1,\ u=1,2,\mathrm{h},z. \qquad (16)$$

Then the rates of getting out from the reliability state subset $\{u,u+1,...,z\}$ of remaining components of the *i*-th, $i=1,2,\mathrm{h},k_n,$ multi-state subsystem are given by

$$\lambda^{(v)}(u)=\frac{l_i-v}{l_i}\lambda(u),\ v=0,1,2,\mathrm{h},l_i-1, \qquad (17)$$
$$u=1,2,\mathrm{h},z.$$

Proposition 2. If in a homogeneous regular multi-state parallel-series system

(i) components failure in dependent way according to (16),

(ii) components have exponential reliability functions given by (12)-(13),

then the multi-state system reliability function is given by the formula

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)=[1,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,1),\mathrm{h},\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z)], \qquad (18)$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)=1 \text{ for } t<0,$$

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)=\left[\sum_{v=0}^{l_n-1}\frac{(l_n\lambda(u)t)^v}{v!}\exp[-l_n\lambda(u)t]\right]^{k_n}$$

for $t\geq 0,\ u=1,\mathrm{h},z.$ (19)

3. Asymptotic approach

The investigation of exact reliability functions of large systems often leads to complicated formulae. Thus, from practical point of view, the asymptotic approach to large systems reliability evaluation is very important. The suggested method allows us to obtain formulae that simplify optimising calculations.

In the paper in the asymptotic approach to the reliability evaluation of multi-state parallel-series systems the linear standardization of their lifetimes in the reliability state subsets is used. This approach relies on an investigation of limit distribution of a standardized random variable $(T(u)-b_n(u))/a_n(u),$ $u=1,2,\mathrm{h},z,$ where $T(u)$ is the system lifetime in the state subset $\{u,u+1,\mathrm{h},z\}$ and $a_n(u)>0,$ $b_n(u)\in(-\infty,\infty),$ are called normalizing constants. For that reason, we assume the following definition [4].

Definition 7. A reliability function

$$\Re(t,\cdot)=[1,\Re(t,1),\mathrm{h},\Re(t,z)],\ t\in(-\infty,\infty),$$

is called a multi-sate limit reliability function of a system with reliability function

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)=[1,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,1),\mathrm{h},\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z)],$$

if there exist normalizing constants $a_n(u)>0,$ $b_n(u)\in(-\infty,\infty)$ such as

$$\lim_{n\to\infty}\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(a_n(u)t+b_n(u),u)=\Re(t,u) \text{ for } t\in C_{\Re(u)},$$

where $C_{\Re(u)}$ is the set of continuity points of $\Re(t,u),\ u=1,2,\mathrm{h},z.$

From *Definition 7* it follows that the knowledge the limit reliability function of a system $\Re(t,\cdot)$ for sufficiently large *n* allows us to estimate the multi-state reliability function of a system $\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)$ according the following approximate formula

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)\cong\Re((t-b_n(u))/a_n(u),\cdot), \qquad (20)$$

i.e.

$$[1, \overline{\boldsymbol{R}}_{k_n,l_n}(t,1), \text{h}, \overline{\boldsymbol{R}}_{k_n,l_n}(t,z)]$$

$$\cong [1, \Re(\frac{t-b_n(1)}{a_n(1)},1), \text{h}, \Re(\frac{t-b_n(z)}{a_n(z)},z)],\ t \in (-\infty,\infty).$$

3.1. Reliability evaluation of large parallel-series systems with independent components

In this point the possibilities of multi-state asymptotic approach to the reliability evaluation of large parallel-series systems are presented. There are formulated two propositions that allow us to find multi-state limit reliability functions of homogeneous parallel-series systems with independent components under the assumption that they have exponential reliability functions.

Proposition 3. If in a homogeneous regular multi-state parallel-series system

(i) components failure independently,

(ii) components have exponential reliability functions given by (12)-(13),

(iii) $k_n = n,\ l_n - c\log n >> s,\ c > 0,\ s > 0,$

(iv) $a_n(u) = \dfrac{1}{\lambda(u)\log n},\ b_n(u) = \dfrac{1}{\lambda(u)}\log\dfrac{l_n}{\log n},$
$u = 1,2,\text{h},z,$

then

$$\overline{\Re}_3(t,\cdot) = [1, \overline{\Re}_3(t,1), \text{h}, \overline{\Re}_3(t,z)],\ t \in (-\infty,\infty), \quad (21)$$

where

$$\overline{\Re}_3(t,u) = \exp[-\exp t] \text{ for } u = 1,\text{h},z, \quad (22)$$

is the multi-state limit reliability function of considered system.

Proposition 4. If in a homogeneous regular multi-state parallel-series system

(i) components failure independently,

(ii) components have exponential reliability functions given by (12)-(13),

(iii) $k_n \to k,\ k > 0,\ l_n \to \infty,$

(iv) $a_n(u) = \dfrac{1}{\lambda(u)},\ b_n(u) = \dfrac{\log l_n}{\lambda(u)},\ u = 1,2,\text{h},z,$

then

$$\overline{\Re}_{10}(t,\cdot) = [1, \overline{\Re}_{10}(t,1), \text{h}, \overline{\Re}_{10}(t,z)],\ t \in (-\infty,\infty), \quad (23)$$

where

$$\overline{\Re}_3(t,u) = [1 - \exp[-\exp[-t]]]^k \text{ for } u = 1,\text{h},z, \quad (24)$$

is the multi-state limit reliability function of considered system.

3.2. Reliability evaluation of large parallel-series systems with dependent component failures

The proofs of presented below propositions are given in [3].

Proposition 5. If in a homogeneous regular multi-state parallel-series system

(i) components failure in dependent way according to (16),

(ii) components have exponential reliability functions given by (12)-(13),

(iii) $\overline{\boldsymbol{R}}_{k_n,l_n}(t,\cdot)$ is a multi-state system reliability function given by (18)-(19),

(iv) $k_n \to k$ = constant as $n \to \infty,$

(v) $l_n = n,$

(vi) $a_n(u) = \dfrac{1}{\lambda(u)\sqrt{n}},\ b_n(u) = \dfrac{1}{\lambda(u)}$ for $u = 1,\text{h},z,$

then

$$\overline{\Re}_2(t,\cdot) = [1, \overline{\Re}_2(t,1), \text{h}, \overline{\Re}_2(t,z)],\ t \in (-\infty,\infty), \quad (25)$$

where

$$\overline{\Re}_2(t,u) = [1 - F_{N(0,1)}(t,u)]^k \text{ for } u = 1,\text{h},z, \quad (26)$$

and

$$F_{N(0,1)}(t,u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp[-\frac{t^2}{2}]dt,\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of considered system.

According to *Proposition 5* and by (20) we can get the approximate formula for the exact multi-state reliability function of the considered parallel-series system

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)=[1,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,1),\ldots,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)\cong\overline{\Re}_2\left(\frac{t-b_n(u)}{a_n(u)},u\right)$$

$$=[1-F_{N(0,1)}(\sqrt{n}\lambda(u)t-\sqrt{n},u)]^k \qquad (27)$$

for $t\in(-\infty,\infty)$, $u=1,\ldots,z$.

Proposition 6. If in a homogeneous regular multi-state parallel-series system

(i) components failure in dependent way according to (16),

(ii) components have exponential reliability functions given by (12)-(13),

(iii) $\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)$ is a multi-state system reliability function given by (18)-(19),

(iv) $k_n\to\infty$ as $n\to\infty$,

(v) $l_n=n$,

(vi) $n^{-1/3}\log k_n\to 0$ as $n\to\infty$,

(vii) $a_n(u)=\dfrac{1}{\lambda(u)\sqrt{2n\log k_n}}$,

$$b_n(u)=\frac{\log(4\pi)+\log\log k_n-4\log k_n}{\lambda(u)\sqrt{8n\log k_n}}+\frac{1}{\lambda(u)}$$

for $u=1,2,\ldots,z$,

then

$$\overline{\Re}_4(t,\cdot)=[1,\overline{\Re}_4(t,1),\ldots,\overline{\Re}_4(t,z)],\quad t\in(-\infty,\infty), \qquad (28)$$

where

$$\overline{\Re}_4(t,u)=\exp[-\exp t],\ u=1,2,\ldots,z, \qquad (29)$$

is the multi-state limit reliability function of considered system.

According to *Proposition 6* and by (20) we can get the approximate formula for the exact multi-state reliability function of the considered parallel-series system

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,\cdot)=[1,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,1),\ldots,\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,z)],\ t\in(-\infty,\infty),$$

where

$$\overline{\boldsymbol{R}}_{\boldsymbol{k}_n,\boldsymbol{l}_n}(t,u)\cong\overline{\Re}_4\left(\frac{t-b_n(u)}{a_n(u)},u\right)$$

$$=\exp[-\exp[(\lambda(u)t-1)\sqrt{2n\log k_n}$$

$$-\frac{1}{2}\log(4\pi)-\frac{1}{2}\log\log k_n+2\log k_n]] \qquad (30)$$

for $t\in(-\infty,\infty)$, $u=1,2,\ldots,z$.

4. Application

The obtained theoretical results can be applied to the reliability evaluation of real technical systems that are composed of large number of components with failure dependency. In this part, the ship-rope elevator used in the Naval Shipyard in Gdynia is considered and its reliability analysis is performed. The elevator is composed of a steel platform-carriage moved with 10 rope-hoisting winches fed by separate electric motors. The exact and limit multi-state reliability functions, the mean values and standard deviations of this system lifetimes in the reliability state subsets, the mean values of the system lifetimes in particular reliability states and the system risk function are determined.

In our further analysis we will discuss the reliability of the rope system only. The system under consideration is in order if all its ropes do not fail. Thus we may assume that it is a series system composed of 10 components. Each of the ropes is composed of 22 strands. According to rope reliability data given in their technical certificates and experts' opinions [5] based on the nature of strand failures the following reliability states have been distinguished:

state 3 – a strand is new, without any defects,

state 2 – the number of broken wires in the strand is greater than 0% and less than 25% of all its wires, or corrosion of wires is greater than 0% and less than 25%,

state 1 – the number of broken wires in the strand is greater than or equal to 25% and less than 50% of all its wires, or corrosion of wires is greater than or equal to 25% and less than 50%,

state 0 – otherwise (a strand is failed).

We consider the strands as basic components of the system. The system of ropes is in the reliability state subset $\{u, u+1, ..., z\}$, $u = 1,2,3$, when all of its ropes are in this state subset and each of the ropes is in the reliability state subset $\{u, u+1, ..., z\}$, $u = 1,2,3$, if at least one of 22 strands is in this state subset. Thus, according to *Definition 6* we conclude that the rope elevator is a regular 4-states parallel-series system composed of k_n = 10 series-linked subsystems (ropes) with l_n = 22 parallel-linked components (strands).
It has been assumed that the strands have exponential reliability functions:

$$R(t,\cdot) = [1, R(t,1), R(t,2), R(t,3)] \text{ for } t \in (-\infty, \infty),$$

$$R(t,u) = 1 \text{ for } t < 0, \ u = 1,2,3,$$

$$R(t,1) = \exp[-0.1613t], \ R(t,2) = \exp[-0.2041t],$$

$$R(t,3) = \exp[-0.2326t] \text{ for } t \geq 0.$$

4.1. The ship-rope elevator as a system with independent components

Discussed in the paper the shipyard rope transportation system under the assumption that its components are independent is also widely described and analysed in [4].
Applying *Proposition 1* the exact multi-state reliability function of the elevator under the assumption that its components are independent is given by the formula

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (31)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,1) = [1 - [1 - \exp[-0.1613t]]^{22}]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) = [1 - [1 - \exp[-0.2041t]]^{22}]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) = [1 - [1 - \exp[-0.2326t]]^{22}]^{10},$$

for $t \in (-\infty, \infty)$.
The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations counted in years, according to (5)-(7), are:

$$M(1) \cong 13.434, \ M(2) \cong 10.617, \ M(3) \cong 9.316,$$

$$\sigma(1) \cong 2.106, \ \sigma(2) \cong 1.597, \ \sigma(3) \cong 1.360.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$$\overline{M}(1) \cong 2.817, \ \overline{M}(2) \cong 1.301, \ \overline{M}(3) \cong 9.316.$$

Assuming that a critical reliability state of the rope elevator is r = 2, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= 1 - [1 - [1 - \exp[-0.2041t]]^{22}]^{10}, \ t \in (-\infty, \infty).$$

The moment when the system risk exceeds the permitted level e.g. $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 9 \text{ years and 212 days.}$$

Since the number of parallel subsystems in the system is $k_n = 10$ and the number of components in each subsystem is $l_n = 22$, then taking into account that $l_n = 22 >> \log k_n = \log 10 \cong 2.3$, it seems reasonable to apply in the elevator's reliability evaluation either *Proposition 2* or *Proposition 3*. First applying *Proposition 2* we conclude that multi-state limit reliability function of the elevator is of the form

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (32)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,1) \cong \exp[-\exp[0.3714t - 5.1969]],$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) \cong \exp[-\exp[0.4699t - 5.1969]],$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) \cong \exp[-\exp[0.5356t - 5.1970]],$$

for $t \in (-\infty, \infty)$.

The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations counted in years, according to (5)-(7), are:

$$M(1) \cong 12.453, \ M(2) \cong 9.843, \ M(3) \cong 8.636,$$

$$\sigma(1) \cong 3.199, \ \sigma(2) \cong 2.487, \ \sigma(3) \cong 2.158.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$\overline{M}(1) \cong 2.610,\ \overline{M}(2) \cong 1.207,\ \overline{M}(3) \cong 8.636.$

If a critical reliability state of the rope elevator is $r = 2$, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= 1 - \exp[-\exp[0.4699t - 5.1969]],\ t \in (-\infty, \infty).$$

The moment when the system risk exceeds the permitted level e.g. $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 6 \text{ years}.$$

Next applying *Proposition 3* we get

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (33)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,1) \cong [1 - \exp[-\exp[-0.1613t + 3.0911]]]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) \cong [1 - \exp[-\exp[-0.2041t + 3.0910]]]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) \cong [1 - \exp[-\exp[-0.2326t + 3.0911]]]^{10},$$

for $t \in (-\infty, \infty)$.

The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations counted in years, according to (5)-(7), are:

$$M(1) \cong 13.027,\ M(2) \cong 10.295,\ M(3) \cong 9.034,$$

$$\sigma(1) \cong 2.300,\ \sigma(2) \cong 1.758,\ \sigma(3) \cong 1.506.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$$\overline{M}(1) \cong 2.732,\ \overline{M}(2) \cong 1.261,\ \overline{M}(3) \cong 9.034.$$

If a critical reliability state of the rope elevator is $r = 2$, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= 1 - [1 - \exp[-\exp[-0.2041t + 3.0910]]]^{10},$$

for $t \in (-\infty, \infty)$.

The moment when the system risk exceeds the permitted level e.g. $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 8 \text{ years } 310 \text{ days}.$$

4.2. The ship-rope elevator as a system with dependent component failures

From practical point of view it seems reasonable to consider the shipyard rope transportation system assuming components' dependence. Indeed, while failing some of strands in a rope the load of the remaining not failed ones may be getting larger. Thus, the assumption about dependence of strands is natural and justified.

After considering *Proposition 4* the exact multi-state elevator reliability function is given by the formula

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (34)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,u) = 1 \text{ for } t < 0,\ u = 1,2,3,$$

$$\overline{\boldsymbol{R}}_{10,22}(t,1) = [\sum_{j=0}^{21} \frac{(3.5486t)^j}{j!} \exp[-3.5486t]]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) = [\sum_{j=0}^{21} \frac{(4.4902t)^j}{j!} \exp[-4.4902t]]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) = [\sum_{j=0}^{21} \frac{(5.1172t)^j}{j!} \exp[-5.1172t]]^{10},$$

for $t \geq 0$.

The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations, according to (5)-(7), in years are:

$$M(1) \cong 4.335,\ M(2) \cong 3.426,\ M(3) \cong 3.006,$$

$$\sigma(1) \cong 0.773,\ \sigma(2) \cong 0.472,\ \sigma(3) \cong 0.414.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$$\overline{M}(1) \cong 0.909,\ \overline{M}(2) \cong 0.420,\ \overline{M}(3) \cong 3.006.$$

If a critical reliability state of the rope elevator is $r = 2$, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= \begin{cases} 0, & t < 0, \\ 1 - [\sum_{j=0}^{21} \frac{(4.4902t)^j}{j!} \exp[-4.4902t]]^{10}, & t \geq 0. \end{cases}$$

The moment when the system risk exceeds the permitted level e.g. $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2 \text{ years and 230 days.}$$

In the asymptotic approach to the reliability evaluation of the rope elevator assuming component failure dependency similarly as in the first case we can apply either *Proposition 5* or *Proposition 6*. Applying *Proposition 5* we can find the approximate multi-state reliability function of the rope elevator system.

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (35)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,1) \cong [1 - F_{N(0,1)}(0.7566t - 4.6904)]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) \cong [1 - F_{N(0,1)}(0.9573t - 4.6904)]^{10},$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) \cong [1 - F_{N(0,1)}(1.0901t - 4.6904)]^{10},$$

for $t \in (-\infty, \infty)$.

The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations counted in years, according to (5)-(7), are:

$$M(1) \cong 4.166, \quad M(2) \cong 3.292, \quad M(3) \cong 2.891,$$

$$\sigma(1) \cong 0.776, \quad \sigma(2) \cong 0.613, \quad \sigma(3) \cong 0.538.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$$\overline{M}(1) \cong 0.873, \quad \overline{M}(2) \cong 0.401, \quad \overline{M}(3) \cong 2.891.$$

Assuming that a critical reliability state of the rope elevator is $r = 2$, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= 1 - [1 - F_{N(0,1)}(0.9573t - 4.6904)]^{10}, \; t \in (-\infty, \infty).$$

The moment when the system risk exceeds the permitted level assuming as before $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2 \text{ years 80 days.}$$

Now the system multi-state reliability function is estimated from the formula (13) as an application of *Proposition 6*.

$$\overline{\boldsymbol{R}}_{10,22}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{10,22}(t,1), \overline{\boldsymbol{R}}_{10,22}(t,2), \overline{\boldsymbol{R}}_{10,22}(t,3)], \quad (36)$$

where

$$\overline{\boldsymbol{R}}_{10,22}(t,1) = \exp[-\exp[(0.1613t - 1)\sqrt{44\log 10} + \log(50/\sqrt{\pi \log 10})]],$$

$$\overline{\boldsymbol{R}}_{10,22}(t,2) = \exp[-\exp[(0.2041t - 1)\sqrt{44\log 10} + \log(50/\sqrt{\pi \log 10})]],$$

$$\overline{\boldsymbol{R}}_{10,22}(t,3) = \exp[-\exp[(0.2326t - 1)\sqrt{44\log 10} + \log(50/\sqrt{\pi \log 10})]],$$

for $t \in (-\infty, \infty)$.

The expected values of the elevator lifetimes $T(1)$, $T(2)$, $T(3)$ in the state subsets {1,2,3}, {2,3}, {3} and their standard deviations, according to (5)-(7), in years are:

$$M(1) \cong 4.044, \quad M(2) \cong 3.196, \quad M(3) \cong 2.805,$$

$$\sigma(1) \cong 0.787, \quad \sigma(2) \cong 0.622, \quad \sigma(3) \cong 0.546.$$

Hence, from (9), the elevator mean lifetimes in the particular states in years are

$$\overline{M}(1) \cong 0.848, \quad \overline{M}(2) \cong 0.392, \quad \overline{M}(3) \cong 2.805.$$

If a critical reliability state of the rope elevator is $r = 2$, then from (10) its risk function takes the following form

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}}_{10,22}(t,2)$$

$$= 1 - \exp[-\exp[(0.2041t - 1)\sqrt{44\log 10}$$

$$+ \log(50/\sqrt{\pi \log 10})]] \text{ for } t \in (-\infty, \infty).$$

The moment when the system risk exceeds the permitted level e.g. $\delta = 0.05$, according to (11), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2 \text{ years and 11 days.}$$

Comparing the expected values of the elevator lifetimes in the state subset and the elevator mean lifetimes in the particular states in the case when strands failure in dependent and independent way we can conclude that these values are lower in the first case for about 68% percent for exact reliability functions and for about 67% and 69% for approximate reliability functions.

The obtained results illustrate that the increased load of remaining un-failed components causes shortening the lifetime of these components in a significant way. That fact can be interpreted as a decrease of their reliability much faster then for the systems with independent components. Taking into account the presented ship-rope elevator we can notice that the lifetime in the reliability state subset of the elevator under the assumption that strand failure in dependent way is even about 70% shorten then in the case when strands are independent.

5. Conclusion

In the paper the exact reliability analysis and asymptotic approach to the reliability evaluation of homogeneous multi-state parallel-series systems are presented. For these systems the exact and limit reliability functions and other characteristics both in the case when their components are independent and when they are dependent are determined under the assumption that components of systems have exponential reliability functions.

Introduced in the paper the method of reliability evaluation of large systems relies on application of some approximate methods based on classical asymptotic approach to this issue. The obtained results are concerned with typical systems with regular structure. Applied in the paper analytical methods are successful rather for not very complex systems. In this background it seems to be justified the extension of this issue for systems with less regular structures and use of any other reliability analysis methods.

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Appendix

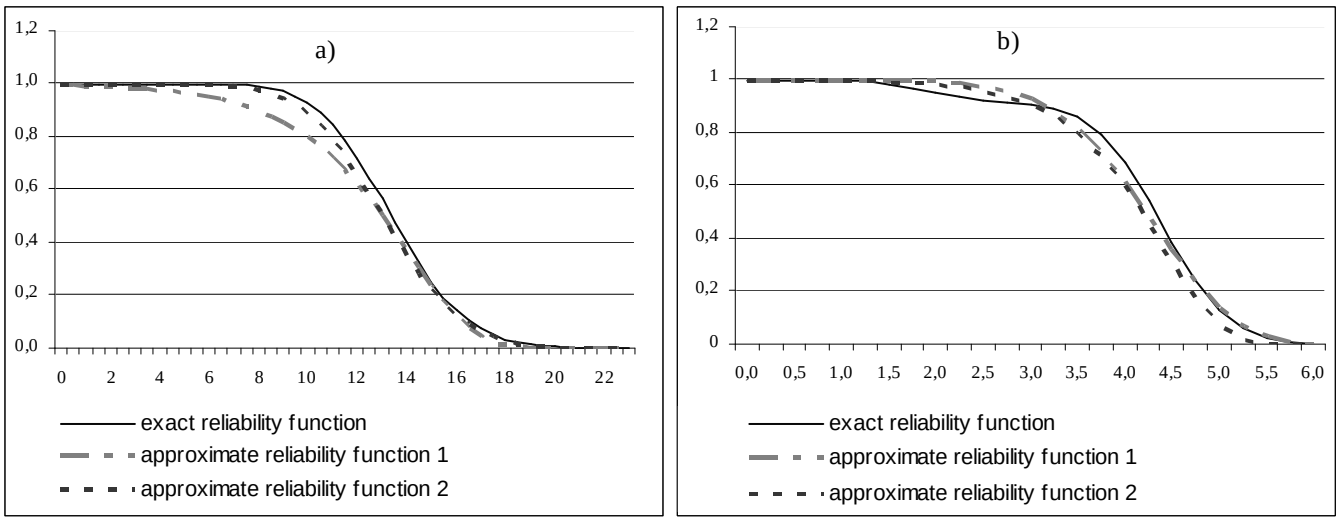

Figure 1. Graphs of the rope elevator exact and approximate reliability functions in the state subset $u \geq 1$
a) in the case when components are independent b) in the case when components fail dependently

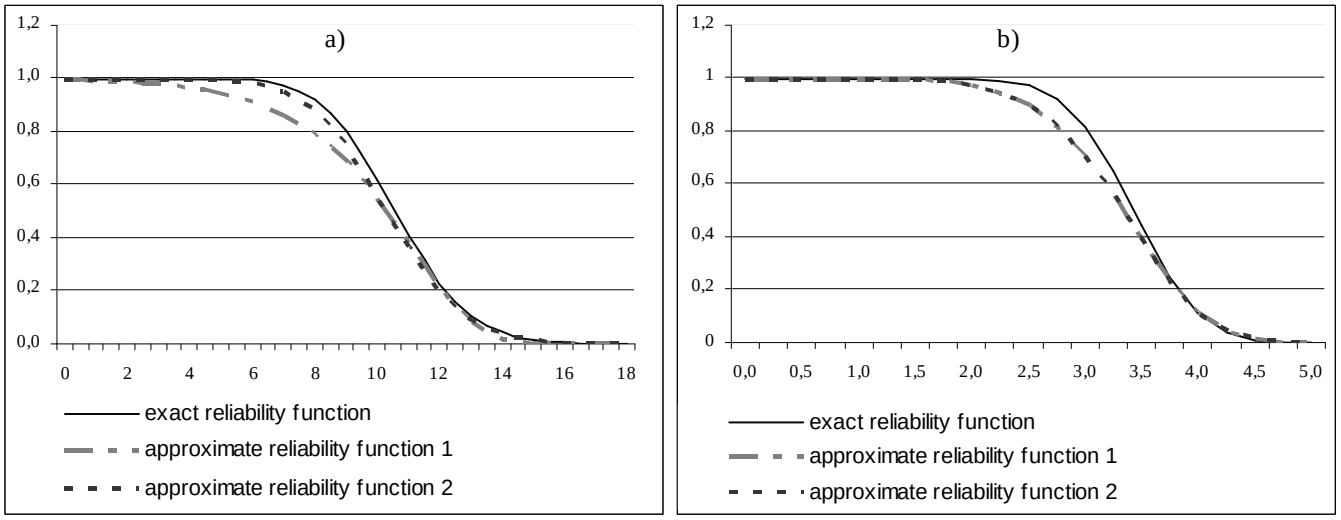

Figure 2. Graphs of the rope elevator exact and approximate reliability functions in the state subset $u \geq 2$
a) in the case when components are independent b) in the case when components fail dependently

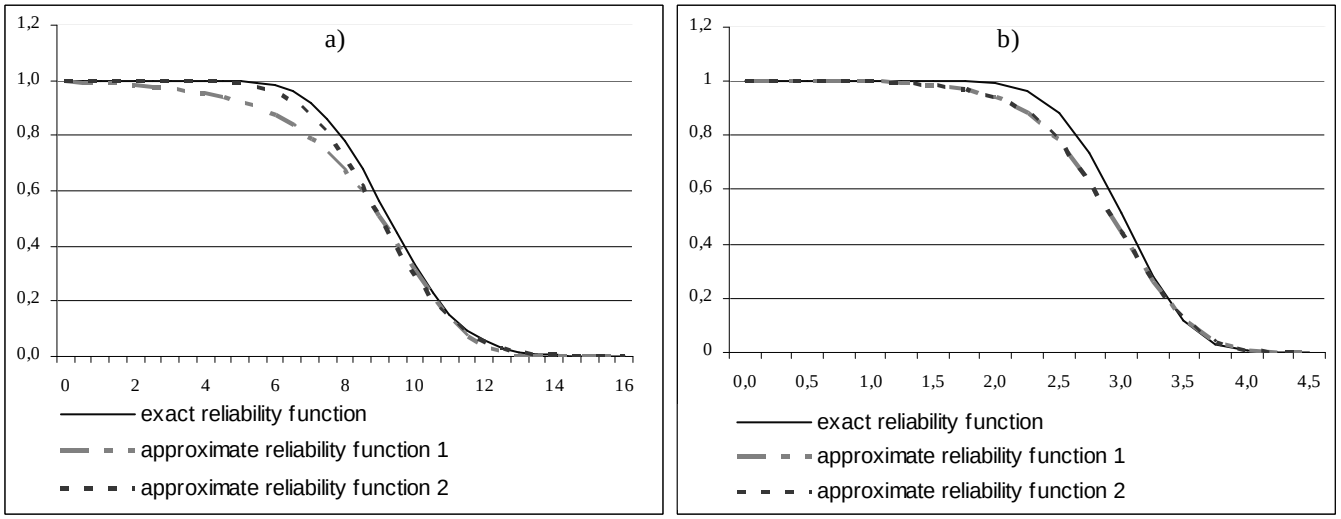

Figure 3. Graphs of the rope elevator exact and approximate reliability functions in the state subset $u = 3$
a) in the case when components are independent b) in the case when components fail dependently

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Stochastic ageing models – extensions of the classic renewal theory

Keywords

renewal theory, alternating renewal theory, maintenance processes, ageing, unavailability coefficient

Abstract

Exact knowledge of the reliability characteristics as the time dependent unavailability coefficient for example, under influence of different ageing processes as well as under different failure types is very useful to the practitioners who have to find the optimal maintenance policy for their equipment. In this paper found models and their solutions have potential to face the optimisation task under the conflicting issues of safety and economics. Most of the solved models take into account ageing processes. An increasing tendency lately exists to include aging effects into the risk assessment models to evaluate its contribution. We developed different renewal models taking into account different ageing distributions of failures (Weibull, Erlang, log-normal): models with negligible renewal time, models with periodical preventive maintenance, alternating renewal process with lognormal distribution of failure time, and with two types of failures.

1. Introduction

This paper mainly concentrates on the modelling of various types of renewal processes and on the computation of principal characteristics of these processes – the time dependent coefficient of availability, possibly unavailability. The aim is to generate models, most often found in practice, which describe the processes of ageing, further the occurrence of dormant failures that are eliminated by periodical inspections as well as monitored failures which are detectable immediately after their occurrence.

Renewal theory seems to be a feasible option to quantify time-dependent effects on component unavailability due to ageing, periodical inspections, or repairs [1]. Closed form solutions for the asymptotic the failure rate and unavailability can be obtained using Laplace transform. Obtaining the detailed time behaviour may not be a trivial numerical task.

Basic information from renewal theory brings Appendix [4], [3]. The following chapter 2 is devoted to models with a negligible renewal time in which a main impact is given on flexible models with the Erlang and Weibull distribution. The solution of these models is received from a Laplace and discrete Fourier transformation. In the following chapter 3 we introduce different models with maintenance. Main attention is paid to models with periodical preventive maintenance - basic equations for the model are formulated. The solution of a system of equations is demonstrated for the situations with an exponential and Weibull distribution. In the next chapter the alternating model with an inconsiderable renewal time is solved, this is demonstrated for lognormal distribution of time to failure. The final part involves generally formulated alternating models with the occurrence of two types of independent failures.

Time-dependent unavailability of components under maintenance and ageing processes can exhibit mathematically complex behaviour [5]. The unavailability may be also dependent on maintenance history. First failure distributions may not be continuous functions. Within this paper we can say that renewal theory provides a feasible approach in selected cases to implement and evaluate interventions given by maintenance and aging processes.

A lot of notable asymptotic results on availability analyses are focused on the situation that the components have exponential lifetime distributions. Using so-called phase-type approach, author in [2] shows that the multi-state model also provides a framework for covering other types of distributions, but with limitations - the approach makes use of the fact that a distribution function can be approximated by a mixture of Erlang distributions (with the same scale parameter). Asymptotic analysis of highly

available systems has been carried out by a number of researchers. A survey is given by Gertsbakh [7], with emphasis on results related to the convergence of the distribution of the first system failure to the exponential distribution.
If the lifetimes are distributed arbitrarily, then the system can be described by a semi-Markov process or Markov renewal process. Semi-Markov processes and Markov renewal processes are based on a marriage of renewal processes and Markov chains. Pyke [8] gave a careful definition and discussion of Markov renewal processes in detail. In reliability, these processes are one of the most powerful mathematical techniques for analysing maintenance and random models. A detailed analysis of the non-exponential case (non-regenerative case) is however outside of the scope of the introduction part. Further research is needed to present formally proved results for the general case. Presently, the literature covers only some particular cases, what is also the case of this presentation.

2. Models with a negligible renewal period

In some cases we can take into account a renewal period equal to zero. For example the situation when a time to a renewal is substantially smaller than a time to a failure and its implementation would not influence an expected result. This case was intensively studied in [6]. Basic relationships for Poisson process are derived in [4]:
Renewal function and renewal density are given as follows

$$H(t) = EN_t = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda t.$$

Basic definitions from renewal theory see in Appendix. Renewal density is constant

$$h(t) = H'(t) = \lambda.$$

Methodology based on Laplace transforms was dramatically extended in [6] for the case when a time to failure is modelled by the Erlang distribution, which has a probability density function

$$f(t) = \frac{\lambda(\lambda t)^a e^{-\lambda t}}{\Gamma(a)}, \quad t \geq 0, \ \lambda \text{ ¤ } 0, \ a \text{ ¤ } 0.$$

After the backward transformation a renewal density is equal to

$$h(t) = \frac{\lambda}{a} + \sum_{k=1}^{a-1} \frac{\lambda + s_k}{a} e^{s_k t}, \ t \text{ ¤ } 0,$$

in the expression there is $s_k \in C$,

which is k[th] nonzero root of the equation

$$(s + \lambda)^a = \lambda^a \qquad s \in C,$$

For example for $a = 4$ nonzero roots are equal to

$$s_1 = \lambda(e^{\frac{i\pi}{2}} - 1) = (-1 + i)\lambda,$$

$$s_2 = \lambda(e^{i\pi} - 1) = -2\lambda,$$

$$s_3 = \lambda(e^{\frac{i3\pi}{2}} - 1) = (-1 - i)\lambda,$$

and a renewal density

$$h(t) = \frac{\lambda}{4} + \sum_{k=1}^{3} \frac{\lambda + s_k}{4} e^{s_k t}$$

$$= \frac{\lambda}{4}[1 - e^{-2\lambda t} - e^{-\lambda t} \sin(\lambda t)].$$

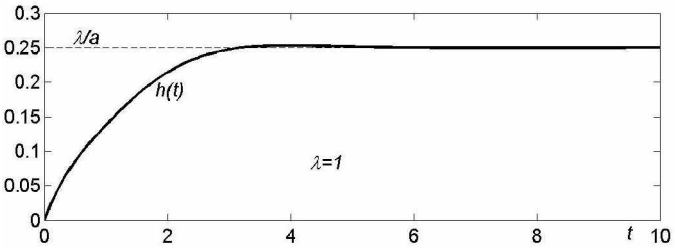

Figure 1. Renewal density for Erlang distribution

Another calculation method was applied for Weibull distribution of time to failure, which has a probability density

$$f(t) = \alpha\lambda(\lambda t)^{\alpha - 1} e^{-(\lambda t)^{\alpha}}, \ t \geq 0,$$

$\alpha > 0$ is a parameter of the shape, $\lambda > 0$ is a parameter of a scale.
A probability density $f_n(t)$, $n = 2,3,\ldots$, or a probability density of time to n[th] failure can be calculated as a convolution of the function ($f_{n-1} * f$). We can express it numerically e.g. with the help of discrete Fourier transformation [6].
By a numerical integration we can determine a vector of a distribution function $\mathbf{F_n}$.
A formula (1) for a calculation of the renewal function

$$H(t) = \sum_{n=0}^{\infty} F_n(t) \qquad (1)$$

is necessary to substitute by a finite sum of the first K computed terms at the numerical calculation. It can be conducted because these terms converge quickly to a zero at a definite interval [0, T]. Equally, $S_n = X_1 + X_2 + \ldots + X_n$ has an asymptotic normal distribution $N(n\mu, n\sigma^2)$ where μ and σ^2 are definite expected value and a dispersion of X_i.
Considering that Weibull distribution of time to failure for $\alpha > 1$ has an increasing failure rate

$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}$$

a distribution function $F_n(t)$ can be upper estimated by the function

$$F_n(t) \le 1 - \sum_{i=0}^{n-1} \frac{(\frac{t}{\mu})^i}{i!} e^{-\mu t} = G_n(t), \; t \to n\mu,$$

where μ is an expected value of a time to failure [3].

$$\mu = \frac{\Gamma(1+\frac{1}{\alpha})}{\lambda}.$$

We can estimate in this way an error of a finite sum

$$H(t) = \sum_{n=0}^{K} F_n(t),$$

because a remainder is limited

$$\sum_{n=K+1}^{\infty} F_{n(t)} \le \sum_{n=K+1}^{\infty} G_n(t), \; t \to n\mu.$$

Exact relationship for the reminder is derived in [6]. The behaviour of the renewal function estimated for different number of members in the above finite sum we can see in *Figure 2*.

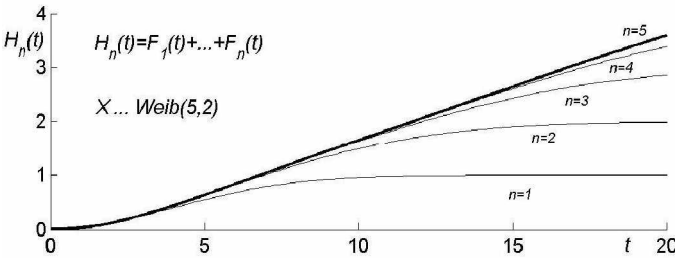

Figure 2. Renewal function for Weibull distribution

3. Models with maintenance

In many situations, failure of a unit during actual operation is costly or dangerous. If the unit is characterized by a failure rate that increases with age, it may be wise to replace it before it has aged too greatly. In this section we shall concentrate on the operating characteristics of some commonly employed replacement policies.
A commonly considered replacement policy is the policy based on age (age replacement). Such a policy is in force if a unit is always replaced at the time of failure or τ_c hours after its installation, whichever occurs first; τ_c is a constant unless otherwise specified. If τ_c is a random variable, we shall refer to the policy as a random age replacement policy. Under a policy of block replacement the unit is replaced at times $k\tau_c$ (k = 1,2,…), and at failure. This replacement policy derives its name from the commonly employed practice of replacing a block or group of units in a system at prescribed times $k\tau_c$(k = 1,2,…) independent of the failure history of the system.

3.1. Replacement based on age

A unit is replaced τ_c hours after its installation or at failure, whichever occurs first; τ_c is considered constant. Let $R(t)$ denote the probability that an item does not fail in service before time t. Then

$$R(t) = R(\tau_c)^n R(t - n\tau_c),$$

$$n \in N \cup \{0\} : n\tau_c \le t < (n+1)\tau_c.$$

The distribution function of a time to failure X is

$$F(t) = 1 - R(t) = 1 - R(\tau_c)^n R(t - n\tau_c),$$

$$t \ge 0, \; n \in N \cup \{0\} : n\tau_c \le t < (n+1)\tau_c.$$

Expected time to failure E(X) is

$$E(x) = \int_0^{\infty} R(x)dx$$

$$= \sum_{n=0}^{\infty} \int_{n\tau_c}^{(n+1)\tau_c} R(\tau_c)^n R(t - n\tau_c)dt$$

$$= \sum_{n=0}^{\infty} R(\tau_c)^n \int_0^{\tau_c} R(t)dt = \frac{1}{F(\tau_c)} \int_0^{\tau_c} R(t)dt,$$

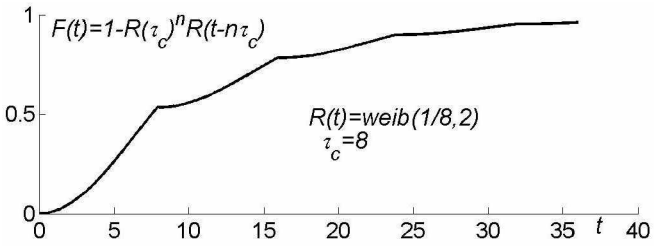

Figure 3. The Weibull distribution function for a unit with the replacement based on age.

When the time to failure is exponentially distributed $X \sim \exp(1/\mu)$, then we have

$$F(t) = 1 - R(\tau_c)^n R(t - n\tau_c)$$

$$= 1 - e^{-n\mu\tau_c} e^{-m\mu(t-n\tau_c)} = 1 - e^{-\mu t},$$

which means that distribution function is independent on replacements. Other words, the unit does not age.

3.2. Block replacement policy

Under a policy of block replacement all components of a given type are replaced simultaneously at times $k\tau_c$ (k = 1,2,…) independent of the failure history of the system. If X_i is time of *i-th* failure of a unit which has distribution function F_i and probability density f_i and $R_i(t)=1- F_i(t)$, then

$$R_1(t) = R(\tau_c)^n R(t - n\tau_c),$$

$$n \in N \cup \{0\}: n\tau_c \le t < (n+1)\tau_c,$$

$$f_1(t) = R(\tau_c)^n \frac{d}{dt}[1 - R(t - n\tau_c)].$$

Distribution of time to *i-th* failure for $i > 1$ we can derive on the basis of conditional probability:

$$x > 0,\ y > 0,\ \ k = [x/\tau_c],\ l = [(x+y)/\tau_c],\ \ l > k,$$

$$\Pr(X_i > y \setminus X_{i-1} = x)$$

$$= R((k+1)\tau_c - x) R(\tau_c)^{(n-(k+1))} R(x + y - n\tau_c).$$

Then

$$\Pr(X_i > y) = \int_0^\infty R((k+1)\tau_c - x) R(\tau_c)^{(l-(k+1)} \cdot R(x + y - n\tau_c) f_{i-1}(x)dx$$

$$= \sum_{k=0}^{\infty} \int_{k\tau_c}^{(k+1)\tau_c} R((k+1)\tau_c - x) R(\tau_c)^{l-(k+1)} \cdot R(x + y - l\tau_c) f_{i-1}(x)dx$$

$$z = x - k\tau_c,\ n = [(z+y)/\tau_c]$$

$$= \sum_{k=0}^{\infty} R_{i-1}(\tau_c)^k \int_0^{\tau_c} R(\tau_c - z) R(\tau_c)^{n-1} \cdot R(z + y - n\tau_c) f_{i-1}(z)dz$$

$$= \frac{R(\tau_c)^{n-1}}{1 - R_{i-1}(\tau_c)} \int_0^{\tau_c} R(\tau_c - z) R(z + y - n\tau_c) f_{i-1}(z)dz.$$

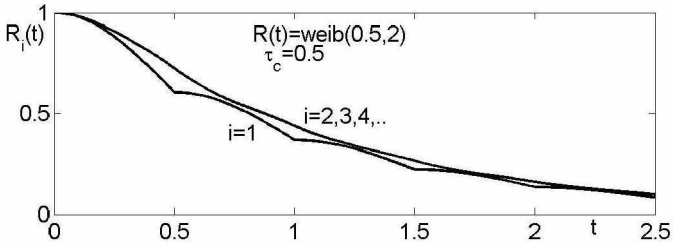

Figure 4. The Weibull distribution function for a unit with block replacement

In *Figure 4*, we can see time dependencies for $R_i(t)$ for Weibull distribution of time to failure.

3.3. Periodical preventive maintenance

May a device goes through a periodical maintenance after a time interval of the operation τ_c, whose intention is a detection of possible dormant flaws and their possible elimination. The period of device maintenance is τ_d and after this period the device starts operating again. $F(t)$ is here a time distribution to a failure X. Then in the interval $[0, \tau_c + \tau_d)$ there is a probability that the device appears in the not operating state equal to

$$P(t) = F(t),\ \ t \to \tau_c$$

$$= 1,\ t \ge \tau_c.$$

The state of a failure is considered then both a time to the maintenance after a possible failure and the time when the device is under maintenance. The probability $P(t)$ (also a coefficient of unavailability) for $t \in [0, \infty)$ is generated by following system of equations:

$$P(t) = P_n(t),$$
$$n \in N \cup \{0\}: n(\tau_c + \tau_d) \to t \le (n+1)(\tau_c + \tau_d),$$

…

$$P_i(t) = P_{i-1}(t) - P_{i-1}(i\tau_c)[1 - P_0(t - i(\tau_c + \tau_d))]$$

$$i = 1,2,...,n-1,$$

…

$$P_0(t) = F(t), \quad t \text{ ¤ } 0.$$

Here $P_i(t)$ stands for a probability that a device exists in the failure state provided that before it had gone through i inspections. A term $P_{i-1}(t)$ - $P_{i-1}(i\tau_c)$ represents a probability that a device was all right at the previous inspection and it failed in the interval $(i\tau_c, t)$, $P_{i-1}(i\tau_c)$ $P_0(t - i(\tau_c + \tau_d))$ is a probability that it had failed in the previous inspection and since then it failed again.

3.3.1. Exponential distribution of time to failure

May

$$F(t) = 1 - e^{-\lambda t}.$$

For the time $t \in [0, \infty)$ is then a probability $P(t)$ equal to

$$P(t) = F(t - n(\tau_c + \tau_d)),$$

$$t \in [n(\tau_c + \tau_d), n(\tau_c + \tau_d) + \tau_c),$$

$$P(t) = 1, \; t \in [n(\tau_c + \tau_d) + \tau_c, (n+1)(\tau_c + \tau_d)),$$

If $\tau_d = 0$, the expression for $P(t)$ can be further simplified into the form

$$P(t) = F(t - n\tau_c),$$

$$n \in N \cup \{0\} : n(\tau_c + \tau_d) \le t \text{ ® } (n+1)(\tau_c + \tau_d).$$

3.3.2. Weibull distribution of time to failure

Let the intensity of failures of the given distribution of time to failure is not constant, it is then a function of time past since the last renewal. In this case it is necessary for the given t and n, related with it which sets a number of done inspections to solve above mentioned system of n equations and the solution of the given system is not eliminated anyhow as it is at an exponential distribution.
In the *Figures 5* and *Figure* 6 a slightly marked curve draws points of the local extremes in the case of shortening a time to the first inspection.

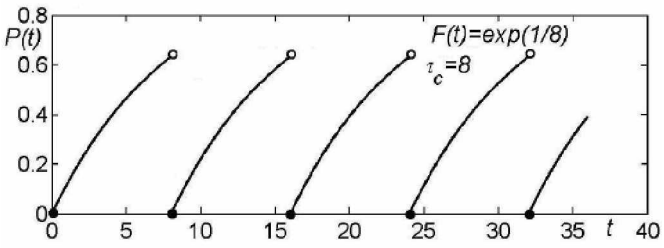

Figure 5. Coefficient of unavailability for exponential distribution

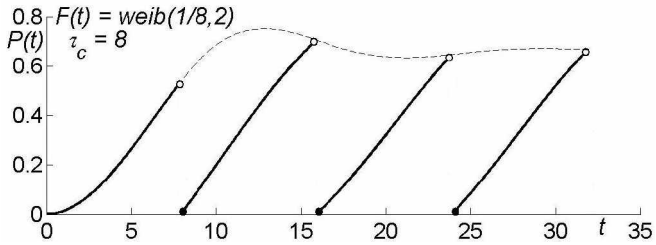

Figure 6. Coefficient of unavailability for Weibull distribution.

4. Alternating renewal models

Alternating models are those where two of the significantly diverse states appear, between which a model converts from one to another. A faulty device is the example of the alternating model where a time to a repair is compared with a time to failure and it cannot be neglected.
In the case that both a time to failure and a time to a repair follows an exponential distribution, general solution for a calculation of a coefficient of availability can be found in [4].

4.1. Lognormal distribution of a time to failure

If a distribution to a failure X_f has a lognormal distribution, then a probability density is in the form

$$f_f(t) = \frac{1}{\sigma t}\varphi\left(\frac{\ln t - \lambda}{\delta}\right), \; t \text{ ¤ } 0$$

where $\varphi(x)$ is standard normal density.
In this case a numerical calculation is offered again for the computation of the coefficient of availability.
We can compute a probability density of a sum of random quantities X_f *and* X_r (X_r is an exponential time to a repair) from a discrete Fourier transformation [6], equally as a convolution in the equation

$$K(t) = R_f(t) + \int_0^t h(x) R_f(t - x)dx.$$

The calculation of a renewal density is substituted by a finite sum

$$h(t) = \sum_{n=1}^{N} f_n(t).$$

An example: In the following example a calculation for parameter values σ=1/4, λ=8σ, τ=1/2, is done. In the *Figure 7* there is a renewal density. The asymptotic value is marked by dots, which is in this case equal to

$$\lim_{t\to\infty} h(t) = \frac{1}{EX_f + EX_r} = \frac{1}{e^{-\lambda+\frac{1}{2}\sigma^2} + \tau} \cong 0.123$$

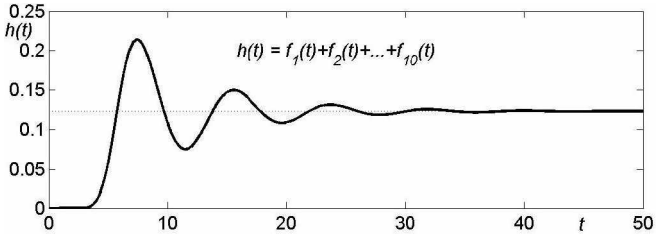

Figure 7. A renewal density for lognormal distribution

Figure 8 shows a procedure of the coefficient of availability *K(t)*, the asymptotic value is marked by dots again and it is given by the following formula:

$$K = \lim_{t\to\infty} K(t) = \frac{1}{EX_f + EX_r} \cong 0.938$$

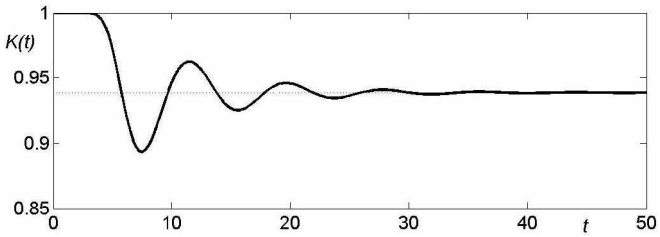

Figure 8. Coefficient of availability for a lognormal distribution

5. Alternating renewal models with two types of failures

The following part presents models, which consider an appearance of two different independent failures. These failures can be described by an equal distribution with different parameters or by different distributions.

5.1. Common repair

A device composed of two serial elements can be an example whereas a failure of one of them causes a failure of the whole device. A time to a renewal is common for both the failures and begins immediately after one of them. It is described by an exponential distribution with a mean value 1/τ.

A failure occurrence in the renewal time is not taken into account, after the renewal both the parts are considered to be new.

May X_{f1} and X_{f2} are independent random values describing time of failures with probability densities $f_{f1}(t)$ and $f_{f2}(t)$, further a time to a repair is X_r with a density $f_r(t)$. A probability that no failure occurs in the interval [0,t) is equal to

$$R_f(t) = P(X_{f1} \geq t \wedge X_{f2} \geq t)$$

$$= [1 - F_{f1}(t)][1 - F_{f2}(t)] = R_{f_1}(t) R_{f2}(t).$$

and is a reliability function of the time to failure X_f of the whole device. Then $X_{f\,has}$ a probability density

$$f_f(t) = -\frac{d}{dt} R_f(t).$$

With the knowledge $f_f(t)$ we can calculate the functions describing this alternating process.

If the time to failure has an exponential distribution with mean values 1/λ and 1/μ, then

$$f_f(t) = -\frac{d}{dt} e^{-(\lambda+\mu)t} = (\lambda+\mu) e^{-(\lambda+\mu)t}.$$

Then $f_f(t)$ has an exponential distribution with a mean value 1/(λ+μ) and the coefficient of availability is equal

$$K(t) = \frac{\tau}{\lambda+\mu+\tau} + \frac{\lambda+\mu}{\lambda+\mu+\tau} e^{-(\lambda+\mu+\tau)t},\ t \text{ ¤ } 0.$$

If the analytical procedure is uneasy or impossible, a numerical calculation can be used. For a renewal density computation is desirable instead of the equation

$$h(t) = f_f(t) + \int_0^\infty h(x) f_f(t-x) dx$$

use a renewal equation for a renewal density

$$h(t) = \sum_{n=1}^{\infty} f_n(t)$$

and conduct a sum of the only definite number of elements with a fault stated above. $f_n(t)$ is a probability density of time to n^{th} failure. Then for the calculation of convolutions is used for example a quick discrete Fourier's transformation.

In the *Figure 9* there is a graph of a coefficient of availability in the case that a time to a failure X_{f1} and

X_{f2} have Weibull distribution. Expected value to the failure EX_f is equal to

$$EX_f = \sqrt{\frac{EX_{f1}^2 EX_{f2}^2}{EX_{f1}^2 + EX_{f2}^2}} = \sqrt{\frac{2^2 6^2}{2^2 + 6^2}} = \sqrt{3.6}$$

and that is why an asymptotic coefficient of availability is equal to

$$K = \frac{EX_f}{EX_f + EX_r} = 0.79$$

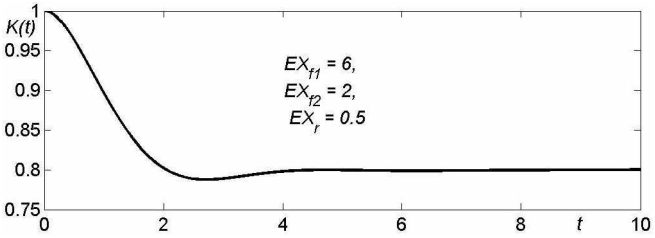

Figure 9. Coefficient of availability for Weibull distribution

5.2. Two independent parts

Supposing the device consists of two independent parts. The behaviour of each one is described by its alternating model with a given time to a failure and a time to a repair. Maintenance proceeds for both differently and independently. Equally, the failure of one of them can appear regardless of the state of the other part, even in the state of a failure.

Let us consider the whole device to be in the state of a failure when at least one of the parts is in the state of a failure. $K_a(t)$ is a coefficient of availability of the first part and $K_bt)$ is a coefficient of availability of the second one. For the whole device $K(t)$ is equal to

$$K(t) = K_a(t) K_b(t).$$

May dormant faults occur in the first part, with Weibull's distribution and with an expected value $EX = 2$ and a parameter of the form $\alpha = 2$ which are eliminated by periodical inspections with a period $\tau_c = 2$ (See Models with periodical preventive maintenance) and the second part is equal as in the previous model. The course of the coefficient of availability as the product of already computed partial ones is designed in the *Figure 10*.

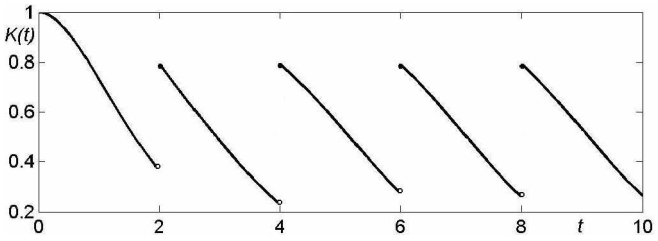

Figure 10. Coefficient of availability for independent parts

6. Conclusion

In this paper a few types of renewal processes, which differentiate in a renewal course and a type of probability distribution of a time to failure, were described. These processes were mathematically modelled by the means of a renewal theory and these models were subsequently solved.

In the cases, when the solving of integral equations was not analytically feasible, numerical computations were successfully applied. It was known from the theory that the cases with the exponential probability distribution are analytically easy to solve.

With the gained results and gathered experience it would be possible to continue in modelling and solving more complex mathematical models which would precisely describe real problems. For example by the involvement of certain relations which would specify the emergence, or a possible renewal of individual types of failures which in reality do not have to be independent on each other. Equally, it would be practically efficient to continue towards the calculation of optimal maintenance strategies with the set costs connected with failures, exchanges and inspections of individual components of the system and determination of the expected number of these events at a given time interval.

Acknowledgements

This research is supported by The Ministry of Education, Youth and Sports of the Czech Republic. Project CEZ MSM6198910007.

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Appendix

Renewal Process

Renewal process serves for example to model mathematically a device behaviour which is maintained in such a way that it stays running as most effectively and longest as possible. May a file of components or the whole device with a time to a failure *X* (non-negative absolutely continuous random variable) with a dispersion given by a probability density *f(t)* exists and may a symbol *t* denotes for clearness a time. The first component is put into operation at time $t = 0$. Further, X_1 is a period when the first component comes to the failure and at the same time it is substituted by a new identical component from a given file. It means that a renewal period (in this case a change period) is negligible, or equal to zero. This second component breaks down after the period X_2 since it started to operate. At the time $X_1 + X_2$ the second component is renewed by the exchange for the third one and the process continues further in such a way. The *r*-th renewal will happen at the time $S_r = X_1 + X_2 + \ldots + X_r$.
If $X_1, X_2 \ldots$ are independent non-negative equally distributed random variables with a finite expected value and dispersion,

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, n \in N,$$

then a random process $\{S_n\}_{n=0}^{\infty}$ is called a renewal process in a renewal theory. Sometimes an order of stated random variables $\{X_n\}^{\infty}{}_{n=0}$ is denoted in this way. In the case when a time distribution until the failure is exponential, we speak about Poisson process. A function $F_n(t)$ indicates a distribution function of a random variable S_n. There are a few other random variables connected with the renewal process, which describe its behaviour (at time). Let we call N_t *a number of renewals in the interval* [0, t] for a firm $t \geq 0$, it means

$$N_t = \max\{n : S_n \leq t\}$$

From this we also get that $S_{Nt} \leq t < S_{Nt+1}$. Regarding the fact that the interval [0, t] contains *n* failures (as well as renewals) only if n^{th} failure happens at the latest at the time *t*

$$P\{N_t \geq n\} = P\{S_n \leq t\} = F_n(t)$$

and the probability that at the time *t* there are *n* renewals in the given renewal process can be described in the following way

$$P\{N_t = n\} = P\{S_n \leq t \wedge S_{n+1} \geq t\}$$

$$= F_n(t)[1 - F_{n+1}(t)] = F_n(t) - F_{n+1}(t)$$

Provided that $X_1, X_2 \ldots$ are independent non-negative equally distributed random variables and $P_r(X_1 = 0) < 1$, then a random variable N_t has finite moments of all the series (Stein`s theorem).
And if N_t , $t \geq 0$ gives a number of renewals in the interval [0, t], then a function

$$H(t) = EN_t, \quad t \geq 0$$

is called *a renewal function.* As it is apparent it gives an expected number of renewals in the interval [0, t]. The expected number of renewals in the interval $[t_1, t_2]$, $0 < t_1 < t_2$ can be quantified from $H(t_1) - H(t_2)$, because a number of renewals in this interval is $N_{t2} - N_{t1}$.
A renewal function can be also expressed from distributional functions $F_n(t)$ of random variables S_n

$$H(t) = \sum_{n=0}^{\infty} nP(N_t = n)$$

$$= \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)] = \sum_{n=1}^{\infty} F_n(t).$$

A renewal equation is important for the renewal function computation *H(t)*. It provides a mutual unique relation between distributional function of a time to a renewal and a renewal function: if a distributional function of a time to the renewal *F(t)* is continuous, then a renewal function *H(t)* is convenient with an integral equation

$$H(t) = F(t) + \int_0^t H(t-u)F(u)du.$$

This equation can be easily derived from the previous equation with help of its integral transformation (e.g. Laplace).
An asymptotic behaviour of a renewal process is substantial. An *asymptotic* behaviour of a renewal process is discussed in an *Elementary theorem about a renewal:* if a time distribution to a renewal has a finite expected value μ, then

$$\lim_{t\to\infty}\frac{H(t)}{t}=\frac{1}{\mu}.$$

It is *a Blackwell theorem, which* testifies about a limited behaviour of an expected number of renewals at a finite interval $(t, t+\Delta t]$: if a time to a renewal has a non-lattice distribution with a definite positive expected value μ, then $\forall h \¤ 0$ is

$$\lim_{t\to 0}\left[H(t+h)-H(t)\right]=\frac{h}{\mu}.$$

If a derivation of a renewal function exists (i.e. X_1, X_2...are absolutely continuous random variables), then for the arbitrary time $\boldsymbol{t > 0}$ a function $h(t)$ that is defined by a relation

$$h(t)=\lim_{\Delta t\to 0+}\frac{H(t)-H(t+\Delta t)}{\Delta t}=H'(t)$$

is a *renewal density*. Then with a help of a probability density $f_n(t) = F'_n(t)$ we have

$$h(t)=\sum_{n=1}^{\infty} f_n(t).$$

A renewal density most often appears in the following integral equation

$$h(t)=f(t)+\int_0^t h(t-u)\,f(u)du,$$

so called a *renewal equation for a renewal density*. Here $f(t)$ is a probability density of a absolutely continuous non-negative time to the renewal X.
We can describe the equation approximately by words in such a way that for $\Delta t\to 0$ renewal probability $h(t)\Delta t$ in the interval $(t, t + \Delta t]$ is equal to a probability sum $f(t)\,\Delta t$ *that* in the interval $(t, t + \Delta t]$ the first renewal happens and the sum of probabilities for $\forall u \in (0,t)$ that the renewal happens at the time $t - u$ followed by a time to the failure of the length u.

Alternating Renewal Process

Provided that there are two kinds of components with various independent time to a failure X,Y , respectively adequate distributional functions $F(t)$, $G(t)$ (densities $f(t)$, $g(t)$), at the time $t = 0$ the component of the first type is activated and every time at the time of failure is substituted by the component of the opposite type, resulting process is named *Alternating renewal process.*
We can simulate a renewal process with a definite time to a renewal with such a model. At the time $t = 0$ the component begins to work to the moment of failure X_1. The final time to the renewal Y_1 follows. At the moment $X_1 + Y_1$ the renewal ends and a new (or repaired) component is activated with a time to a failure X_2. X_1,X_2...resp. Y_1,Y_2...are independent non-negative random variables with a distr. function $F(t)$ resp. $G(t)$.The n^{th} failure happens at the moment

$$S_n = X_1 + Y_1 + ... + X_{n-1} + Y_{n-1} + X_n ,$$

for n^{th} renewal we have

$$T_n = X_1 + Y_1 + ... + X_{n-1} + Y_{n-1} + X_n + Y_n .$$

A random process $\{S_1, T_1, S_2, T_2.....\}$ is then an alternating renewal process. A coefficient of availability $K(t)$ (or also $A(t)$ - availability) is a basic characteristic of a renewal process with a finite time to a renewal. It determines a probability that at the time t the component will work. It is consequently equal to a sum of probabilities that $X_1 > t$, it means that the first component has a time to a failure greater than t, and that the renewal happens in the interval $(u, u + \Delta u]$, $\Delta u \to 0$, $0 < u < t$ and a renewed component will have a time to a failure greater than $t - u$.
Written by an integral equation:

$$\begin{aligned}K(t)&=1-F(t)+\int_0^t h(x)\left[1-F(t-x)\right]dx\\&=R(t)+\int_0^t h(x)R(t-x)dx,\end{aligned}$$

$h(x)$ is a renewal process density of a renewal $\{T_n\}^{\infty}_{n=0}$, $F(t)$ is a distribution function of the time to a failure, resp. $1 - F(t) = R(t)$ is reliability function.
In particular, *an asymptotic coefficient of availability* of the alternating renewal process is important practical reliability characteristics,

$$K=\lim_{t\to\infty}K(t).$$

It describes behaviour of the alternating renewal process in the situation when the system is stabilized in "a distant time moment t", i.e. a stationary case, when the influence of the beginning configuration subsides.

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Two various approaches to VTS Zatoka radar system reliability analysis

Keywords

VTS system, system reliability, shipping safety

Abstract

In the paper we propose two ways of reliability calculation of radar system in Vessel Traffic Services Zatoka. Reliability and availability of the system were calculated on the base of reliability of the system components. In the first approach there was assumed that system is series, in the second approach system is treated as a series-"*m* out of *n*". We obtain different results. Conclusion is that choosing proper method of approach to system reliability and availability analysis is decisive in appropriate evaluation of those properties.

1. Introduction

One of the most important properties of the devices and technical systems is their reliability. Reliability is extremely important when concerns systems, which assure people safety or/and natural environment protection. Vessel Traffic Services System – VTS Zatoka is that type of system. Its main task is to assure safe navigation for all ships that sails to ports of Gdynia and Gdansk. The most important part of the VTS Zatoka system are shore based maritime radars. Reliability of that system can be evaluated in different ways. In the paper there are proposed two possible approaches to calculate that reliability [1].

2. Systems' definitions

We assume that [2]

$$E_i,\ i = 1,2,...,n,\ n \in N,$$

are two-state components of the system having reliability functions

$$R_i(t) = P(T_i > t),\ t \in (-\infty,\infty),\ i = 1,2,...n,$$

where

$$T_i,\ i = 1,2,...,n,$$

are independent random variables representing the lifetimes of components E_i with distribution functions

$$F_i(t) = P(T_i \le t),\ t \in (-\infty,\infty),\ i = 1,2,...,n.$$

Definition 1. A two-state system is called series if its lifetime T is given by

$$T = \min_{1 \le i \le n}\{T_i\}.$$

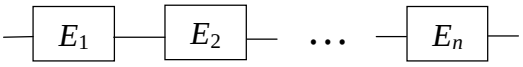

Figure 1. The scheme of a series system

The above definition means that the series system is not failed if and only if all its components are not failed, and therefore its reliability function is given by

$$R(t) = \prod_{i=1}^{n} R_i(t),\ t \in (-\infty,\infty). \quad (1)$$

Definition 2. A two-state series system is called non-homogeneous if it is composed of a, $1 \le a \le n$, different types of components and the fraction of the ith type component in the system is equal to q_i, where $q_i > 0$, $\sum_{i=1}^{a} q_i = 1$. Moreover

$$R^{(i)}(t) = 1 - F^{(i)}(t),\ t \in (-\infty,\infty),\ i = 1,2,...,a, \quad (2)$$

is the reliability function of the ith type component.

The scheme of a non-homogeneous series system is given in *Figure 2*.

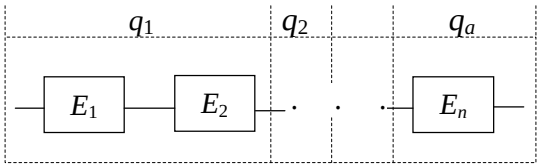

Figure 2. The scheme of a non-homogeneous series system

It is easy to show that the reliability function of the non-homogeneous two-state series system is given by

$$\overline{\mathbf{R}}'_{k_n l_n}(t) = \prod_{i=1}^{a} (R^{(i)}(t))^{q_i n}, t \in (-\infty, \infty). \qquad (3)$$

A two-state system is called an "*m* out of *n*" system if its lifetime *T* is given by

$$T = T_{(n-m+1)}, \ m = 1,2,...,n,$$

where $T_{(n-m+1)}$ is the mth maximal order statistic in the sequence of component lifetimes $T_1, T_2, ..., T_n$.

The above definition means that the two-state "*m* out of *n*" system is not failed if and only if at least m out of its n components are not failed. The two-state "*m* out of *n*" system becomes a parallel system if *m* = 1, whereas it becomes a series system if *m* = *n*. The reliability function of the two-state "*m* out of *n*" system is given either by

$$\mathbf{R}_n^{(m)}(t) = 1 - \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \le m-1}}^{1} \prod_{i=1}^{n} [R_i(t)]^{r_i} [F_i(t)]^{1-r_i}, \qquad (4)$$

$$t \in (-\infty, \infty),$$

or by

$$\overline{\mathbf{R}}_n^{(\overline{m})}(t) = \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \le \overline{m}}}^{1} \prod_{i=1}^{n} [F_i(t)]^{r_i} [R_i(t)]^{1-r_i}, \qquad (5)$$

$$t \in (-\infty, \infty), \ \overline{m} = n - m.$$

Definition 3. A two-state "*m* out of *n*" system is called non-homogeneous if it is composed of *a*, $1 \le a \le n$, different types of components and the fraction of the ith type component in the system is equal to q_i, where qi > 0, $\sum_{i=1}^{a} q_i = 1$. Moreover

$$R^{(i)}(t) = 1 - F^{(i)}(t), \ t \in (-\infty, \infty), \ \text{i} = 1,2,...,\text{a}, \qquad (6)$$

The scheme of a non-homogeneous "*m* out of *n*" system is given in *Figure 3*, where $i_1, i_2, ..., i_n \in \{1,2,...,n\}$ and $i_j \ne i_k$ for $j \ne k$.

The reliability function of the non-homogeneous two-state "*m* out of *n*" system is given either by

$$\overline{\mathbf{R}}'^{(m)}_n(t) = 1 - \sum_{\substack{0 \le r_i \le q_i n \\ r_1 + r_2 + ... + r_a \le m-1}} \prod_{i=1}^{a} \binom{q_i n}{r_i} [R^{(i)}(t)]^{r_i} [F^{(i)}(t)]^{q_i n - r_i}, \qquad (7)$$

$$t \in (-\infty, \infty),$$

or by

$$\overline{\mathbf{R}}'^{(\overline{m})}_n(t) = \sum_{\substack{0 \le r_i \le q_i n \\ r_1 + r_2 + ... + r_a \le \overline{m}}} \prod_{i=1}^{a} \binom{q_i n}{r_i} [F^{(i)}(t)]^{r_i} [R^{(i)}(t)]^{q_i n - r_i}, \qquad (8)$$

$$t \in (-\infty, \infty),$$

where $\overline{m} = n - m$.

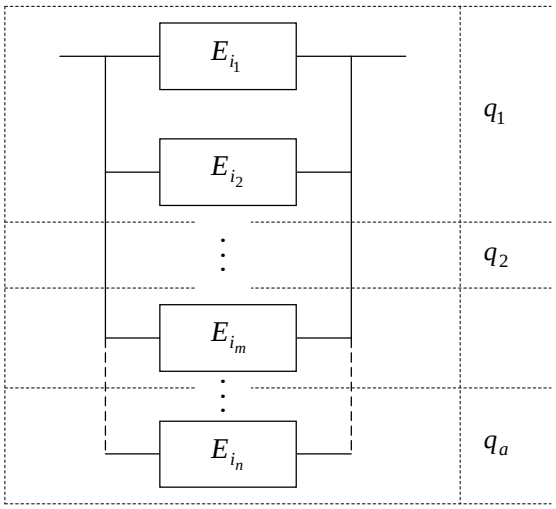

Figure 3. The scheme of a non-homogeneous "*m* out of *n*" system

Definition 4. A multi-state system is called series- "*m* out of k_n" if its lifetime *T* is given by

$$T = T_{(k_n - m + 1)}, \ m = 1,2,...,k_n,$$

where $T_{(k_n - m + 1)}$ is *m*-th maximal statistics in the random variables set

$$T_i = \min_{1 \le j \le l_i} \{T_{ij}\}, \ \text{i} = 1,2,...,k_n.$$

The above definition means that series-" m out of k_n" system is composed of k_n series subsystems and it is not failed if and only if at least m out of its k_n series subsystems are not failed.

The reliability of the series-"m out of k_n" system is given either by

$$R^{(m)}_{k_n,l_1,l_2,\ldots,l_{k_n}}(t) = 1 - \sum_{\substack{r_1,r_2,\ldots,r_{k_n}=0 \\ r_1+r_2+\ldots+r_{k_n}\le m-1}}^{1} \prod_{i=1}^{k_n}[\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[1-\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}, \qquad (9)$$

$$t \in (-\infty, \infty),$$

or by

$$\overline{R}^{(\overline{m})}_{k_n,l_1,l_2,\ldots,l_{k_n}}(t) = \sum_{\substack{r_1,r_2,\ldots,r_{k_n}=0 \\ r_1+r_2+\ldots+r_{k_n}\le \overline{m}}}^{1} \prod_{i=1}^{k_n}[1-\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}, \qquad (10)$$

$$t \in (-\infty, \infty),$$

where $\overline{m} = k_n - m$.

2. System of VTS Zatoka radars

Radars' system is the basic subsystem of whole VTS system and also part of identification and watching system at the Gulf of Gdańsk region. The purpose of

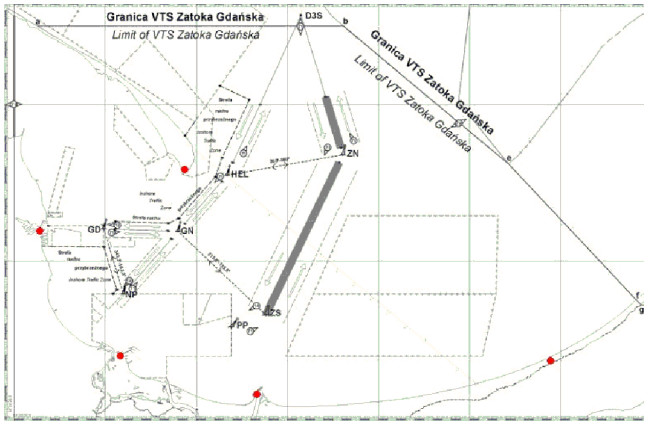

Figure 4. Positions of VTS Zatoka shore based radars (dots)

that system is assuring real time information about ships' traffic in that region [3]. VTS Zatoka system works involving five shores based radars, which are put in following places:

- Lighthouse Hel, (radar height 42,5 m a.s.l.);
- Port of Gdynia Harbourmaster Office building (HMO), (32,5 m a.s.l.);
- Northern Port of Gdansk Harbourmaster Office building, (66 m a.s.l.);
- Western Hills?, (17,5 m a.s.l.);
- Lighthouse Krynica Morska, (26 m a.s.l.).

Radars work permanently and their range cover whole responsibility area assigned to VTS Zatoka. Two, or even three, radars cover most part of Gulf of Gdansk simultaneously. That situation has great matter in case of failure of single radar.

For the VTS Zatoka systems' radars, apart from standard equipment, additional Radar Data Processor (RDP) has been installed. RDP changes radar data from analogue to digital form. This digital information is next transferred to VTS centre by wireless line or light cable, which connect two Harbourmaster's offices of Ports in Gdynia and Gdansk. Signals from radars, after preliminary treatment, are transferred to the VTS Centre. Then after final processing signals are sending out and visualized (with use of computer program ARAMIS) at VTS Centre itself, Harbourmaster's offices of Gdynia and Gdansk ports and at Harbourmaster's office of Krynica Morska port.

Scheme of the radar' subsystem and data transmission is showed on *Figure 5*.

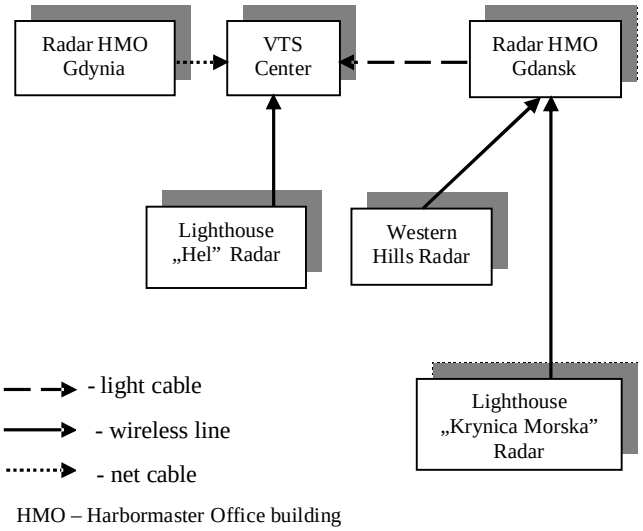

Figure 5. Scheme of VTS Zatoka radars' system

3. VTS Zatoka radar system reliability

In order to analyse the considered system reliability we will firstly calculate reliability parameters of single radar.

3.1. Single radar reliability

VTS Zatoka system has been designed and constructed by Holland Institute of Traffic Technology (HITT). It

came into service on the 1st of May 2003. Used radars were produced by Danish corporation Terma Electronic AS (frequency 9735 MHz). Mean time between failure and mean time to repair given by HITT have been used to construct VTS Zatoka radars' subsystem reliability model (*Table 1*). Times to repair given in *Table 1*, concern the situation in which service team is near damaged radar. In fact, sustaining service team near each radar is very expensive. For further consideration we assume that service team is in Tri-city i.e. in Gdynia, Sopot or Gdansk. Taking into account access time, time needed to fix what device is damaged and mean time of device' exchange, mean time to repair of the single radar ($MTTR_S$) is about 3 and a half hours. After considering frequency of failures of particular parts of radars, the mean time of exchanging damaged part is about 40 minutes.

Table 1. $MTBF_i$ – mean time between failure and $MTTR_i$ – mean time to repair [5]

Device/component	$MTBF_i$ [h]	λ_i	$MTTR_i$ [h]
Antenna	50 000	0,00002	3
Single radar transmitter	13 000	0,000077	0,5
Receiver	17 390	0,000058	0,34
Video processor	40 000	0,000025	0,25
Radar processor	20 000	0,00005	0,25
Data transmitter	87 500	0,000011	0,5

To find single radar reliability we assume that the radar is a series system. This means that the failure of one component of radar causes the failure of the whole radar. Two radars placed at Harbourmaster office buildings have five elements (antenna, single radar transmitter, receiver, video processor, radar processor), three others additionally have data transmitter. We assume that component reliability functions are exponential and given by the equation

$$R_i(t) = \exp(-\lambda_i \cdot t), t \geq 0. \quad (11)$$

When we put data from *Table 1* to (11), and then to equation (1), as a result we obtain the single radar reliability function for radars at harbourmaster office buildings

$$R_H(t) = \exp(-0{,}000229 \cdot t), t \geq 0, \quad (12)$$

and for three others radars

$$R_O(t) = \exp(-0{,}000235 \cdot t), t \geq 0. \quad (13)$$

Mean time between failures of single radar is given by equation

$$MTBF = \int_0^\infty R(t)\,dt. \quad (14)$$

According to equations (12)-(14) and to *Table 1*, we obtain mean time between failures of single radar at harbourmaster office buildings

$$MTBF_H = \frac{1}{0{,}000229} = 4366h \approx 182 \text{ days},$$

and mean time between failures of three other radars

$$MTBF_O = \frac{1}{0{,}000235} = 4255h \approx 177 \text{ days}.$$

3.2. Reliability of radar system

In order to evaluate radars system reliability, we can use different approaches. First, we can assume that subsystem is series. VTS Zatoka radars' subsystem is working when all five radars are working. According to equation (3) with parameters

$$n = 5, q_1 = \frac{2}{5}, q_2 = \frac{3}{5},$$

and equations (12) and (13), we obtain the system reliability function

$$R_S(t) = [\exp[-0{,}000229t]]^2 [\exp[-0{,}000235t]]^3$$

$$= \exp[-0{,}001163t], \; t \geq 0. \quad (15)$$

Mean time between failures of that system is given by equation

$$MTBF_s = \int_0^\infty R_S(t)\,dt = \frac{1}{0{,}001163}$$

$$= 860h \approx 36 \text{ days}. \quad (16)$$

System availability is given by equation [1]

$$G = \frac{MTBF_S}{MTBF_S + MTTR_S} \quad (17)$$

and after substituting $MTBF_S$ = 860h, $MTTR_S$ = 3,5h, amounts

$G = 0{,}9959$.

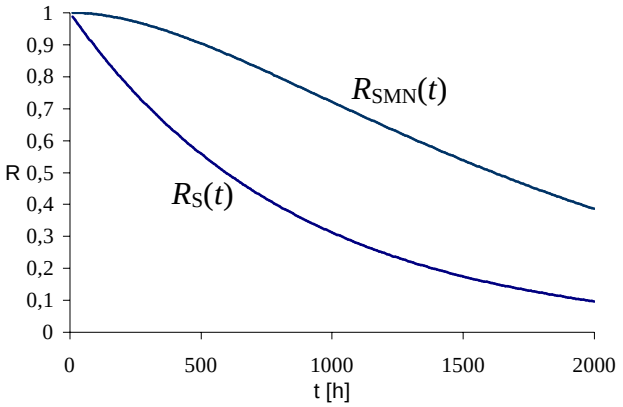

Figure 6. Radars' subsystem's reliability functions

Another way of describing reliability of radars' subsystem is assumption that system is „*m* out of *n*". We can assume that system is working if at least four out of five radars are working. If we take into account that particular radars are series systems we obtain a non-homogenous series-„4 out of 5" system.

The above assumption is acceptable because ranges of any four radars covered fairways to ports in Gdansk and Gdynia and most traffic (nearly entire) is concentrated in those fairways.

According to equations (10) and *Table 1*, reliability function of such system is given by equation

$$R_{SMN} = \overline{R}^{(1)}_{5,5,5,6,6,6}(t) = 2\exp(-0{,}000934t) + 3\exp(-0{,}000928t) - 4\exp(-0{,}001163t). \quad (18)$$

$R_{SMN}(t)$ function is showed on *Figure 4*.

Mean time to failure of series-„*m* out of *n*" system $MTBF_{SMN}$ is given by equation

$$MTBF_{SMN} = \int_0^\infty R_{SMN}(t)\,dt. \quad (19)$$

The mean time to failure of above described system according to equations (18) and (19) equals

$$MTBF_{SMN} = \int_0^\infty R^{(1)}_{5,5,5,6,6,6}(t)\,dt = 2\cdot\frac{1}{0{,}000934} + 3\cdot\frac{1}{0{,}000928} - 4\cdot\frac{1}{0{,}001163} = 1935h \cong 81\,\text{days}. \quad (20)$$

The availability of the series-„*m* out of *n*" system is given by equation

$$G_{SMN} = \frac{MTBF_{SMN}}{MTBF_{SMN} + MTTR_S} \quad (21)$$

and hence

$$G_{SMN} = \frac{1935}{1935 + 3{,}5} = 0{,}9982.$$

As we can see system defined as series-„*m* out of *n*" has both higher reliability and availability than a series system.

4. Conclusion

As we can see from the performed analysis evaluation of the system reliability depends on taken assumptions. Reliability functions are significantly different one from the other (*Figure 6*), so choosing proper method of describing of system reliability structure is very important.

Whatever method was chosen, thanks to reliability of components of radars, VTS Zatoka radars system is highly reliable. Access to spare parts and organization of service has significant matter for availability of the subsystem. In order to sustain acceptable availability it is necessary to provide the service support located in Tri-City. It allows for quick reaction in case of failures and for repairing damaged parts of radars.

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Optimisation of the preventive maintenance plan of a series components system with Weibull hazard function

Keywords

availability, hazard function, Weibull distribution, preventive maintenance, series components

Abstract

In this paper we propose an algorithm to calculate the optimum frequency to perform preventive maintenance in equipment that exhibits Weibull hazard function and constant repair rate in order to ensure its availability. Based on this algorithm we have developed another one to solve the problem of maintenance management of a series system based on preventive maintenance over the different system components. We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value

1. Introduction

Throughout the years, there has been tremendous pressure on manufacturing and service organizations to be competitive and provide timely delivery of quality products. In many industries, heavily automated and capital intensive, any loss of production due to equipment unavailability strongly impairs the company profit. This new environment has forced managers and engineers to optimise all sectors involved in their organizations.

Maintenance, as a system, plays a key role in achieving organizational goals and objectives. It contributes to reducing costs, minimizing equipment downtime, improving quality, increasing productivity, and providing reliable equipment that are safe and well configured to achieve timely delivery of orders to costumers. In addition, a maintenance system plays an important role in minimizing equipment life cycle cost.

To achieve the target rate of return on investment, plant availability and equipment effectiveness have to be maximized.

Grag and Deshmukh [38] had recently review the literature on maintenance management and points out that, next to the energy costs, maintenance costs can be the largest part of any operational budget.

A brief bibliographic review, (Andrews & Moss [1], Elsayed [5], McCormick [9] and Modarres *et al* [10]), is enough to conclude that the discipline known as reliability was developed to provide methods that can guarantee that any product or service will function efficiently when its user needs it. From this point of view, reliability theory incorporates techniques to determine what can go wrong, what should be done in order to prevent that something goes wrong, and, if something goes wrong, what should be done so that there is a quick recovery and consequences are minimal.

So, reliability has several meanings. However it is usually associated to the ability of a system to perform successfully a certain function. To measure quantitatively the reliability of a system it is used a probabilistic metric, which we state next.

Reliability of a system is the probability that a system will operate without failure for a stated period of time under specified conditions.
Another measure of the performance of a system is its availability that reflects the proportion of time that we expect it to be operational. Availability of a system is the probability to guarantee the intended function, that is, the probability that the system is normal at time *t*. The availability of a system is a decreasing function of the failure rate and it is an increasing function of the repair rate.
According to Elsayed [5], reliability of a system depends mainly in the quality and reliability of its components and in the implementation and accomplishment of a suitable preventive maintenance and inspection program. If failures, degradation and aging are characteristics of any system, however, it is possible to prolong its useful lifetime and, consequently, to delay the wear-out period carrying out maintenance and monitoring programs.
This type of programs leads necessarily to expenses and so we are taken to a maintenance optimisation problem.

Basic maintenance approaches can be classified as:

- Unplanned (corrective): this amounts to the replacement or repair of failed units;
- Planned (preventive):
  - Scheduled: this amounts to performing inspections, and possibly repair, following a predefined schedule;
  - Conditioned: this amounts to monitor the health of the system and to decide on repair actions based on the degradation level assessed.

In the unplanned, corrective strategy, no maintenance action is carried out until the component or structure breaks down. Upon failure, the associated repair time is typically relatively large, thus leading to large downtimes and high costs. In this approach, efforts are undertaken to achieve small mean times to repair (MTTRs).
To avoid failures at occasions that have high cost consequences preventive maintenance is normally chosen. This allows that inspections and upgrading can be planned for periods, which have the lowest impact on production or availability of the systems.
The main function of planned maintenance is to restore equipment to the "as good as new" condition; periodical inspections must control equipment condition and both actions will ensure equipment availability. In order to do so it is necessary to determine:

- Frequency of the maintenance, substitutions and inspections
- Rules of the components replacements
- Effect of the technological changes on the replacement decisions
- The size of the maintenance staff
- The optimum inventory levels of spare parts

There are several strategies for maintenance; the one we have just described and that naturally frames in what has been stated is known as Reliability Centered Maintenance - RCM. Gertsbakh [7] reviews some of the most popular models of preventive maintenance.
In theory, maintenance management, facing the problems stated above, could have benefited from the advent of a large area in operations research, called maintenance optimisation. This area was founded in the early sixties by researchers like Barlow and Proschan. Basically, a maintenance optimisation model is a mathematical model in which both costs and benefits of maintenance are quantified and in which an optimal balance between both is obtained. Well-known models originating from this period are the so-called age and the block replacement models.
Valdez-Flores & Feldman [12] presents a comprehensive review of these approaches. Dekker [2] gives an overview of applications of maintenance optimisation models published so far and Duffuaa [4] describes various advanced mathematical models in this area that have "high potential of being applied to improve maintenance operations".
More recently, Nakagawa [11] summarizes the results of maintenance policies and Garg and Deshmukh [6] describe the existence of 24 papers on the quantitative approach to maintenance optimisation
As we have already mentioned, one of the most critical problems in preventive maintenance is the determination of the optimum frequency to perform preventive maintenance in equipment, in order to ensure its availability.
The Preventive Maintenance policies are adapted to slow the degradation process of the system while the system is operating and to extend the system life. A number of Preventive Maintenance policies have been proposed in the literature. These policies are typically to determine the optimum interval between preventive maintenance tasks to minimize the average cost over a finite time span. But in many areas one complains about the gap between theory and practice. Practitioners say maintenance optimisation models are difficult to understand and to interpret [2]. Vatn *et al* [13] claim there exists a vast number of academic papers describing narrow maintenance models, which are rarely, if ever, used in practice. Most of these papers are too abstract, and the models that could be useful are difficult to identify among this large number of suggested models.

In this paper we propose an algorithm to solve the previous problem for equipment that exhibits Weibull hazard function and constant repair rate. Based on this algorithm we have developed another one to optimise maintenance management of a series system based on preventive maintenance over the different system components.
This is a problem with many applications in real systems and there are not many practical solutions for it. The main objective of this paper is to present an optimisation model understandable by practitioners, simple and useful for practical applications.
Duarte and Craveiro [3] have already outlined a solution for this problem for equipment that exhibit linearly increasing hazard rate and constant repair rate.
We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. We define a cost function for maintenance tasks (preventive and corrective) for the system. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value.

2. Previous concepts and results

In this section we present the classical concept of availability, while describing how to calculate it.
Point-wise availability of a system at time *t, A(t), is* the probability of the system being in a working state (operating properly) at time *t.* The unavailability of the system, *Q(t)*, is *Q(t) = 1 – A(t).*
The pointwise availability of a system that has constant failure rate λ and constant repair rate μ is

$$A(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\exp\left[-(\mu+\lambda)t\right]. \qquad (1)$$

and the limiting availability is

$$A = \lim_{t\to+\infty} A(t) = \frac{\mu}{\mu+\lambda}. \qquad (2)$$

The second parcel in formula (1) decreases rapidly to zero as time *t* increases; so, we can state

$$A(t) \approx \frac{\mu}{\mu+\lambda}$$

and this means that the availability of such a system is almost constant.

Example. A system is found to exhibit a constant failure rate of 0,000816 failures per hour and a constant repair rate of 0,02 repairs per hour.

Using formula 1, the availability of such a system (see *Figure 1*) is obtained as

$$A(t) = 0{,}9608 + 3.9201\times10^{-2}\exp(-2{,}0816\times10^{-2}t).$$

and the limiting availability is

$$\lim_{t\to\infty} A(t) = 0{,}9608.$$

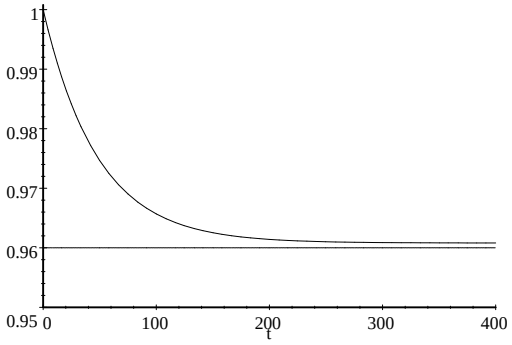

Figure 1. The availability function

It should be notice that, in this in case, we do not have almost any variation in the value of component's availability for *t > 200.*
We can therefore conclude that, to guarantee a value of availability *A*, known the constant repair rate, μ, the value of the constant failure rate of the system it will have to satisfy the relationship

$$A \approx \frac{\mu}{\mu+\lambda} \Leftrightarrow \lambda \approx \frac{\mu(1-A)}{A}. \qquad (3)$$

3. Model and assumptions

Suppose a system is found to exhibit an increasing hazard rate,

$$h(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \theta > 0, \beta > 0, t \geq 0,$$

and a constant repair rate μ.
Our goal is to determine the interval time between preventive maintenance tasks (we assume that the system is restored to the "as good as new" condition after each maintenance operation) in such a way that the availability of the system is no lesser than *A*.
The main idea for the solution of this problem consists of determining the time interval during which the

increasing hazard rate can be substituted by a constant failure rate in order to guarantee the pre-determinate availability level.
Applying the mean value theorem of integral calculus to the function

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0,$$

we obtain

$$\int_0^x \frac{\beta}{\theta^{\beta}} t^{\beta-1}dt=\lambda x \Rightarrow \left[\frac{t^{\beta}}{\theta^{\beta}}\right]_0^x=\lambda x$$

$$\Rightarrow \frac{x^{\beta}}{\theta^{\beta}}=\lambda x \qquad (4)$$

$$\Rightarrow x=0 \vee x=\sqrt[\beta-1]{\lambda\theta^{\beta}}.$$

Substituting λ in (4) by its approximate value given in formula 3, we have

$$x=\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^{\beta}}.$$

We can therefore conclude that in the time interval

$$\left[0,\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^{\beta}}\right],$$

the hazard functions,

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0,$$

and

$$h(t)=\frac{\mu(1-A)}{A}$$

guarantee approximately the same value of availability.
What we have just demonstrated can formally be stated on the following form:
Proposition: Let *S* be a system exhibiting an increasing hazard rate,

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0$$

and a constant repair rate μ. To guarantee an availability for the system equal or greater than *A* the interval of time between two consecutive preventive maintenance tasks must be equal or lesser than

$$\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^{\beta}}.$$

Example. A system is found to exhibit an increasing hazard rate, $h(t)=5\times10^{-8}\times t^{1,25}$, and a constant repair rate $m(t)=4\times10^{-2}$.
What should be the greatest time interval between preventive maintenance tasks (we assume that the system is restored to the "as good as new" condition after each maintenance operation) in such a way that the availability of the system is at least *98%*?
If the system had a constant failure rate, to guarantee such availability it should be

$$\lambda=\frac{4\times10^{-2}(1-0{,}98)}{0{,}98}=0{,}0008163.$$

We want to calculate the instant *x* in order to satisfy the following condition

$$\int_0^x 5\times10^{-8}\times t^{1,25}dt=0{,}0008163x$$

$$\Rightarrow \frac{5\times10^{-8}\times x^{2,25}}{2{,}25}=0{,}0008163x$$

$$\Rightarrow x=0 \vee x=4488.$$

We can therefore conclude that the system must be restored to the "as good as new" condition after each maintenance task every *4488* hours in order to achieve the availability target of *98%*.
Figure 2 illustrates this example.

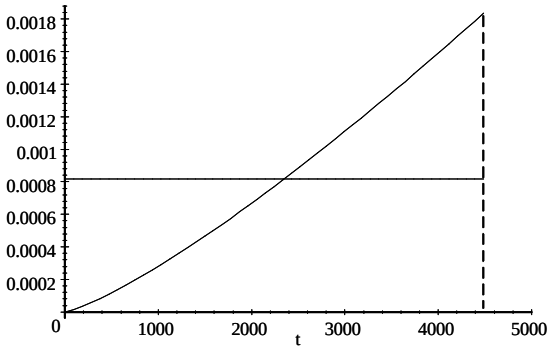

Figure 2. Hazard functions $h(t)=5\times10^{-8}t^{1,25}$ and $h(t)=0{,}0008163$ over the interval [0, 4488]

4. Optimisation of the preventive maintenance plan of a series components system

In this section we will present a model for the preventive maintenance management of a series system.
The system is composed by a set of n components in series as *Figure 3* shows.

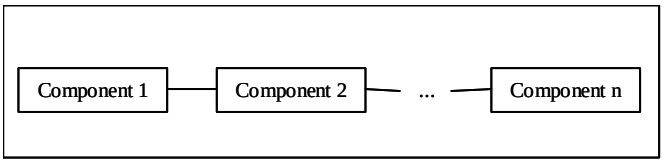

Figure 3. A series system of n components

Let τ_1, τ_2, …, τ_n be the time units between preventive maintenance tasks on components *1, 2, …, n*, respectively (*Figure 4*); assuming that these actions will restore periodically the components to the "as good as new" condition, they will have, therefore, consequences at the reliability and availability levels of the system.

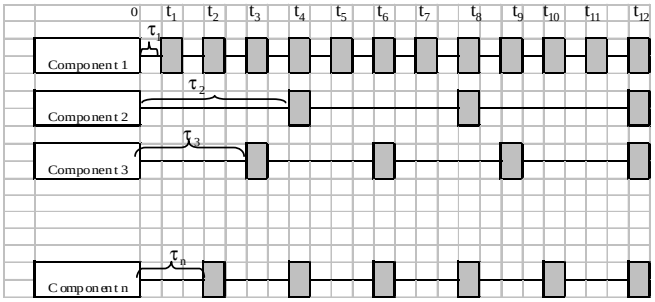

Figure 4. A preventive maintenance plan.

Our goal is to calculate the vector

$$[\tau_1 \quad \tau_2 \quad \tau_3 \quad \text{i} \quad \tau_n]^T$$

in such a way that the total down time in a certain period of time does not exceed a predetermined value, that is to say, that it guarantees the specified service level and simultaneously minimizes the maintenance costs.
We assume that each component has a Weibull hazard function,

$$h_i(t) = \frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i - 1}, \theta_i > 0, \beta_i > 0, t \geq 0$$

and a constant repair rate

$$m_i(t) = \mu_i .$$

The cost of each preventive maintenance task is cmp_i and the cost of each corrective maintenance task is cmc_i.
Since the availability of the system consisting of n components in series requires that all units must be available (assuming that components' failures are independent), system availability A is

$$A = \prod_{i=1}^{n} A_i$$

where A_i is the availability of component i.
Applying proposition presented in section 3 we can write that the availability of each component i is A_i over the interval

$$\left[0, \sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}}\right],$$

and its hazard function can be approximated by the constant function

$$h_i(t) = \frac{\mu_i(1-A_i)}{A_i}$$

Then, the expected number of failures in that time interval is

$$\sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}} \times \frac{\mu_i(1-A_i)}{A_i}$$

The objective function (defined as a cost function per unit time) is

$$c(A_1, A_2, \text{h}, A_n) = \sum_{i=1}^{n}\left[\frac{cmp_i}{\sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}}} + \frac{cmc_i}{\frac{\mu(1-A_i)}{A_i}}\right]$$

subject to

$$\begin{cases} \prod_{i=1}^{n} A_i \geq A, \\ 0 < A_i < 1, i = 1,2, \text{h}, n. \end{cases}$$

5. Numerical example

The model described on section 4 was implemented to a three components series system.

We assume that each component has a Weibull hazard function and a constant repair rate. Components are maintained preventively at periodic times.
Data is presented on *Table 1*.
First we present the nomenclature.
θ_i , β_i – parameters of hazard function.
TTR – Mean Time to Repair (corrective maintenance).
TTP – Time of one preventive maintenance action.
PMC – Preventive maintenance cost.
CMC – Corrective maintenance cost.
τ - time between two consecutive preventive maintenance tasks.

Table 1. Initial conditions

Components	θ_i	β_i	TTR	TTP	PM Cost	CM Cost	τ_i
1	4472,136	2	100	10	2000	4000	2000
2	1873,1716	2	50	40	2500	5000	1500
3	500,94	2	80	10	1000	2000	250

With this preventive maintenance plan the availability achieved is about *90,30%* and the life cycle cost is *122055,79*.
The target for availability is 90%.
The objective function was slightly modified in order to include the cost of down time.
MATLAB was used to optimise the objective function. Table 2 shows the results. With this new preventive maintenance policy we have a reduction of 5,5% in Life Cycle Cost (LCC) and simultaneously the availability *A* achieved (*92,70%*) is greater than the existing one (*90,30%*).

Table 2. Results of MatLab optimisation

MatLab Optimisation	τ_1	1600.2
	τ_2	1246.8
	τ_3	170.7535
	A - %	92.70
	LCC	115345.22
	Δ LCC - %	-5.5

With these results as initial conditions we have applied the tool "SOLVER" of Excel and we got a better solution (Table 3).

Table 3. Results of MatLab + Excel optimisation

MatLab + Excel Optimisation	τ_1	1606.498
	τ_2	1255.498
	τ_3	175.4996
	A - %	93.02
	LCC	113809.75
	Δ LCC - %	-6.8

5. Conclusion

This paper deals with a maintenance optimisation problem for a series system. First we have developed an algorithm to determine the optimum frequency to perform preventive maintenance in systems exhibiting Weibull hazard function and constant repair rate, in order to ensure its availability. Based on this algorithm we have developed another one to optimise maintenance management of a series system based on preventive maintenance over the different system components. We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. We define a cost function for maintenance tasks (preventive and corrective) for the system. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value. The maintenance interval of each component depends on factors such as failure rate, repair and maintenance times of each component in the system. In conclusion, the proposed analytical method is a feasible technique to optimise preventive maintenance scheduling of each component in a series system.
Currently we are developing a software package for the implementation of the algorithms presented in this paper.

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On multi-state safety analysis in shipping

Keywords

safety, multi-state system, operation process, complex system, shipping

Abstract

A multi-state approach to defining basic notions of the system safety analysis is proposed. A system safety function and a system risk function are defined. A basic safety structure of a multi-state series system of components with degrading safety states is defined. For this system the multi-state safety function is determined. The proposed approach is applied to the evaluation of a safety function, a risk function and other safety characteristics of a ship system composed of a number of subsystems having an essential influence on the ship safety. Further, a semi-markov process for the considered system operation modelling is applied. The paper also offers a general approach to the solution of a practically important problem of linking the multi-state system safety model and its operation process model. Finally, the proposed general approach is applied to the preliminary evaluation of a safety function, a risk function and other safety characteristics of a ship system with varying in time its structure and safety characteristics of the subsystems it is composed of.

1. Introduction

Taking into account the importance of the safety and operating process effectiveness of technical systems it seems reasonable to expand the two-state approach to multi-state approach in their safety analysis [2]. The assumption that the systems are composed of multi-state components with safety states degrading in time gives the possibility for more precise analysis and diagnosis of their safety and operational processes' effectiveness. This assumption allows us to distinguish a system safety critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system safety characteristic is the time to the moment of exceeding the system safety critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state safety function that is a basic characteristic of the multi-state system. Determining the multi-state safety function and the risk function of systems on the base of their components' safety functions is then the main research problem. Modelling of complicated systems operations' processes is difficult mainly because of large number of operations states and impossibility of precise describing of changes between these states. One of the useful approaches in modelling of these complicated processes is applying the semi-markov model [3]. Modelling of multi-state systems' safety and linking it with semi-markov model of these systems' operation processes is the main and practically important research problem of this paper. The paper is devoted to this research problem with reference to basic safety structures of technical systems [9], [10] and particularly to safety analysis of a ship series system [5] in variable operation conditions. This new approach to system safety investigation is based on the multi-

state system reliability analysis considered for instance in [1], [4], [6], [7], [8], [11] and on semi-markov processes modelling discussed for instance in [3].

2. Basic notions

In the multi-state safety analysis to define systems with degrading components we assume that:
- n is the number of system's components,
- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the safety state set $\{0,1,...,z\}$, $z \geq 1$,
- the safety state indexes are ordered, the state 0 is the worst and the state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
- the system and its components safety states degrade with time t,
- $E_i(t)$ is a component E_i safety state at the moment t, $t \in <0,\infty)$.
- $S(t)$ is a system safety state at the moment t, $t \in <0,\infty)$.

The above assumptions mean that the safety states of the system with degrading components may be changed in time only from better to worse. The way in which the components and the system safety states change is illustrated in *Figure 1*.

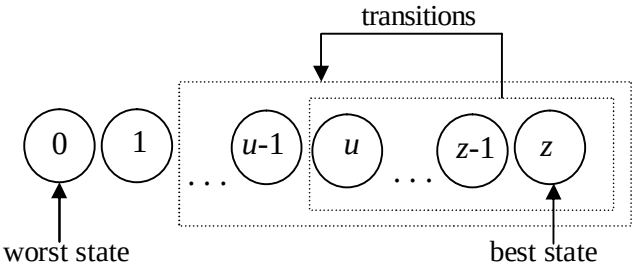

Figure 1. Illustration of a system and components safety states changing

The basis of our further considerations is a system component safety function defined as follows.

Definition 1. A vector

$$s_i(t,\cdot) = [s_i(t,0), s_i(t,1),..., s_i(t,z)],\ t \in <0,\infty),\quad i = 1,2,...,n, \tag{1}$$

where

$$s_i(t,u) = P(E_i(t) \geq u \mid E_i(0) = z) = P(T_i(u) > t) \tag{2}$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, $i = 1,2,...,n$, is the probability that the component E_i is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state safety function of a component E_i.

Similarly, we can define a multi-state system safety function.

Definition 2. A vector

$$\boldsymbol{s}_n(t,\cdot) = [\boldsymbol{s}_n(t,0), \boldsymbol{s}_n(t,1),..., \boldsymbol{s}_n(t,z)],\ t \in <0,\infty), \tag{3}$$

where

$$\boldsymbol{s}_n(t,u) = P(S(t) \geq u \mid S(0) = z) = P(T(u) > t) \tag{4}$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, is the probability that the system is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state safety function of a system.

Definition 3. A probability

$$\boldsymbol{r}(t) = P(S(t) < r \mid S(0) = z) = P(T(r) \leq t),\quad t \in <0,\infty), \tag{5}$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the state z at the moment $t = 0$ is called a risk function of the multi-state system.

Under this definition, considering (4) and (5), we have

$$\boldsymbol{r}(t) = 1 - P(S(t) \geq r \mid S(0) = z) = 1 - \boldsymbol{s}_n(t,r),\quad t \in <0,\infty), \tag{6}$$

and, if τ is the moment when the risk exceeds a permitted level δ, $\delta \in <0,1>$, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{7}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$ given by (6).

3. Basic system safety structures

The proposition of a multi-state approach to definition of basic notions, analysis and diagnosing of systems' safety allowed us to define the system safety function and the system risk function. It also allows us to define basic structures of the multi-state systems of components with degrading safety states. For these

basic systems it is possible to determine their safety functions. Further, as an example, we will consider a series system.

Definition 4. A multi-state system is called a series system if it is in the safety state subset $\{u, u+1,..., z\}$ if and only if all its components are in this subset of safety states.

Corollary 1. The lifetime $T(u)$ of a multi-state series system in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1\le i\le n}\{T_i(u)\}, \; u = 1,2,...,z.$$

The scheme of a series system is given in *Figure 2.*

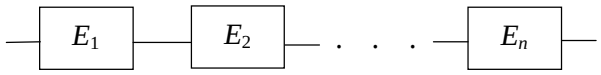

Figure 2. The scheme of a series system

It is easy to work out the following result.

Corollary 2. The safety function of the multi-state series system is given by

$$\bar{s}_n(t,\cdot) = [1, \bar{s}_n(t,1),..., \bar{s}_n(t,z)], \; t \in <0,\infty), \tag{8}$$

where

$$\bar{s}_n(t,u) = \prod_{i=1}^{n} s_i(t,u), t \in <0,\infty), \; u = 1,2,...,z. \tag{9}$$

Corollary 3. If components of the multi-state series system have exponential safety functions, i.e., if

$$s_i(t,\cdot) = [1, s_i(t,1),..., s_i(t,z)], \; t \in <0,\infty),$$

where

$$s_i(t,u) = \exp[-\lambda_i(u)t] \text{ for } t \in <0,\infty), \; \lambda_i(u) > 0,$$
$$u = 1,2,...,z, \; i = 1,2,...,n,$$

then its safety function is given by

$$\bar{s}_n(t,\cdot) = [1, \bar{s}_n(t,1),..., \bar{s}_n(t,z)], \tag{10}$$

where

$$\bar{s}_n(t,u) = \exp[-\sum_{i=1}^{n}\lambda_i(u)t] \text{ for } t \in <0,\infty), \tag{11}$$
$$u = 1,2,...,z.$$

4. Basic system safety structures in variable operation conditions

We assume that the system during its operation process has v different operation states. Thus we can define $Z(t), \; t \in <0,+\infty>,$ as the process with discrete operation states from the set

$$Z = \{z_1, z_2, \ldots, z_v\},$$

In practice a convenient assumption is that $Z(t)$ is a semi-markov process [3] with its conditional lifetimes θ_{bl} at the operation state z_b when its next operation state is z_l, $b, l = 1,2,...,v, \; b \neq l.$ In this case the process $Z(t)$ may be described by:

- the vector of probabilities of the process initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of the process transitions between the operation states $[p_{bl}]_{vxv}$, where $p_{bb}(t) = 0$ for $b = 1,2,...,v.$
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the process lifetimes θ_{bl}, $b \neq l,$ in the operation state z_b when the next operation state is z_l, where $H_{bl}(t) = P(\theta_{bl} < t)$ for $b, l = 1,2,...,v, \; b \neq l,$ and $H_{bb}(t) = 0$ for $b = 1,2,...,v.$

Under these assumptions, the lifetimes θ_{bl} mean values are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t), \; b, l = 1,2,...,v, \; b \neq l. \tag{12}$$

The unconditional distribution functions of the lifetimes θ_b of the process $Z(t)$ at the operation states z_b, $b = 1,2,...,v,$ are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \; b = 1,2,...,v.$$

The mean values $E[\theta_b]$ of the unconditional lifetimes θ_b are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}, \; b = 1,2,...,v,$$

where M_{bl} are defined by (12).

Limit values of the transient probabilities at the operation states

$$p_b(t) = P(Z(t) = z_b), \; t \in <0,+\infty), \; b = 1,2,...,v,$$

are given by

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v}\pi_l M_l}, \quad b=1,2,...,v, \tag{13}$$

where the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases}[\pi_b]=[\pi_b][p_{bl}]\\ \sum_{l=1}^{v}\pi_l=1.\end{cases}$$

We assume that the system is composed of n components E_i, $i=1,2,...,n$, the changes of the process $Z(t)$ operation states have an influence on the system components E_i safety and on the system safety structure as well. Thus, we denote the conditional safety function of the system component E_i while the system is at the operational state z_b, $b=1,2,...,v$, by

$$s_i^{(b)}(t,\cdot)=[1,\ s_i^{(b)}(t,1),\ s_i^{(b)}(t,2),\ ...,\ s_i^{(b)}(t,z)],$$

where

$$s_i^{(b)}(t,u)=P(T_i^{(b)}(u)>t\,|\,Z(t)=z_b)$$

for $t\in<0,\infty)$, $b=1,2,...,v$, $u=1,2,...,z$, and the conditional safety function of the system while the system is at the operational state z_b, $b=1,2,...,v$, by

$$\boldsymbol{s}_{n_b}^{(b)}(t,\cdot)=[1,\ \boldsymbol{s}_{n_b}^{(b)}(t,1),\ \boldsymbol{s}_{n_b}^{(b)}(t,2),\ ...,\ \boldsymbol{s}_{n_b}^{(b)}(t,z)],$$
$$n_b\in\{1,2,...,n\},$$

where n_b are numbers of components in the operation states z_b and

$$\boldsymbol{s}_{n_b}^{(b)}(t,u),=P(T^{(b)}(u)>t\,|\,Z(t)=z_b)$$

for $t\in<0,\infty)$, $n_b\in\{1,2,...,n\}$, $b=1,2,...,v$,
$u=1,2,...,z$.

The safety function $s_i^{(b)}(t,u)$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the state subset $\{u,u+1,...,z\}$ is not less than t, while the process $Z(t)$ is at the operation state z_b. Similarly, the safety function $\boldsymbol{s}_{n_b}^{(b)}(t,u)$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the state subset $\{u,u+1,...,z\}$ is not less than t, while the process $Z(t)$ is at the operation state z_b.

In the case when the system operation time is large enough, the unconditional safety function of the system is given by

$$\boldsymbol{s}_n(t,\cdot)=[1,\ \boldsymbol{s}_n(t,1),\ \boldsymbol{s}_n(t,2),\ ...,\ \boldsymbol{s}_n(t,z)],\ t\ge 0,$$

where

$$\boldsymbol{s}_n(t,u)=P(T(u)>t)\cong\sum_{b=1}^{v}p_b\boldsymbol{s}_{n_b}^{(b)}(t,u) \tag{14}$$

for $t\ge 0$, $n_b\in\{1,2,...,n\}$, $u=1,2,...,z$, and $T(u)$ is the unconditional lifetime of the system in the safety state subset $\{u,u+1,...,z\}$.

The mean values and variances of the system lifetimes in the safety state subset $\{u,u+1,...,z\}$ are

$$m(u)=E[T(u)]\cong\sum_{b=1}^{v}p_b m^{(b)}(u),\ u=1,2,...,z, \tag{15}$$

where [2]

$$m^{(b)}(u)=\int_0^\infty \boldsymbol{s}_{\boldsymbol{n_b}}^{(b)}(t,u)dt,\ n_b\in\{1,2,...,n\}, \tag{16}$$
$$u=1,2,...,z,$$

and

$$[\sigma^{(b)}(u)]^2=2\int_0^\infty t\boldsymbol{s}_{n_b}(t,u)dt-[m^{(b)}(u)]^2, \tag{17}$$
$$u=1,2,...,z,$$

for $b=1,2,...,v$, and

$$[\sigma(u)]^2=2\int_0^\infty t\boldsymbol{s}_n(t,u)dt-[m(u)]^2,\quad u=1,2,...,z. \tag{16}$$

The mean values of the system lifetimes in the particular safety states u, are [2]

$$\overline{m}(u)=m(u)-m(u+1),\ u=1,2,...,z-1,$$
$$\overline{m}(z)=m(z). \tag{19}$$

5. Ship safety Model in constant operation conditions

We preliminarily assume that the ship is composed of a number of main technical subsystems having an essential influence on its safety. There are distinguished her following technical subsystems:

S_1 - a navigational subsystem,

S_2 - a propulsion and controlling subsystem,

S_3 - a loading and unloading subsystem,

S_4 - a hull subsystem,

S_5 - a protection and rescue subsystem,

S_6 - an anchoring and mooring subsystem.

According to *Definition 1*, we mark the safety functions of these subsystems respectively by vectors

$$s_i(t,\cdot) = [s_i(t,0), s_i(t,1),..., s_i(t,z)],\ t \in <0,\infty),\quad i = 1,2,...,6, \tag{20}$$

with co-ordinates

$$s_i(t,u) = P(S_i(t) \geq u \mid S_i(0) = z) = P(T_i(u) > t) \tag{21}$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, $i = 1,2,...,6$, where $T_i(u)$, $i = 1,2,...,6$, are independent random variables representing the lifetimes of subsystems S_i in the safety state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$ and $S_i(t)$ is a subsystem S_i safety state at the moment t, $t \in <0,\infty)$.

Further, assuming that the ship is in the safety state subset $\{u,u+1,...,z\}$ if all its subsystems are in this subset of safety states and considering *Definition 4*, we conclude that the ship is a series system of subsystems S_1, S_2, S_3, S_4, S_5, S_6 with a scheme presented in *Figure 3*.

Figure 3. The scheme of a structure of ship subsystems

Therefore, the ship safety is defined by the vector

$$\bar{s}_6(t,\cdot) = [\bar{s}_6(t,0),\ \bar{s}_6(t,1),...,\ \bar{s}_6(t,z)],\quad t \in <0,\infty), \tag{22}$$

with co-ordinates

$$\bar{s}_6(t,u) = P(S(t) \geq u \mid S(0) = z) = P(T(u) > t) \tag{23}$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, where $T(u)$ is a random variable representing the lifetime of the ship in the safety state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$ and $S(t)$ is the ship safety state at the moment t, $t \in <0,\infty)$, according to *Corollary 2*, is given by the formula

$$\bar{s}_6(t,\cdot) = [1,\ \bar{s}_6(t,1),...,\ \bar{s}_6(t,z)],\ t \in <0,\infty), \tag{24}$$

where

$$\bar{s}_6(t,u) = \prod_{i=1}^{6} s_i(t,u),\ t \in <0,\infty),\ u = 1,2,...,z. \tag{25}$$

6. Ship operation process

Technical subsystems S_1, S_2, S_3, S_4, S_5, S_6 are forming a general ship safety structure presented in *Figure 3*. However, the ship safety structure and the ship subsystems safety depend on her changing in time operation states.

Considering basic sea transportation processes the following operation ship states have been specified:

z_1 - loading of cargo,

z_2 - unloading of cargo,

z_3 - leaving the port,

z_4 - entering the port,

z_5 - navigation at restricted water areas,

z_6 - navigation at open sea waters.

In this case the process $Z(t)$ may be described by:

- the vector of probabilities of the initial operation states $[p_b(0)]_{1x6}$,
- the matrix of the probabilities of its transitions between the operation states $[p_{bl}]_{6x6}$, where $p_{bb}(t) = 0$ for $b = 1,2,...,6$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{6x6}$ of the lifetimes θ_{bl}, $b \neq l$, where $H_{bl}(t) = P(\theta_{bl} < t)$ for $b,l = 1,2,...,6$, $b \neq l$, and $H_{bb}(t) = 0$ for $b = 1,2,...,6$.

Under these assumptions, the lifetimes θ_{bl} mean values are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t),\ b,l = 1,2,...,6,\ b \neq l. \tag{26}$$

The unconditional distribution functions of the lifetimes θ_b of the process $Z(t)$ at the operation states z_b, $b = 1,2,...,6$, are given by

$$H_b(t) = \sum_{l=1}^{6} p_{bl} H_{bl}(t),\ b = 1,2,...,6.$$

The mean values $E[\theta_b]$ of the unconditional lifetimes θ_b are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{6} p_{bl} M_{bl},\ b = 1,2,...,6, \tag{27}$$

where M_{bl} are defined by (26).

Limit values of the transient probabilities at the operation states

$$p_b(t) = P(Z(t) = z_b),\ t \in <0,+\infty),\ b = 1,2,...,6,$$

are given by

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{6} \pi_l M_l},\quad b = 1,2,...,6, \tag{28}$$

where the probabilities π_b of the vector $[\pi_b]_{1x6}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{6} \pi_l = 1. \end{cases} \tag{29}$$

7. Safety model of ship in variable operation conditions

We assume as earlier that the ship is composed of $n = 6$ subsystems S_i, $i = 1,2,...,6$, and that the changes of the process $Z(t)$ of ship operation states have an influence on the system subsystems S_i safety and on the ship safety structure as well. Thus, we denote the conditional safety function of the ship subsystem S_i while the ship is at the operational state z_b, $b = 1,2,...,6$, by

$$s_i^{(b)}(t,\cdot) = [1,\ s_i^{(b)}(t,1),\ s_i^{(b)}(t,2),\ ...,\ s_i^{(b)}(t,z)],$$

where

$$s_i^{(b)}(t,u) = P(T_i^{(b)}(u) > t \,|\, Z(t) = z_b)$$

for $t \in <0,\infty)$, $b = 1,2,...,6$, $u = 1,2,...,z$, and the conditional safety function of the ship while the ship is at the operational state z_b, $b = 1,2,...,6$, by

$$\boldsymbol{s}_{n_b}^{(b)}(t,\cdot) = [1,\ \boldsymbol{s}_{n_b}^{(b)}(t,1),\ \boldsymbol{s}_{n_b}^{(b)}(t,2),\ ...,\ \boldsymbol{s}_{n_b}^{(b)}(t,z)],$$

where

$$\boldsymbol{s}_{n_b}^{(b)}(t,u), = P(T^{(b)}(u) > t \,|\, Z(t) = z_b)$$

for $t \in <0,\infty)$, $b = 1,2,...,6$, $n_b \in \{1,2,3,4,5,6\}$, $u = 1,2,...,z$.

The safety function $s_i^{(b)}(t,u)$ is the conditional probability that the subsystem S_i lifetime $T_i^{(b)}(u)$ in the state subset $\{u,u+1,...,z\}$ is not less than t, while the process $Z(t)$ is at the ship operation state z_b. Similarly, the safety function $\boldsymbol{s}_{n_b}^{(b)}(t,u)$ is the conditional probability that the ship lifetime $T^{(b)}(u)$ in the state subset $\{u,u+1,...,z\}$ is not less than t, while the process $Z(t)$ is at the ship operation state z_b.

In the case when the ship operation time is large enough, the unconditional safety function of the system is given by

$$\boldsymbol{s}_6(t,\cdot) = [1,\ \boldsymbol{s}_6(t,1),\ \boldsymbol{s}_6(t,2),\ ...,\ \boldsymbol{s}_6(t,z)],\ t \ge 0,$$

where

$$\boldsymbol{s}_6(t,u) = P(T(u) > t) \cong \sum_{b=1}^{6} p_b \boldsymbol{s}_{n_b}^{(b)}(t,u) \tag{30}$$

for $t \ge 0$, $n_b \in \{1,2,3,4,5,6\}$, $u = 1,2,...,z$, and $T(u)$ is the unconditional lifetime of the ship in the safety state subset $\{u,u+1,...,z\}$.

The mean values and variances of the ship lifetimes in the safety state subset $\{u,u+1,...,z\}$ are

$$m(u) = E[T(u)] \cong \sum_{b=1}^{6} p_b m^{(b)}(u),\ u = 1,2,...,z, \tag{31}$$

where

$$m^{(b)}(u) = \int_0^{\infty} \boldsymbol{s}_{n_b}^{(b)}(t,u)dt, \tag{32}$$

for $b = 1,2,...,6$, $n_b \in \{1,2,3,4,5,6\}$, $u = 1,2,...,z$, and

$$[\sigma(u)]^2 = D[T(u)] = 2\int_0^{\infty} t\boldsymbol{s}_6(t,u)dt - [m(u)]^2, \tag{33}$$
$$u = 1,2,...,z,$$

The mean values of the system lifetimes in the particular safety states u, are

$$\overline{m}(u) = m(u) - m(u+1),\ u = 1,2,...,z-1,$$

$$\overline{m}(z) = m(z). \tag{34}$$

8. Preliminary application of general safety model of ship in variable operation conditions

According to expert opinions [5] in the ship operation process, $Z(t)$, $t \ge 0$, we distinguished seven operation

states: z_1, z_2, z_3, z_4, z_5, z_6. On the basis of data coming from experts, the probabilities of transitions between the operation states are approximately given by

$$[p_{bl}]_{6x6} = \begin{bmatrix} 0.00 & 0.00 & 0.96 & 0.00 & 0.02 & 0.02 \\ 0.48 & 0.00 & 0.48 & 0.00 & 0.02 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.02 & 0.96 & 0.02 \\ 0.49 & 0.49 & 0.02 & 0.00 & 0.00 & 0.00 \\ 0.02 & 0.02 & 0.00 & 0.48 & 0.00 & 0.48 \\ 0.02 & 0.02 & 0.00 & 0.01 & 0.95 & 0.00 \end{bmatrix},$$

and the distributions of the ship conditional lifetimes in the operation states are exponential of the following forms:

$$H_{13}(t) = 1 - \exp[-0.5t],\ H_{15}(t) = 1 - \exp[-1.0t],$$

$$H_{16}(t) = 1 - \exp[-1.0t],\ H_{21}(t) = 1 - \exp[-0.5t],$$

$$H_{23}(t) = 1 - \exp[-0.5t],\ H_{25}(t) = 1 - \exp[-1.0t],$$

$$H_{26}(t) = 1 - \exp[-1.0t],\ H_{34}(t) = 1 - \exp[-25.0t],$$

$$H_{35}(t) = 1 - \exp[-25.0t],\ H_{36}(t) = 1 - \exp[-12.5t],$$

$$H_{51}(t) = 1 - \exp[-0.33t],\ H_{52}(t) = 1 - \exp[-0.33t],$$

$$H_{54}(t) = 1 - \exp[-0.5t],\ H_{56}(t) = 1 - \exp[-0.5t],$$

$$H_{61}(t) = 1 - \exp[-0.2t],\ H_{62}(t) = 1 - \exp[-0.2t],$$

$$H_{64}(t) = 1 - \exp[-0.25t],\ H_{65}(t) = 1 - \exp[-0.25t]$$

for $t \geq 0$.

Hence, by (26), the conditional mean values of lifetimes in the operation states are

$$M_{13} = 2,\ M_{15} = 1,\ M_{16} = 1,$$

$$M_{21} = 2,\ M_{23} = 2,\ M_{25} = 1,\ M_{26} = 1,$$

$$M_{34} = 0.04,\ M_{35} = 0.04,\ M_{36} = 0.08,$$

$$M_{41} = 0.08,\ M_{42} = 0.08,\ M_{43} = 0.04,$$
$$M_{51} = 3,\ M_{52} = 3,\ M_{54} = 2,\ M_{56} = 2,$$

$$M_{61} = 5,\ M_{62} = 5,\ M_{64} = 4,\ M_{65} = 4.$$

Whereas, by (27), the unconditional mean lifetimes in the operation states are

$$M_1 = E[\theta_1] = p_{13}M_{13} + p_{15}M_{15} + p_{16}M_{16}$$
$$= 0.96 \cdot 2 + 0.02 \cdot 1 + 0.02 \cdot 1 = 1.96,$$

$$M_2 = E[\theta_2]$$
$$= p_{21}M_{21} + p_{23}M_{23} + p_{25}M_{25} + p_{26}M_{26}$$
$$= 0.48 \cdot 2 + 0.48 \cdot 2 + 0.02 \cdot 1 + 0.02 \cdot 1 = 1.96,$$

$$M_3 = E[\theta_3] = p_{34}M_{34} + p_{35}M_{35} + p_{36}M_{36}$$
$$= 0.02 \cdot 0.04 + 0.96 \cdot 0.04 + 0.02 \cdot 0.08 = 0.0408,$$

$$M_4 = E[\theta_4] = p_{41}M_{41} + p_{42}M_{42} + p_{43}M_{43}$$
$$= 0.49 \cdot 0.08 + 0.49 \cdot 0.08 + 0.02 \cdot 0.04 = 0.0792,$$

$$M_5 = E[\theta_5]$$
$$= p_{51}M_{51} + p_{52}M_{52} + p_{54}M_{54} + p_{56}M_{56}$$
$$= 0.02 \cdot 3 + 0.02 \cdot 3 + 0.48 \cdot 2 + 0.48 \cdot 2 = 2.04,$$

$$M_6 = E[\theta_6]$$
$$= p_{61}M_{61} + p_{62}M_{62} + p_{64}M_{64} + p_{65}M_{65}$$
$$= 0.02 \cdot 5 + 0.02 \cdot 5 + 0.01 \cdot 4 + 0.95 \cdot 4 = 4.04.$$

Since from the system of equations

$$\begin{cases} [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6] \\ = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6][p_{bl}]_{6x6} \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1, \end{cases}$$

we get

$$\pi_1 = 0.126,\ \pi_2 = 0.085,\ \pi_3 = 0.165,$$

$$\pi_4 = 0.155,\ \pi_5 = 0.312,\ \pi_6 = 0.157,$$

then the limit values of the transient probabilities $p_b(t)$ at the operational states z_b, according to (28), are given by

$$p_1 = 0.145,\ p_2 = 0.098,\ p_3 = 0.004,\ p_4 = 0.007,$$

$$p_5 = 0.374, \quad p_6 = 0.372. \tag{35}$$

We assume that the ship subsystems S_i, $i = 1,2,...,6$, are its five-state components, i.e. $z = 4$, with the multi-state safety functions

$$s_i^{(b)}(t,\cdot) = [1,\ s_i^{(b)}(t,1),\ s_i^{(b)}(t,2),\ ...,\ s_i^{(b)}(t,z)],$$
$$b = 1,2,...,6,\ i = 1,2,...,6,$$

with exponential co-ordinates different in various ship operation states z_b, $b = 1,2,...,6$.

At the operation states z_1 and z_2, i.e. at the cargo loading and un-loading state the ship is built of $n_1 = n_2 = 4$ subsystems S_3, S_4, S_5 and S_6 forming a series structure shown in *Figure 4*.

Figure 4. The scheme of the ship structure at the operation states z_1 and z_2

We assume that the ship subsystems S_i, $i = 3,4,5,6$, are its five-state components, i.e. $z = 4$, having the multi-state safety functions

$$s_i^{(b)}(t,\cdot) = [1, s_i^{(b)}(t,1),\ s_i^{(b)}(t,2),\ s_i^{(b)}(t,3),\ s_i^{(b)}(t,4)],$$
$$i = 3,4,5,6,\ b = 1,2,$$

with exponential co-ordinates, for $b = 1,2$, respectively given by:

- for the loading subsystem S_3

$$s_3^{(b)}(t,1) = \exp[-0.06t],\ s_3^{(b)}(t,2) = \exp[-0.07t],$$

$$s_3^{(b)}(t,3) = \exp[-0.08t],\ s_3^{(b)}(t,4) = \exp[-0.09t],$$

- for the hull subsystem S_4

$$s_4^{(b)}(t,1) = \exp[-0.03t],\ s_4^{(b)}(t,2) = \exp[-0.04t],$$

$$s_4^{(b)}(t,3) = \exp[-0.06t],\ s_4^{(b)}(t,4) = \exp[-0.07t],$$

- for the protection and rescue subsystem S_5

$$s_5^{(b)}(t,1) = \exp[-0.10t],\ s_5^{(b)}(t,2) = \exp[-0.12t],$$

$$s_5^{(b)}(t,3) = \exp[-0.15t],\ s_5^{(b)}(t,4) = \exp[-0.16t],$$

- for the anchor and mooring subsystem S_6

$$s_6^{(b)}(t,1) = \exp[-0.06t],\ s_6^{(b)}(t,2) = \exp[-0.08t],$$

$$s_6^{(b)}(t,3) = \exp[-0.10t],\ s_6^{(b)}(t,4) = \exp[-0.12t].$$

Assuming that the ship is in the safety state subsets $\{u, u+1,..., z\}$, $u = 1,2,3,4$, if all its subsystems are in this safety state subset, according to *Definition 1* and *Definition 4*, the considered system is a five-state series system. Thus, by *Corollary 3*, after applying (10)–(11), we have its conditional safety functions in the operation states z_1 and z_2 respectively for $b = 1,2$, given by

$$\bar{s}_4^{(b)}(t,\cdot)$$

$$= [1, \bar{s}_4^{(b)}(t,1),\ \bar{s}_4^{(b)}(t,2),\ \bar{s}_4^{(b)}(t,3),\ \bar{s}_4^{(b)}(t,4)],$$
$$t \geq 0,\ b = 1,2,$$

where

$$\bar{s}_4^{(b)}(t,1) = \exp[-(0.06 + 0.03 + 0.10 + 0.06)t]$$
$$= \exp[-0.25t],$$

$$\bar{s}_4^{(b)}(t,2) = \exp[-(0.07 + 0.04 + 0.12 + 0.08)t]$$
$$= \exp[-0.31t],$$

$$\bar{s}_4^{(b)}(t,3) = \exp[-(0.08 + 0.06 + 0.15 + 0.10)t]$$
$$= \exp[-0.39t],$$

$$\bar{s}_4^{(b)}(t,4) = \exp[-(0.09 + 0.07 + 0.16 + 0.12)t]$$
$$= \exp[-0.44t] \text{ for } t \geq 0,\ b = 1,2.$$

The expected values and standard deviations of the ship conditional lifetimes in the safety state subsets calculated from the above result, according to (16)-(17), for $b = 1,2$, are:

$$m^{(b)}(1) \cong 4.00,\ m^{(b)}(2) \cong 3.26,\ m^{(b)}(3) \cong 2.56,$$

$$m^{(b)}(4) \cong 2.27 \text{ years},$$

$$\sigma^{(b)}(1) \cong 4.00,\ \sigma^{(b)}(2) \cong 3.26,\ \sigma^{(b)}(3) \cong 2.56,$$

$\sigma^{(b)}(4) \cong 2.27$ years,

and further, from (10), the ship conditional lifetimes in the particular safety states, for $b = 1,2,$ are:

$\overline{m}^{(b)}(1) \cong 0.74,\ \overline{m}^{(b)}(2) \cong 0.70,\ \overline{m}^{(b)}(3) \cong 0.29,$

$\overline{m}^{(b)}(4) \cong 2.27$ years.

At the operation states z_3 and z_4, i.e. at the leaving and entering state the ship is built of $n_3 = n_4 = 5$ subsystems S_1, S_2, S_4, S_5 and S_6 forming a series structure shown in *Figure 5*.

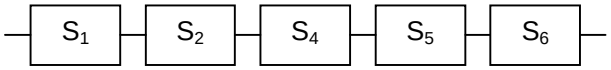

Figure 5. The scheme of the ship structure at the operation states z_3 and z_4

We assume that the ship subsystems S_i, $i = 1,2,4,5,6,$ are its five-state components, i.e. z = 4, having the multi-state safety functions

$$s_i^{(b)}(t,\cdot) = [1,\ s_i^{(b)}(t,1), s_i^{(b)}(t,2), s_i^{(b)}(t,3), s_i^{(b)}(t,4)],$$
$$i = 1,2,4,5,6,\ b = 3,4,$$

with exponential co-ordinates, for $b = 3,4,$ respectively given by:

- for the navigational subsystem S_1

$s_1^{(b)}(t,1) = \exp[-0.15t],\ s_1^{(b)}(t,2) = \exp[-0.20t],$

$s_1^{(b)}(t,3) = \exp[-0.22t],\ s_1^{(b)}(t,4) = \exp[-0.25t],$

- for the propulsion and controlling subsystem S_2

$s_2^{(b)}(t,1) = \exp[-0.05t],\ s_2^{(b)}(t,2) = \exp[-0.06t],$

$s_2^{(b)}(t,3) = \exp[-0.07t],\ s_2^{(b)}(t,4) = \exp[-0.08t],$

- for the hull subsystem S_4

$s_4^{(b)}(t,1) = \exp[-0.04t],\ s_4^{(b)}(t,2) = \exp[-0.05t],$

$s_4^{(b)}(t,3) = \exp[-0.07t],\ s_4^{(b)}(t,4) = \exp[-0.08t],$

- for the protection and rescue subsystem S_5

$s_5^{(b)}(t,1) = \exp[-0.12t],\ s_5^{(b)}(t,2) = \exp[-0.14t],$

$s_5^{(b)}(t,3) = \exp[-0.16t],\ s_5^{(b)}(t,4) = \exp[-0.18t],$

- for the anchor and mooring subsystem S_6

$s_6^{(b)}(t,1) = \exp[-0.02t],\ s_6^{(b)}(t,2) = \exp[-0.04t],$

$s_6^{(b)}(t,3) = \exp[-0.06t],\ s_6^{(b)}(t,4) = \exp[-0.08t].$

Assuming that the ship is in the safety state subsets $\{u, u+1, ..., z\}$, $u = 1,2,3,4,$ if all its subsystems are in this safety state subset, according to *Definition 1* and *Definition 4*, the considered system is a five-state series system. Thus, by *Corollary 3*, after applying (10)–(11), we have its conditional safety functions in the operation states z_3 and z_4 respectively for $b = 3,4,$ given by

$$\bar{s}_5^{(b)}(t,\cdot)$$
$$= [1, \bar{s}_5^{(b)}(t,1),\ \bar{s}_5^{(b)}(t,2),\ \bar{s}_5^{(b)}(t,3),\ \bar{s}_5^{(b)}(t,4)],$$
$$t \geq 0,\ b = 3,4,$$

where

$$\bar{s}_5^{(b)}(t,1) = \exp[-(0.15 + 0.05 + 0.04 + 0.12 + 0.02)t]$$
$$= \exp[-0.38t],$$
$$\bar{s}_5^{(b)}(t,2) = \exp[-(0.20 + 0.06 + 0.05 + 0.14 + 0.04)t]$$
$$= \exp[-0.49t],$$
$$\bar{s}_5^{(b)}(t,3) = \exp[-(0.22 + 0.07 + 0.07 + 0.16 + 0.06)t]$$
$$= \exp[-0.58t],$$
$$\bar{s}_5^{(b)}(t,4) = \exp[-(0.25 + 0.08 + 0.08 + 0.18 + 0.08)t]$$
$$= \exp[-0.67t] \text{ for } t \geq 0,\ b = 3,4.$$

The expected values and standard deviations of the ship conditional lifetimes in the safety state subsets calculated from the above result, according to (16)-(17), for $b = 3,4,$ are:

$m^{(b)}(1) \cong 2.63,\ m^{(b)}(2) \cong 2.04,\ m^{(b)}(3) \cong 1.72,$

$m^{(b)}(4) \cong 1.49$ years,

$\sigma^{(b)}(1) \cong 2.63,\ \sigma^{(b)}(2) \cong 2.04,\ \sigma^{(b)}(3) \cong 1.72,$

$\sigma^{(b)}(4) \cong 1.49$ years,

and further, from (10), the ship conditional lifetimes in the particular safety states, for $b = 1,2,$ are:

$\overline{m}^{(b)}(1) \cong 0.59,\ \overline{m}^{(b)}(2) \cong 0.32,\ \overline{m}^{(b)}(3) \cong 0.23,$

$\overline{m}^{(b)}(4) \cong 1.49$ years.

At the operation state z_5, i.e. at the navigation at restricted areas state the ship is built of $n_5 = 5$ subsystems S_1, S_2, S_4, S_5 and S_6 forming a series structure shown in *Figure 6*.

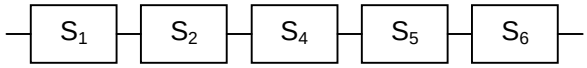

Figure 6. The scheme of the ship structure at the operation state z_5

We assume that the ship subsystems S_i, $i = 1,2,4,5,6,$ are its five-state components, i.e. z = 4, having the multi-state safety functions

$$s_i^{(5)}(t,\cdot) = [1, s_i^{(5)}(t,1), s_i^{(5)}(t,2), s_i^{(5)}(t,3), s_i^{(5)}(t,4)],$$
$$i = 1,2,4,5,6,$$

with exponential co-ordinates respectively given by:

- for the navigational subsystem S_1

$s_1^{(5)}(t,1) = \exp[-0.18t],\ s_1^{(5)}(t,2) = \exp[-0.22t],$

$s_1^{(5)}(t,3) = \exp[-0.24t],\ s_1^{(5)}(t,4) = \exp[-0.26t],$

- for the propulsion and controlling subsystem S_2

$s_2^{(5)}(t,1) = \exp[-0.06t],\ s_2^{(5)}(t,2) = \exp[-0.07t],$

$s_2^{(5)}(t,3) = \exp[-0.08t],\ s_2^{(5)}(t,4) = \exp[-0.09t],$

- for the hull subsystem S_4

$s_4^{(5)}(t,1) = \exp[-0.06t],\ s_4^{(5)}(t,2) = \exp[-0.08t],$

$s_4^{(5)}(t,3) = \exp[-0.09t],\ s_4^{(5)}(t,4) = \exp[-0.10t],$

- for the protection and rescue subsystem S_5

$s_5^{(5)}(t,1) = \exp[-0.14t],\ s_5^{(5)}(t,2) = \exp[-0.15t],$

$s_5^{(5)}(t,3) = \exp[-0.17t],\ s_5^{(5)}(t,4) = \exp[-0.20t],$

- for the anchor and mooring subsystem S_6

$s_6^{(5)}(t,1) = \exp[-0.02t],\ s_6^{(5)}(t,2) = \exp[-0.03t],$

$s_6^{(5)}(t,3) = \exp[-0.04t],\ s_6^{(5)}(t,4) = \exp[-0.05t].$

Assuming that the ship is in the safety state subsets $\{u, u+1,..., z\}$, $u = 1,2,3,4,$ if all its subsystems are in this safety state subset, according to *Definition 1* and *Definition 4*, the considered system is a five-state series system. Thus, by *Corollary 3*, after applying (10)–(11), we have its safety function given by

$$\bar{\boldsymbol{s}}_5^{(5)}(t,\cdot)$$
$$= [1, \bar{\boldsymbol{s}}_5^{(5)}(t,1), \bar{\boldsymbol{s}}_5^{(5)}(t,2), \bar{\boldsymbol{s}}_5^{(5)}(t,3), \bar{\boldsymbol{s}}_5^{(5)}(t,4)],\quad t \geq 0,$$

where

$$\bar{\boldsymbol{s}}_5^{(5)}(t,1) = \exp[-(0.18 + 0.06 + 0.06 + 0.14 + 0.02)t]$$
$$= \exp[-0.46t],$$

$$\bar{\boldsymbol{s}}_5^{(5)}(t,2) = \exp[-(0.22 + 0.07 + 0.08 + 0.15 + 0.03)t]$$
$$= \exp[-0.55t],$$

$$\bar{\boldsymbol{s}}_5^{(5)}(t,3) = \exp[-(0.24 + 0.08 + 0.09 + 0.17 + 0.04)t]$$
$$= \exp[-0.62t],$$

$$\bar{\boldsymbol{s}}_5^{(5)}(t,4) = \exp[-(0.26 + 0.09 + 0.10 + 0.20 + 0.05)t]$$
$$= \exp[-0.70t] \text{ for } t \geq 0.$$

The expected values and standard deviations of the ship lifetimes in the safety state subsets calculated from the above result, according to (16)-(17), are:

$m^{(6)}(1) \cong 2.17,\ m^{(6)}(2) \cong 1.82,\ m^{(6)}(3) \cong 1.61,$

$m^{(6)}(4) \cong 1.43$ years,

$\sigma^{(6)}(1) \cong 2.17$, $\sigma^{(6)}(2) \cong 1.82$, $\sigma^{(6)}(3) \cong 1.61$,

$\sigma^{(6)}(4) \cong 1.43$ years,

and further, from (10), the ship lifetimes in the particular safety states are:

$\overline{m}^{(6)}(1) \cong 0.35$, $\overline{m}^{(6)}(2) \cong 0.21$, $\overline{m}^{(6)}(3) \cong 0.18$,

$\overline{m}^{(6)}(4) \cong 1.43$ years.

At the operation state z_6, i.e. at the navigation at open sea state the ship is built of $n_6 = 4$ subsystems S_1, S_2, S_4, and S_5 forming a series structure shown in *Figure 7*.

Figure 7. The scheme of the ship structure at the operation state z_6

We assume that the ship subsystems S_i, $i = 1,2,4,5$, are its five-state components, i.e. $z = 4$, having the multi-state safety functions

$$s_i^{(6)}(t,\cdot) = [1, s_i^{(6)}(t,1), s_i^{(6)}(t,2), s_i^{(6)}(t,3), s_i^{(6)}(t,4)], \quad i = 1,2,4,5,$$

with exponential co-ordinates respectively given by:

- for the navigational subsystem S_1

$s_1^{(6)}(t,1) = \exp[-0.18t]$, $s_1^{(6)}(t,2) = \exp[-0.22t]$,

$s_1^{(6)}(t,3) = \exp[-0.24t]$, $s_1^{(6)}(t,4) = \exp[-0.26t]$,

- for the propulsion and controlling subsystem S_2

$s_2^{(6)}(t,1) = \exp[-0.06t]$, $s_2^{(6)}(t,2) = \exp[-0.07t]$,
$s_2^{(6)}(t,3) = \exp[-0.08t]$, $s_2^{(6)}(t,4) = \exp[-0.09t]$,

- for the hull subsystem S_4

$s_4^{(6)}(t,1) = \exp[-0.05t]$, $s_4^{(6)}(t,2) = \exp[-0.06t]$,

$s_4^{(6)}(t,3) = \exp[-0.07t]$, $s_4^{(6)}(t,4) = \exp[-0.08t]$,

- for the protection and rescue subsystem S_5

$s_5^{(6)}(t,1) = \exp[-0.15t]$, $s_5^{(6)}(t,2) = \exp[-0.16t]$,

$s_5^{(6)}(t,3) = \exp[-0.18t]$, $s_5^{(6)}(t,4) = \exp[-0.22t]$.

Assuming that the ship is in the safety state subsets $\{u, u+1, \ldots, z\}$, $u = 1,2,3,4$, if all its subsystems are in this safety state subset, according to *Definition 1* and *Definition 4*, the considered system is a five-state series system. Thus, by *Corollary 3*, after applying (10)–(11), we have its safety function given by

$$\bar{s}_4^{(7)}(t,\cdot) = [1, \bar{s}_4^{(7)}(t,1), \bar{s}_4^{(7)}(t,2), \bar{s}_4^{(7)}(t,3), \bar{s}_4^{(7)}(t,4)], t \geq 0,$$

where

$$\bar{s}_4^{(6)}(t,1) = \exp[-(0.18 + 0.06 + 0.05 + 0.15)t] = \exp[-0.44t],$$

$$\bar{s}_4^{(6)}(t,2) = \exp[-(0.22 + 0.07 + 0.06 + 0.16)t] = \exp[-0.51t],$$

$$\bar{s}_4^{(6)}(t,3) = \exp[-(0.24 + 0.08 + 0.07 + 0.18)t] = \exp[-0.57t],$$

$$\bar{s}_4^{(6)}(t,4) = \exp[-(0.26 + 0.09 + 0.08 + 0.22)t] = \exp[-0.67t] \text{ for } t \geq 0.$$

The expected values and standard deviations of the ship lifetimes in the safety state subsets calculated from the above result, according to (16)-(17), are:

$m^{(6)}(1) \cong 2.27$, $m^{(6)}(2) \cong 1.96$, $m^{(6)}(3) \cong 1.75$,

$m^{(6)}(4) \cong 1.49$ years,

$\sigma^{(6)}(1) \cong 2.27$, $\sigma^{(6)}(2) \cong 1.96$, $\sigma^{(6)}(3) \cong 1.75$,

$\sigma^{(6)}(4) \cong 1.49$ years,

and further, from (18), the ship lifetimes in the particular safety states are:

$\overline{m}^{(6)}(1) \cong 0.31,\ \overline{m}^{(6)}(2) \cong 0.21,\ \overline{m}^{(6)}(3) \cong 0.26,$

$\overline{m}^{(6)}(4) \cong 1.49$ years.

In the case when the system operation time is large enough, the unconditional safety function of the ship is given by the vector

$$\boldsymbol{s}_6(t,\cdot)$$

$$= [1, \boldsymbol{s}_6(t,1),\ \boldsymbol{s}_6(t,2),\ \boldsymbol{s}_6(t,3),\ \boldsymbol{s}_6(t,4)],\ t \geq 0,$$

where, according to (14), the co-ordinates are

$$\boldsymbol{s}_6(t,1) = p_1 \overline{\boldsymbol{s}}_4^{(1)}(t,1) + p_2 \overline{\boldsymbol{s}}_4^{(2)}(t,1) + p_3 \overline{\boldsymbol{s}}_5^{(3)}(t,1)$$

$$+ p_4 \overline{\boldsymbol{s}}_5^{(4)}(t,1) + p_5 \overline{\boldsymbol{s}}_5^{(5)}(t,1) + p_6 \overline{\boldsymbol{s}}_4^{(6)}(t,1)$$

$$= 0.145 \cdot \exp[-0.25t] + 0.098 \cdot \exp[-0.25t]$$

$$+ 0.004 \cdot \exp[-0.38t] + 0.007 \cdot \exp[-0.38t]$$

$$+ 0.374 \cdot \exp[-0.46t] + 0.374 \cdot \exp[-0.44t],$$

$$\boldsymbol{s}_6(t,2) = p_1 \overline{\boldsymbol{s}}_4^{(1)}(t,2) + p_2 \overline{\boldsymbol{s}}_4^{(2)}(t,2) + p_3 \overline{\boldsymbol{s}}_5^{(3)}(t,2)$$

$$+ p_4 \overline{\boldsymbol{s}}_5^{(4)}(t,2) + p_5 \overline{\boldsymbol{s}}_5^{(5)}(t,2) + p_6 \overline{\boldsymbol{s}}_4^{(6)}(t,2)$$

$$= 0.145 \cdot \exp[-0.31t] + 0.098 \cdot \exp[-0.31t]$$

$$+ 0.004 \cdot \exp[-0.49t] + 0.007 \cdot \exp[-0.49t]$$

$$+ 0.3.74 \cdot \exp[-0.55t] + 0.372 \cdot \exp[-0.51t],$$

$$\boldsymbol{s}_6(t,3) = p_1 \overline{\boldsymbol{s}}_4^{(1)}(t,3) + p_2 \overline{\boldsymbol{s}}_4^{(2)}(t,3) + p_3 \overline{\boldsymbol{s}}_5^{(3)}(t,3)$$

$$+ p_4 \overline{\boldsymbol{s}}_5^{(4)}(t,3) + p_5 \overline{\boldsymbol{s}}_5^{(5)}(t,3) + p_6 \overline{\boldsymbol{s}}_4^{(6)}(t,3)$$

$$= 0.145 \cdot \exp[-0.39t] + 0.098 \cdot \exp[-0.39t]$$

$$+ 0.004 \cdot \exp[-0.58t] + 0.007 \cdot \exp[-0.58t]$$

$$+ 0.0374 \cdot \exp[-0.62t] + 0.372 \cdot \exp[-0.57t],$$

$$\boldsymbol{s}_6(t,4) = p_1 \overline{\boldsymbol{s}}_4^{(1)}(t,4) + p_2 \overline{\boldsymbol{s}}_4^{(2)}(t,4) + p_3 \overline{\boldsymbol{s}}_5^{(3)}(t,4)$$

$$+ p_4 \overline{\boldsymbol{s}}_5^{(4)}(t,4) + p_5 \overline{\boldsymbol{s}}_5^{(5)}(t,4) + p_6 \overline{\boldsymbol{s}}_4^{(6)}(t,4)$$

$$= 0.145 \cdot \exp[-0.44t] + 0.098 \cdot \exp[-0.44t]$$

$$+ 0.004 \cdot \exp[-0.67t] + 0.007 \cdot \exp[-0.67t]$$

$$+ 0.374 \cdot \exp[-0.70t] + 0.372 \cdot \exp[-0.67t] \text{ for } t \geq 0.$$

The mean values and variances of the system unconditional lifetimes in the safety state subsets, according to (31) and (33), respectively are

$$m(1) = p_1 m^{(1)}(1) + p_2 m^{(2)}(1) + p_3 m^{(3)}(1)$$

$$+ p_4 m^{(4)}(1) + p_5 m^{(5)}(1) + p_6 m^{(6)}(1),$$

$$\cong 0.145 \cdot 4.00 + 0.098 \cdot 4.00 + 0.004 \cdot 2.63$$

$$+ 0.007 \cdot 2.63 + 0.374 \cdot 2.17 + 0.372 \cdot 2.27 = 2.66.$$

$$[\sigma(1)]^2 \cong 2[0.145 \cdot [4.00]^2 + 0.098 \cdot [4.00]^2$$

$$+ 0.004 \cdot [2.63]^2 + 0.007 \cdot [2.63]^2 + 0.374 \cdot [2.17]^2$$

$$+ 0.372 \cdot [2.27]^2] - [2.66]^2 = [2.87]^2,\ \sigma(1) \cong 2.87,$$

$$m(2) = p_1 m^{(1)}(2) + p_2 m^{(2)}(2) + p_3 m^{(3)}(2)$$

$$+ p_4 m^{(4)}(2) + p_5 m^{(5)}(2) + p_6 m^{(6)}(2),$$

$$\cong 0.145 \cdot 3.26 + 0.098 \cdot 3.26 + 0.004 \cdot 2.04$$

$$+ 0.007 \cdot 2.04 + 0.374 \cdot 1.82 + 0.372 \cdot 1.96 = 2.22,$$

$$[\sigma(2)]^2 \cong 2[0.145 \cdot [3.26]^2 + 0.098 \cdot [3.26]^2$$

$$+ 0.004 \cdot [2.04]^2 + 0.007 \cdot [2.04]^2 + 0.374 \cdot [1.82]^2$$

$$+ 0.372 \cdot [1.96]^2] - [2.22]^2 = [2.38]^2,\ \sigma(2) \cong 2.38,$$

$$m(3) = p_1 m^{(1)}(3) + p_2 m^{(2)}(3) + p_3 m^{(3)}(3)$$

$$+ p_4 m^{(4)}(3) + p_5 m^{(5)}(3) + p_6 m^{(6)}(3),$$

$$\cong 0.145 \cdot 2.56 + 0.098 \cdot 2.56 + 0.004 \cdot 1.72$$

$$+ 0.007 \cdot 1.72 + 0.374 \cdot 1.61 + 0.372 \cdot 1.75 = 1.89,$$

$$[\sigma(3)]^2 \cong 2[0.145 \cdot [2.56]^2 + 0.098 \cdot [2.56]^2$$

$$+ 0.004 \cdot [1.72]^2 + 0.007 \cdot [1.72]^2 + 0.374 \cdot [1.61]^2$$

$$+ 0.372 \cdot [1.75]^2] - [1.89]^2 = [1.97]^2, \ \sigma(3) \cong 1.97,$$

$$m(3) = p_1 m^{(1)}(4) + p_2 m^{(2)}(4) + p_3 m^{(3)}(4)$$

$$+ p_4 m^{(4)}(4) + p_5 m^{(5)}(4) + p_6 m^{(6)}(4),$$

$$\cong 0.145 \cdot 2.27 + 0.098 \cdot 2.27 + 0.004 \cdot 1.49$$

$$+ 0.007 \cdot 1.49 + 0.374 \cdot 1.43 + 0.372 \cdot 1.49 = 1.66,$$

$$[\sigma(4)]^2 \cong 2[0.145 \cdot [2.27]^2 + 0.098 \cdot [2.27]^2$$

$$+ 0.004 \cdot [1.49]^2 + 0.007 \cdot [1.49]^2 + 0.374 \cdot [1.43]^2$$

$$+ 0.372 \cdot [1.49]^2] - [1.66]^2 = [1.73]^2, \ \sigma(4) \cong 1.3.$$

The mean values of the system lifetimes in the particular safety states, by (34), are

$$\overline{m}(1) = m(1) - m(2) = 0.44,$$

$$\overline{m}(2) = m(2) - m(3) = 0.33,$$

$$\overline{m}(3) = m(3) - m(4) = 0.23,$$

$$\overline{m}(4) = m(4) = 1.66.$$

If the critical safety state is $r = 2$, then the system risk function, according to (6), is given by

$$\boldsymbol{R}(t) = 1 - \boldsymbol{s}_6(t, 2)$$

$$= 0.145 \cdot \exp[-0.31t] + 0.098 \cdot \exp[-0.31t]$$

$$+ 0.004 \cdot \exp[-0.49t] + 0.007 \cdot \exp[-0.49t]$$

$$+ 0.3.74 \cdot \exp[-0.55t] + 0.372 \cdot \exp[-0.51t] \text{ for } t \geq 0.$$

Hence, the moment when the system risk function exceeds a permitted level, for instance $\delta = 0.05$, from (7), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 0.11 \text{ years.}$$

9. Conclusion

In the paper the multi-state approach to the safety analysis and evaluation of systems related to their variable operation processes has been considered. Theoretical definitions and preliminary results have been illustrated by the example of their application in the safety evaluation of a ship transportation system with changing in time its operation states. The ship safety structure and its safety subsystems characteristics are changing in different states what makes the analysis more complicated but also more precise than the analysis performed in [2]. However, the varying in time ship safety structure used in the application is very general and simplified and the subsystems safety data are either not precise or not real and therefore the results may only be considered as an illustration of the proposed methods possibilities of applications in ship safety analysis. Anyway, the obtained evaluation may be a very useful example in simple and quick ship system safety characteristics evaluation, especially during the design and when planning and improving her operation processes safety and effectiveness.

The results presented in the paper suggest that it seems reasonable to continue the investigations focusing on the methods of safety analysis for other more complex multi-state systems and the methods of safety evaluation related to the multi-state systems in variable operation processes [9], [10] and their applications to the ship transportation systems [5].

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Improvement of manufacturing cells with unreliable machines

Keywords

manufacturing cell, intercellular transfer, markov chains, availability, simulation, performances

Abstract

The performance of cellular manufacturing (CM) is conditioned by disruptive events, such as failure of machines, which randomly occur and penalize the performance of the cells and disturb seriously the smooth working of the factory. To overcome the problems caused by the breakdowns, we develop a solution, based on the principle of virtual cell (VC) and the notion of intercellular transfer that can improve performances of the system. In this context, we use an analytical method based on Markov chains to model the availability of the cell. The found results are validated using simulation. The proposed solution in this paper confirmed that it is possible to reduce the severity of breakdowns in the CM system and improve the performances of the cells through an intercellular transfer. Simulation allowed a validation of the analytical model and showed the contribution of the suggested solution.

1. Introduction

Group technology (GT) is a manufacturing philosophy that has attracted a lot of attention because of its positive impacts in the batch-type production. Cellular manufacturing (CM) is an application of GT to manufacturing. It has emerged in the last two decades as an innovative manufacturing strategy that collects the advantages of both product and process oriented system for a medium-volume and medium-variety production. By applying the GT concept and CM system, manufacturing companies can achieve many benefits including reduced set-up times, reduced work-in-process, less material handling cost, higher throughput rates.

The performance of a cellular manufacturing system is conditioned by disruptive events (e.g., failures of machines) that randomly occur and penalize the performance of the system. Therefore, equipment that falls in breakdown generates eventually the interruption of the whole cell. Consequently, failure of the machine implies total loss of cell capabilities and it leads to the partial deterioration of the performance of the total system [1]. Therefore, the application of an efficient strategy against these perturbations permits to improve the performance of those production systems.

Few researches can be found related to the effect of the failure on the operation of CM system. Some of these discussed the efficient maintenance politics to improve the performance of the cellular manufacturing [1], [4] and [5]. Others developed new coefficients of similarity that consider a number of alternative ways during the machine breakdown [3].

This paper is concerned with problems of the availability of production cells facing random event due to an internal disruption of breakdown-machine type. It uses intercellular transfer as a policy to surmount this type of disruption. The proposed solution is based on the external routing flexibility: the ability to release parts to alternative cell. This policy is assessed through modelling of the production cell and its simulation with Arena software.

The remainder of this paper is organized as follows. In section 2 and 3, we formulate a comprehensive idea of intercellular transfer policy and we give the method for modelling the availability of the cell. Section 4 presents a comprehensive simulation model to validate the analytical model and evaluate the policy of intercellular transfer. Finally, we recapitulate in section 5 the main conclusion of this work and we make recommendations for future research.

2. Description of the manufacturing cell

The shop consists of several machines that are grouped into different group technology cells operating in a static environment. Each cell is characterized by a classically structured flow line with (m) machines in series. These machines are unreliable, with operation dependent failures, and have a constant failure and repair rates λ_i and μ_i ($i = 1\ldots m$).

In this section, we introduce the parameters that characterize the behaviour of machines in a material flow model. Each machine M_i is characterized by three parameters:

- Average utilization rate Tu_i: This is the rate at which material flows gets processed through the machine M_i in the absence of failure.
- Average failure rate λ_i: This is the rate at which machine M_i fails when working at its maximum processing rate (100%).
- Average repair rate μ_i: This is the rate at which machine Mi gets repaired if it is down for a failure.

Using theses parameters Tu_i, λ_i and μ_i, we can define the basic parameters of an isolated machine M_i:

- There is the average failure rate $\lambda_{i.Fc}$: This is the rate at which machine Mi fails when working at its utilization rate Tu_i.

$$\lambda_{i.Fc} = \lambda_i \cdot F_c \tag{1}$$

where

$$F_c = Tu_i . \tag{2}$$

- Isolated efficiency $A_i(\infty)$: that is the average proportion of time during which machine Mi would be operational. It is equal to the steady-state availability. $A_i(\infty)$ is defined in terms of parameters as follows:

$$A_i(\infty) = \frac{\mu_i}{\lambda_{i.Fc} + \mu_i}. \tag{3}$$

If the single machine, whose parameters are defined as above, were part of a production cell, additional parameters would be needed to characterize it. These parameters are given below.

- Average utilization rate Tu_i^*: in a production cell environment, a cell allows the manufacturing of family products that made of several types, which forms various batches, having different sequences of operations. We regard $Tu_{i.k}$ as the utilisation rate of the machine M_i during the manufacture of the batch k. Therefore to determine the machine utilisation rate, it is enough to calculate its utilisation rate during the manufacturing of family products (various batches allocated to cell) without taking account of the effect of the breakdowns of the other machines. This rate is given by the following expression:

$$Tu_i = \frac{\sum_{k=1}^{n} Tu_{i.k} \cdot t_{i.k}}{tt_i} \tag{4}$$

with:

- $t_{i.k}$: the time put by the machine (i) to manufacture the batch k,
- tt_i: the total time to manufacture family products on the machine (i),
- n: number of batches treated in manufacturing cell.

Taken account of the condition which says that the breakdown of a machine generates the interruption of the cell, i.e. that we do not have a breakdown at the same time inside a cell; the utilization rate will be approximated by the following expression:

$$Tu_i^* = Tu_i \cdot E , \tag{5}$$

- where E is the efficiency of the cell given by the following expression [2]

$$E = \frac{1}{1 + \sum_{i=1}^{m} \frac{\lambda_{i.Fc}}{\mu_i}}. \tag{6}$$

- Failure rate $\lambda_{i\ Fc*}$: This is the rate at which machine M_i fails when it work in a cell.

$$\lambda_{i.Fc*} = \lambda_i \cdot F_c* \tag{7}$$

where

$$F_c* = Tu_i \cdot \frac{1}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{\mu_j}{\lambda_{j.Fc}}}, \tag{8}$$

with: m represents the number of machine in the cell.

Then the stationary availability will be given by the expression (9).

$$A_{cell}(\infty) = \frac{1}{1+\sum_{i=1}^{m} \frac{\lambda_{i.Fc^*}}{\mu_i}}. \quad (9)$$

3. Problem definition and solution technique

The failure of only one machine in the cellular manufacturing system can disrupt the product flow in the whole system. Indeed, this failure is going to generate the interruption of different machine in the corresponding cell. It implies a reduction of the machine utilization rate, a reduction of the production capacity and a dissatisfaction of customers. In this study we are interested in a solution based on the external routing flexibility. This solution has the tendency to apply a strategy that permits to reduce the severity of the failure by the application of an intercellular transfer policy in case of breakdown of a machine of the cell. For a production cell treating a type of product, the breakdown of a machine doesn't imply the interruption of the production in this cell. Sometimes the continuity of the production will be assured by the transfer of the product flow toward a neighbouring cell admitting an inactive machine capable to treat this product type. By this action, it will be possible to continue the process of manufacturing in presence of the fault. The cell will be formed by machines of the first cell and the standby machine of the second cell. The creation of this intercellular transfer is the origin of the formation of the virtual cells. Then virtual cells are created periodically, for instance at machine breakdown, depending on the presence of the failure and the standby machine. It is necessary to note that the realization of intercellular transfer can bring advantages at the level of performance of the system taking into account the transfer duration, the inactivity delay of the standby machine and the repair duration.

Therefore, for the studied system the strategy consists in applying the intercellular transfer of the cell (a) toward the cell (b) in case of failure of one of machines of the first cell (see *Figure 1*).

The production cell can be assimilated to a repairable system operating according to a set structure composed of (M) independent modules. The number of modules is equal to the number of machines constituting the studied cell (a). Then the block diagram of cell reliability (a) is given by *Figure 2*.

We study the system in steady state; that is where the probability of the system being in a given state does not depend on the initial conditions. In the case of an application of a transfer policy, the availability of the cell is determined from different module availability. The availability of module (2) will be determined with the process of Markov chains.

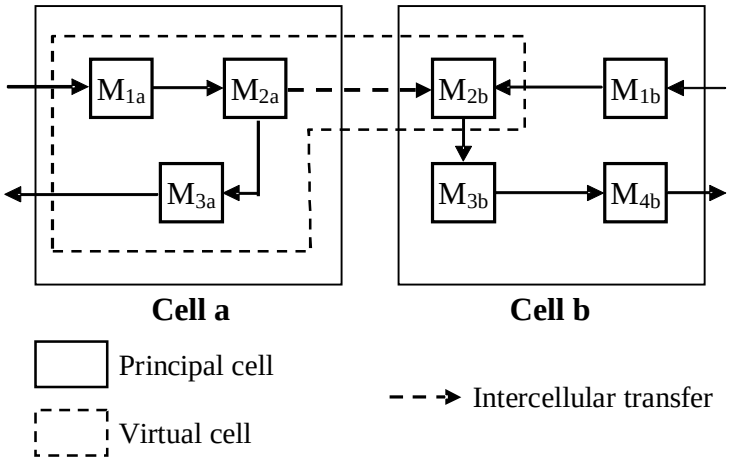

Figure 1. Manufacturing system with intercellular transfer

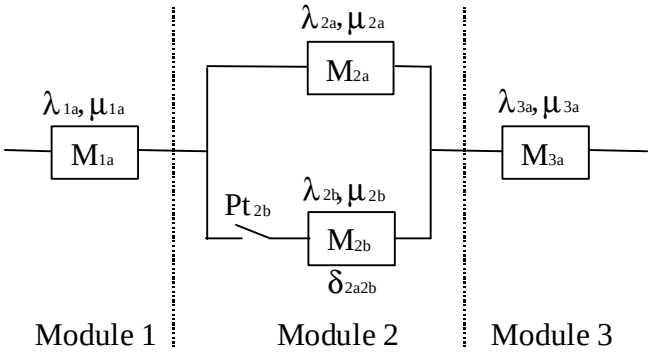

Figure 2. Function block diagram of cell reliability (a)

For this raison, we are considering the module represented by both machines: the main machine and the standby one. The main machine M_1 belonging to the studied cell, exhibits breakdown and repair rates λ_1 and μ_1 respectively. The standby machine M_2 is characterized respectively by breakdown and repair rates λ_2 and μ_2. In case of a breakdown of the main machine, the probability of transfer toward the replacement machine M_2 is equal to Pt_2 with a transfer rate equal to δ_{12}.

The Markov process describing the evolution of the stochastic behaviour of the module is given by the state diagram depicted in *Figure 3*.

Given a set of differential equations developed from the diagram, we can determine the module availability represented in the following operational states 1, 2_1, and 2_2. Therefore, the stationary availability is valued by the expression (10).

$$A_{Module}(\infty) = \frac{\delta_{12}\cdot(\mu_1\cdot\mu_2\cdot(\mu_1+\lambda_1+\mu_2+\lambda_2)+\lambda_1\cdot Pt_2\cdot(\mu_1\cdot\lambda_2+\mu_1\cdot\mu_2+\lambda_1\cdot\mu_2+\mu_2^{\,2}))}{\left(\begin{array}{l}\lambda_1\cdot\lambda_2\cdot\delta_{12}\cdot Pt_2(\mu_1+\lambda_1)+\delta_{12}\cdot\mu_2\cdot(\lambda_1\cdot(\mu_2+\lambda_1)+\lambda_2\cdot(\mu_1+\lambda_1))\\ +\mu_1\cdot\mu_2\cdot\lambda_1\cdot Pt_2\cdot(\lambda_2+\mu_2+\mu_1+\lambda_1)+\mu_1\cdot\mu_2\cdot\delta_{12}\cdot(\mu_1+2\cdot\lambda_1+\mu_2)\end{array}\right)} \quad (10)$$

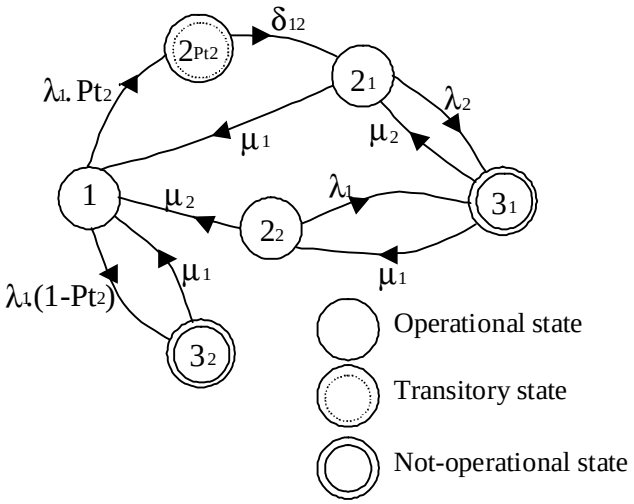

Figure 3. Diagram of module state

Besides, the expression (10) shows that intercellular transfer in case of failure permits to improve the stationary availability of the module, only in the case where the transfer rate is superior to a limit value (δ_{12}*). This value is given by the expression (11).

$$\delta_{12}^{*} = \frac{\mu_1^{\,2} \cdot (\mu_1 + \lambda_1 + \mu_2 + \lambda_2)}{\mu_1^{\,2} + 2 \cdot \mu_1 \cdot \lambda_1 + \lambda_1^{\,2} + \mu_1 \cdot \mu_2 + \mu_2 \cdot \lambda_1} . \qquad (11)$$

With the help of a good team of maintenance arranging some necessary logistical means, the time of preparation of an intercellular transfer can be reduced. In these conditions, times of transfer preparation can be disregarded in front of times between failings and repair machine times. Therefore, the expression of the availability of the module will be given by the expression (12).

$$A_{Module}(\infty) = \frac{\left(\mu_1 \cdot \mu_2 \cdot (\mu_1 + \lambda_1 + \mu_2 + \lambda_2) + \lambda_1 \cdot Pt_2 \cdot \left(\mu_1 \cdot \lambda_2 + \mu_1 \cdot \mu_2 + \lambda_1 \cdot \mu_2 + \mu_2^{\,2}\right)\right)}{\left(\begin{array}{l} \lambda_1 \cdot \lambda_2 \cdot Pt_2 (\mu_1 + \lambda_1) + \mu_2 \cdot (\lambda_1 \cdot (\mu_2 + \lambda_1) + \lambda_2 \cdot (\mu_1 + \lambda_1)) \\ + \mu_1 \cdot \mu_2 \cdot (\mu_1 + 2 \cdot \lambda_1 + \mu_2) \end{array}\right)} \qquad (12)$$

4. Simulation of manufacturing cells

To conduct our simulation, we defined first the problem and stated our objectives. The problem facing cellular manufacturing is the effect of breakdown machines. The objectives of this simulation were to examine the performance of the system with and without the policy of intercellular transfer in the event of breakdowns and to validate the analytical model.
We consider the system shown in the *Figure 4*. The cell (a) is constituted of three different machines dedicated to the manufacturing of two product types. The cell (b) is formed of four different machines capable to manufacture three product types.

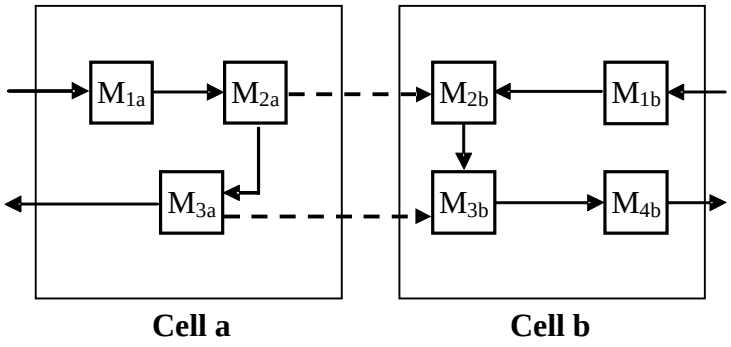

Figure 4. Manufacturing system with intercellular transfer

Table 1 summarizes the routing and the processing time information of each part

Table 1. The routing and processing times of each part

	Product type	Batch size	Machine (Processing times)
Cell a	1	19	M_1(19) à M_2(12) à M_3(13)
	2	30	M_2(10) à M_3(9)
Cell b	3	16	M_1(14) à M_2(11) à M_4(13)
	4	25	M_1(16) à M_3(14) à M_4(11)
	5	20	M_2(17) à M_3(13) à M_4(7)

The mean time to failure (*MTTF*) and the mean time to repair (*MTTR*) for all machines are shown in *Table 2*.

Table 2. *MTTF* and *MTTR* of each machine in the system

Machine	M_{1a}	M_{2a}	M_{3a}	M_{b1}	M_{b2}	M_{b3}	M_{b4}
MTTF	6000	3500	3000	5500	4000	3500	6500
MTTR	500	420	350	400	400	360	380

In this paper, the batches arrival is considered cyclic. Indeed, the manufacturing of a new batch is only permitted if the previous batch is finished. In addition, products are generated in a cyclic manner. For a given batch, the time between the arrivals of two products is equal to the time of execution of the first operation.

We perform discrete flow simulation using simulation software called ARENA. We simulate the system during a 12 years horizon; during the first 60000 minutes, statistics are not collected. This warm-up period (the first 60000 minutes) is used to avoid transient effects on the final results.

4.1. Study of the system without intercellular transfer

In this section we simulate our system without the intercellular transfer policy in order to validate our model of cellular system and the developed expressions. The following table summarizes the values of the parameters developed according to the analytical model and that of simulation.

Table 3. Results from the approximate method and simulation

Machines	M_{1a}	M_{2a}	M_{3a}	M_{b1}	M_{b2}	M_{b3}	M_{b4}
$Tu^*_{analytic}$ (%)	42.53	62.21	60.91	50.77	41.98	49.63	50.69
$Tu^*_{simulation}$ (%)	42.56	62.26	60.96	50.61	41.85	49.47	50.53
Error (%)	< 1 %						
$A(\infty)_{analytic}$ (%)	96.46	92.54	92.89	96.31	95.8	94.9	97.04
$A(\infty)_{simulation}$ (%)	96.53	92.67	92.84	96.01	95.68	94.79	97.37
Error (%)	< 0.5 %						

The obtained results show that the error between the values given by the analytical formulation and simulation is relatively small. Then the approximation of the cell to a production line working under a continuous and constant load is sufficiently robust to estimate the availability of the autonomous cells.

4.2. Study of the system with intercellular transfer

For the studied system, the strategy consists in applying the intercellular transfer of the cell (a) toward the cell (b) in case of failure of one of machines of the first cell (see *Figure 4*). We assume that preparation times of intercellular transfer are negligible compared to times between failings and repair machine times.
The values of the availability governed by the two types of policy with and without transfer are calculated according to the analytical model. The obtained results show the improvement made by the application of the intercellular transfer policy in term of cell availability (see *Figure 5*).
To evaluate the performance of our policy with simulation tool and to support the results of the analytical model, we select the productivity of the cell, presented in the number of produced pieces, and machine utilization rate as performance criteria (see *Table 4*).

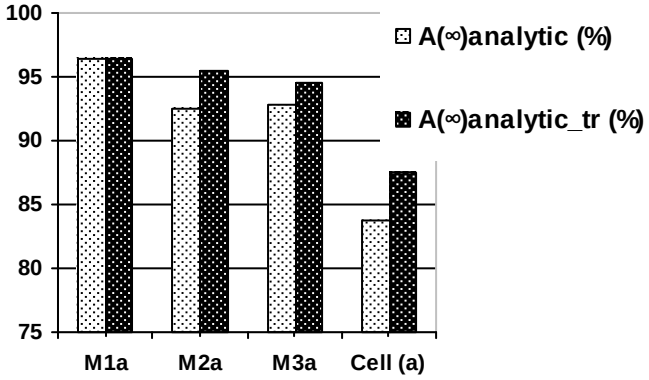

Figure 5. Improvement of the availability by the intercellular transfer policy predicted by the analytical model

Table 4. Performance of the cell (a) with and without intercellular transfer policy from simulation

Machine		Cell (a)		
		M_{1a}	M_{2a}	M_{3a}
Utilization rate (%)	Without transfer	42.56	62.26	60.96
	With transfer	44.12	62.95	62.40
Number of manufactured products	Without transfer	346650		
	With transfer	359337		

The simulation results show that intercellular transfer policy improves the machines utilisation rate and the productivity of the cell. This improvement reflects the augmentation of cell availability.

5. Conclusion

In this paper, an analysis of cellular manufacturing system is presented. The notion of virtual cells and intercellular transfer allowed the development of a solution, which overcomes the effect of failures by continuing the process with the machine of the adjacent cell. Analytical modelling makes possible to determine the expression of the availability of the cell and to explain the improvement obtained by applying the intercellular transfer policy. The simulation results validated our analytical model and proved the effectiveness of the applied policy in the improvement of the system performance.

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Applications of semi-Markov processes in reliability

Keywords

semi-Markov processes, reliability, random failure rate, cold standby system with repair

Abstract

The basic definitions and theorems from the semi-Markov processes theory are discussed in the paper. The semi-Markov processes theory allows us to construct the models of the reliability systems evolution within the time frame. Applications of semi-Markov processes in reliability are considered. Semi-Markov model of the cold standby system with repair, semi-Markov process as the reliability model of the operation with perturbations and semi-Markov process as a failure rate are presented in the paper.

1. Introduction

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy, W.L. Smith, and L.Takacs in 1954-55. The essential developments of semi-Markov processes theory were proposed by Cinlar [3], Koroluk & Turbin [13], Limnios & Oprisan [14]. We would apply only semi-Markov processes with a finite or countable state space. The semi-Markov processes are connected to the Markov renewal processes.
The semi-Markov processes theory allows us to construct many models of the reliability systems evolution through the time frame.

2. Definition of semi-Markov processes with a discrete state space

Let *S* be a discrete (finite or countable) state space and let $R_+ = [0,\infty)$, $N_0 = \{0,1,2,...\}$. Suppose, that ξ_n, ϑ_n, $n = 0,1,2,...$ are the random variables defined on a joint probabilistic space (Ω, F, *P*) with values on *S* and R_+ respectively. A two-dimensional random sequence $\{(\xi_n, \vartheta_n), n = 0,1,2,...\}$ is called a Markov renewal chain if for all
$i_0,....,i_{n-1}, i \in S, t_0,...,t_n \in R_+, \quad n \in N_0$:

$$1.\ P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \,|\, \xi_n = i, \vartheta_n = t_n,...,\xi_0 = i_0, \vartheta_0 = t_0\}$$

$$= P\{\xi_{n+1} = j, \vartheta_{n+1} \le t \,|\, \xi_n = i\} = Q_{ij}(t), \qquad (1)$$

$$2.\ P\{\xi_0 = i_o, \vartheta_0 = 0\} = P\{\xi_0 = i_0\} = p_{i_o} \qquad (2)$$

hold.
From the above definition it follows that a Markov renewal chain is a homogeneous two-dimensional Markov chain such that the transition probabilities do not depend on the second component. It is easy to notice that a random sequence $\{\xi_n : n = 0,1,2,...\}$ is a homogeneous one-dimensional Markov chain with the transition probabilities

$$p_{ij} = P\{\xi_{n+!} = j \,|\, \xi_n = i\} = \lim_{t\to\infty} Q_{ij}(t). \qquad (3)$$

The matrix

$$\mathbf{Q}(t) = \left\lfloor Q_{ij}(t): \quad i, j \in S \right\rfloor, \qquad (4)$$

Is called a Markov renewal kernel. Both Markov renewal kernel and the initial distribution define the Markov renewal chain. This fact allows us to construct a semi-Markov process.
Let

$$\tau_0 = \vartheta_0 = 0,$$

$$\tau_n = \vartheta_1 + ... + \vartheta_n, \tau_\infty = \sup\{\tau_n : n \in N_0\}$$

A stochastic process $\{X(t) : t \ge 0\}$ given by the following relation

$$X(t)=\xi_n \quad \text{for} \quad t\in[\tau_n,\tau_{n+1}) \tag{5}$$

is called a semi-Markov process on S generated by the Markov renewal chain related to the kernel $Q(t)$, $t\geq 0$ and the initial distribution *p*.
Since the trajectory of the semi-Markov process keeps the constant values on the half-intervals $[\tau_n,\tau_{n+1})$ and it is a right-continuous function, from equality $X(\tau_n)=\xi_n$, it follows that the sequence $\{X(\tau_n): n=0,1,2,...\}$ is a Markov chain with the transition probabilities matrix

$$\mathbf{P}=[p_{ij}: i,j\in S] \tag{6}$$

The sequence $\{X(\tau_n): n=0,1,2,...\}$ is called an embedded Markov chain in a semi-Markov process $\{X(t): t\geq 0\}$.
The function

$$F_{ij}(t)=P\{\tau_{n+1}-\tau_n\leq t \mid X(\tau_n)=i, X(\tau_{n+1})=j\}$$

$$=\frac{Q_{ij}(t)}{p_{ij}} \tag{7}$$

is a cumulative probability distribution of a random variable T_{ij} that is called holding time of a state i, if the next state will be j. From (11) we have

$$Q_{ij}(t)=p_{ij}F_{ij}(t). \tag{8}$$

The function

$$G_i(t)=P\{\tau_{n+1}-\tau_n\leq t \mid X(\tau_n)=i\}=\sum_{j\in S}Q_{ij}(t) \tag{9}$$

is a cumulative probability distribution of a random variable T_i that is called waiting time of the state i. The waiting time T_i means the time being spent in state i *when* we do not know the successor state.
A stochastic process $\{N(t): t\geq 0\}$ defined by

$$N(t)=n \quad for \quad t\in[\tau_n,\tau_{n+1}) \tag{10}$$

is called a counting process of the semi-Markov process $\{X(t): t\geq 0\}$.
The semi-Markov process $\{X(t): t\geq 0\}$ is said to be regular if for all $t\geq 0$

$$P\{N(t)<\infty\}=1$$

It means that the process $\{X(t): t\geq 0\}$ *has* the finite number of state changes on a finite period.
Every Markov process $\{X(t): t\geq 0\}$ with the discrete space S and the right-continuous trajectories keeping constant values on the half-intervals, with the generating matrix of the transition rates $A=[\alpha_{ij}: \ i,j\in S]$, $0<-\alpha_{ii}=\alpha_i<\infty$ is the semi-Markov process with the kernel

$$\mathbf{Q}(t)=[Q_{ij}(t): i,j\in S],$$

Where

$$Q_{ij}(t)=p_{ij}(1-e^{-\alpha_{ij}t}),\ t\geq 0,$$

$$p_{ij}=\frac{\alpha_{ij}}{\alpha_i} \text{ for } i\neq j \text{ and } p_{ii}=0$$

In the reliability models the parameters and characteristics of a semi-Markov process are interpreted as the reliability characteristics and parameters of the system.

3. Transition probabilities of a semi-Markov process

The transition probabilities of the semi-Markov process are introduced as follows:

$$P_{ij}(t)=P\{X(t)=j \mid X(0)=i\},\ i,j\in S. \tag{11}$$

Applying the Markov property of the semi-Markov process at the jump moments, as a result, we obtain Markov renewal equation for the transitions probabilities, [4], [12]

$$P_{ij}(t)=\delta_{ij}[1-G_i(t)]+\sum_{k\in S}\int_0^t P_{kj}(t-x)dQ_{ik}(x), \tag{12}$$
$$i,j\in S.$$

Using Laplace-Stieltjes transformation we obtain the system of linear equation

$$\tilde{p}_{ij}(s)=\delta_{ij}[1-\tilde{g}_i(s)]+\sum_{k\in S}\tilde{q}_{ik}(s)\tilde{p}_{kj}(s), \tag{13}$$
$$i,j\in S$$

where the transforms

$$\tilde{p}_{ij}(s)=\int_0^\infty e^{-st}dP_{ij}(t)$$

are unknown while the transforms

$$\tilde{q}_{ik}(s) = \int_0^\infty e^{-st} dQ_{ik}(t), \quad \tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t)$$

are given.
Passing to matrices we obtain the following equation

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{g}}(s)] + \tilde{\mathbf{q}}(s)\tilde{\mathbf{p}}(s), \tag{14}$$

where

$$\tilde{\mathbf{p}}(s) = [\tilde{p}_{ij}(s) : i, j \in S], \quad \tilde{\mathbf{q}}(s) = [\tilde{q}_{ij}(s) : i, j \in S],$$

$$\tilde{\mathbf{g}}(s) = [\delta_{ij}(1 - \tilde{g}_i(s)) : i, j \in S].$$

In many cases the transitions probabilities $P_{ij}(t)$ and the states probabilities

$$P_j(t) = P\{X(t) = j\}, \quad j \in S, \tag{15}$$

approach constant values for large t

$$P_{ij} = \lim_{t\to\infty} P_{ij}(t), \quad P_j = \lim_{t\to\infty} P_j(t) \ . \tag{16}$$

To formulate the appropriate theorem, we have to introduce a random variable

$$\Delta_j = \min\{n \in N : X(\tau_n) = j\}, \tag{17}$$

That denotes the time of first arrival at state *j*. A number

$$f_{ij} = P\{\Delta_j < \infty \mid X(\tau_n) = i\} \tag{18}$$

is the probability that the chain that leaves state *i* will sooner or later achieve the state *j*.
As a conclusion of theorems presented by Korolyuk and Turbin [13], we have obtained following theorem

Theorem 1.
Let $\{X(t) : t \geq 0\}$ be a semi-Markov process with a discrete state space S and continuous kernel $\mathbf{Q}(t) = [Q_{ij}(t): \ i, j \in S]$. *If* the embedded Markov chain $\{X(\tau_n) : n = 0,1,2,...\}$, contains one positive recurrent class *C*, such that for each state $i \in S, \ j \in C, \quad f_{ij} = 1$ and $0 < E(T_i) < \infty, i \in S$, then

$$P_{ij} = \lim_{t\to\infty} P_{ij}(t) = P_j = \lim_{t\to\infty} P_j(t) = \frac{\pi_j E(T_j)}{\sum_{i\in S} \pi_j E(T_j)} \tag{19}$$

where $\pi = [\pi_j, \quad j \in S]$ is the unique stationary distribution of the embedded Markov chain that satisfies system of equations.

$$\sum_{i\in S} \pi_i p_{ij} = \pi_j, \ j \in S, \ \sum_{i\in S} \pi_i = 1. \tag{20}$$

4. First passage time from the state *i* to the states subset *A*.

The random variable

$$\Theta_A = \tau_{\Delta_A},$$

where

$$\Delta_A = \min\{n \in N : X(\tau_n) \in A\},$$

denotes the time of first arrival of semi-Markov process, at the set of states *A*.
The function

$$\Phi_{iA}(t) = P\{\Theta_A \leq t \mid X(0) = i\}. \tag{21}$$

is the cumulative distribution of the random variable Θ_{iA} that denotes the first passage time from the state *i* to the states subset *A*.

Theorem 2. [4], [13]
For the regular semi-Markov processes such that,

$$f_{iA} = P\{\Delta_A < \infty \mid X(0) = i\} = 1, \quad i \in A', \tag{22}$$

the distributions $\Phi_{iA}(t), \quad i \in A'$ are proper and they are the unique solutions of the system of equations

$$\Phi_{iA}(t) = \sum_{j\in A} Q_{ij}(t) + \sum_{k\in S} \int_0^t \Phi_{kA}(t - x) dQ_{ik}(x),$$
$$i \in A'$$

Applying Laplace-Stieltjes transformation we obtain the system of linear equations

$$\tilde{\phi}_{iA}(s) = \sum_{j\in A} \tilde{q}_{ij}(s) + \sum_{k\in A'} \tilde{\phi}_{kA}(s)\tilde{q}_{ik}(s), \ i \in A' \tag{23}$$

with unknown transforms

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t).$$

Generating matrix form we get equation

$$\left(I-\tilde{\mathbf{q}}_{A'}(s)\right)\tilde{\mathbf{u}}_{A'}(s)=\tilde{\mathbf{b}}(s), \tag{24}$$

where

$$\mathbf{I}=[\delta_{ij}: i, j\in A'],\ \tilde{\mathbf{q}}_{A'}(s)=[\tilde{q}_{ij}(s): i, j\in A]$$

are the square matrices and

$$\tilde{\mathbf{u}}_{A'}(s)=\left[\tilde{u}_{iA}(s): i\in A'\right]^T,$$

$$\tilde{\mathbf{b}}(s)=\left[\sum_{j\in A}\tilde{q}_{ij}(s): i\in A'\right]^T$$

are the one-column matrices of transforms. The formal solution of the equation is

$$\tilde{\mathbf{u}}_{A'}(s)=\left(I-\tilde{\mathbf{q}}_{A'}(s)\right)^{-1}\tilde{\mathbf{b}}(s).$$

To solve this equation we use any computer programs, for example MATHEMATICA. Obtaining the inverse Laplace transform is much more complicated.
It is essentially simpler to find the expected values and the second moment of the random variables $\Theta_{iA}, i\in A'$. If the second moments of the waiting times $T_i, i\in A'$ are positive and commonly bounded, and $f_{iA}=1, \quad i\in A'$, then the expected values of the random variables $\Theta_{iA}, i\in A'$ are the unique solution of equation

$$\left(I-\mathbf{P}_{A'}\right)\overline{\mathbf{\Pi}}_{A'}=\overline{\mathbf{T}}_{A'}, \tag{25}$$

where

$$\mathbf{I}=\left[\delta_{ij}: i, j\in A'\right],\ \mathbf{P}_{A'}=\left[p_{ij}: i, j\in A'\right],$$

$$\overline{\mathbf{\Pi}}_{A'}=\left[E(\Theta_{iA}):\ i\in A'\right]^T,\ \overline{\mathbf{T}}_{A'}=\left[E(T_i): i\in A'\right]^T$$

and the second moments of the time to failure are the unique solution of equation

$$\left(\mathbf{I}-\mathbf{P}_{A'}\right)\overline{\mathbf{\Pi}}^2_{A'}=\mathbf{B}_{A'} \tag{26}$$

where

$$\mathbf{I}=\left[\delta_{ij}: i, j\in A'\right],\ \mathbf{P}_{A'}=\left[p_{ij}: i, j\in A'\right],$$

$$\overline{\mathbf{\Pi}}^2_{A'}=\left[E(\Theta^2{}_{iA}):\ i\in A'\right]^T,\ \mathbf{B}_{A'}=\left[b_i: i\in A'\right]^T,$$

$$b_i=E(T_i)+2\sum_{k\in A'}p_{ik}E(T_{ik})E(\Theta_{kA}).$$

5. Semi-Markov model of the cold standby system with repair

The problem is well known in reliability theory (Barlow & Proschan [1]). The model presented here is some modification of the model that was considered by Brodi & Pogosian [2].

5.1. Description and assumptions

A system consists of one operating component, an identical stand-by component and a switch, (*Figure1*).

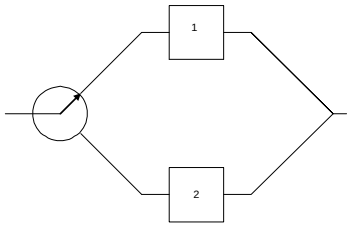

Figure 1. Diagram of the system

When the operating component fails, the spare is put in motion by the switch immediately. The failed unit (component) is repaired. There is a single repair facility. The repairs fully restore the components i.e. the components repairs means their renewals. The system fails when the operating component fails and component that was sooner failed in not repaired yet or when the operating units fail and the switch fails. We assume that the time to failure of the operating components are represented by the independent copies of a non-negative random variable ς with distribution given by a probability density function (pdf) $f(x), x\geq 0$. We suppose that the lengths of the repair periods of the components are represented by the identical copies of the non-negative random variables γ with cumulative distribution function (CDF) $G(x)=P(\gamma\leq x)$. Let U be a random variable having binary distribution

$$b(k)=P(U=k)=a^k(1-a)^{1-k}, k=0,1,\ 0<a<1,$$

where $U=0$, when a switch is failed at the moment of the operating component failure, and $U=1$, when the switch work at that moment. We suppose that the whole failed system is replaced by the new identical system. The replacing time is a non-negative random variable η with CDF $H(x)=P(\eta\leq x)$.

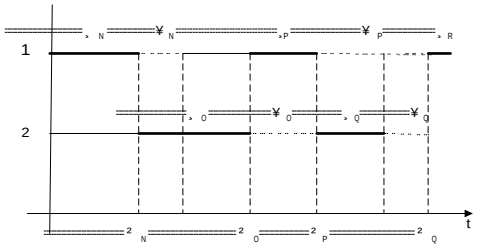

Figure 2. Reliability evolution of the standby system

Moreover we assume that the all random variables, mentioned above are independent.

5.2. Construction of the semi-Markov model

To describe reliability evolution of the system, we have to define the states and the renewal kernel. We introduce the following states:
0 - the system is failed

1 - the failed component is repaired, spare is operated

2 - both operating component and spare are "up".

Let $0=\tau_0^*,\tau_1^*,\tau_2^*,...$ denote the instants of the states changes, and $\{Y(t):t\geq 0\}$ be a random process with the state space $S=\{0,1,2\}$, which keeps constant values on the half-intervals $[\tau_n^*,\tau_{n+1}^*)$, 0,1,... and is right-continuous. The realization of this process is shown in *Figure 1*. This process is not semi-Markov, because the condition (1) of definition (2) is not satisfied for all instants of the state changes of the process.
Let us construct a new random process a following way. Let $0=\tau_0$ and $\tau_1,\tau_2,...$ denote the instants of the system components failures or the instants of whole system renewal. The random process $\{X(t):t\geq 0\}$ defined by equation

$$X(0)=0,\ X(t)=Y(\tau_n) \text{ for } t\in[\tau_n,\tau_{n+1})$$

is the semi-Markov process.
To have semi-Markov process as a model we must define its initial distribution and all elements of its kernel

$$\mathbf{Q}(t)=\begin{bmatrix} 0 & 0 & Q_{02}(t) \\ Q_{10}(t) & Q_{11}(t) & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 \end{bmatrix}$$

For $t\geq 0$ we obtain

$$Q_{02}(t)=P(\eta\leq t)=H(t),$$

$$Q_{10}(t)=P(\varsigma\leq t,\gamma>\zeta)$$

$$+P(U=0,\varsigma\leq t,\gamma<\zeta)$$

$$=\int_0^t[1-G(x)]dF(x)+(1-a)\int_0^t G(x)dF(x)$$

$$=F(t)-a\int_0^t G(x)dF(x),$$

$$Q_{11}(t)=P(U=1,\varsigma\leq t,\gamma<\zeta)=a\int_0^t G(x)dF(x),$$

$$Q_{20}(t)=P(U=0,\varsigma\leq t)=(1-a)F(t),$$

$$Q_{21}(t)=P(U=1,\varsigma\leq t)=aF(t).$$

We assume that, the initial state is 2. It means that an initial distribution is

$$\mathbf{p}(0)=[0\quad 0\quad 1].$$

Hence, the semi-Markov model is constructed.

5.3. The reliability characteristics

The random variable Θ_{iA}, that denotes the first passage time from the state i to the states subset A, for $i=2$ and $A=\{0\}$ in our model, represents the time to failure of the system. The function

$$R(t)=P(\Theta_{20}>t)=1-\Phi_{20}(t),\quad t\geq 0 \tag{27}$$

is the reliability function of the considered cold standby system with repair.
System of linear equation (23) for the Laplace-Stieltjes transforms of the functions

$$\Phi_{i0}(t),\ t\geq 0,\ i=1,2,$$

in this case is

$$\tilde{\phi}_{10}(s)=\tilde{q}_{10}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{11}(s)$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{21}(s)$$

The solution is

$$\tilde{\phi}_{10}(s)=\frac{\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)},$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\frac{\tilde{q}_{21}(s)\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}. \tag{28}$$

Hence, we obtain the Laplace transform of the reliability function

$$\tilde{R}(s) = \frac{1-\tilde{\phi}_{20}(s)}{s}. \tag{29}$$

The transition probabilities matrix of the embedded Markov chain in the semi-Markov process $\{X(t): t \geq 0\}$ is

$$\mathrm{P} = \begin{bmatrix} 0 & 0 & 1 \\ p_{10} & p_{11} & 0 \\ p_{20} & p_{21} & 0 \end{bmatrix} \tag{30}$$

Where

$$p_{10} = 1 - p_{11}$$

$$p_{11} = P(U = 1, \gamma < \zeta) = a\int_0^\infty G(x)dF(x)\,,$$

$$p_{20} = 1 - a, \quad p_{21} = P(U = 1) = a.$$

Using formula (9) we obtain the CDF of the waiting times of $T_i, i = 0,1,2.$

$$G_0(t) = H(t),\ G_1(t) = F(t),\ G_2(t) = F(t)\,.$$

Hence

$$E(T_0) = E(\eta),\ E(T_1) = E(\varsigma),\ E(T_3) = E(\varsigma)\,.$$

The equation (25) in this case has form

$$\begin{bmatrix} 1-p_{11} & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} E(\Theta_{10}) \\ E(\Theta_{20}) \end{bmatrix} = \begin{bmatrix} E(\varsigma) \\ E(\varsigma) \end{bmatrix}$$

The solution is

$$E(\Theta_{10}) = \frac{E(\varsigma)}{1-p_{11}},$$

$$E(\Theta_{20}) = E(\varsigma) + \frac{a\,E(\varsigma)}{1-p_{11}}\,. \tag{31}$$

We will apply theorem 1 to calculate the limit probability distribution of the state. Now, the system of linear equation (20) is

$$\pi_1 p_{10} + \pi_2 p_{20} = \pi_0,$$

$$\pi_1 p_{12} + \pi_2 p_{21} = \pi_1,$$

$$\pi_0 = \pi_2,$$

$$\pi_0 + \pi_1 + \pi_2 = 1.$$

Since, the stationary distribution of the embedded Markov chain is

$$\pi_0 = \frac{p_{11}}{2p_{11} + p_{21}},$$

$$\pi_1 = \frac{p_{21}}{2p_{11} + p_{21}},$$

$$\pi_2 = \frac{p_{11}}{2p_{11} + p_{21}}.$$

Using formula (19) we obtain the limit distribution of semi-Markov process

$$P_0 = \frac{p_{11}E(\eta)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)} \tag{32}$$

$$P_1 = \frac{p_{21}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}$$

$$P_2 = \frac{p_{11}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}$$

5.4. Conclusion

The expectation $E(\Theta_{20})$ denoting the mean time to failure is

$$E(\Theta_{20}) = E(\varsigma) + \frac{a\,E(\varsigma)}{1-p_{11}},$$

where

$$p_{11} = a\int_0^\infty G(x)dF(x)\,.$$

Let us notice, that the cold standby determines increase the meantime to failure $1 + \frac{a}{1-p_{11}}$ times.

The limiting availability coefficient of the system is

$$A = P_1 + P_2 = \frac{p_{21}E(\varsigma) + p_{11}E(\varsigma)}{p_{11}E(\eta) + p_{21}E(\varsigma) + p_{11}E(\varsigma)}\,.$$

6. Semi-Markov process as the reliability model of the operation with perturbation

Semi-Markov process as the reliability model of multi-stage operation was considered by F. Grabski in [8] and [10]. Many operations consist of some elementary tasks, which are realized in turn. Duration of the each task realization is assumed to be positive random variable. Each elementary operation may be perturbed or failed. The perturbations increase the time of operation and the probability of failure as well.

6.1. Description and assumptions

Suppose, that the operation consists of *n* stages which following in turn. We assume that duration of an *i*-th stage, ($i = 1, \ldots, n$) is a nonnegative random variable $\xi_i; i = 1, \ldots, n$ with a cumulative probability distribution

$$F_i(t) = P(\xi_i \le t) = \int_0^t f_i(x)dx, \quad i = 1, \ldots, n,$$

where $f_i(x)$ denotes its probability density function in an extended sense.
Time to failure of the operation on the *i*-th stage (component) is the nonnegative random variable η_i, $i = 1, \ldots, n$ with exponential distribution

$$P(\eta_i \le t) = 1 - e^{-\lambda_i t}; \quad i = 1, \ldots, n.$$

The operation on each step may be perturbed. We assume that no more then one event causing perturbation on each stage of the operation may occur. Time to event causing of an operation perturbation on *i*-th stage is a nonnegative random variable $\zeta_i; i = 1, \ldots, n$ with exponential distribution

$$P(\zeta_i \le t) = 1 - e^{-\alpha_i t}; \quad i = 1, \ldots, n.$$

The perturbation degreases the probability of the operation fail. We suppose that time to failure of the perturbed operation on the i-th stage is the nonnegative random variable $\nu_i, i = 1,2\ldots,n$ that has the exponential distribution with a parameter $\beta_i > \lambda_i$

$$P(\nu_i \le t) = 1 - e^{-\beta_i t}; \quad i = 1, \ldots, n.$$

We assume that the operation is cyclical.
We assume that random variables $\xi_i, \eta_i, \nu_i, \zeta_i, \quad i = 1,\ldots,n$ are mutually independent.

6.2. Semi-Markov model

To construct reliability model of operation, we have to start from definition of the process states.
Let e_{ij}, *i=1,...,n, j=0,1* denotes *j-th* reliability state on *i-th* step of the operation where, *j=0* denotes perturbation and *j=1* denotes success
e_{2n+1} - failure (un-success) of the operation
e_{11} - an initial state.
For convenience we numerate the states

$$e_{i1} \leftrightarrow i, \quad i = 1,\ldots,n$$

$$e_{i0} \leftrightarrow i+n, \quad i = 1,\ldots,n$$

$$e_{2n+1} \leftrightarrow 2n+1,$$

Under the above assumptions, stochastic process describing of the overall operation in reliability aspect, is a semi-Markov process $\{X(t): \ t \ge 0\}$ with a space of states $S = \{1, 2, \ldots, 2n, 2n+1\}$ and flow graph shown in *Figure 3.*

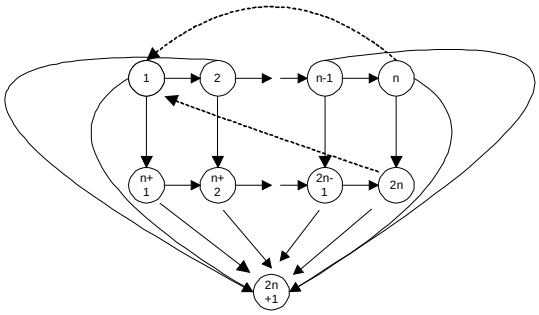

Figure 3. Transition graph for n-stage cyclic operation

To obtain a semi-Markov model we have to define all nonnegative elements of semi- Markov kernel

$$\mathbf{Q}(t) = \lfloor Q_{ij}(t) : i, j \in S \rfloor$$

$$Q_{ij}(t) = P\{X(\tau_{n+1}) = j, \tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i\}$$

First, we define transition probabilities from the state *i to* the state *j* for time not greater than t for *i=1,…,n-1.*

$$Q_{i\,i+1}(t) = P(\xi_i \le t, \eta_i > \xi_i, \zeta_i > \xi_i)$$
$$= \iiint_D \alpha_i e^{-\alpha_i y} \lambda_i e^{-\lambda_i z} f_i(x)dx\,dy\,dz$$

where

$$D = \{(x, y, z): \quad x \ge 0, \ y \ge 0, \ z \ge 0,$$
$$x \le t, \quad z > x, \quad x > y\}$$

Since, we have

$$Q_{i\,i+1}(t)=\int_0^t f_i(x)dx\int_0^x \alpha_i e^{-\alpha_i y}dy\int_x^\infty \lambda_i e^{-\lambda_i z}dz$$

$$=\int_0^t e^{-(\lambda_i+\alpha_i)x}\,f_i(x)\,dx\ .$$

For $i=n+1,...,2n-1$ we obtain

$$Q_{i\,i+n}(t)=P(\zeta_i\le t,\eta_i>\zeta_i,\zeta_i>\xi_i)$$

$$=\int_0^t \alpha_i e^{-(\lambda_i+\alpha_i)u}[1-F_i(u)]du\ ,\ i=1,...,n-1.$$

For $i=1,...,n$ we get

$$Q_{i\,2n+1}(t)=P(\eta_i\le t,\eta_i<\zeta_i,\eta_i<\xi_i)$$

$$=\int_0^t \lambda_i e^{-(\lambda_i+\alpha_i)u}[1-F_i(u)]du\ ,$$

$i=1,...,n-1.$

If on *i*-th stage a perturbation has happened the transition probability to next state for time less then or equal to t is

$$Q_{n+i\,n+i+1}(t)=P(\xi_i-\zeta_i\le t,\nu_i>\xi_i-\zeta_i\,|\,\xi_i>\zeta_i)$$

$$=\frac{\iiint_D \alpha_i e^{-\alpha_i y}\beta_i e^{-\beta_i z}\,f_i(x)dx\,dy\,dz}{\iint_E \alpha_i e^{-\alpha_i y}\,f_i(x)dx\,dy},$$

$i=1,...,n-1,$

where

$$D=\{(x,y,z):\quad x\ge 0,\ y\ge 0,\ z\ge 0,$$

$$0\le x-y\le t,\quad z>x-y,\quad x>y\}$$

$$E=\{(x,y):\quad x\ge 0,\ y\ge 0,\ x>y\}.$$

To find the triple integral over the region *D*, we apply change of coordinates:

$$u=x-y,\quad v=y,\quad w=z.$$

Hence

$$x=u+v,\quad y=v,\quad z=w.$$

This mapping assigns to points from set

$$\Delta=\{(u,v,w):0\le u\le t,\ v\ge 0,\ w>u\}$$

the points from plane region *D*. The Jacobian of this mapping is

$$J(u,v,w)=1.$$

Since, we get

$$\iiint_D \alpha_i e^{-\alpha_i y}\beta_i e^{-\beta_i z}\,f_i(x)dx\,dy\,dz$$

$$=\iiint_\Delta \alpha_i e^{-\alpha_i v}\beta_i e^{-\beta_i w}\,f_i(u+v)J(u,v,w)du\,dv\,dw$$

$$=\int_0^\infty \alpha_i e^{-\alpha_i v}dv\int_u^\infty \beta_i e^{-\beta_i w}dw\int_0^t f_i(u+v)du$$

$$=\int_0^t e^{-\beta_i u}du\int_0^\infty \alpha_i e^{-\alpha_i v}\,f_i(u+v)dv\ .$$

Let us notice that

$$\iint_E \alpha_i e^{-\alpha_i y}\,f_i(x)dx\,dy=\int_0^\infty \alpha_i e^{-\alpha_i y}\int_y^\infty f_i(x)dx$$

$$=\int_0^\infty \alpha_i e^{-\alpha_i y}[1-F_i(y)]dy$$

$$=1-\int_0^\infty \alpha_i e^{-\alpha_i y}F_i(y)dy.$$

Finally, we obtain

$$Q_{n+i\,n+i+1}(t)=\frac{\int_0^t e^{-\beta_i u}[\int_0^\infty \alpha_i e^{-\alpha_i v}\,f_i(u+v)dv]du}{\int_0^\infty \alpha_i e^{-\alpha_i y}[1-F_i(y)]dy},\ .$$

$i=1,...,n-1.$

In the same way we get

$$Q_{n+i\,2n+1}(t) = P(\nu_i \le t, \nu_i < \xi_i - \zeta_i \mid \xi_i > \zeta_i)$$

$$= \frac{\iiint_D \alpha_i e^{-\alpha_i y} \beta_i e^{-\beta_i z} f_i(x)\, dx\, dy\, dz}{\iint_E \alpha_i e^{-\alpha_i y} f_i(x) dx\, dy},$$

$i = 1,..., n$

where

$$D = \{(x, y, z): \quad x \ge 0, y \ge 0,$$

$$0 \le z \le t, \quad z < x - y, \quad x > y\}$$

$$E = \{(x, y): \quad x \ge 0, y \ge 0, \ x > y\}.$$

Hence

$$Q_{n+i\,2n+1}(t) = \frac{\beta_i \int_0^t e^{-\beta_i w} dw \int_w^\infty \alpha_i e^{-\alpha_i v} dv \int_0^\infty f_i(u+v) du}{\int_0^\infty \alpha_i e^{-\alpha_i y}[1 - F_i(y)] dy}$$

$i = 1,..., n.$

Similar way we obtain

$$Q_{n1}(t) = P(\xi_n \le t, \eta_n > \xi_n\ \zeta_n > \xi_n)$$

$$= \int_0^t e^{-(\lambda_n + \alpha_n)u} f_n(u)\, du$$

$$Q_{2n1}(t) = P(\xi_n - \zeta_n \le t, \nu_n > \xi_n - \zeta_n \mid \xi_n > \zeta_n)$$

$$= \frac{\int_0^t e^{-\beta_n u} du \int_0^\infty \alpha_n e^{-\alpha_i v} f_n(u+v) dv}{\int_0^\infty \alpha_n e^{-\alpha_n x}[1 - F_n(x)]\, dx}.$$

Therefore the semi-Markov reliability model of operation has been constructed.

6.3. Two-stage cyclical operation

We will investigate particular case of that model, assuming $n = 2$. A transition matrix for the semi-Markov model of the *2*-stage cyclic operation in reliability aspect takes the following form

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) & 0 & Q_{15}(t) \\ Q_{21}(t) & 0 & 0 & Q_{24}(t) & Q_{25}(t) \\ 0 & 0 & 0 & Q_{34}(t) & Q_{35}(t) \\ Q_{41}(t) & 0 & 0 & 0 & Q_{45}(t) \\ 0 & 0 & 0 & 0 & Q_{55}(t) \end{bmatrix},$$

where

$$Q_{12}(t) = \int_0^t e^{-(\lambda_1 + \alpha_1)u} dF_1(u),$$

$$Q_{13}(t) = \int_0^t \alpha_1 e^{-(\lambda_{1i} + \alpha_1)u}[1 - F_1(u)] du,$$

$$Q_{15}(t) = \int_0^t \lambda_1 e^{-(\lambda_{1i} + \alpha_1)u}[1 - F_1(u)] du,$$

$$Q_{21}(t) = \int_0^t e^{-(\lambda_2 + \alpha_2)u} dF_2(u),$$

$$Q_{24}(t) = \int_0^t \alpha_2 e^{-(\lambda_2 + \alpha_2)u}[1 - F_2(u)] du,$$

$$Q_{25}(t) = \int_0^t \lambda_2 e^{-(\lambda_2 + \alpha_2)u}[1 - F_2(u)] du,$$

$$Q_{34}(t) = \frac{\int_0^t e^{-\beta_1 u}[\int_0^\infty \alpha_1 e^{-\alpha_1 v} f_1(u+v) dv] du}{\int_0^\infty \alpha_1 e^{-\alpha_i y}[1 - F_1(y)] dy},$$

$$Q_{35}(t) = \frac{\beta_1 \int_0^t e^{-\beta_i w} dw \int_w^\infty \alpha_1 e^{-\alpha_1 v} dv \int_0^\infty f_1(u+v) du}{\int_0^\infty \alpha_1 e^{-\alpha_1 x}[1 - F_1(x)] dx},$$

$$Q_{41}(t) = \frac{\int_0^t e^{-\beta_2 u} du \int_0^\infty \alpha_2 e^{-\alpha_2 v} f_2(u+v)\ ] dv}{\int_0^\infty \alpha_2 e^{-\alpha_2 x}[1 - F_2(x)] dx},$$

$$Q_{45}(t) = \frac{\beta_2 \int_0^t e^{-\beta_2 w} dw \int_w^\infty \alpha_2 e^{-\alpha_2 v} dv \int_0^\infty f_2(u+v) du}{\int_0^\infty \alpha_2 e^{-\alpha_2 x}[1 - F_2(x)] dx},$$

$$Q_{55}(t) = U(t).$$

We will consider 3

That model allows us to obtain some reliability characteristics of the operation. The random variable Θ_{15} denoting the first passage time from state 1 to state 5 in our model, means time to failure of the operation. The Laplace-Stieltjes transform for the cumulative distribution function of that random variable we will obtain from a matrix equation (25).

In this case we have $A' = \{1,2,3,4\}$, $A = \{5\}$ and

$$\tilde{\mathbf{u}}_{A'}(s) = \begin{bmatrix} \tilde{\phi}_{15}(s) \\ \tilde{\phi}_{25}(s) \\ \tilde{\phi}_{35}(s) \\ \tilde{\phi}_{45}(s) \end{bmatrix}, \quad \tilde{\mathbf{b}}(s) = \begin{bmatrix} \tilde{q}_{15}(s) \\ \tilde{q}_{25}(s) \\ \tilde{q}_{35}(s) \\ \tilde{q}_{45}(s) \end{bmatrix},$$

$$\tilde{\mathbf{q}}_{A'}(s) = \begin{bmatrix} 0 & \tilde{q}_{12}(s) & \tilde{q}_{13}(s) & 0 \\ \tilde{q}_{21}(s) & 0 & 0 & \tilde{q}_{24}(s) \\ 0 & 0 & 0 & \tilde{q}_{34}(s) \\ \tilde{q}_{41}(s) & 0 & 0 & 0 \end{bmatrix}.$$

From the solution of equation (24) we obtain Laplace-Stieltjes transform of the cumulative distribution function of the random variable Θ_{15} denoting time to failure of the operation

$$\tilde{\phi}_{15}(s) = \frac{\tilde{a}(s)}{\tilde{b}(s)}, \tag{33}$$

$$\tilde{a}(s) = \tilde{q}_{15}(s) + \tilde{q}_{12}(s)\tilde{q}_{25}(s) + \tilde{q}_{13}(s)\tilde{q}_{35}(s)$$

$$+ \tilde{q}_{12}(s)\tilde{q}_{24}(s)\tilde{q}_{45}(s) + \tilde{q}_{13}(s)\tilde{q}_{34}(s)\tilde{q}_{45}(s)$$

$$\tilde{b}(s) = 1 - \tilde{q}_{12}(s)\tilde{q}_{21}(s) - \tilde{q}_{12}(s)\tilde{q}_{24}(s)\tilde{q}_{41}(s)$$

$$- \tilde{q}_{13}(s)\tilde{q}_{34}(s)\tilde{q}_{41}(s).$$

The Laplace transform of the reliability function is given by the formula

$$\tilde{R}(s) = \frac{1 - \tilde{\phi}_{15}(s)}{s}. \tag{34}$$

6.4. Examples

Example 1
We suppose that

$$F_i(t) = 1 - e^{-\kappa_i t}, \; t \geq 0, \; i = 1,2.$$

Then

$$Q_{12}(t) = \frac{\kappa_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{13}(t) = \frac{\alpha_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{15}(t) = \frac{\lambda_1}{\lambda_1 + \alpha_1 + \kappa_1}\left(1 - e^{-(\lambda_1+\alpha_1+\kappa_1)t}\right),$$

$$Q_{21}(t) = \frac{\kappa_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{24}(t) = \frac{\alpha_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{25}(t) = \frac{\lambda_2}{\lambda_2 + \alpha_2 + \kappa_2}\left(1 - e^{-(\lambda_2+\alpha_2+\kappa_2)t}\right),$$

$$Q_{34}(t) = \frac{\kappa_1}{\beta_1 + \kappa_1}\left(1 - e^{-(\beta_1+\kappa_1)t}\right),$$

$$Q_{35}(t) = \frac{\beta_1}{\beta_1 + \kappa_1}\left(1 - e^{-(\beta_1+\kappa_1)t}\right),$$

$$Q_{41}(t) = \frac{\kappa_2}{\beta_2 + \kappa_2}\left(1 - e^{-(\beta_2+\kappa_2)t}\right),$$

$$Q_{45}(t) = \frac{\beta_2}{\beta_2 + \kappa_2}\left(1 - e^{-(\beta_2+\kappa_2)t}\right),$$

Laplace-Stieltjes transform of these functions are:

$$\tilde{q}_{12}(s) = \frac{\kappa_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{13}(s) = \frac{\alpha_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{15}(s) = \frac{\lambda_1}{s + \lambda_1 + \alpha_1 + \kappa_1},$$

$$\tilde{q}_{21}(s) = \frac{\kappa_2}{s + \lambda_2 + \alpha_2 + \kappa_2},$$

$$\tilde{q}_{24}(s) = \frac{\alpha_2}{s + \lambda_2 + \alpha_2 + \kappa_2},$$

$$\tilde{q}_{25}(s)=\frac{\lambda_2}{s+\lambda_2+\alpha_2+\kappa_2},$$

$$\tilde{q}_{34}(s)=\frac{\kappa_1}{s+\beta_1+\kappa_1},$$

$$\tilde{q}_{35}(s)=\frac{\beta_1}{s+\beta_1+\kappa_1},$$

$$\tilde{q}_{41}(s)=\frac{\kappa_2}{s+\beta_2+\kappa_2},$$

$$\tilde{q}_{45}(s)=\frac{\beta_2}{s+\beta_2+\kappa_2}.$$

For $\kappa_1=0{,}1$; $\kappa_2=0{,}12$; $\lambda_1=0{,}002$; $\lambda_2=0{,}001$; $\alpha_1=0{,}02$; $\alpha_2=0{,}04$; $\beta_1=0{,}01$; $\beta_1=0{,}01$, applying (33) and (34), with help of MATHEMATICA computer program, we obtain the density function and the reliability function as inverse Laplace transforms .
The density function is given by the formula

$$\phi_{15}(t)$$

$$=0.00712201\,e^{-0.213357\,t}-0.00872613\,e^{-0.180053\,t}$$

$$-0.000144434\,e^{-0.125901\,t}+0.00374856\,e^{-0.00368869\,t}$$

This function is shown in *Figure 4.*
The reliability function is

$$R(t)=3.79823\times10^{-16}+0.0333808\,e^{-0.213357\,t}$$

$$-0.0484641\,e^{-0.180053\,t}-0.0011472\,e^{-0.125901\,t}$$

$$+1.01623\,e^{-0.00368869\,t}$$

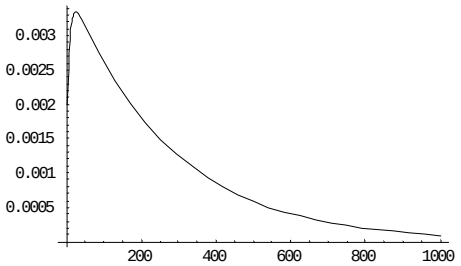

Figure 4. The density function the time to failure of 2-stage cyclic operation

This function is shown on *Figure 5*.

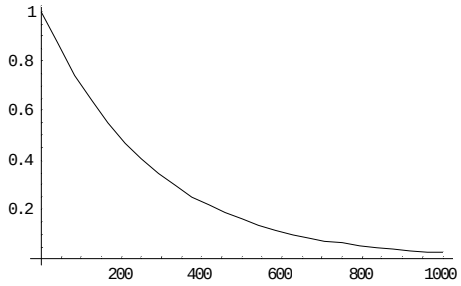

Figure 5. The reliability function of 2-stage cyclic operation

Mean time to failure we can find solving the matrix equation

$$\left(I-P_{A'}\right)\overline{\Theta}_{A'}=\overline{T}_{A'} \tag{35}$$

where

$$\mathbf{P}_{A'}=\begin{bmatrix}0 & p_{12} & p_{13} & 0\\ p_{21} & 0 & 0 & p_{24}\\ 0 & 0 & 0 & p_{34}\\ p_{41} & 0 & 0 & 0\end{bmatrix},$$

$$\overline{\mathbf{T}}=\begin{bmatrix}E(T_1)\\ E(T_2)\\ E(T_3)\\ E(T_4)\end{bmatrix},\qquad \overline{\Theta}_{A'}=\begin{bmatrix}E(\Theta_{15})\\ E(\Theta_{25})\\ E(\Theta_{35})\\ E(\Theta_{45})\end{bmatrix}.$$

From this equation we obtain the mean time to failure

$$E(\Theta_{15})=275{,}378$$

Example 2
Now we assume that

$$F_i\ (t)=\begin{cases}0 & \text{dla}\quad t\le L_i\\ 1 & \text{dla}\quad t>L_i\end{cases},\ i=1{,}2\ .$$

It means that the duration of the stages are determined and they are equal $\xi_i=L_i$ for $i=1{,}2$.
In this case the elements of $\boldsymbol{Q}(t)$ are:

$$Q_{12}(t)=\begin{cases}0 & \text{for}\quad t\le L_1\\ e^{-(\lambda_1+\alpha_1)L_1} & \text{for}\quad t>L_1\end{cases}$$

$$Q_{13}(t)=\begin{cases}\dfrac{\alpha_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)t}\right) & \text{for} \quad t\le L_1\\ \dfrac{\alpha_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)L_1}\right) & \text{for} \quad t>L_1\end{cases}$$

$$Q_{15}(t)=\begin{cases}\dfrac{\lambda_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)t}\right) & \text{for} \quad t\le L_1\\ \dfrac{\lambda_1}{\lambda_1+\alpha_1}\left(1-e^{-(\lambda_1+\alpha_1)L_1}\right) & \text{for} \quad t>L_1\end{cases}$$

$$Q_{21}(t)=\begin{cases}0 & \text{for} \quad t\le L_2\\ e^{-(\lambda_2+\alpha_2)L_2} & \text{for} \quad t>L_2\end{cases}$$

$$Q_{24}(t)=\begin{cases}\dfrac{\alpha_2}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)t}\right) & \text{for} \quad t\le L_2\\ \dfrac{\alpha_1}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)L_2}\right) & \text{for} \quad t>L_2\end{cases}$$

$$Q_{25}(t)=\begin{cases}\dfrac{\lambda_2}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)t}\right) & \text{for} \quad t\le L_2\\ \dfrac{\lambda_1}{\lambda_2+\alpha_2}\left(1-e^{-(\lambda_2+\alpha_2)L_2}\right) & \text{for} \quad t>L_2\end{cases}$$

$$Q_{34}(t)=\begin{cases}\dfrac{\alpha_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)(1-e^{-a_1L_1})}\left(1-e^{-(\beta_1-\alpha_1)t}\right), & 0\le t\le L_1\\ \dfrac{\alpha_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)(1-e^{-a_1L_1})}\left(1-e^{-(\beta_1-\alpha_1)L_1}\right), & t>L\end{cases},$$

$$Q_{35}(t)=\begin{cases}\dfrac{1}{(1-e^{-a_1L_1})}\left[\left(1-e^{-\beta_1 t}\right)-\dfrac{\beta_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)}\left(1-e^{-(\beta_1-\alpha_1)t}\right)\right], & 0\le t\le L_1\\ \dfrac{1}{(1-e^{-a_1L_1})}\left[\left(1-e^{-\beta_1L_1}\right)-\dfrac{\beta_1 e^{-a_1L_1}}{(\beta_1-\alpha_1)}\left(1-e^{-(\beta_1-\alpha_1)L_1}\right)\right], & t>L_1\end{cases}$$

$$Q_{41}(t)=\begin{cases}\dfrac{\alpha_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)(1-e^{-a_2L_2})}\left(1-e^{-(\beta_2-\alpha_2)t}\right), & 0\le t\le L_2\\ \dfrac{\alpha_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)(1-e^{-a_2L_2})}\left(1-e^{-(\beta_2-\alpha_2)L_2}\right), & t>L_2\end{cases}$$

$$Q_{45}(t)=\begin{cases}\dfrac{1}{(1-e^{-a_2L_2})}\left[\left(1-e^{-\beta_2 t}\right)-\dfrac{\beta_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)}\left(1-e^{-(\beta_2-\alpha_2)t}\right)\right], & 0\le t\le L_2\\ \dfrac{1}{(1-e^{-a_2L_2})}\left[\left(1-e^{-\beta_2L_1}\right)-\dfrac{\beta_2 e^{-a_2L_2}}{(\beta_2-\alpha_2)}\left(1-e^{-(\beta_2-\alpha_2)L_1}\right)\right], & t>L_2\end{cases}$$

The Laplace'a-Stieltjes transform of these functions are:

$$\tilde{q}_{12}(s)=e^{-(\lambda_1+\alpha_1+s)L_1}$$

$$\tilde{q}_{13}(s)=\frac{\alpha_1}{s+\lambda_1+\alpha_1}-\frac{\alpha_1 e^{-(\lambda_1+\alpha_1+s)L_1}}{s+\lambda_1+\alpha_1}$$

$$\tilde{q}_{15}(s)=\frac{\lambda_1}{s+\lambda_1+\alpha_1}-\frac{\lambda_1 e^{-(\lambda_1+\alpha_1+s)L_1}}{s+\lambda_1+\alpha_1}$$

$$\tilde{q}_{21}(s)=e^{-(\lambda_2+\alpha_2+s)L_2},$$

$$\tilde{q}_{24}(s)=\frac{\alpha_2}{s+\lambda_2+\alpha_2}-\frac{\alpha_2 e^{-(\lambda_2+\alpha_2+s)L_2}}{s+\lambda_2+\alpha_2}$$

$$\tilde{q}_{25}(s)=\frac{\lambda_2}{s+\lambda_2+\alpha_2}-\frac{\lambda_2 e^{-(\lambda_2+\alpha_2+s)L_{21}}}{s+\lambda_2+\alpha_2}$$

$$\tilde{q}_{34}(s)=\frac{\alpha_1 e^{-\alpha_1L_1}}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1-\alpha_1+s)L_1}}{s+\beta_1-\alpha_1}\right)$$

$$\tilde{q}_{35}(s)=\frac{\beta_1}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1+s)L_1}}{s+\beta_1}-\frac{e^{-\alpha_1L_1}(1-e^{-(\beta_1-\alpha_1+s)L_1})}{s+\beta_1-\alpha_1}\right)$$

$$\tilde{q}_{41}(s)=\frac{\alpha_2 e^{-\alpha_2L_2}}{1-e^{-\alpha_2L_2}}\left(\frac{1-e^{-(\beta_2-\alpha_2+s)L_2}}{s+\beta_2-\alpha_2}\right)$$

$$\tilde{q}_{45}(s)=\frac{\beta_2}{1-e^{-\alpha_2L_2}}\left(\frac{1-e^{-(\beta_2+s)L_2}}{s+\beta_2}-\frac{e^{-\alpha_2L_2}(1-e^{-(\beta-\alpha_2+s)L_2})}{s+\beta_2-\alpha_2}\right)$$

The mean time to failure we can find solving the matrix equation (35), where

$$p_{12}=e^{-(\lambda_1+\alpha_1)L_1}$$

$$p_{13}=\frac{\alpha_1(1-e^{-(\lambda_1+\alpha_1)L_1})}{\lambda_1+\alpha_1} \qquad p_{15}=\frac{\lambda_1(1-e^{-(\lambda_1+\alpha_1)L_1})}{\lambda_1+\alpha_1}$$

$$p_{21}=e^{-(\lambda_2+\alpha_2)L_2}$$

$$p_{24}=\frac{\alpha_2(1-e^{-(\lambda_2+\alpha_2)L_2})}{\lambda_2+\alpha_2}$$

$$p_{25}=\frac{\lambda_2(1-e^{-(\lambda_2+\alpha_2)L_{21}})}{\lambda_2+\alpha_2}$$

$$p_{34}=\frac{\alpha_1 e^{-\alpha_1L_1}}{1-e^{-\alpha_1L_1}}\left(\frac{1-e^{-(\beta_1-\alpha_1)L_1}}{\beta_1-\alpha_1}\right)$$

$$p_{35} = \frac{\beta_1}{1-e^{-\alpha_1 L_1}}\left(\frac{1-e^{-\beta_1 L_1}}{\beta_1}\right.$$

$$\left. - \frac{e^{-\alpha_1 L_1}(1-e^{-(\beta_1-\alpha_1)L_1}}{\beta_1-\alpha_1}\right)$$

$$p_{41} = \frac{\alpha_2 e^{-\alpha_2 L_2}}{1-e^{-\alpha_2 L_2}}\left(\frac{1-e^{-(\beta_2-\alpha_2)L_2}}{\beta_2-\alpha_2}\right)$$

$$p_{45} = \frac{\beta_2}{1-e^{-\alpha_2 L_2}}$$

$$\left(\frac{1-e^{-\beta_2 L_2}}{\beta_2} - \frac{e^{-\alpha_2 L_2}(1-e^{-(\beta-\alpha_2)L_2})}{\beta_2-\alpha_2}\right)$$

$$p_{51} = 1$$

$$E(T_1) = \frac{1}{\alpha_1+\lambda_1} - \frac{e^{-(\alpha_1+\lambda_1)L_1}}{\alpha_1+\lambda_1}$$

$$E(T_2) = \frac{1}{\alpha_2+\lambda_2} - \frac{e^{-(\alpha_2+\lambda_2)L_2}}{\alpha_2+\lambda_2}$$

$$E(T_3) = -\frac{d(\tilde{q}_{34}(s)+\tilde{q}_{35}(s))}{ds}\Big|_{s=0}$$

$$E(T_4) = -\frac{d(\tilde{q}_{41}(s)+\tilde{q}_{45}(s))}{ds}\Big|_{s=0}$$

For the same parameters $\kappa_1 = 0{,}1$; $\kappa_2 = 0{,}12$; $\lambda_1 = 0{,}002$; $\lambda_2 = 0{,}001$; $\alpha_1 = 0{,}02$; $\alpha_2 = 0{,}04$; $\beta_1 = 0{,}01$; $\beta_1 = 0{,}01$, and $L_1 = \frac{1}{\kappa_1}$ i $L_2 = \frac{1}{\kappa_1}$, the mean time to failure of the operation is

$$E(\Theta_{15}) = 366.284 .$$

In previous case the mean time to failure is

$$E(\Theta_{15}) = 275{,}378 .$$

6.5. Conclusion

It means that for the determined duration of the stages mean time to failure of the operation is essentially greater than for exponentially distributed duration of the stages with the same expectations.

To assess reliability of the many stage operation we can apply a semi-Markov process. Construction of the semi-Markov model consist in defining a kernel of that process. A way of building the kernel for the semi-Markov model of the many stage operation is presented in this paper. From Semi-Markov model we can obtain many interesting parameters and characteristics for analysing reliability of the operation.
From presented examples we get conclusion that for the determined duration of the stages, mean time to failure of the operation is essentially greater than for exponentially distributed duration of the stages with the same expectations.

7. Semi-Markov process as a failure rate

The reliability function with semi-Markov failure rate was considered by Kopociński & Kopocińska [11], Kopocińska [12] and by Grabski [4], [6], [9]. Suppose that the failure rate $\{\lambda(t): t \geq 0\}$ is the semi-Markov process with the discrete state space $S = \{\lambda_j : j \in J\}$, $J = \{0,1,...,m\}$ or $J = \{0,1,2,...\}$, $0 \leq \lambda_0 < \lambda_1 < ...$ with the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in J]$$

and the initial distribution $p = [p_i : i \in J]$.
We define a conditional reliability function as

$$R_i(t) = E\left[\exp\left(-\int_0^t л(u)du\right)\middle|\lambda(0) = \lambda_i\right], \quad t \geq 0, \quad i \in J$$

In [6] it is proved, that for the regular semi-Markov process $\{\lambda(t): t \geq 0\}$ the conditional reliability functions $R_i(t)$, $t \geq 0$, $i \in J$ defined by (17), satisfy the system of equations

$$R_i(t) = e^{-\lambda_i t}[1-G_i(t)] + \sum_j \int_0^t e^{-\lambda_i x} R_j(t-x)dQ_j(x), i \in J$$

Applying the Laplace transformation we obtain the system of linear equations

$$\tilde{R}_i(s) = \frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) + \sum_j \tilde{R}_j(s)\tilde{q}_{ij}(s+\lambda_i), i \in J,$$

where

$$\tilde{R}_i(s) = \int_0^\infty e^{-st} R_i(t)dt, \ \tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t)dt,$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t) .$$

In matrix notation we have

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]\tilde{\mathbf{R}}(s) = \tilde{\mathbf{H}}(s),$$

where

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]$$

$$= \begin{bmatrix} 1-\tilde{q}_{00}(s+\lambda_0) & -\tilde{q}_{01}(s+\lambda_0) & -\tilde{q}_{02}(s+\lambda_0) & \mathrm{i} \\ -\tilde{q}_{10}(s+\lambda_1) & 1-\tilde{q}_{11}(s+\lambda_1) & -\tilde{q}_{12}(s+\lambda_1) & \mathrm{i} \\ -\tilde{q}_{20}(s+\lambda_2) & -\tilde{q}_{21}(s+\lambda_2) & 1-\tilde{q}_{22}(s+\lambda_2) & \mathrm{i} \\ \mathrm{j} & \mathrm{j} & \mathrm{j} & \mathrm{I} \end{bmatrix}$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \\ \tilde{R}_2(s) \\ \mathrm{j} \end{bmatrix} \qquad \tilde{\mathbf{H}}(s) = \begin{bmatrix} \dfrac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) \\ \dfrac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1) \\ \dfrac{1}{s+\lambda_2} - \tilde{G}_1(s+\lambda_2) \\ \mathrm{j} \end{bmatrix}$$

The conditional mean times to failure we obtain from the formula

$$\mu_i = \lim_{p\to 0^+} \tilde{R}_i(p),\ p \in (0,\infty), \quad i \in J \tag{21}$$

The unconditional mean time to failure has a form

$$\mu = \sum_{i=1}^{N} P(\lambda(0) = \lambda_i)\, \mu_i .$$

7.1. Alternating random process as a failure rate

Assume that the failure rate is a semi-Markov process with the state space $S = \{\lambda_0, \lambda_1\}$ and the kernel

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & G_0(t) \\ G_1(t) & 0 \end{bmatrix},$$

where $G_0(t), G_1(t),$ are the cumulative probability distribution functions with nonnegative support. Suppose that at least one of the functions is absolutely continuous with respect to the Lebesgue measure. Let $p = [p_0,\ p_1]$ be an initial probability distribution of the process. That stochastic process is called the *alternating random process*. In that case the matrices from the equation (20) are

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] = \begin{bmatrix} 1 & -\tilde{g}_0(s+\lambda_0) \\ -\tilde{g}_1(s+\lambda_1) & 1 \end{bmatrix},$$

where

$$\tilde{g}_0(s) = \tilde{q}_{01}(s) = \int_0^\infty e^{-st} dG_0(t) ,$$

$$\tilde{g}_1(s) = \tilde{q}_{10}(s) = \int_0^\infty e^{-st} dG_1(t) ,$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \end{bmatrix},$$

$$\tilde{\mathbf{H}}(s) = \begin{bmatrix} \frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) \\ \frac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1) \end{bmatrix}.$$

The solution of (20) takes the form

$$\tilde{R}_0(s)$$

$$= \frac{\frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0) + \tilde{g}_0(s+\lambda_0)\left[\frac{1}{s+\lambda_1} - \tilde{G}_1(s+\lambda_1)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)},$$

$$\tilde{R}_1(s)$$

$$= \frac{\frac{1}{s+\lambda_0} - \tilde{G}_1(s+\lambda_1) + \tilde{g}_1(s+\lambda_1)\left[\frac{1}{s+\lambda_0} - \tilde{G}_0(s+\lambda_0)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)}.$$

The Laplace transform of the unconditional reliability function is

$$\tilde{R}(s) = p_0 \tilde{R}_0(s) + p_1 \tilde{R}_1(s).$$

Example 3.
Assume that

$$G_0(t) = \int_0^t g_0(x)dx ,\quad G_1(t) = \int_0^t g_1(x)dx ,$$

where

$$g_0(x) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} x^{\alpha_0 - 1} e^{-\beta_0 x}, \quad x \ge 0 ,$$

$$g_1(t) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_2)} x^{\alpha_1 - 1} e^{-\beta_1 x}, \quad x \ge 0.$$

Suppose that an initial state is λ_0. Hence the initial distribution is $\mathbf{p}(0)=[1 \quad 0]$ and the Laplace transform of the unconditional reliability function is $\tilde{R}(s)=\tilde{R}_0(s)$. Now the equation (20) takes the form of

$$\begin{bmatrix} 1 & -\dfrac{\beta_0^{\alpha_0}}{(s+\beta_0+\lambda_0)^{\alpha}} \\ -\dfrac{\beta_1^{\alpha_1}}{(s+\beta_1+\lambda_1)^{\alpha_1}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s+\lambda_0}-\dfrac{\beta_1^{\alpha_0}}{(s+\lambda_0)(s+\beta_0+\lambda_0)^{\alpha_0}} \\ \dfrac{1}{s+\lambda_1}-\dfrac{\beta_1^{\alpha_1}}{(s+\lambda_1)(s+\beta_1+\lambda_1)^{\alpha_1}} \end{bmatrix}$$

For

$$\alpha_0=2, \alpha_1=3, \beta_0=0.2, \beta_1=0.5, \lambda_0=0, \lambda_1=0.2,$$

we have

$$\tilde{R}_0(s)=\frac{\dfrac{1}{s}-\dfrac{0.04}{s(s+0.2)^2}+\dfrac{0.04}{(s+0.2)^2}\left[\dfrac{1}{s+0.2}-\dfrac{0.125}{(s+0.2)(s+0.7)^3}\right]}{1-\dfrac{0.04}{(s+0.2)^2}\dfrac{0.125}{(s+0.7)^3}}$$

Using the MATHEMATICA computer program we obtain the reliability function as the inverse Laplace transform.

$$R(t)=1.33023\exp(-0.0614293\,t)$$

$$+\exp(-0.02\,t)(1.34007\cdot 10^{-14}+9.9198\cdot 10^{-15}\,t)$$

$$-2\exp(-0.843935\,t)[0.0189459\cos(0.171789\,t)$$

$$+0.00695828\sin(0.171789\,t)]$$

$$-2\exp(-0.37535\,t)[0.146168\cos(0.224699\,t)$$

$$+0.128174\sin(0.224699\,t)]$$

Figure 6 shows the reliability function.

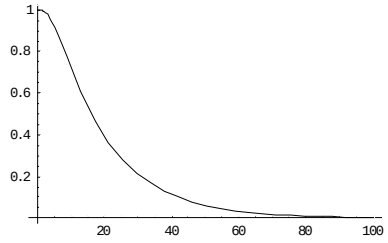

Figure 6. The reliability function from example 2

The corresponding density function

$$f(t)=-R'(t)$$

is shown in *Figure 7*

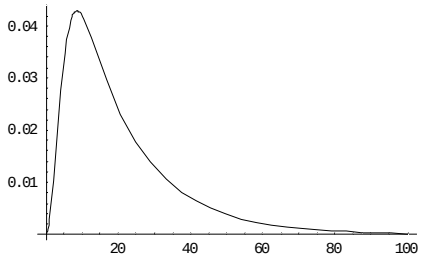

Figure 7. The density function from example 2

8. Conclusion

The semi-Markov processes theory is convenient for description of the reliability systems evolution through the time. The probabilistic characteristics of semi-Markov processes are interpreted as the reliability coefficients of the systems. If A represents the subset of failing states and i is an initial state, the random variable Θ_{iA} designating the first passage time from the state i to the states subset A, denotes the time to failure of the system. Theorems of semi-Markov processes theory allows us to find the reliability characteristic, like the distribution of the time to failure, the reliability function, the mean time to failure, the availability coefficient of the system and many others. We should remember that semi-Markov process might be applied as a model of the real system reliability evolution, only if the basic properties of the semi-Markov process definition are satisfied by the real system.

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The random failure rate

Keywords

reliability, random failure rate, semi-Markov process

Abstract

A failure rate of the object is assumed to be a stochastic process with nonnegative, right continuous trajectories. A reliability function is defined as an expectation of a function of a random failure rate process. The properties and examples of the reliability function with the random failure rate are presented in the paper. A semi-Markov process as the random failure rate is considered in this paper.

1. Introduction

Often, the environmental conditions are randomly changeable and they cause a random load of an object. Thus, the failure rate depending on the random load is a random process. The reliability function with semi-Markov failure rate was considered in the following papers Kopociński & Kopocińska [5], [6], Grabski [3], [4].

2. Reliability function with random failure rate

Let $\{л(t): \ t \geq 0\}$ be a random failure rate of an object. We assume that the stochastic process has the nonnegative, right continuous trajectories. The reliability function is defined as

$$R(t) = E\left[\exp\left(-\int_0^t л(x)dx\right)\right], \ t \geq 0. \tag{1}$$

It means that the reliability function is an expectation of the process $\{o(t): t \geq 0\}$, where

$$o(t) = \exp\left(-\int_0^t л(x)dx\right), \ t \geq 0. \tag{2}$$

Let

$$\underline{\underline{R}}(t) = \exp\left(-\int_0^t E[л(x)]dx\right), \ t \geq 0. \tag{3}$$

From Jensen's inequality we get very important result

$$R(t) = E\left[\exp\left(-\int_0^t л(x)dx\right)\right] \geq \exp\left(-\int_0^t E[л(x)]dx\right) = \underline{\underline{R}}(t), \ t \geq 0. \tag{4}$$

The above mentioned inequality means that the reliability function defined by the stochastic process $\{л(t): \ t \geq 0\}$ is greater than or equal to the reliability function with the deterministic failure rate, equal to the expectation $\bar{\lambda}(t) = E[л(t)]$.

It is obvious, that for the stationary stochastic process $\{л(t): \ t \geq 0\}$, that has a constant mean value $\bar{\lambda}(t) = E[л(t)] = \lambda$, the reliability function defined by (3) is

$$\underline{\underline{R}}(t) = \exp\left(-\lambda\int_0^t dx\right) = \exp(-\lambda\, t), \ t \geq 0. \tag{5}$$

Hence, we come to conclusion: for each stationary random failure rate process, the according reliability function for each $t \geq 0$, has values greater than or equal to the exponential reliability function with parameter λ.

Example 1.
Suppose that, the failure rate of an object is a stochastic process $\{л(t): t \geq 0\}$, given by $л(t) = C\,t,\ t \geq 0,$ where C is a nonnegative random variable. Trajectories of the process $\{o(t): t \geq 0\}$, are

$$\xi(t) = \exp(-c\frac{t^2}{2}),\ t \geq 0,$$

where c is a value of the random variable C. Assume that the random variable C has the exponential distribution with parameter β :

$$P(C \leq u) = 1 - e^{-\beta u},\ u \geq 0.$$

Then, according to (1), we compute the reliability function

$$R(t) = E\left[\exp\left(-\int_0^t Cx\,dx\right)\right] = \int_0^\infty e^{-u\frac{t^2}{2}}\beta e^{-\beta u}du$$

$$= \beta\int_0^\infty e^{-u\left(\frac{t^2}{2}+\beta\right)}du = \frac{2\beta}{t^2+2\beta}$$

Figure 1 shows that function.

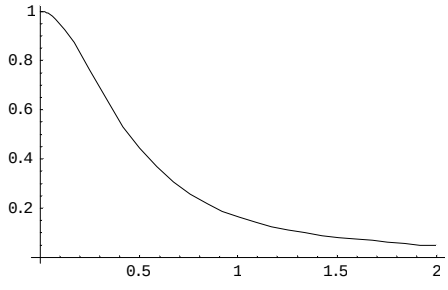

Figure 1. Reliability function $R(t)$

In that case the function (3) is

$$\breve{R}(t) = \exp\left(-\int_0^t E[Cx]dx\right) = \exp\left(-\frac{t^2}{2\beta}\right),\ t \geq 0.$$

Figure 2 shows that function.

Suppose that a failure rate process $\{л(t): t \geq 0\}$ is a linear function of a random load process $\{u(t): t \geq 0\}$:

$$л(t) = \varepsilon\,u(t).$$

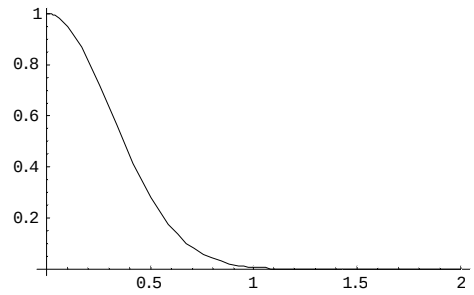

Figure 2. Reliability function $\breve{R}(t)$

Assume that the process $\{u(t): t \geq 0\}$ has an ergodic mean, i.e.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(x)dx = E[u(t)] = \bar{u}\,.$$

Then, [2], [3]

$$\lim_{\varepsilon\to 0} R(\tfrac{t}{\varepsilon}) = \exp[-\bar{u}t]\,.$$

It means, that for small ε

$$R(x) \approx \exp[-\varepsilon\,\bar{u}x]\,.$$

3. Semi-Markov process as a random failure rate

The semi-Markov process as a failure rate and the reliability function with that failure rate was introduced by Kopociński & Kopocińska [5]. Some extensions and developments of the results from [3] were obtained by Grabski [3], [4].

3.1. Semi-Markov processes with a discrete state space

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy, W.L. Smith, and L.Takacs in 1954-55. The essential developments of semi-Markov processes theory were achieved by Cinlar [1], Koroluk & Turbin [8], Limnios & Oprisan [7], Silvestrov [9]. We will apply only semi-Markov processes with a finite or countable state space. The semi-Markov processes are connected to the Markov renewal processes.

Let S be a discrete (finite or countable) state space and let $R_+ = [0,\infty)$, $N_0 = \{0,1,2,...\}$. Suppose, that $\xi_n, \vartheta_n, n = 0,1,2,...$ are the random variables defined on a joint probabilistic space (Ω, F, P) with values on S and R_+ respectively. A two-dimensional random sequence $\{(\xi_n, \vartheta_n), n = 0,1,2,...\}$ is called a Markov renewal chain if for all $i_0,....,i_{n-1}, i \in S, t_0,...,t_n \in R_+, n \in N_0$.

The equalities

$$1.\ P\{\xi_{n+1}=j,\vartheta_{n+1}\le t\,|\,\xi_n=i,\vartheta_n=t_n,...,\xi_0=i_0,\vartheta_0=t_0\}$$

$$=P\{\xi_{n+1}=j,\vartheta_{n+1}\le t\,|\,\xi_n=i\}=Q_{ij}(t) \tag{6}$$

$$2.\ P\{\xi_0=i_o,\vartheta_0=0\}=P\{\xi_0=i_0\}=p_{i_o} \tag{7}$$

hold.

It follows from the above definition that a Markov renewal chain is a homogeneous two-dimensional Markov chain such that the transition probabilities do not depend on the second component. It is easy to notice that a random sequence $\{\xi_n : n=0,1,2,...\}$ is a homogeneous one-dimensional Markov chain with the transition probabilities

$$p_{ij}=P\{\xi_{n+!}=j\,|\,\xi_n=i\}=\lim_{t\to\infty}Q_{ij}(t). \tag{8}$$

A matrix

$$Q(t)=\lfloor Q_{ij}(t):\ i,j\in S\rfloor$$

Is called a Markov renewal kernel. The Markov renewal kernel and the initial distribution $p=[p_i : i\in S]$ define the Markov renewal chain. That chain allows us to construct a semi-Markov process.

Let

$$\tau_0=\vartheta_0=0,\tau_n=\vartheta_1+...+\vartheta_n,\tau_\infty=\sup\{\tau_n : n\in N_0\}$$

A stochastic process $\{X(t):t\ge 0\}$ given by the following relation

$$X(t)=\xi_n \quad \text{for} \quad t\in[\tau_n,\tau_{n+1}) \tag{9}$$

is called a semi-Markov process on S generated by the Markov renewal chain related to the kernel $Q(t),t\ge 0$ and the initial distribution p.

Since the trajectory of the semi-Markov process keeps the constant values on the half-intervals $[\tau_n,\tau_{n+1})$ and it is a right-continuous function, from equality $X(\tau_n)=\xi_n$, it follows that the sequence $\{X(\tau_n):n=0,1,2,...\}$ is a Markov chain with the transition probabilities matrix

$$P=[p_{ij} : i,j\in S]. \tag{10}$$

The sequence $\{X(\tau_n):n=0,1,2,...\}$ is called an embedded Markov chain in a semi-Markov process $\{X(t):t\ge 0\}$.

The function

$$F_{ij}(t)=P\{\tau_{n+1}-\tau_n\le t\,|\,X(\tau_n)=i,X(\tau_{n+1})=j\}$$

$$=\frac{Q_{ij}(t)}{p_{ij}} \tag{11}$$

is a cumulative probability distribution of a holding time of a state i, if the next state will be j. From (11) we have

$$Q_{ij}(t)=p_{ij}F_{ij}(t). \tag{12}$$

The function

$$G_i(t)=P\{\tau_{n+1}-\tau_n\le t\,|\,X(\tau_n)=i\}=\sum_{j\in S}Q_{ij}(t) \tag{13}$$

is a cumulative probability distribution of an occupation time of the state i.

A stochastic process $\{N(t):t\ge 0\}$ defined by

$$N(t)=n \ \text{ for } \ t\in[\tau_n,\tau_{n+1}) \tag{14}$$

is called a counting process of the semi-Markov process $\{X(t):t\ge 0\}$.

The semi-Markov process $\{X(t):t\ge 0\}$ is said to be regular if for all $t\ge 0$

$$P\{N(t)<\infty\}=1. \tag{15}$$

It means that the process $\{X(t):t\ge 0\}$ has the finite number of state changes on a finite period.

Every Markov process $\{X(t):t\ge 0\}$ with the discrete space S and the right-continuous trajectories keeping constant values on the half-intervals, with the generating matrix of the transition rates $A=[\alpha_{ij}:i,j\in S]$, $0<-\alpha_{ii}=\alpha_i<\infty$ is the semi-Markov process with the kernel

$$\mathbf{Q}(t)=[Q_{ij}(t):i,j\in S],$$

where

$$Q_{ij}(t)=p_{ij}(1-e^{-\alpha_{ii}t}),\ t\ge 0,$$

$$p_{ij}=\frac{\alpha_{ij}}{\alpha_i} \text{ for } i\ne j$$

and

$p_{ii} = 0$.

3.2. Semi-Markov failure rate

Suppose that the random failure rate $\{\lambda(t) : t \geq 0\}$ is the semi-Markov process with the discrete state space $S = \{\lambda_j : j \in J\}$, $J = \{0,1,...,m\}$ or $J = \{0,1,2,...\}$, $0 \leq \lambda_0 < \lambda_1 < ...$ with the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in J]$$

and the initial distribution $p = [p_i : i \in J]$.
We define a conditional reliability function as

$$R_i(t) = E\left[\exp\left(-\int_0^t л(u)du\right) \middle| \lambda(0) = \lambda_i\right], \ t \geq 0, \ i \in J. \quad (16)$$

In [3] it is proved, that for the regular semi-Markov process $\{\lambda(t) : t \geq 0\}$ the conditional reliability functions $R_i(t), t \geq 0, \ i \in J$ defined by (16), satisfy the system of equations

$$R_i(t) = e^{-\lambda_i t}[1 - G_i(t)] + \sum_j \int_0^t e^{-\lambda_i x} R_j(t-x) dQ_{ij}(x), \quad (17)$$
$$i \in J.$$

Using the Laplace transform we obtain the system of linear equations

$$\tilde{R}_i(s) = \frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) + \sum_j \tilde{R}_j(s)\tilde{q}_{ij}(s+\lambda_i), \ i \in J \quad (18)$$

where

$$\tilde{R}_i(s) = \int_0^\infty e^{-st} R_i(t) dt,$$

$$\tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t) dt,$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t).$$

In matrix notation we have

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)]\tilde{\mathbf{R}}(s) = \tilde{\mathbf{H}}(s), \quad (19)$$

where

$$\tilde{\mathbf{R}}(s) = \left[\tilde{R}_i(s) : \ i \in J\right]^T,$$

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] = \left[\delta_{ij} - \tilde{q}_{ij}(s+\lambda_i) : i, j \in J\right],$$

$$\tilde{\mathbf{H}}(s) = \left[\frac{1}{s+\lambda_i} - \tilde{G}_i(s+\lambda_i) : i \in J\right].$$

The conditional mean times to failure we obtain from the formula

$$\mu_i = \lim_{p \to 0^+} \tilde{R}_i(p), \ p \in (0,\infty), \ i \in J \quad (20)$$

The unconditional mean time to failure has a form

$$\mu = \sum_{i \in J} P(\lambda(0) = \lambda_i)\, \mu_i. \quad (21)$$

3.3. 3-state random walk process as a failure rate

Assume that the failure rate is a semi-Markov process $\{л(t) : t \geq 0\}$ with the state space $S = \{\lambda_0, \lambda_1, \lambda_2\}$ and the kernel

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & G_0(t) & 0 \\ aG_1(t) & 0 & (1-a)G_1(t) \\ 0 & G_2(t) & 0 \end{bmatrix},$$

where $G_0(t), G_1(t), G_2(t)$ are the cumulative probability distribution functions with nonnegative support. Suppose that at least one of the functions is absolutely continuous with respect to the Lebesgue measure. Let $p = [p_0, p_1, p_2]$ be an initial probability distribution of the process. That stochastic process is called the 3-state random walk process. In that case the matrices from the equation (19) are

$$[\mathbf{I} - \tilde{\mathbf{q}}_\lambda(s)] =$$

$$= \begin{bmatrix} 1 & -\tilde{g}_0(s+\lambda_0) & 0 \\ -a\tilde{g}_1(s+\lambda_1) & 1 & -(1-a)\tilde{g}_1(s+\lambda_1) \\ 0 & -\tilde{g}_2(s+\lambda_2) & 1 \end{bmatrix}, \quad (22)$$

where

$$\tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t), i = 0,1,2.$$

$$\tilde{\mathbf{R}}(s) = \begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \\ \tilde{R}_2(s) \end{bmatrix},$$

$$\tilde{\mathbf{H}}(s)=\begin{bmatrix}\frac{1}{s+\lambda_0}-\tilde{G}_0(s+\lambda_0)\\ \frac{1}{s+\lambda_1}-\tilde{G}_1(s+\lambda_1)\\ \frac{1}{s+\lambda_2}-\tilde{G}_2(s+\lambda_2)\end{bmatrix}. \tag{23}$$

The Laplace transform of unconditional reliability function is

$$\tilde{R}(s)=p_0\tilde{R}_0(s)+p_1\tilde{R}_1(s)+p_1\tilde{R}_1(s)$$

Example 2.
Assume that

$$p_0=1,\quad p_1=0,\quad p_2=0$$

and

$$G_0(t)=1-(1+\alpha t)e^{-\alpha t},$$

$$G_1(t)=1-e^{-\beta t},$$

$$G_2(t)=1-(1+\gamma t)e^{-\gamma t},\ t\geq 0.$$

The corresponding Laplace transforms are

$$\tilde{G}_0(s)=\frac{\alpha^2}{s(s+\alpha)^2},$$

$$\tilde{G}_1(s)=\frac{\beta}{s(s+\beta)},$$

$$\tilde{G}_2(s)=\frac{\gamma^2}{s(s+\gamma)^2},$$

$$\tilde{g}_0(s)=\frac{\alpha^2}{(s+\alpha)^2},$$

$$\tilde{g}_1(s)=\frac{\beta}{s+\beta},$$

$$\tilde{g}_2(s)=\frac{\gamma^2}{(s+\gamma)^2}.$$

Let

$$p=[1,0,0],\quad a=0.4$$

and

$$\alpha=0.4,\ \beta=0.04,\gamma=0.02,\ \lambda_0=0,\ \lambda_1=0.1,\ \lambda_2=0.2\ .$$

Since the matrices (22) and (23) are

$$[\mathbf{I}-\tilde{\mathbf{q}}_\lambda(s)]=$$

$$=\begin{bmatrix}1 & -\frac{0.0025}{(0.05+s)^2} & 0\\ -0.4\frac{0.04}{0.14+s} & 1 & -0.6\frac{0.04}{0.14+s}\\ 0 & -\frac{0.0004}{(0.22+s)^2} & 1\end{bmatrix},$$

$$\tilde{\mathbf{H}}(s)=\begin{bmatrix}\frac{1}{s}-\frac{0.0025}{s(0.05+s)^2}\\ \frac{1}{s+0.1}-\frac{0.04}{(s+0.1)(0.14+s)}\\ \frac{1}{s+\lambda_2}-\frac{0.0004}{(s+0.2)(0.22+s)^2}\end{bmatrix}.$$

From solution of equation (19), in this case, we obtain

$$\tilde{R}(s)=\tilde{R}_0(s)=\frac{\tilde{a}(s)}{\tilde{b}(s)}$$

where

$$\tilde{a}(s)=(0.01623+0.23349s+s^2)$$

$$\cdot(0.05002+0.44655s+s^2)$$

$$\tilde{b}(s)=(0.03083+s)(0.07486+s)(0.13292+s)$$

$$\cdot(0.04882+0.44138s+s^2)$$

Using the MATHEMATICA computer program we obtain the reliability function as the inverse Laplace transform

$$R(t)=0.51646e^{-0.13292t}+0.23349e^{-0.07486t}$$

$$+2.28565e^{-0.13292t}$$

$$-2\cdot 0.01539e^{-0.22069t}\cos(0.01075t)$$
$$-2\cdot 0.01343e^{-0.22069t}\cos(0.01075t).$$

Figure 3 shows this reliability function.

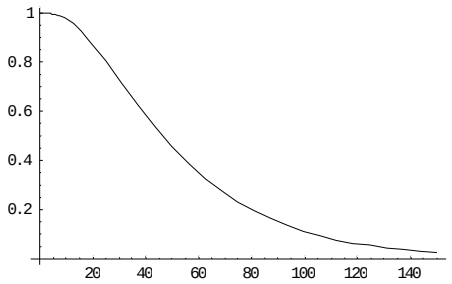

Figure 3. The reliability function from example 2

The corresponding density function is shown in *Figure 4.*

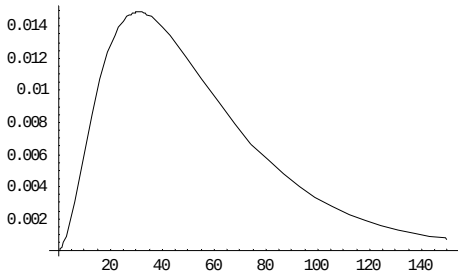

Figure 4. The density function from example 2

3.4. The Poisson process as a failure rate

Suppose that the random failure rate $\{\lambda(t): t \geq 0\}$ is the Poisson process with parameter $\lambda > 0$. Of course, the Poisson process is the Markov process with the counting state space $S = \{0,1,2,...\}$. That process can be treated as the semi-Markov process defined on by the initial distribution $p = [1,0,0,...]$ and the kernel

$$Q(t) = \begin{bmatrix} 0 & G_0(t) & 0 & 0 & ... \\ 0 & 0 & G_1(t) & 0 & ... \\ 0 & 0 & 0 & G_2(t) & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \end{bmatrix},$$

where

$$G_i(t) = 1 - e^{-\lambda t}, t \geq 0, \ i = 0,1,2,...$$

The Poisson process is of course a Markov process too.

Applying equation (19), Grabski [3] proved the following theorem:

If the random failure rate $\{\lambda(t): t \geq 0\}$ is the Poisson process with parameter $\lambda > 0$, than the reliability function defined by (16) takes form

$$R(t) = \exp\{-\lambda [t - 1 + \exp(-t)]\}, t \geq 0.$$

The corresponding density function is given by the formula

$$f(t) = \lambda \exp\{-\lambda [t - 1 + \exp(-t)]\}[1 - \exp(-t)], t \geq 0.$$

Those functions with parameter $\lambda = 0.2$ are shown in *Figure 5* and *Figure 6.*

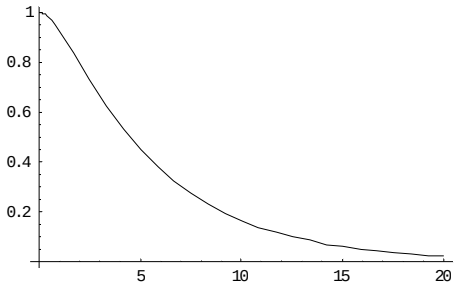

Figure 5. The reliability function for the Poisson process

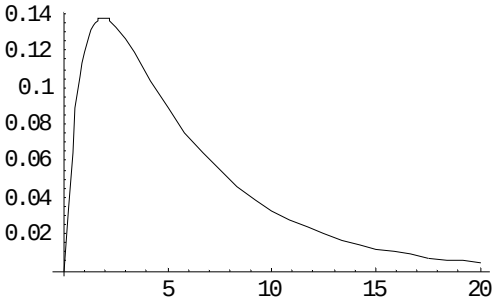

Figure 6. The density function for the Poisson process

3.5. The Furry-Yule process as a failure rate

The Furry-Yule is the semi-Markov process on the counting state space $S = \{0,1,2,...\}$ with the initial distribution $p = [1,0,0,...]$ and the kernel similar to the Poison process

$$Q(t) = \begin{bmatrix} 0 & G_0(t) & 0 & 0 & ... \\ 0 & 0 & G_1(t) & 0 & ... \\ 0 & 0 & 0 & G_2(t) & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \end{bmatrix},$$

where

$$G_i(t) = 1 - e^{-\lambda(i+1)t}, t \geq 0, \ i = 0,1,2,...$$

The Furry-Yule process is also the Markov process.

Assume that the random failure rate $\{\lambda(t): t \geq 0\}$ is the Furry-Yule process with parameter $\lambda > 0$. The following theorem is proved by Grab ski [4]:

If the random failure rate $\{\lambda(t): t \geq 0\}$ is the Furry-Yule process with parameter $\lambda > 0$, then the reliability function defined by (1) is given by

$$R(t) = \frac{(\lambda+1)\exp(-\lambda t)}{1+\lambda\exp[-(\lambda+1)t]}, t \geq 0.$$

The corresponding density function is

$$f(t) = \frac{\lambda(\lambda+1)\exp[1-(\lambda+1)t]}{\{1+\lambda\exp[-(\lambda+1)t]\}^2}, t \geq 0.$$

Those functions with parameter $\lambda = 0.2$ are shown in *Figure 7* and *Figure 8*.

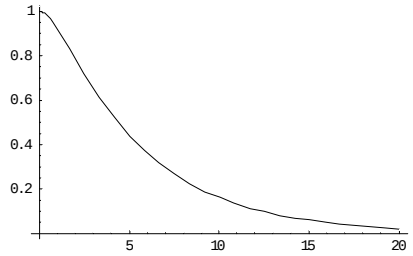

Figure 7. The reliability function for the Furry-Yule process

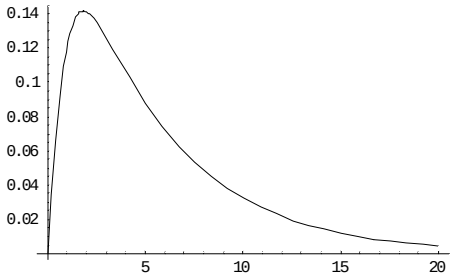

Figure 8. The density function for the Furry-Yule process

4. Conclusion

Frequently, because of the randomly changeable environmental conditions and tasks, the assumption that a failure rate of an object is a random process seems to be proper and natural. We obtain the new interesting classes of reliability functions for the different stochastic failure rate processes.

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The model of non-renewal reliability systems with dependent time lengths of components

Keywords

reliability, dependent components, series systems, parallel systems

Abstract

The models of the non-renewal reliability systems with dependent times to failure of components are presented. The dependence arises from some common environmental stresses and shocks. It is assumed that the failure occurs only because of two independent sources common for two neighbour components. The reliability function of series and parallel systems with components depending on common sources are computed. The reliability functions of the systems with dependent and independent life lengths of components are compared.

1. Introduction

The problem of determining the reliability function of the system with dependent components is important but difficult to solve. Many papers are devoted to it i.e. [1], [2], [3], [4]. In the Barlow & Proshan book [1975], there is defined, based on the reliability theory, multivariate exponential distribution as a distribution of a random vector, the coordinates of which are dependent random variables defining life lengths of the components. Their dependence arises from some environmental common sources of shocks. Using that idea we are going to present some examples of systems with dependent components, giving up the assumption that the joint survival probability is exponential and accepting the assumption that the failure occurs only because of two independent sources common for two neighbour components.
Assume that due to reliability there are n ordered components

$$E = (e_1, e_2, ..., e_n).$$

Assume also that n+1 independent sources of shocks are present in the environment

$$Z = (z_1, z_2, ..., z_n, z_{n+1})$$

and each component e_i can be destroyed only because of shocks from two sources z_i and z_{i+1}.
Let U_i be non-negative random variable defining the time to failure of the component caused by the shock from the source z_i. Thus the life length of the object depends on the random vector

$$U = (U_1, U_2, ..., U_n, U_{n+1}). \tag{1}$$

Admit that the coordinates of the vector are independent random variables with distributions defined as follows

$$G_i(u_i) = P(U_i \le u_i), \quad i = 1, 2, ..., n+1. \tag{2}$$

The life length of the component e_i is a random variable satisfy

$$T_i = \min(U_i, U_{i+1}), \quad i = 1, 2, ..., n. \tag{3}$$

Notice that two neighbour components in the sequence $(e_1, e_2, ..., e_n)$ have one common source of shock – depend on the same random variable. The random variables T_1, T_2,…,T_n are *dependent*. Their joint distribution is expressed by means of the multivariate reliability function and it can be easily determined:

$$R(t_1,t_2,...,t_n)=P(T_1>t_1,T_2>t_2,...,T_n>t_n)$$

$$=P(\min(U_1,U_2)>t_1,\min(U_2,U_3)>t_2,$$

$$...,\min(U_n,U_{n+1})>t_n)$$

$$=P(U_1>t_1,U_2>\max(t_1,t_2),$$

$$...,U_n>\max(t_{n-1},t_n),U_{n+1}>t_n)$$

$$=P(U_1>t_1)P(U_2>\max(t_1,t_2))$$

$$...P(U_n>\max(t_{n-1},t_n))P(U_{n+1}>t_n).$$

Thus

$$R(t_1,t_2,...,t_n)=\overline{G}_1(t_1)\,\overline{G}_2(\max(t_1,t_2))...$$
$$\overline{G}_n(\max(t_{n-1},t_n)\,\overline{G}_{n+1}(t_n) \quad (4)$$

where

$$\overline{G}_i(u_i)=P(U_i>u_i)=1-G_i(u_i)=P(T\le u_i),$$
$$i=1,2,...,n+1.$$

The reliability functions of the components can be obtained as marginal distributions computing the limit of the function (4), when

$$t_1\to 0^+,...,t_{i-1}\to 0^+,\quad t_{i+1}\to 0^+,...,t_n\to 0^+ \quad .$$

$$R_i(t_i)=P(T_i>t_i)=\overline{G}_i(t_i)\,\overline{G}_{i+1}(t_i),\quad i=1,2,...,n. \quad (5)$$

The bivariate reliability functions can be determined by computing the limit of (4), when

$$t_1\to 0^+,...,t_{i-1}\to 0^+,\quad t_{i+1}\to 0^+,...,t_{j-1}\to 0^+,$$
$$t_{j+1}\to 0^+,...,t_n\to 0^+.$$

$$R_{ij}(t_i,t_j)=P(T_i>t_i,T_j>t_j)$$

$$=\overline{G}_i(t_i)\,\overline{G}_{i+1}(t_i)\,\overline{G}_j(t_j)\,\overline{G}_{j+1}(t_j), \quad (6)$$
$$i+1<j,\quad i,j=1,2,...,n-1,$$

$$R_{ij}(t_i,t_j)=P(T_i>t_i,T_j>t_j)$$

$$=\overline{G}_i(t_i)\,\overline{G}_{i+1}(\max(t_i,t_{i+1}))\overline{G}_{i+2}(t_{i+1}), \quad (7)$$
$$i+1=j,\quad i,j=1,2,...,n-1,$$

it could be proved that

$$P(T_1>t_1\,|\,T_2>t_2,...,T_n>t_n)$$
$$=P(T_1>t_1\,|\,T_2>t_2) \quad (8)$$

and generally

$$P(T_i>t_1\,|\,T_{i+1}>t_{i+1},...,T_n>t_n)$$
$$=P(T_i>t_i\,|\,T_{i+1}>t_{i+1}),\ i=1,2,...,n-1. \quad (9)$$

That property asserts that the life length of e_i depends only on the life length of the next component e_{i+1}, does not depend on the life lengths of the rest of the components. That is a certain kind of Markov property.

2. Reliability of the object with the series structure

If the object has a series reliability structure then its life length T is the random variable defined by the formula

$$T=\min(T_1,T_2,...,T_n). \quad (10)$$

Using (4) we can determine the reliability function:

$$\mathbf{R}(t)=P(T>t)=P(T_1>t,T_2>t,...,T_n>t)$$
$$=R(t,t,...,t) \quad (11)$$
$$=\overline{G}_1(t)\,\overline{G}_2(t)...\overline{G}_n(t)\,\overline{G}_{n+1}(t).$$

Let us compare the function with the reliability function of a series system in which the life lengths of the components T_1, T_2, …,T_n are independent and their reliability functions are defined by (5). Let $\tilde{\mathbf{R}}(t),t\ge 0$ be a reliability function of that system. It satisfies

$$\tilde{\mathbf{R}}(t)=P(T>t)=P(T_1>t,T_2>t,\,\mathsf{h}\,,T_n>t)$$
$$=R_1(t)R_2(t)\,\mathsf{h}\,R_n(t)$$

$$= \overline{G}_1(t)\overline{G}_2(t)\overline{G}_2(t)\overline{G}_3(t) \,\mathsf{h}\, \overline{G}_n(t)\overline{G}_n(t)\overline{G}_{n+1}(t)$$

$$= \overline{G}_2(t)\overline{G}_3(t) \,\mathsf{h}\, \overline{G}_n(t)\mathbf{R}(t) \qquad (12)$$

Thus, for $t \geq 0$

$$\tilde{\mathbf{R}}(t) \leq \mathbf{R}(t)$$

holds.

The inequality means that the reliability of a series system with dependent (in the considering sense) life lengths of components is greater than (or equal) to the reliability of that system with independent life lengths of components and the same distributions as the marginals of $T_1, T_2, \ldots, T_n$.

Accepting the assumption about independence of the life lengths of the components even though the random variables describing the life lengths are dependent, we make an obvious mistake but that error is „safe" because the real series system has a greater reliability. That estimation is very conservative.

Example 1.

Assume that a non-negative random variable *Ui*, describing time to failure of the component caused by the shock from source z_i has a Weibull distribution with parameters

$$\alpha_i, \lambda_i, \; i = 1,2,\ldots,n+1$$

for $u_i > 0$

$$\overline{G}_i(u_i) = P(U_i > u_i) = e^{-\lambda_i u_i^{\alpha_i}}, \quad i = 1, 2, \ldots, n+1.$$

The reliability function of a series system with dependent components satisfies

$$\mathbf{R}(t) = P(T > t) = \overline{G}_1(t)\,\overline{G}_2(t) \ldots \overline{G}_n(t)\,\overline{G}_{n+1}(t)$$

$$= e^{-(\lambda_1 t^{a_1} + \ldots + \lambda_{n+1} t^{a_{n+1}})}$$

For $n = 3$ and

$$\alpha_1 = 1.2 \quad , \lambda_1 = 0.1, \; \alpha_2 = 2, \quad \lambda_2 = 0.2,$$

$$\alpha_3 = 2.2, \; \lambda_3 = 0.1, \quad \alpha_4 = 3, \; \lambda_4 = 0.2$$

we get

$$\mathbf{R}(t) = P(T > t) = e^{-(0.1t^{1.2} + 0.2t^2 + 0.1t^{2.2} + 0.2t^3)}$$

The graph of the function is presented in *Figure 1.*

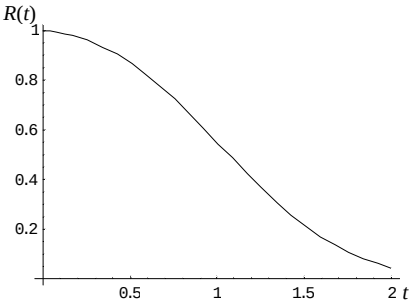

Figure 1. The graph of the series reliability function with dependent components

The reliability function $\tilde{\mathbf{R}}(t), t \geq 0$ of the series system with independent life lengths of components, the same marginals satisfies

$$\tilde{\mathbf{R}}(t) = P(T > t) = e^{-(0.1t^{1.2} + 0.4t^2 + 0.2t^{2.2} + 0.2t^3)}.$$

3. Reliability of the object of the parallel structure

The life length of the object of a parallel structure is a random variable defined by

$$T = \max(T_1, T_2, \ldots, T_n). \qquad (13)$$

Let us compute the reliability function of the object:

$$\mathbf{R}(t) = P(T > t) = 1 - P(T \leq t)$$

$$= 1 - P(T_1 \leq t, T_2 \leq t, \ldots, T_n \leq t) \qquad (14)$$

$$= P(\{T_1 > t\} \cup \{T_2 > t\} \cup \ldots \cup \{T_n > t\}).$$

Using the formula of probability of a sum of events we obtain

$$\mathbf{R}(t) = P(T > t) = \sum_{i=1}^{n} P(T_i > t) - \sum_{\substack{i,j=1 \\ i<j}}^{n} P(T_i > t, T_j > t)$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} P(T_i > t, T_j > t, T_k > t) - \ldots$$

$$+ (-1)^{n+1} P(T_1 > t, T_2 > t, \ldots, T_n > t).$$

Hence and from (4), (6), (7) we get

$$\mathbf{R}(t)=\sum_{i=1}^{n}\overline{G}_i(t)\,\overline{G}_{i+1}(t)-\sum_{\substack{i,j=1\\ i+1<j}}^{n}\overline{G}_i(t)\,\overline{G}_{i+1}(t)\,\overline{G}_j(t)\,\overline{G}_{j+1}(t)$$
$$-\sum_{i=1}^{n-1}\overline{G}_i(t)\,\overline{G}_{i+1}(t)\overline{G}_{i+2}(t) \qquad (15)$$
$$\ldots+(-1)^{n+1}\overline{G}_1(t)\,\overline{G}_2(t)\ldots\overline{G}_n(t)\,\overline{G}_{n+1}(t).$$

Particularly, for $n=3$ we have,

$$\begin{aligned}\mathbf{R}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)-\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\&+\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\&=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\&+\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)-\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t).\end{aligned} \qquad (16)$$

If $T_1, T_2, \ldots, T_n$ are independent then

$$\begin{aligned}\overline{\mathbf{R}}(t)&=1-P(T_1\le t,T_2\le t,\ldots,T_n\le t)\\&=1-P(T_1\le t)P(T_2\le t)\ldots P(T_n\le t)\\&=1-[1-R_1(t)][1-R_2(t)]\ldots[1-R_n(t)]\\&=1-[1-\overline{G}_1(t)\,\overline{G}_2(t)][1-\overline{G}_2(t)\,\overline{G}_3(t)]\\&\ldots[1-\overline{G}_n(t)\,\overline{G}_{n+1}(t)].\end{aligned}$$

For $n = 3$

$$\begin{aligned}\overline{\mathbf{R}}(t)&=1-[1-\overline{G}_1(t)\,\overline{G}_2(t)][1-\overline{G}_2(t)\,\overline{G}_3(t)]\\&\cdot[1-\overline{G}_3(t)\,\overline{G}_4(t)].\end{aligned}$$

After multiplication we get

$$\begin{aligned}\overline{\mathbf{R}}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)+\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_2(t)\,\overline{G}_3(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\&+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t).\end{aligned}$$

Notice that

$$\begin{aligned}\overline{\mathbf{R}}(t)-\mathbf{R}(t)&=\overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t)+\overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)\\&-\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t)+\\&+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_2(t)\,\overline{G}_3(t)\overline{G}_3(t)\,\overline{G}_4(t)\\&=\overline{G}_2(t)\,\overline{G}_3(t)[\overline{G}_1(t)+\overline{G}_4(t)\\&-\overline{G}_1(t)\,\overline{G}_2(t)-\overline{G}_1(t)\,\overline{G}_4(t)\\&-\overline{G}_3(t)\,\overline{G}_4(t)+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)].\end{aligned}$$

Let A_i, $i=1, 2, 3, 4$ be independent events with probabilities defined by

$$P(A_i)=\overline{G}_i(t),\ i=1,2,3,4.$$

The expression

$$\begin{aligned}&\overline{G}_1(t)+\overline{G}_4(t)-\overline{G}_1(t)\,\overline{G}_2(t)-\overline{G}_1(t)\,\overline{G}_4(t)\\&-\overline{G}_3(t)\,\overline{G}_4(t)+\overline{G}_1(t)\,\overline{G}_2(t)\overline{G}_3(t)\,\overline{G}_4(t)\end{aligned}$$

can be rewritten as

$$\begin{aligned}&P(A_1)+P(A_4)-P(A_1)P(A_2)\\&-P(A_1)P(A_4)-P(A_3)P(A_4)\\&+P(A_1)P(A_2)P(A_3)P(A_4)\\&=P(A_1\cup A_4)-P((A_1\cap A_2)\cup P(A_3\cap A_4)).\end{aligned}$$

As

$$A_1 \cup A_4 \supset (A_1 \cap A_2) \cup (A_3 \cap A_4),$$

So

$$P(A_1 \cup A_4) \geq P((A_1 \cap A_2) \cup (A_3 \cap A_4)).$$

Thus

$$\breve{\mathbf{R}}(t) \geq \mathbf{R}(t).$$

The inequality can be proved for any n by the induction. It assures that the reliability of the parallel system with the independent components is greater than (or equal) to the reliability of that system with dependent components.

Computing the reliability of the real systems we often assume that the components life lengths are independent even though the random variables describing the life lengths are dependent. That example shows that such assumption leads towards careless conclusions. The real parallel system may have significantly lower reliability. Moreover, we come to the similar conclusions if we take under consideration more general assumption about the association of the random variables $T_1, T_2, \ldots, T_n$ [1].

Example 2.

Assume as previously that a non-negative random variable U_i , describing time to failure of the component caused by the source z_i has a Weibull distribution with parameters

$$\alpha_i, \lambda_i, \ i = 1,2,...,n+1.$$

Let $n = 3$. Then for $u_i > 0$

$$\overline{G}_i(u_i) = P(U_i > u_i) = e^{-\lambda_i u_i^{\alpha_i}}, \quad i = 1,2,3,4.$$

As previously

$$\alpha_1 = 1.2 \quad , \lambda_1 = 0.1, \ \alpha_2 = 2, \quad \lambda_2 = 0.2,$$

$$\alpha_3 = 2.2, \quad \lambda_3 = 0.1, \quad \alpha_4 = 3, \quad \lambda_4 = 0.2.$$

Using (16) we obtain the reliability function of the parallel system with dependent components. For $t > 0$ it satisfies

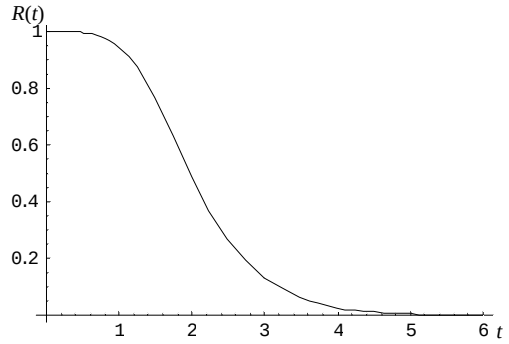

Figure 2. The graph of the reliability function of the parallel system with dependent components

$$\mathbf{R}(t) = \overline{G}_1(t)\,\overline{G}_2(t) + \overline{G}_2(t)\,\overline{G}_3(t) + \overline{G}_3(t)\,\overline{G}_4(t)$$

$$+ \overline{G}_1(t)\,\overline{G}_2(t)\,\overline{G}_3(t) - \overline{G}_2(t)\,\overline{G}_3(t)\,\overline{G}_4(t)$$

$$= e^{-(0.1t^{1.2} + 0.2t^2)} + e^{-(0.2t^2 + 0.1t^{2.2})}$$

$$+ e^{-(0.1t^{2.2} + 0.2t^3)} - e^{-(0.1t^{1.2} + 0.2t^2 + 0.1t^{2.2})}$$

$$- e^{-(0.2t^2 + 0.1t^{2.2} + 0.2t^3)}.$$

Figure 2. Presents its graph.

4. Conclusion

The reliability of a series system with dependent (in the considered sense) life lengths of components is greater than (or equal) to the reliability of that system with independent life lengths of components.

Assuming the independence of the life lengths of the components even though the random variables describing the life lengths are dependent, we make a mistake but that error is „safe" because the real series system has a greater reliability. The estimation of the reliability function is very conservative.

The reliability of the parallel system with the independent components is greater than (or equal) to the reliability of that system with dependent components.

Computing the reliability of the real systems we often assume that the components life lengths are independent even though the random variables describing the life lengths are dependent. The examples presented here show that such assumption leads towards careless conclusions. The real parallel system may have significantly lower reliability.

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An univariate DEMR modelling on repair effects

Keywords

grey theory, DEMR, repair effect, credibility measure theory, random fuzzy variable

Abstract

Repairable system analysis is in nature an evaluation of repair effects. Recent tendency in reliability engineering literature was estimating system repair effects or linking repair to certain covariate to extract repair impacts by imposing repair regimes during system reliability analysis. In this paper, we develop a differential equation motivated regression (abbreviated as DEMR) model with a random fuzzy error term based on the axiomatic framework of self-dual fuzzy credibility measure theory proposed by Liu [5] and grey differential equation models. The fuzzy variable indexes the random fuzzy error term will be used to facilitate the evaluation of repair effects. We further propose a parameter estimation approach for the fuzzy variable (repair effect) under the maximum entropy principle.

1. Introduction

Repairable system analysis is in nature an evaluation of repair effects. Recent tendency in reliability engineering literature was estimating system repair effects or linking repair to certain covariate to extract repair impacts by imposing repair regimes to the system. Guo [3], [4] proposed an approach to isolate repair effects in terms of grey differential equation modelling, particularly, the one-variable first order differential equation model, abbreviated as GM (1,1) model, initiated by Deng [2]. The efforts of modelling of system repair effects in terms of grey differential equation models has attracted attention from because it is easy to calculated, for example, in Microsoft Excel. However, there were two fundamental problems necessary to be addressed. The first issue is the nature of the GM(1,1) model. In The second fundamental problem is GM(1,1) model is a deterministic approach and is just a delicate approximation approach and in nature ignores the regression error structure, which may be very reasonable if the sample size is too small, however, in general, Deng's approach results in information loss, particularly he used the adjective word "grey", implying grey uncertainty involved, but there was not uncertainty structure build up to describe "grey uncertainty". In other words, the existing GM(1,1) model has a good idea without a convincingly rigorous mathematical foundation yet.

In this paper, we will review the coupling principle materialization in GM(1,1) model in section 2. In section 3, will propose a families of first order differential equation motivated regression models under unequal-gaped data, which is suitable for the usages in system functioning time analysis. In section 4, we argue that the differential equation motivated regression model is a coupling regression model with random fuzzy error terms in nature. In section 5, review Liu's [5] fuzzy credibility measure theory and then discuss the random fuzzy variable theory in order to establish the differential equation motivated regression models as a coupling regression with random fuzzy error terms. In section 6, we will discuss the parameter estimation for the fuzzy variable repair effect indexing the random fuzzy error terms of the differential equation motivated regression modelling on system functioning time sequence under maximum entropy principle. Section 7 concludes the paper.

2. An univariate DEMR model

The success of GM(1,1) model lies on the following two aspects: data accumulative generation operator (abbreviated as AGO), which is the partial sum operation in algebra, and a simple regression model

coupled with a first-order linear constant coefficient differential equation model, which Deng [2] called is as whitening differential equation or the shadow differential equation. Let $X^{(0)}$ =($x^{(0)}$ (1), $x^{(0)}$ (2),…, $x^{(0)}$ (n)) be a data sequence, and the partial sum with respect to $X^{(0)}$

$$x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i),\ \ k=2,3,4,\mathsf{h},n, \tag{1}$$

and the mean of two consecutive partial sums, which is used as an approximation to the primitive function of $x^{(1)}(t)$

$$z^{(1)}(k)=\frac{1}{2}\left[x^{(1)}(k)+x^{(1)}(k-1)\right]. \tag{2}$$

Definition 1. Given a (strictly positive) discrete real-valued data sequence $X^{(0)}$ =($x^{(0)}$(1), $x^{(0)}$(2),…, $x^{(0)}$(n)), the equation

$$x^{(0)}(k)=\alpha+\beta\left(-z^{(1)}(k)\right)+\varepsilon_k, \quad k=2,3,4,\mathsf{h},n, \tag{3}$$

"coupled" with the first-order constant coefficient linear ordinary differential equation.

$$\begin{cases}\dfrac{dx^{(1)}(t)}{dt}+\beta x^{(1)}(t)=\alpha\\ x^{(0)}(k)=\alpha+\beta\left(-z^{(1)}(k)\right)+\varepsilon_k,\ k=2,3,4,\mathsf{h},n\end{cases} \tag{4}$$

is called a univariate DEMR model with respect to the data sequence $X^{(0)}$ = ($x^{(0)}$(1), $x^{(0)}$ (2), …, $x^{(0)}$(n)) . Parameter β is called the developing coefficient, parameter a is the grey input, term $x^{(0)}$ is called a *grey derivative* and term $x^{(1)}(k)$ is called the k^{th} 1-AGO of $X^{(0)}$ value (partial sum in fact). Furthermore, the differential equation $dx^{(1)}/dt + \beta x^{(1)} = \alpha$ in Eq. (4) is called the whitening differential equation or the *shadow* equation of the grey differential equation Eq. (3) by Deng [2]. The unknown parameter values (α,β) can be estimated in terms of a standard regression. Note that Eq. (3) can be re-written as in a simple regression form,

$$y_k=\alpha+\beta x_k+\varepsilon_k,\ \ k=2,3,4,\mathsf{h},n, \tag{5}$$

where

$$y_k=x^{(0)}(k),\ \ x_k=-z^{(1)}(k),\ \ k=2,3,4,\mathsf{h},n. \tag{6}$$

The estimate for regression parameter pair(α,β), denoted as (a,b), can be calculated by,

$$(a,b)^T=\left(X^TX\right)^{-1}X^TY \tag{7}$$

where

$$X=\begin{bmatrix}1 & -z^{(1)}(2)\\ 1 & -z^{(1)}(3)\\ \mathsf{j} & \mathsf{j}\\ 1 & -z^{(1)}(n)\end{bmatrix},\ \ Y=\begin{bmatrix}x^{(0)}(2)\\ x^{(0)}(3)\\ \mathsf{j}\\ x^{(0)}(n)\end{bmatrix}. \tag{8}$$

The grey filtering-prediction equation is thus

$$\hat{x}^{(0)}(k)=\hat{x}^{(1)}(k)-\hat{x}^{(1)}(k-1), \tag{9}$$

where

$$\hat{x}^{(1)}(k+1)=\left[x^{(0)}(1)-\frac{a}{b}\right]e^{-bk}+\frac{a}{b}. \tag{10}$$

Note that Eq. (10) is the discrete version of the solution to the differential equation (Eq. (4))

$$x^{(1)}=\left[x^{(1)}(0)-\frac{\alpha}{\beta}\right]e^{-\beta t}+\frac{\alpha}{\beta}. \tag{11}$$

The typical goodness-of-fit measure of GM(1,1) model is the (absolute) relative error described by Deng [2], i.e.

$$e(k)=\frac{\left|x^{(0)}(k)-\hat{x}^{(0)}(k)\right|}{x^{(0)}(k)},\ \ k=2,3,4,\mathsf{h},n, \tag{12}$$

and the model efficiency is defined as

$$E=\frac{1}{n-1}\sum_{i=2}^{n}e(i). \tag{13}$$

The nature of the univariate DEMR model can be identified as that the model couples a differential equation model and a simple regression model together organically. The form of the motivated differential equation (i.e., Deng's whitening differential equation) in Eq. (4) determines the form of the coupling regression (i.e., CREG) in Eq. (3). The data assimilated parameter pair (a,b) in CREG determines the system parameter pair(α,β). The coupling translation rule is listed in *Table 1*.

Table 1. Coupling Rule in Univariate DEMT Model

Term	Motivated DE	Coupling REG
Translation between MDE and CREG		
Intrinsic feature	Continuous	Discrete
Independent Variable	t	k
1st-order Derivative	$dx^{(1)}(t)/dt$	$x^{(0)}(k)$
2st-order Derivative	$d^{(2)}x^{(1)}(t)/dt^2$	$x^{(-1)}(k)$
Primitive function	$x^{(1)}(t)$	$z^{(1)}(k)$
Model Formation	$\frac{dx^{(1)}(t)}{dt}+\beta x^{(1)}(t)=\alpha$	$x^{(0)}(k)+bz^{(1)}(k)=a$
Parameter Coupling		
Parameter	(α,β)	(a,b)
Dynamics (Solution)	$x^{(1)}(t)=$ $\left[x^{(1)}(0)-\frac{\alpha}{\beta}\right]e^{-\beta t}+\frac{\alpha}{\beta}$	$\hat{x}^{(1)}(k+1)=$ $\left[x^{(0)}(1)-\frac{a}{b}\right]e^{-bk}+\frac{a}{b}$
Filtering (Prediction)	$x^{(0)}(t)=$ $\left[\alpha-\beta x^{(0)}(1)\right]e^{-\beta t}$	$\hat{x}^{(0)}(t)=$ $\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$

In DEMR modelling, the motivated differential equation and the coupling regression model are not separable but are organic integration. The DEMR models are differential equation motivated but defined by system data. A DEMR model starts with a motivated differential equation, then the coupling regression model is specified in the form "translated" from the form of the motivated differential equation, in return, in terms of coupling regression model, the parameters specifying the motivated differential equation are estimated under L_2-optimality, and finally, the solution to the motivated differential equation (or the discredited solution) equipped with data-assimilated parameters is used for system analysis or prediction. In nature a DEMR model is a coupling of a motivated differential equation and a regression formed by the discredited version of the motivated differential equation. We call the "translation" rule in grey differential equation modelling as a coupling principle.

3. Unequal-gapped differential equation motivated regression model with term of product of exponential and sine function

The basic form of the first order linear differential equation with constant term in right side is

$$\frac{dx}{dt}+\beta x=\alpha e^{\delta t}\sin(\omega t+\varpi) \tag{14}$$

Note here, the proposal of the motivated differential equation in Eq. (14) is featured by the term $\alpha e^{\delta t}\sin\omega t$ to replacing the constant term α in Eq. (4) with an intention that the fluctuating pattern of $e^{\delta t}\sin(\omega t+\varpi)$ will help the model goodness-of-fit.

Then the solution to Eq. (14) is

$$x=x_h+x_p \tag{15}$$

where

$$x_h=c_0e^{-\beta t}, \tag{16}$$

is the solution to the homogeneous equation

$$\frac{dx}{dt}+\alpha x=0 \tag{17}$$

while a particular solution to the motivated differential equation Eq. (14) takes a form

$$x_p=e^{\delta t}\left(A_0\sin(\omega t+\varpi)+B_0\cos(\omega t+\varpi)\right). \tag{18}$$

Note that x_p satisfies Eq. (14), thus substitute the particular solution into Eq. (14), we obtain

$$\begin{aligned}&\frac{dx_p}{dt}+\beta x_p\\&=A_0\delta e^{\delta t}\sin(\omega t+\varpi)+B_0\delta e^{\delta t}\cos(\omega t+\varpi)\\&\quad+A_0\omega e^{\delta t}\cos(\omega t+\varpi)+B_0\omega e^{\delta t}\sin(\omega t+\varpi)\\&\quad+A_0\beta e^{\delta t}\sin(\omega t+\varpi)+B_0\beta e^{\delta t}\cos(\omega t+\varpi)\\&=\alpha e^{\delta t}\sin(\omega t+\varpi),\end{aligned} \tag{19}$$

which leads to an equation system by comparing the coefficients of term $e^{\delta t}\sin(\omega t+\varpi)$ and term $e^{\delta t}(\omega t+\varpi)$ respectively,

$$\begin{cases}A_0(\delta+\beta)-B_0\omega=\alpha\\A_0\omega+B_0(\delta+\beta)=0\end{cases} \tag{20}$$

Solving the linear equation Eq. (19), we obtain the coefficients A_0 and B_0 respectively as follows

$$\begin{cases} A_0 = \dfrac{(\delta+\beta)\alpha}{\omega^2+(\beta+\delta)^2} \\ B_0 = -\dfrac{\alpha\omega}{\omega^2+(\beta+\delta)^2} \end{cases} \quad (21)$$

In theory, the expressions of A_0 and B_0 will determine the particular solution x_p

$$x_p = A_0 e^{\delta t}\sin(\omega t+\varpi) + B_0 e^{\delta t}\cos(\omega t+\varpi) \quad (22)$$

which will result in the general solution to Eq. (14) as

$$x = c_1 e^{-\beta t} + A_0 e^{\delta t}\sin(\omega t+\varpi) + B_0 e^{\delta t}\cos(\omega t+\varpi) \quad (23)$$

Note that for the unequal-gapped data sequence, $X^{(0)} = \left(x^{(0)}(t_1), x^{(0)}(t_2), \mathrm{i}, x^{(0)}(t_n)\right)$, the coupling (or translation) rule is slightly different from the equal-gapped data sequence.

Table 2. Coupling Principle in unequal gapped GM(1,1) Model.

Term	Motivated DE	Coupling REG
Model Formation		
Intrinsic feature	Continuous	Discrete
Independent Variable	t	t_k
Response	$x^{(0)}(t)$	$x^{(0)}(t_k)$
1st-order Derivative	$dx^{(1)}(t)/dt$	$x^{(0)}(t_k)$
2nd-order Derivative	$d^2x^{(1)}(t)/dt^2$	$\dfrac{x^{(0)}(t_k)-x^{(0)}(t_{k-1})}{t_k-t_{k-1}}$
Primitive function	$x^{(1)}(t)$	$z^{(1)}(t_k)$
Data Assimilation in Model		
Parameter	(α,β)	(a,b)
Dynamic law	$\frac{dx}{dt}+\beta x = \alpha e^{\delta t}\sin(\omega t+\varpi)$	$x^{(0)}(t_k) = \alpha e^{\delta t_k}\sin(\omega t_k+\varpi)+\beta\left(-z^{(1)}(t_k)\right)$
Dynamics (Solution)	$x^{(1)}(t) = c_1 e^{-\beta t} + A_0 e^{\delta t}\sin(\omega t+\varpi) + B_0 e^{\delta t}\cos(\omega t+\varpi)$	$x^{(1)}(t_k) = c_1 e^{-\beta t_k} + A_0 e^{\delta t_k}\sin(\omega t_k+\varpi) + B_0 e^{\delta t_k}\cos(\omega t_k+\varpi)$
Filtering (Prediction)	$x^{(0)}(t) = -\beta c_1 e^{-\beta t} + (A_0\omega+B_0\delta)e^{\delta t}\cos(\omega t+\varpi) + (A_0\delta-B_0\omega)e^{\delta t}\sin(\omega t+\varpi)$	$x^{(0)}(t_k) = -\beta c_1 e^{-\beta t_k} + (A_0\omega+B_0\delta)e^{\delta t_k}\cos(\omega t_k+\varpi) + (A_0\delta-B_0\omega)e^{\delta t_k}\sin(\omega t_k+\varpi)$

The coupling regression is

$$x^{(0)}(t_k) = \alpha e^{\delta t_k}\sin(\omega t_k+\varpi) + \beta\left(-z^{(1)}(t_k)\right)+\varepsilon_k,\ k=2,3,4,\mathrm{h},n, \quad (24)$$

where

$$z^{(1)}(t_1) = z^{(0)}(t_1)t_1$$

$$z^{(1)}(t_k) = z^{(1)}(t_{k-1}) + z^{(0)}(t_k)(t_k - t_{k-1})$$

$$k = 2,3,4,\mathrm{h},n. \quad (25)$$

The parameter pair (α,β) is obtained by least-square estimation $(a,b)^T = \left(X^T X\right)^{-1}X^T Y$, where

$$X = \begin{bmatrix} e^{\delta t_2}\sin(\omega t_1+\varpi) & -z^{(1)}(t_1) \\ e^{\delta t_3}\sin(\omega t_2+\varpi) & -z^{(1)}(t_2) \\ \mathrm{j} & \mathrm{j} \\ e^{\delta t_n}\sin(\omega t_n+\varpi) & -z^{(1)}(t_n) \end{bmatrix},$$

$$Y = \begin{bmatrix} z^{(0)}(t_1) \\ z^{(0)}(t_2) \\ \mathrm{j} \\ z^{(0)}(t_n) \end{bmatrix} \quad (26)$$

since δ and ω are given (in a manner by trials and errors).

Formally, we have a DEMR model as

$$\begin{cases} \dfrac{dx}{dt}+\beta x = \alpha e^{\delta t}\sin(\omega t+\varpi) \\ x^{(0)}(t_k) = \alpha e^{\delta t}\sin(\omega t+\varpi)+\beta\left(-z^{(1)}(t_k)\right)+\varepsilon_k. \end{cases} \quad (27)$$

4. Fuzzy repair effect structure

In standard regression modelling exercises, it is often to assume that the error terms ε_i, $i=1,2,\mathrm{i},n$ are random with zero mean and constant variance, i.e., $\mathrm{E}[\varepsilon_i]=0$ and $\mathrm{VAR}[\varepsilon_i]=\sigma^2$, $i=1,2,\mathrm{i},n$. It is typically assuming a normal distribution with zero mean and constant variance, i.e., $N(0,\sigma^2)$.

Furthermore, as we pointed out that a grey differential equation model is a motivated differential equation motivated regression, which takes the form translated from the motivated differential equation, as shown in *Table 1* for GM(1,1) case. However, we should be fully aware that translation back and forward between the motivated differential equation and the coupling

regression will bring in new error which is different from the random sampling error $N\left(0,\sigma^2\right)$. The errors brought in come from the steps of the usage of difference $x^{(0)}(k)=x^{(1)}(k)-x^{(1)}(k-1)$ to replace the derivative $\left(dx/dt\right)_{t=k}$ and the usage of the average accumulated partial sum $z^{(1)}(t_k)$ to replace the primitive function $x^{(1)}(t_k)$ during the translation between the motivated differential equation and the coupling regression.

Our simulation studies have shown that the coupling-introduced error is dependent upon the grids size Δ, or equivalent to the total number of approximation *N*. The simulation evidences have shown that the larger the number of approximating grid, or equivalently, the smaller the approximating grid, the coupling translation error is smaller. However, the coupling translation error and the approximating grid do not hold a deterministic functional relation. What we can see is the functional relation has a certain degree of belongingness. In other words, the coupling translation process induces a fuzzy error term, denoted as ς with a membership function.

We perform a simulation study of the error occurrence frequencies of approximating $\cos(\pi/2)$ by $\left(\sin(\pi/2)-\sin(\pi/2+\Delta x)\right)/\Delta x$.

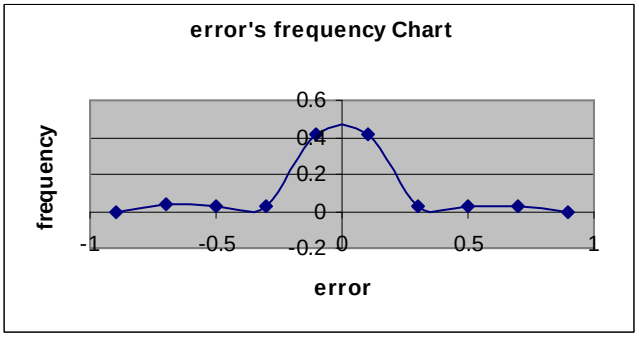

Figure 1. Error occurrence frequency

Therefore, in general the error terms of a differential equation motivated regression model (i.e., grey differential equation in current grey theory literature) is fuzzy because the vague nature of the error occurrences.

As a standard exercise, the fuzzy error component e_i may be assumed as triangular fuzzy variable with a membership function

$$\mu_e(s)=\begin{cases}\dfrac{s+o}{o} & if\ -o\le s<0\\ \dfrac{o-s}{o} & if\ 0\le s\le o\\ 0 & otherwise\end{cases} \tag{28}$$

which has a fuzzy mean zero.

However, in the modelling of system functioning times, we further note that the repair will reset the system dynamic rule so that the repair impact may be understood as a fuzzy variable having a triangular membership

$$\mu_r(z)=\begin{cases}\dfrac{z-a}{b-a} & if\ a\le z<b\\ \dfrac{c-z}{c-b} & if\ b\le z<c\\ 0 & otherwise\end{cases} \tag{29}$$

The fuzzy mean of the fuzzy repair effect is thus

$$E_\mu(r)=\frac{1}{4}(a+2b+c), \tag{30}$$

which provides a repair effect structure. Therefore, the "composite" fuzzy "error" term appearing in the differential equation motivated regression for modelling a system function time will be

$$\zeta_i=e_i+r_i,\ \ i=2,3,\ldots,n, \tag{31}$$

with a triangular membership function, i.e.,

$$\mu_\zeta(w)=\begin{cases}\dfrac{w-a+\varpi}{b-a+\varpi} & if\ a-\varpi\le w<b\\ \dfrac{c+\varpi-w}{c+\varpi-b} & if\ b\le w<c+\varpi\\ 0 & otherwise\end{cases} \tag{32}$$

because the sum of two triangular fuzzy variables is still a triangular fuzzy variable. The total error

$$\xi_i=\zeta_i+\varepsilon_i=(r_i+e_i)+\varepsilon_i,\ \ i=2,3,\ldots,n, \tag{33}$$

which is a sequence of random fuzzy variables because the summation nature of a random fuzzy variable and a fuzzy variable according to Liu [5]. Now, we reach a point that the random fuzzy variable concept is involved and therefore it is necessary to

have a quick review on the relevant theoretical foundation.

5. A random fuzzy variable foundation

First we need to review the fuzzy credibility measure theory foundation proposed by Liu [5], then we will establish the normal random fuzzy variable theory for a facilitation of error analysis in the differential equation motivated regression models.

Let Θ be a nonempty set, and 2^{Θ} the power set on Θ. Each element, let us say, $A \subset \Theta$, $A \in 2^{\Theta}$ is called an event. A number denoted as $\mathrm{Cr}\{A\}$, $0 \le \mathrm{Cr}\{A\} \le 1$, is assigned to event $A \in 2^{\Theta}$, which indicates the credibility grade with which event $A \in 2^{\Theta}$ occurs. $\mathrm{Cr}\{A\}$ satisfies following axioms given by Liu [5]:

Axiom 1. $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2. $\mathrm{Cr}\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$.

Axiom 3. $\mathrm{Cr}\{\cdot\}$ is self-dual, i.e., for any $A \in 2^{\Theta}$, $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$.

Axiom 4. $\mathrm{Cr}\{\mathrm{r}_i A_i\} \wedge 0.5 = \sup_i \left[\mathrm{Cr}\{A_i\}\right]$ for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\} \le 0.5$.

Axiom 5. Let set functions $Cr_k\{\cdot\}: 2^{\Theta_k} \to [0,1]$ satisfy *Axioms 1-4*, and $\Theta = \Theta_1 \times \Theta_2 \times \mathrm{i} \times \Theta_p$, then:

$$Cr\{\theta_1, \theta_2, \mathrm{h}, \theta_p\} = Cr\{\theta_1\} \wedge Cr\{\theta_2\} \wedge \mathrm{i} \wedge Cr\{\theta_p\} \quad (34)$$

for each $\{\theta_1, \theta_2, \mathrm{h}, \theta_p\} \in 2^{\Theta}$.

Definition 5.1. Liu [5] Any set function $\mathrm{Cr}: 2^{\mathrm{Q}} \to [0,1]$ satisfies *Axioms 1-4* is called a $(\vee, \wedge)$-credibility measure (or classical credibility measure). The triple $(\Theta, 2^{\Theta}, Cr)$ is called the $(\vee, \wedge)$-credibility measure space.

Definition 5.2. Liu [5] A fuzzy variable ξ is a mapping from credibility space $(\Theta, 2^{\Theta}, Cr)$ to the set of real numbers, i.e., $\xi: (\Theta, 2^{\Theta}, \mathrm{Cr}) \to \mathbf{R}$.

Definition 5.3. Liu [5] The (induced) membership function of a fuzzy variable ξ on $(\Theta, 2^{\Theta}, Cr)$ is:

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, \quad x \in \mathbf{R} \quad (35)$$

Conversely, for given membership function the credibility measure is determined by the credibility inversion theorem.

Theorem 5.4. Liu [5] Let ξ be a fuzzy variable with membership function m. Then for $\forall B \subset \mathbf{R}$,

$$Cr\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^C} \mu(x) \right) \quad (36)$$

As an example, if the set *B* is degenerated into a point *x*, then

$$Cr\{\xi = x\} = \frac{1}{2}\left(\mu(x) + 1 - \sup_{y \ne x} \mu(y) \right), \ \forall x \in \mathbf{R} \quad (37)$$

Definition 5.5. Liu [5] The credibility distribution $\Phi: \mathrm{R} \to [0,1]$ of a fuzzy variable ξ on $(\Theta, 2^{\Theta}, Cr)$ is

$$\Phi(x) = Cr\{\theta \in \Theta \,|\, \xi(\theta) \le x\}. \quad (38)$$

The credibility distribution $\Phi(x)$ is the accumulated *credibility grade* that the fuzzy variable ξ takes a value less than or equal to a real number $x \in \mathrm{R}$. Generally speaking, the credibility distribution Φ is neither left-continuous nor right-continuous.

Theorem 5.6. Liu [5] Let ξ be a fuzzy variable on $(\Theta, 2^{\Theta}, Cr)$ with membership function μ. Then its credibility distribution,

$$\Phi(x) = \frac{1}{2}\left(\sup_{y \le x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right), \ \forall x \in \mathbf{R} \quad (39)$$

Definition 5.7. Liu [5] Let Φ be the credibility distribution of the fuzzy variable ξ. Then function $\phi: \mathrm{R} \to [0, +\infty)$ of a fuzzy variable ξ is called a credibility density function such that,

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)dy, \ \forall x \in \mathbf{R}. \quad (40)$$

Now we are ready to state the normal random fuzzy variable theory for the error analysis in the repairable system modelling.

Liu [5] defines a random fuzzy variable as a mapping from the credibility space $(\Theta, 2^{\Theta}, Cr)$ to a set of random variables. We would like to present a definition similar to that of stochastic process in probability theory and expect readers who are familiar

with the basic concept of stochastic processes can understand the comparative definition.

Definition 5.8. A random fuzzy variable, denoted as $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, \mathsf{A}, \Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, Cr)$.

Similar to the interpretation of a stochastic process, $X = \{X_t, t \in \mathbf{R}^+\}$, a random fuzzy variable is a bivariate mapping from $(\Omega \times \Theta, \mathsf{A} \times 2^\Theta)$ to the space (R, B). As to the index, in stochastic process theory, index used is referred to as time typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the "index" is a fuzzy variable, say, β. Using uncertain parameter as index is not starting in random fuzzy variable definition. In stochastic process theory we already know that the stochastic process $X = \{X_{\tau(\omega)}, \omega \in \Omega\}$ uses stopping time $t(w)$, w OW, which is an (uncertain) random variable as its index.

In random fuzzy variable theory, we may say that that average chance measure, denoted as ch, plays a similar role similar to a probability measure, denoted as Pr, in probability theory.

Definition 5.9. Liu and Liu [6] Let x be a random fuzzy variable, then the average chance measure denoted by ch{Ω}, of a random fuzzy event $\{\xi \le x\}$, is

$$ch\{\xi \le x\} = \int_0^1 Cr\{\theta \in \Theta \,|\, \Pr\{\xi(\theta) \le x\} \ge \alpha\} d\alpha . \quad (41)$$

Then function Y(Ψ) is called as average chance distribution if and only if

$$\Psi(x) = ch\{\xi \le x\}. \quad (42)$$

Liu [5] stated that if a random variable η has zero mean and a fuzzy variable ζ, then the sum of the two, $\eta + \zeta$, results in a random fuzzy variable ξ. Now, it is time to find the average chance distribution for a normal random fuzzy variable $\mathrm{x}: {}^{d} N(\mathrm{z}, \mathrm{s}^2)$, where ζ is a triangular fuzzy variable and σ^2 is a given positive real number. Note that fuzzy event

$$\{\theta \in \Theta : \Pr\{\xi(\theta) \le x\} \ge \alpha\}$$

$$\Leftrightarrow \left\{\theta \in \Theta : \Phi\left(\frac{x - \zeta(\theta)}{\sigma}\right) \ge \alpha\right\} \quad (43)$$

$$\Leftrightarrow \{\theta \in \Theta : \mathrm{x} \ge \zeta(\theta) + \sigma\Phi^{-1}(\alpha)\}$$

$$\Leftrightarrow \{\theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha)\} \quad (43)$$

The fuzzy mean is assumed to have a triangular membership function

$$\mu_\zeta(w) = \begin{cases} \dfrac{w - a_\zeta}{b_\zeta - a_\zeta} & a_\zeta \le w \le b_\zeta \\ \dfrac{c_\nu - w}{c_\nu - b_\nu} & b_\zeta \le w \le a_\zeta \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

and

$$\Phi(w) = Cr\{\zeta \le w\} = \begin{cases} 0 & w < a_\nu \\ \dfrac{w - a_\zeta}{2(b_\zeta - a_\zeta)} & a_\zeta \le w \le b_\zeta \\ \dfrac{w + c_\zeta - 2b_\zeta}{2(c_\zeta - b_\zeta)} & b_\zeta \le w < c_\zeta \\ 1 & w \ge c_\zeta \end{cases} \quad (45)$$

which gives the credibility distribution for the fuzzy mean, ζ.

Then the critical step is to derive the expression of $Cr\{\zeta(\theta) \in \Theta \,|\, \Pr\{\xi(\omega, \theta) \le x\} \ge \alpha\}$. For normal random fuzzy variable with a triangular fuzzy mean,

$$\{\zeta(\theta) : \Pr\{\xi(\omega, \theta) \le x\} \ge \alpha\}$$

$$\Leftrightarrow \{\theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha)\}. \quad (46)$$

Then the range for the integration of the integrand $Cr\{\theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha)\}$ with respect to α is listed in *Table 3*.

$g(\alpha)$	Range for α	$Cr\{\theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha)\}$

$-\infty < g(\alpha) < a_\zeta$	$\Phi\left(\frac{x-a_\zeta}{\sigma}\right) < \alpha < 1$	0
$a_\zeta \le g(\alpha) < b_\zeta$	$\Phi\left(\frac{x-b_\zeta}{\sigma}\right) < \alpha < \Phi\left(\frac{x-a_\zeta}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\alpha)-a_\zeta}{2(b_\zeta-a_\zeta)}$
$b_\zeta \le g(\alpha) < c_\zeta$	$\Phi\left(\frac{x-c_\zeta}{\sigma}\right) < \alpha < \Phi\left(\frac{x-b_\zeta}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\alpha)+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}$
$g(\alpha) \ge c_\zeta$	$0 < \alpha < \Phi\left(\frac{x-c_\zeta}{\sigma}\right)$	1

Table 3. Integration range with respect to α

where $\zeta = g(\alpha) = x - \sigma\Phi^{-1}(\alpha)$.
Then we obtain the average chance measure for the event $\{\xi(\omega,\theta) \le x\}$

$$ch\{\xi(\omega,\theta) \le x\} = \int_{\Phi\left(\frac{x-b_\zeta}{\sigma}\right)}^{\Phi\left(\frac{x-a_\zeta}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\alpha)-a_\zeta}{2(b_\zeta-a_\zeta)}d\alpha + \int_{\Phi\left(\frac{x-c_\zeta}{\sigma}\right)}^{\Phi\left(\frac{x-b_\zeta}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\alpha)+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}d\alpha + \int_0^{\Phi\left(\frac{x-c_\zeta}{\sigma}\right)} 1x \quad (47)$$

which leads to the average chance distribution

$$\Psi(x) = \frac{x-a_\zeta}{2(b_\zeta-a_\zeta)}\left(\Phi\left(\frac{x-a_\zeta}{\sigma}\right)-\Phi\left(\frac{x-b_\zeta}{\sigma}\right)\right) + \frac{x+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}\left(\Phi\left(\frac{x-b_\zeta}{\sigma}\right)-\Phi\left(\frac{x-c_\zeta}{\sigma}\right)\right) + \Phi\left(\frac{x-c_\zeta}{\sigma}\right) - \frac{\sigma}{2(b_\zeta-a_\zeta)}\int_{\frac{x-b_\zeta}{\sigma}}^{\frac{x-a_\zeta}{\sigma}} u\phi(u)du - \frac{\sigma}{2(c_\zeta-b_\zeta)}\int_{\frac{x-c_\zeta}{\sigma}}^{\frac{x-b_\zeta}{\sigma}} u\phi(u)du \quad (48)$$

6. Fuzzy repair effect estimation under fuzzy maximum entropy principle

Entropy is a measure of uncertainty. The entropy of De Luca and Termini [1] characterizes uncertainty resulting primarily from the linguistic vagueness rather than resulting from information deficiency, and vanishes when the fuzzy variable takes all the values with membership degree 1. However, we hope that the degree of uncertainty is 0 when the fuzzy variable degenerates to a crisp number, and is maximum when the fuzzy variable is an equi-possible one, i.e., all values have the same possibility. In order to address such a requirement, Li and Liu [6] provided a new definition based on credibility measure.

Definition 6.1.(Fuzzy Entropy) Let x be a continuous fuzzy variable defined on a credibility space $(\Theta, 2^\Theta, Cr)$, then the fuzzy entropy, $H[x]$, is defined by

$$H[\xi] = \int_{-\infty}^{\infty} S(Cr(\{\theta : \xi(\theta) = u\}))du \quad (49)$$

where

$$S(t) = -t\ln t - (1-t)\ln(1-t) \quad (50)$$

For convenience, we name $S(t)$ as entropy density at point *t*.
The maximum entropy principle provides a route such that it is possible to select the parameter(s) $\underline{\lambda}$ that maximizes the value of entropy function and satisfies certain given constraints for specifying a membership function with a given form. However, what we aim at is not obtaining parameters from the theoretical entropy function rather we must determine the parameters based on observations of the fuzzy variable, say, ξ. In other words, we need to develop a criterion to obtain data-assimilated membership function. Therefore, we suggest an *empirical* fuzzy entropy function for parameter searching since the optimal value of the data-dependent object function has to reflect the constraints specified by observational data implicitly. The data assimilated object function is the average of entropy densities evaluated at $\{z_1, z_2, \mathrm{L}\ , z_n\}$ respectively, i.e.,

$$J[-L_1, L_2] = \frac{1}{n}\sum_{i=1}^{N} S(Cr\{Z(\theta) = z_i; (\delta,\eta)\}) \quad (51)$$

where a finite interval $[-L_1, L_2]$, $L_2 > L_1 \ge 0$ is defined for the domain of the entropy. Note that with the finiteness of empirical entropy, $J[-L_1, L_2] \to H[Z;\lambda]$ asymptotically with parameter constrained by the data structure and $Z \in [-L_1, L_2]$, $L_2 > L_1 \ge 0$ which guarantees the theoretical entropy $H[Z]$ exists and finite in general.

Then, we can estimate the parameter $(a_\zeta, b_\zeta, c_\zeta)$ of the membership of fuzzy composite error in terms of maximum entropy principle. Furthermore, we can isolate a few repair as bad-as-old regime and thus

repair effect is zero for estimation parameter o for specifying ε_i, the translation error because under triangular membership assumption, the empirical membership can be defined and satisfies the asymptotical requirements.

7. Conclusion

In this paper, we argue that a differential equation motivated regression model will result in a regression model with random fuzzy error terms and thus complete our mission for solidifying a rigorous mathematical foundation for the grey modelling on system repair effects proposed by Guo [3], [4]. The maximum entropy principle facilitates a way for fuzzy parameter estimation. However, the average change distribution is also providing a way for parameter data-assimilation.

Acknowledgements

This work was supported partially by South Africa Natural Research Foundation Grant FA2006042800024. We are grateful to Professor B.D. Liu for his constant encouragements of applying his fuzzy credibility measure theory. The error frequency in *Figure 1* is provided by my MSc student, Mr Li Xiang.

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Numerical approach to reliability evaluation of two-state consecutive "*k* out of *n*: F" systems

Keywords

two-state system, consecutive "k out of n : F" system, reliability, algorithm

Abstract

An approach to reliability analysis of two-state systems is introduced and basic reliability characteristics for such systems are defined. Further, a two-state consecutive "k out of n: F" system composed of two-state components is defined and the recurrent formulae for its reliability function are proposed. The algorithm for numerical approach to reliability evaluation is given. Moreover, the application of the proposed reliability characteristics and formulae to reliability evaluation of the system of pump stations composed of two-state components is illustrated.

1. Introduction

The assumption that the systems are composed of two-state components gives the possibility for basic analysis and diagnosis of their reliability. This assumption allows us to distinguish two states of system reliability. The system works when its reliability state is equal to 1 and is failed when its reliability state is equal to 0. In the stationary case the system reliability is the independent of time probability that the system is in the reliability state 1. The main results determining the stationary reliability and the algorithms for numerical approach to this reliability evaluation for consecutive "*k* out of *n*: F" systems are given for instance in [1], [5]-[6]. An exemplary technical consecutive "*k* out of *n*: F" system can be found in [3]. There is considered the ordered sequence of *n* relay stations $E_1, E_2, \ldots, E_n$, which have to reroute a signal from a source station E_0 to a target station E_{n+1}. A range of each station is equal to *k*. It means, when the station E_i, $i = 0,1,...,n$, is operating, it sends a signal directly to a station $E_{i+1},...,E_{\min(i+k,n)}$. The failed station does not send any signal. The probability of efficiency of the stations E_0 and E_{n+1} is equal to 1. The signal from E_0 to E_{n+1} cannot be sent, if at least *k* consecutive stations out of $E_1, E_2, \ldots, E_n$, are damaged.

The paper is devoted to extension of these stationary results to the non-stationary case and applying them in transmitting then for two-state consecutive "k out of n: F" systems with dependent of time reliability functions of system components ([3]). Then, the reliability function, the lifetime mean value and the lifetime standard deviation are basic characteristics of the system.

2. Reliability of two-state consecutive "k out of n: F" systems

In the non-stationary case of two-state reliability analysis of consecutive "*k* out of *n*: F" systems we assume that ([3]):

- n is the number of system components,
- E_i, $i = 1,2,...,n$, are components of a system,
- T_i are independent random variables representing the lifetimes of components E_i, $i = 1,2,...,n$,
- $R_i(t) = P(T_i > t), t \in < 0, \infty)$, is *a* reliability function of component E_i, $i = 1,2,...,n$,
- $F_i(t) = 1 - R_i(t) = P(T_i \le t), t \in < 0, \infty)$, is the

distribution function (unreliability function) of component E_i, $i=1,2,...,n$.

Definition 1. A two-state system is called a two-state consecutive "k out of n: F" system if it is failed if and only if at least its k neighbouring components out of n its components arranged in a sequence of E_1, E_2, ..., E_n, are failed.

The following auxiliary theorem is proved in [3], [6].

Lemma 1. The stationary reliability of the two-state consecutive "k out of n: F" system composed of components with independent failures is given by the following recurrent formula

$$\boldsymbol{R}_{k,n}=\begin{cases}1 & \text{for } n<k,\\ 1-\prod_{j=1}^{n} q_j & \text{for } n=k,\\ p_n\boldsymbol{R}_{k,n-1} & \\ +\sum_{i=1}^{k-1} p_{n-i}\boldsymbol{R}_{k,n-i-1} & \\ \cdot \prod_{j=n-i+1}^{n} q_j & \text{for } n>k,\end{cases} \tag{1}$$

where

- p_i is a stationary reliability coefficient of component E_i, $i=1,2,...,n$,
- q_i is a stationary unreliability coefficient of component E_i, $i=1,2,...,n$,
- $R_{k,n}$ is the stationary reliability of consecutive "k out of n: F" system.

After assumption that:

- $T_{k,n}$ is a random variable representing the lifetime of a consecutive "k out of n: F" system,
- $R_{k,n}(t)=P(T_{k,n}>t), t\in<0,\infty)$, is the reliability function of consecutive "k out of n: F" system,
- $F_{k,n}(t)=1-R_{k,n}(t)=P(T_{k,n}\le t), t\in<0,\infty)$, is the distribution function of consecutive "k out of n: F" system,

we can formulate the following result.

Lemma 2. The reliability function of the two-state consecutive "k out of n: F" system composed of components with independent failures is given by the following recurrent formula

$$\boldsymbol{R}_{k,n}(t)=\begin{cases}1 & \text{for } n<k,\\ 1-\prod_{j=1}^{n} F_j(t) & \text{for } n=k,\\ R_n(t)\boldsymbol{R}_{k,n-1}(t) & \\ +\sum_{i=1}^{k-1} R_{n-i}(t)\boldsymbol{R}_{k,n-i-1}(t) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t) & \text{for } n>k,\end{cases} \tag{2}$$

for $t\in<0,\infty)$.

Motivation. When we assume in formula (1) that

$$p_i(t)=R_i(t),\ q_i(t)=F_i(t) \text{ for } t\in<0,\infty),\ i=1,2,...,n,$$

we get formula (2).

From the above theorem, as a particular case for the system composed of components with identical reliability functions, we immediately get the following corollary.

Corollary 1. If components of the two-state consecutive "k out of n: F" system are independent and have identical reliability functions, i.e.

$$R_i(t)=R(t),\ F_i(t)=F(t) \text{ for } t\in<0,\infty),\ i=1,2,...,n,$$

then the reliability function of this system is given by

$$\boldsymbol{R}_{k,n}(t)=\begin{cases}1 & \text{for } n<k,\\ 1-[F(t)]^n & \text{for } n=k,\\ R(t)\boldsymbol{R}_{k,n-1}(t) & \\ +R(t)\sum_{i=1}^{k-1} F^i(t) & \\ \cdot \boldsymbol{R}_{k,n-i-1}(t) & \text{for } n>k,\end{cases} \tag{3}$$

for $t\in<0,\infty)$.

In further considerations we will used the following reliability characteristics:

- the mean value of the system lifetime,

$$E[T_{k,n}] = \int_0^\infty \boldsymbol{R}_{k,n}(t)dt, \tag{4}$$

- the second order ordinary moment of the system lifetime,

$$E[T_{k,n}^2] = 2\int_0^\infty t\,\boldsymbol{R}_{k,n}(t)dt, \tag{5}$$

- the standard deviation of the system lifetime,

$$\sigma = \sqrt{D[T_{k,n}]}, \tag{6}$$

where

$$D[T_{k,n})] = E[T_{k,n}^2] - (E[T_{k,n}])^2. \tag{7}$$

3. Algorithm for reliability evaluation of a two-state consecutive „*k* out of *n*: F" system

For numerical approach to evaluation of the reliability characteristics, given by (3)-(6), we use the trapezium rule of numerical integration.

In particular situation, for $t_0 = 0$, step h, we have

$$E[T_{k,n}] = \int_0^\infty \boldsymbol{R}_{k,n}(t)dt$$

$$= \frac{h}{2}\sum_{i=0}^{n-1}\left[R_{k,n}(t_0 + ih) + R_{k,n}(t_0 + (i+1)\cdot h)\right], \tag{8}$$

$$E[T_{k,n}^2] = 2\int_0^\infty t\,\boldsymbol{R}_{k,n}(\mathrm{t})dt$$

$$= h\sum_{i=0}^{n-1}\{(t_0 + ih)\cdot R_{k,n}(t_0 + ih)$$

$$+ (t_0 + (i+1)\cdot h)\cdot R_{k,n}(t_0 + (i+1)\cdot h)\}. \tag{9}$$

Necessary in (7)-(8) values of function $\boldsymbol{R}_{k,n}(\mathrm{t})$ are calculated from (2) using the following algorithm.

Algorithm 1.

1. Given: t, k, n, F(t), R(t);
2. If k > n then $\boldsymbol{R}_{k,n}(t) = 1$
3. else if k = n $\boldsymbol{R}_{k,n}(t) = 1 - [F(t)]^n$
4. else
5. for i = 0 to t do
6. {
7. for j = 1 to k – 1 do
8. temp := temp + $[F(i)]^j \cdot R_{k,n-j-1}(i)$;
9. $\boldsymbol{R}_{k,n}(i) = R(i)\cdot R_{k,n-1}(i) + \text{temp}$;
10. }

where

- *k* is a length of the sequence of consecutive components,
- *n* is a number of all components in sequence,
- *t* is an end of the time interval,
- *F(t)* is a distribution function of components,
- *R(t)* is a reliability function of components.

Example implementation of *Algorithm 1* and formulas (3), (8)-(9) in the D programming language is given in Appendix.

4. Application

From *Corollary 1*, in a particular case, substituting $k = 3$ in (3), we get:

- for $n = 1$

$$\boldsymbol{R}_{3,1}(t) = 1 \text{ for } t \in <0,\infty), \tag{10}$$

- for $n = 2$

$$\boldsymbol{R}_{3,2}(t) = 1, \text{ for } t \in <0,\infty), \tag{11}$$

- for $n = 3$

$$\boldsymbol{R}_{3,3}(t) = 1 - F^3(t) \text{ for } t \in <0,\infty), \tag{12}$$

- for $n \geq 4$

$$\boldsymbol{R}_{3,n}(t) = R(t)\ \boldsymbol{R}_{3,n-1}(t) + R(t)F(t)\ \boldsymbol{R}_{3,n-2}(t)$$

$$+ R(t)[F(t)]^2\ \boldsymbol{R}_{3,n-3}(t) \text{ for } t \in <0,\infty), \tag{13}$$

Example 1. Let us consider the pump stations system with $n = 20$ pump stations $E_1, E_2, ..., E_{20}$. We assume that this system fails when at least 3 consecutive pump stations are down. Thus, the

considered pump stations system is a two-state consecutive "3 out of 20: F" system, and according to (9)-(12), its the reliability function is given by

$$\boldsymbol{R}_{3,20}(t) = R(t)\ \boldsymbol{R}_{3,19}(t) + R(t)F(t)\ \boldsymbol{R}_{3,18}(t) + R(t)[F(t)]^2\ \boldsymbol{R}_{3,17}(t) \quad (14)$$

for $t \in <0,\infty)$.

In the particular case when the lifetimes T_i, of the pump stations E_i, $i = 1,2,\ldots,20$ have exponential distributions of the form

$$F(t) = 1 - e^{-0.01t} \text{ for } t \geq 0,$$

i.e. if the reliability functions of the pump stations E_i, $i = 1,2,\ldots,20$ are given by

$$R(t) = e^{-0.01t} \text{ for } t \geq 0,$$

considering (9)-(12), (13) we get the following recurrent formula for the reliability $\boldsymbol{R}_{3,20}(t)$ of pump stations system

$$\boldsymbol{R}_{3,1}(t) = 1 \text{ for } t \in <0,\infty), \quad (15)$$

$$\boldsymbol{R}_{3,2}(t) = 1 \text{ for } t \in <0,\infty), \quad (16)$$

$$\boldsymbol{R}_{3,3}(t) = 1 - [1 - e^{-0.01t}]^3 \text{ for } t \in <0,\infty), \quad (17)$$

$$\boldsymbol{R}_{3,n}(t) = e^{-0.01t}\ \boldsymbol{R}_{3,n-1}(t) + e^{-0.01t}[1 - e^{-0.01t}]\ \boldsymbol{R}_{3,n-2}(t) + e^{-0.01t}[1 - e^{-0.01t}]^2\ \boldsymbol{R}_{3,n-3}(t) \text{ for } t \in <0,\infty), \quad (18)$$
$$n = 4,5,\ldots,20.$$

The values of reliability function of the system of pump stations given by (14), calculated by the computer programme based on the formulae (10)-(18) and *Algorithm 1*, are presented in *Table 1* and illustrated in *Figure 1*.

Table 1. The values of the two-state reliability function of the pump stations system for $\lambda = 0.01$

t	$\boldsymbol{R}_{3,20}(t)$	$2t\ \boldsymbol{R}_{3,20}(t)$
0.0	1.0000	0.0000
5.0	0.9980	9.9800
10.0	0.9859	19.7189
15.0	0.9583	28.7499
20.0	0.9137	36.5474
25.0	0.8535	42.6743
30.0	0.7811	46.8657
35.0	0.7008	49.0561
40.0	0.6170	49.3614
45.0	0.5337	48.0347
50.0	0.4541	45.4117
55.0	0.3805	41.8584
60.0	0.3144	37.7282
65.0	0.2564	33.3331
70.0	0.2066	28.9274
75.0	0.1647	24.7024
80.0	0.1299	20.7893
85.0	0.1016	17.2662
90.0	0.0787	14.1688
95.0	0.0605	11.5004
100.0	0.0462	9.2416
105.0	0.0350	7.3588
110.0	0.0264	5.8107
115.0	0.0198	4.5531
120.0	0.0148	3.5426
125.0	0.0109	2.7385
130.0	0.0081	2.1044
135.0	0.0060	1.6082
140.0	0.0044	1.2229
145.0	0.0032	0.9255
150.0	0.0023	0.6974
155.0	0.0017	0.5235
160.0	0.0012	0.3916
165.0	0.0009	0.2918
170.0	0.0006	0.2168
175.0	0.0004	0.1607
180.0	0.0003	0.1188
185.0	0.0002	0.0876
190.0	0.0002	0.0644
195.0	0.0001	0.0473
200.0	0.0000	0.0347
205.0	0.0000	0.0253
210.0	0.0000	0.0185
215.0	0.0000	0.0135
220.0	0.0000	0.0098
225.0	0.0000	0.0072
230.0	0.0000	0.0052
235.0	0.0000	0.0038
240.0	0.0000	0.0027
245.0	0.0000	0.0020

250.0	0.0000	0.0014
255.0	0.0000	0.0010
260.0	0.0000	0.0007
265.0	0.0000	0.0005
270.0	0.0000	0.0004
275.0	0.0000	0.0003
280.0	0.0000	0.0002
285.0	0.0000	0.0001

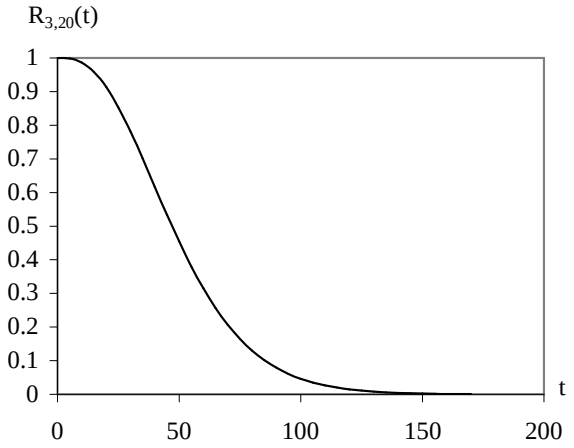

Figure 1. The graph of the pump stations system reliability function

Using the values given in the *Table 1*, the formulae (4)-(9) and numerical integration we find:

- the mean value of the pump stations system lifetime

$$E[T_{3,20}] = \int_0^\infty \boldsymbol{R}_{3,20}(t)dt \cong 50.8639,$$

- the second order ordinary moment of the pump stations system lifetime

$$E[T_{3,20}^2(1)] = 2\int_0^\infty t\,\boldsymbol{R}_{3,20}(t)dt \cong 3246.69,$$

- the standard deviation of the pump stations system lifetime

$$\sigma = \sqrt{D[T_{3,20}]} = \sqrt{659.558} \cong 25.6819.$$

5. Conclusion

Two recurrent formulae for two-state system reliability functions, a general one for non-homogeneous and its simplified form for homogeneous two-state consecutive "k out of n: F" systems have been proposed. The algorithm for reliability evaluation of two-state consecutive "*k* out of *n*: F" system has been shown as well. The formulae and algorithm for two-state reliability function of a homogeneous two-state consecutive "k out of n: F" system have been applied to reliability evaluation of the pump stations system. The considered pump stations system was a two-state consecutive "3 out of 20: F" system composed of components with exponential reliability functions. On the basis of the recurrent formula and the algorithm for two-state pump stations system reliability function the approximate values have been calculated and presented in table and illustrated graphically. On the basis of these values the mean value and standard deviation of the pump stations system lifetime have been estimated. The input structural and reliability data of the considered pump stations system have been assumed arbitrarily and therefore the obtained its reliability characteristics evaluations should be only treated as an illustration of the possibilities of the proposed methods and solutions.
The proposed methods and solutions and the software are general and they may be applied to any two-state consecutive "*k* out of n: F" systems.

Appendix

We present the D programming language code for formulas (3), (8)-(9) and *Algorithm 1*.

```
import std.stdio;
import std.stream;
import std.math;
import std.string;

const real LAMBDA1 = 0.01;

real Ft(real t) {
   return (1-exp(-(LAMBDA1)*t));
}


real Rt(real t) {
   return exp(-(LAMBDA1)*t);
}

real SigmaFi(real ii, real k, real t, real n) {
   real result = 0;
   for(real i = ii; i < k; i++) {
      result += pow(Ft(t),i)*Rkn(t,k,n-i-1);
   }
   return result;
}


real Rkn(real t, real k, real n) {
   if (n < k)
      return 1;
```

```
    if (n == k)
        return 1 – pow(Ft(t),n);
    return Rt(t)*(Rkn(t, k, n-1) + SigmaFi(1, k, t, n));
}


real trapeziumT(real k, real n, uint p, real t){
    real integ = 0;
    real step = 0;

    step=t/p;

    for(real i = 0; i < p; i = i + step){
        integ += (((Rkn(i, k, n) +
          Rkn(i + step, k, n))*step)/2);
    }
    return integ;
}


real trapezium2T(real k, real n, uint p, real t){
    real integ = 0;
    real step = 0;

    step=t/p;

    for(real i=0; i < p; i = i + step){
      integ +=  (((i*Rkn(i, k, n) +
      (i + step)*Rkn(i + step, k, n))*step));
    }
     return integ;
}


int main(char[][] args) {
    real integral = 0;
    real integral1 = 0;
    real dif = 0;
    real sq = 0;

    if (args.length < 3) {
        writefln("Usage:\n   "~ args[0] ~" t k n\n");
        return 0;
    }


    for(real i = 0; i < atoi(args[1]); i = i + 5){
      writefln("%s\t%4s\t%4s\t%s", i, Rkn(i,
                atoi(args[2]), atoi(args[3])), 2*i*Rkn(i,
                atoi(args[2]),atoi(args[3])), 1 - Rkn(i,
                atoi(args[2]), atoi(args[3])) );
    }
    integral=trapeziumT(atoi(args[2]), atoi(args[3]),
     atoi(args[4]) );

    integral1=trapezium2T(atoi(args[2]),  atoi(args[3]),
    atoi(args[4]) );

    diff=(integral1)-pow(integral,2);
    sq=sqrt(diff);

    writefln("The mean value of the system lifetime");
    writefln("%s", integral );
    writefln("The second order ordinary moment of the
        system lifetime");
    writefln("%s", integral1 );
    writefln("%s", diff);
    writefln("The standard deviation of the system
        lifetime");
   writefln("%s",sq);

    return 0;
 }
```

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Reliability analysis of multi-state ageing consecutive „*k* out of *n*: F" systems

Keywords

multi-state system, ageing system, consecutive "k out of n: F" system, reliability

Abstract

A multi-state approach to reliability analysis of systems composed of ageing components is introduced and basic reliability characteristics for such systems are defined. Further, a multi-state consecutive "k out of n: F" system composed of ageing components is defined and the recurrent formulae for its reliability function are proposed. Moreover, the application of the proposed reliability characteristics and formulae to reliability evaluation of the steel cover composed of ageing sheets is illustrated.

1. Introduction

Taking into account the importance of the safety and operating process effectiveness of technical systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis. The assumption that the systems are composed of multi-state components with reliability states degrading in time [4]-[5], [10] gives the possibility for more precise analysis and diagnosis of their reliability and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system safety characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system. The main results determining the multi-state reliability functions and the risk functions of typical series, parallel, series-parallel, parallel-series, series-"*k* out of *n*" and "*k* out of *n*"- series systems with ageing components are given in [4]-[5]. The paper is devoted to transmitting these results on the multi-state ageing consecutive "k out of n: F" systems [1], [2]-[3], [6], [7]-[8], [9].

2. Multi-state system with ageing components

In the multi-state reliability analysis to define systems with degrading components we assume that [4]-[5], [10]:

- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \geq 1$,
- the state indexes are ordered, the state 0 is the worst and the state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$,
- $T_i(u)$, is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
- the system state degrades with time t without repair,
- $e_i(t)$ is a component E_i state at the moment t, $t \geq 0$,
- $s(t)$ is a system state at the moment t, $t \geq 0$.

The above assumptions mean that the reliability states of the system with degrading components may be changed in time only from better to worse. The way in which the components and the system reliability states change is illustrated in *Figure 1*.

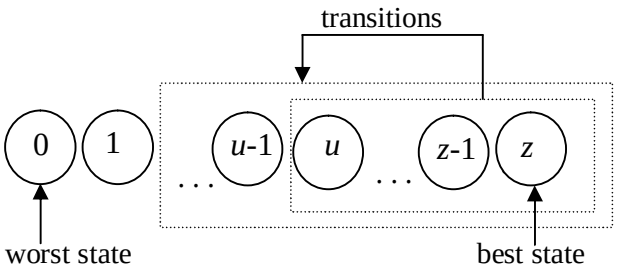

Figure 1. Illustration of reliability states changing in system with ageing components

The basis of our further consideration is a system component reliability function defined as follows.

Definition 1. A vector

$$R_j(t,\cdot) = [R_i(t,0),R_i(t,1),...,R_i(t,z)],\ t \geq 0,$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t)$$

for $t \geq 0,\ u = 0,1,...,z,\ i = 1,2,...,n,$ is the probability that the component E_i is in the reliability state subset $\{u, u+1,..., z\}$ at the moment $t,\ t \geq 0,$ while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Similarly, we can define a multi-state system reliability function.

Definition 2. A vector

$$\boldsymbol{R}_n(t,\cdot) = [1,\boldsymbol{R}_n(t,0),\boldsymbol{R}_n(t,1),...,\boldsymbol{R}_n(t,z)],\ t \geq 0,$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t),$$

for $t \geq 0,\ u = 0,1,...,z,$ is the probability that the system is in the reliability state subset $\{u, u+1,..., z\}$ at the moment $t,\ t \geq 0,$ while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a system.

Under this definition we have

$$\boldsymbol{R}_n(t,0) \geq \boldsymbol{R}_n(t,1) \geq \ldots \geq \boldsymbol{R}_n(t,z),\ t \geq 0,$$

and if

$$p(t) = [p(t,0), p(t,1),..., p(t,z)],\ t \geq 0,$$

where

$$p(t,u) = P(s(t) = u \mid s(0) = z),$$

for $t \geq 0, u = 0,1,...,z,$ is the probability that the system is in the state u at the moment $t,\ t \geq 0,$ while it was in the state z at the moment $t = 0,$ then

$$\boldsymbol{R}_n(t,0) = 1,\ \boldsymbol{R}_n(t,z) = p(t,z),\ t \geq 0, \tag{1}$$

and

$$p(t,u) = \boldsymbol{R}_n(t,u) - \boldsymbol{R}_n(t,u+1),\ u = 0,1,...,z-1,\ t \geq 0. \tag{2}$$

Moreover, if

$$\boldsymbol{R}_n(t,u) = 1 \text{ for } t < 0,\ u = 1,2,..., z,$$

then

$$M(u) = E[T(u)] = \int_0^\infty \boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \tag{3}$$

is the mean lifetime of the system in the state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{D[T(u)]} = \sqrt{N(u) - [M(u)]^2}, \tag{4}$$
$$u = 1,2,...,z,$$

where

$$N(u) = 2\int_0^\infty t\,\boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \tag{5}$$

is the standard deviation of the system lifetime in the state subset $\{u, u+1,..., z\}$ and moreover

$$\overline{M}(u) = \int_0^\infty p(t,u)dt,\ u = 1,2,...,z, \tag{6}$$

is the mean lifetime of the system in the state u while the integrals (3), (4) and (5) are convergent.
Additionally, according to (1), (2), (3) and (6), we get the following relationships

$$\overline{M}(u) = M(u) - M(u+1),\ u = 1,2,...,z-1, \tag{7}$$

$$\overline{M}(z) = M(z).$$

Close to the multi-state system reliability function its basic characteristic is the system risk function defined as follows.

Definition 3. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t),\ t \ge 0,$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the reliability state z at the moment $t = 0$ is called a risk function of the multi-state system.

Considering *Definition 3* and *Definition 2*, we have

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R_n}(t,r),\ \ t \ge 0, \tag{8}$$

and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{9}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

3. Reliability of a multi-state ageing consecutive „*k* out of *n*: F" system

Definition 4. A multi-state system is called an ageing consecutive "k out of n: F" system if it is out of the reliability state subset $\{u,u+1,...,z\}$ if and only if at least its k neighbouring components out of n its components arranged in a sequence of E_1, E_2, ..., E_n, are out of this reliability state subset.

In our further analysis, we denote by $s_{k,n}(t)$ the reliability state of the ageing consecutive "k out of n: F" system at the moment t, $t \in < 0,\infty)$, and by $T_{k,n}(u)$ the lifetime of this system in the reliability subset $\{u,u+1,...,z\}$. Moreover, we denote by

$$\boldsymbol{R}_{k,n}(t,u) = P(s_{k,n}(t) \ge u \mid s(0) = z) = P(T_{k,n}(u) > t)$$

for $t \ge 0$, $u = 0,1,...,z$, the probability that the ageing consecutive "k out of n: F" system is in the reliability state subset $\{u,u+1,...,z\}$ at the moment t, $t \ge 0$, while it was in the reliability state z at the moment $t = 0$ and by

$$\boldsymbol{F}_{k,n}(t,u) = 1 - \boldsymbol{R}_{k,n}(t,u) = P(T_{k,n}(u) \le t)$$

for $t \ge 0$, $u = 0,1,...,z$, the distribution function of the lifetime $T_{k,n}(u)$ of this system in the reliability state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$.

Theorem 1. The reliability function of the ageing consecutive "k out of n: F" system composed of components with independent failures is given by the following recurrent formula

$$\boldsymbol{R}_{k,n}(t,\cdot) = [1, \boldsymbol{R}_{k,n}(t,1),\ \boldsymbol{R}_{k,n}(t,2),\ ...,\ \boldsymbol{R}_{k,n}(t,z)\ ],$$

where

$$\boldsymbol{R}_{k,n}(t,u) = \begin{cases} 1 & \text{for } n < k, \\ 1 - \prod F_j(t,u) & \text{for } n = k, \\ R_n(t,u)\boldsymbol{R}_{k,n-1}(t,u) & \\ + \sum_{i=1}^{k-1} R_{n-i}(t,u)\boldsymbol{R}_{k,n-i-1}(t,u) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t,u) & \text{for } n > k, \end{cases} \tag{10}$$

for $t \in < 0,\infty >$, $u = 1,2,...,z$.

Motivation. Since for each fixed u, $u = 1,2,...,z$, the assumptions of this theorem as the same as the assumptions of *Theorem 2* proved in [2] and the formula (10) is equivalent with the formula (12) from [2], then after considering *Definition 4*, we conclude that this theorem is valid.

From the above theorem, as a particular case for the system composed of components with identical reliability, we immediately get the following corollary.

Corollary 1. If components of the ageing consecutive "k out of n: F" system are independent and have identical reliability functions, i.e.

$$R_i(t,u) = R(t,u),\ \ F_i(t,u) = F(t,u) \ \text{ for } t \in < 0,\infty),$$
$$u = 1,2,...,z,\ i = 1,2,...,n,$$

then the reliability function of this system is given by

$$\boldsymbol{R}_{k,n}(t,\cdot) = [1, \boldsymbol{R}_{k,n}(t,1),\ \boldsymbol{R}_{k,n}(t,2),\ ...,\ \boldsymbol{R}_{k,n}(t,z)\ ],$$

where

$$\boldsymbol{R}_{k,n}(t,u) = \begin{cases} 1 & \text{for } n < k, \\ 1 - [F(t,u)]^n & \text{for } n = k, \\ R(t,u)\boldsymbol{R}_{k,n-1}(t,u) & \\ + R(t,u)\sum_{i=1}^{k-1} F^i(t,u) & \\ \cdot \boldsymbol{R}_{k,n-i-1}(t,u) & \text{for } n > k, \end{cases} \tag{11}$$

for $t \in < 0,\infty)$, $u = 1,2,...,z$.

From *Corollary 1*, in a particular case, substituting $k = 2$ in (11), we get:

- for $n = 1$

$$\boldsymbol{R}_{2,1}(t,\cdot) = [1, \boldsymbol{R}_{2,1}(t,1),\ \boldsymbol{R}_{2,1}(t,2),\ ...,\ \boldsymbol{R}_{2,1}(t,z)], \quad (12)$$

where

$$\boldsymbol{R}_{2,1}(t,u) = 1 \text{ for } t \in <0,\infty),\ u = 1,2,...,z, \quad (13)$$

- for $n = 2$

$$\boldsymbol{R}_{2,2}(t,\cdot) = [1, \boldsymbol{R}_{2,2}(t,1),\ \boldsymbol{R}_{2,2}(t,2),\ ...,\ \boldsymbol{R}_{2,2}(t,z)], \quad (14)$$

where

$$\boldsymbol{R}_{2,2}(t,u) = 1 - F^2(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,...,z, \quad (15)$$

- for $n \geq 3$

$$\boldsymbol{R}_{2,n}(t,\cdot) = [1, \boldsymbol{R}_{2,n}(t,1),\ \boldsymbol{R}_{2,n}(t,2),\ ...,\ \boldsymbol{R}_{2,n}(t,z)], \quad (16)$$

where

$$\boldsymbol{R}_{2,n}(t,u) = R(t,u)\ \boldsymbol{R}_{2,n-1}(t,u)$$

$$+ R(t,u)F(t,u)\ \boldsymbol{R}_{2,n-2}(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,...,z. \quad (17)$$

4. Application

Example 1. Let us consider the steel cover composed of $n = 24$ arranged identical sheets $E_1, E_2, ..., E_{24}$. We assume that $z = 4$, i.e. the cover and the sheets it is composed of may be in the one of the reliability states from the set $\{0,1,2,3,4\}$. The cover is out of the reliability state subset $\{u, u+1, ..., 4\}$ if at least $k = 2$ of its neighbouring sheets is out of this reliability state subset. If the considered steel cover critical reliability state is $r = 2$, then this steel cover is failed if at least 2 neighbouring sheets from 24 sheets are out of the reliability state subset $\{2,3,4\}$. Thus, the considered steel cover is a five-state ageing consecutive "2 out of 24: F" system, and according to (16)-(17), its the reliability function is given by

$$\boldsymbol{R}_{2,24}(t,\cdot) =$$

$$[1, \boldsymbol{R}_{2,24}(t,1),\ \boldsymbol{R}_{2,24}(t,2),\ \boldsymbol{R}_{2,24}(t,3),\ \boldsymbol{R}_{2,24}(t,4)], \quad (18)$$

where

$$\boldsymbol{R}_{2,24}(t,u) = R(t,u)\ \boldsymbol{R}_{2,23}(t,u)$$

$$+ R(t,u)F(t,u)\ \boldsymbol{R}_{2,22}(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,3,4. \quad (19)$$

In the particular case when the lifetimes $T_i(u)$, $u = 1,2,3,4$, of the sheets E_i, $i = 1,2,3,4,5$, in the reliability state subsets have Weibull distributions of the form

$$F(t,u) = 1 - e^{-\lambda(u)t^2} \text{ for } t \geq 0,\ u = 1,2,3,4,$$

where

$$\lambda(1) = 0.01,\ \lambda(2) = 0.02,\ \lambda(3) = 0.05,\ \lambda(4) = 0.10,$$

i.e. if the reliability function of the sheets E_i, $i = 1,2,3,4,5$, is given by

$$R(t,\cdot) = [1, R(t,1), R(t,2), R(t,3), R(t,4)],\ t \in <0,\infty),$$

where

$$R(t,1) = e^{-0.01t^2},\ R(t,2) = e^{-0.02t^2},\ R(t,3) = e^{-0.05t^2},$$

$$R(t,4) = e^{-0.10t^2} \text{ for } t \geq 0,$$

considering (12)-(19), we get the following recurrent formula for the cover reliability

$$\boldsymbol{R}_{2,24}(t,\cdot) =$$

$$[1, \boldsymbol{R}_{2,24}(t,1),\ \boldsymbol{R}_{2,24}(t,2),\ \boldsymbol{R}_{2,24}(t,3),\ \boldsymbol{R}_{2,24}(t,4)], \quad (20)$$

where

- $\boldsymbol{R}_{2,24}(t,1)$ is determined by the formulae

$$\boldsymbol{R}_{2,1}(t,1) = 1 \text{ for } t \in <0,\infty), \quad (21)$$

$$\boldsymbol{R}_{2,2}(t,1) = 1 - [1 - e^{-0.01t^2}]^2 \text{ for } t \in <0,\infty), \quad (22)$$

$$\boldsymbol{R}_{2,n}(t,1) = e^{-0.01t^2}\ \boldsymbol{R}_{2,n-1}(t,1)$$

$$+ e^{-0.01t^2}\,[1-e^{-0.01t^2}]\,\boldsymbol{R}_{2,n-2}(t,1) \text{ for } t\in<0,\infty), \quad (23)$$
$$n=3,4,...,24,$$

- $\boldsymbol{R}_{2,24}(t,2)$ is determined by the formulae

$$\boldsymbol{R}_{2,1}(t,2)=1 \text{ for } t\in<0,\infty), \quad (24)$$

$$\boldsymbol{R}_{2,2}(t,2)=1-[1-e^{-0.02t^2}]^2 \text{ for } t\in<0,\infty), \quad (25)$$

$$\boldsymbol{R}_{2,n}(t,2)=e^{-0.02t^2}\,\boldsymbol{R}_{2,n-1}(t,2)$$

$$+ e^{-0.02t^2}\,[1-e^{-0.02t^2}]\,\boldsymbol{R}_{2,n-2}(t,2) \text{ for } t\in<0,\infty), \quad (26)$$
$$n=3,4,...,24,$$

- $\boldsymbol{R}_{2,24}(t,3)$ is determined by the formulae

$$\boldsymbol{R}_{2,1}(t,3)=1 \text{ for } t\in<0,\infty), \quad (27)$$

$$\boldsymbol{R}_{2,2}(t,3)=1-[1-e^{-0.05t^2}]^2 \text{ for } t\in<0,\infty), \quad (28)$$

$$\boldsymbol{R}_{2,n}(t,3)=e^{-0.05t^2}\,\boldsymbol{R}_{2,n-1}(t,3)$$

$$+ e^{-0.05t^2}\,[1-e^{-0.05t^2}]\,\boldsymbol{R}_{2,n-2}(t,3) \text{ for } t\in<0,\infty), \quad (29)$$
$$n=3,4,...,24,$$

- $\boldsymbol{R}_{2,24}(t,4)$ is determined by the formulae

$$\boldsymbol{R}_{2,1}(t,4)=1 \text{ for } t\in<0,\infty), \quad (30)$$

$$\boldsymbol{R}_{2,2}(t,4)=1-[1-e^{-0.10t^2}]^2 \text{ for } t\in<0,\infty), \quad (31)$$

$$\boldsymbol{R}_{2,n}(t,4)=e^{-0.10t^2}\,\boldsymbol{R}_{2,n-1}(t,4)$$

$$+ e^{-0.10t^2}\,[1-e^{-0.10t^2}]\,\boldsymbol{R}_{2,n-2}(t,4) \text{ for } t\in<0,\infty), \quad (32)$$
$$n=3,4,...,24.$$

The values of the particular vector components of the multi-state reliability function of the steel cover given by (20), calculated by the computer programme based on the formulae (21)-(32), are presented in the *Tables 1-4* and illustrated in *Figure 1*. As earlier we have assumed that $r=2$ is the cover critical reliability state, then according to (8) and (26) its risk function is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}_{2,24}(t,2) = 1-e^{-0.02t^2}\,\boldsymbol{R}_{2,23}(t,2)$$

$$- e^{-0.02t^2}\,[1-e^{-0.02t^2}]\,\boldsymbol{R}_{2,22}(t,2) \text{ for } t\in<0,\infty). \quad (33)$$

The values of the steel cover risk function are given in *Table 5* and illustrated in *Figure 2*.

Table 1. The values of the steel cover multi-state reliability function vector component $u=1$

t	$\boldsymbol{R}_{2,24}(t,1)$	$2t\ \boldsymbol{R}_{2,24}(t,1)$
0.0	1.0000	0.0000
1.0	0.9978	1.9955
2.0	0.9664	3.8657
3.0	0.8531	5.1183
4.0	0.6362	5.0889
5.0	0.3750	3.7499
6.0	0.1664	1.9957
7.0	0.0538	0.7534
8.0	0.0125	0.2001
9.0	0.0021	0.0374
10.0	0.0002	0.0049

Table 2. The values of the steel cover multi-state reliability function vector component $u=2$

t	$\boldsymbol{R}_{2,24}(t,2)$	$2t\ \boldsymbol{R}_{2,24}(t,2)$
0.0	1.0000	0.0000
0.5	0.9994	0.9994
1.0	0.9912	1.9824
1.5	0.9580	2.8742
2.0	0.8802	3.5207
2.5	0.7479	3.7398
3.0	0.5731	3.4388
3.5	0.3876	2.7131
4.0	0.2275	1.8200
4.5	0.1145	1.0307
5.0	0.0491	0.4905
5.5	0.0178	0.1958
6.0	0.0055	0.0655
6.5	0.0014	0.0184
7.0	0.0003	0.0044

Table 3 The values of the steel cover multi-state reliability function vector component $u=3$

t	$\boldsymbol{R}_{2,24}(t,3)$	$2t\ \boldsymbol{R}_{2,24}(t,3)$
0.0	1,0000	0,0000
0.2	0.9999	0.3999
0.4	0.9986	0.7988
0.6	0.9928	1.1914
0.8	0.9781	1.5649
1.0	0.9489	1.8978
1.2	0.9005	2.1613
1.4	0.8302	2.3246
1.6	0.7385	2.3632

1.8	0.6299	2.2675
2.0	0.5122	2.0489
2.2	0.3953	1.7392
2.4	0.2883	1.3837
2.6	0.1980	1.0298
2.8	0.1278	0.7158
3.0	0.0774	0.4642
3.2	0.0438	0.2806
3.4	0.0233	0.1581
3.6	0.0115	0.0830
3.8	0.0053	0.0406
4.0	0.0023	0.0185

Table 4. The values of the steel cover multi-state reliability function vector component $u = 4$

t	$\boldsymbol{R}_{2,24}(t,4)$	$2t\ \boldsymbol{R}_{2,24}(t,4)$
0.0	1.0000	0.0000
0.1	0.9999	0.0399
0.2	0.9996	0.1599
0.3	0.9982	0.3593
0.4	0.9943	0.6364
0.5	0.9864	0.9864
0.6	0.9725	1.4004
0.7	0.9508	1.8636
0.8	0.9195	2.3540
0.9	0.8775	2.8433
1.0	0.8244	3.2975
1.1	0.7605	1.6731
1.2	0.6875	1.6499
1.3	0.6076	1.5799
1.4	0.5242	1.4677
1.5	0.4406	1.3217
1.6	0.3602	1.1528
1.7	0.2862	0.9731
1.8	0.2207	0.7944
1.9	0.1650	0.6269
2.0	0.1195	0.4779
2.1	0.0838	0.3519
2.2	0.0569	0.2502
2.3	0.0373	0.1718
2.4	0.0237	0.1138
2.5	0.0146	0.0728
2.6	0.0086	0.0450
2.7	0.0050	0.0268
2.8	0.0028	0.0154
2.9	0.0015	0.0086
3.0	0.0008	0.0046

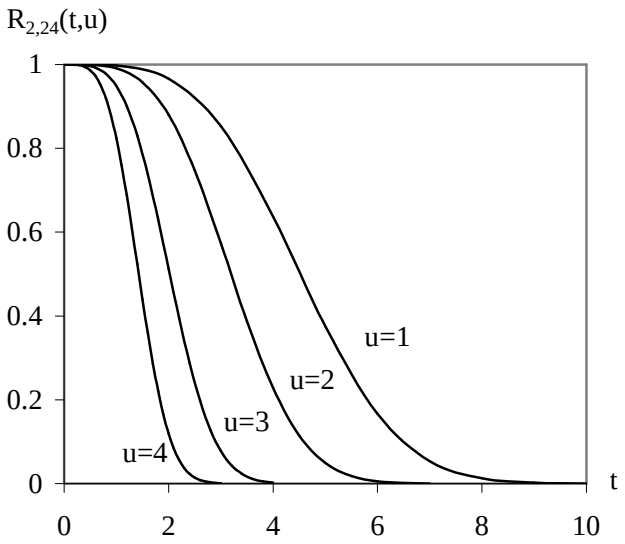

Figure 1. The graphs of the steel cover multi-state reliability function vector components

Table 5. The values of the steel cover multi-state reliability function vector component $u = 2$ and its risk function

t	$\boldsymbol{R}_{2,24}(t,2)$	$\boldsymbol{r}(t) = 1 - \boldsymbol{R}_{2,24}(t,2)$
0.0	1.0000	0.0000
0.5	0.9994	0.0006
1.0	0.9912	0.0088
1.5	0.9581	0.0419
2.0	0.8802	0.1198
2.5	0.7480	0.2520
3.0	0.5731	0.4269
3.5	0.3876	0.6124
4.0	0.2275	0.7725
4.5	0.1145	0.8855
5.0	0.0490	0.9510
5.5	0.0178	0.9822
6.0	0.0055	0.9945
6.5	0.0014	0.9986
7.0	0.0003	0.9997

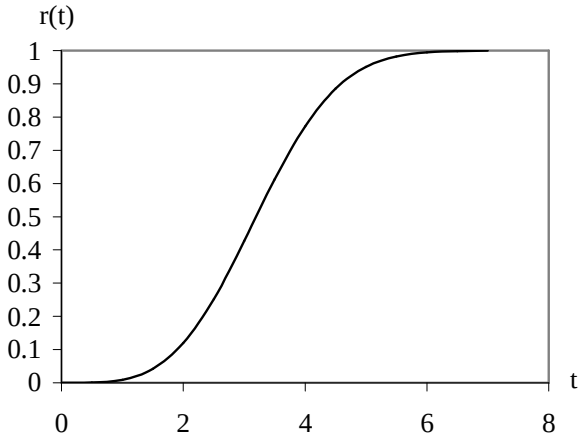

Figure 2. The graphs of the steel cover risk function

Using the values given in these *Tables 1-4*, the formulae (3)-(7) and numerical integration we find:
- the mean values of the cover lifetimes in the reliability state subsets

$$M(1) = E[T_{2,24}(1)] = \int_0^\infty \boldsymbol{R}_{2,24}(t,1)dt \cong 4.5634,$$

$$M(2) = E[T_{2,24}(2)] = \int_0^\infty \boldsymbol{R}_{2,24}(t,2)dt \cong 3.2268,$$

$$M(3) = E[T_{2,24}(3)] = \int_0^\infty \boldsymbol{R}_{2,24}(t,3)dt \cong 2.0408,$$

$$M(4) = E[T_{2,24}(4)] = \int_0^\infty \boldsymbol{R}_{2,24}(t,4)dt \cong 1.4431,$$

- the second ordinary moments of the cover lifetimes in the reliability state subsets

$$N(1) = E[T_{2,24}^2(1)] = 2\int_0^\infty t\,\boldsymbol{R}_{2,24}(t,1)dt \cong 22.9715,$$

$$N(2) = E[T_{2,24}^2(2)] = 2\int_0^\infty t\,\boldsymbol{R}_{2,24}(t,2)dt \cong 11.4879,$$

$$N(3) = E[T_{2,24}^2(3)] = 2\int_0^\infty t\,\boldsymbol{R}_{2,24}(t,3)dt \cong 4.5944,$$

$$N(4) = E[T_{2,24}^2(4)] = 2\int_0^\infty t\,\boldsymbol{R}_{2,24}(t,4)dt \cong 2.2967,$$

- the standard deviations of the cover lifetimes in the reliability state subsets

$$\sigma(1) = \sqrt{N(1) - [M(1)]^2} \cong 1.4651,$$

$$\sigma(2) = \sqrt{N(2) - [M(2)]^2} \cong 1.0370,$$

$$\sigma(3) = \sqrt{N(3) - [M(3)]^2} \cong 0.6553,$$

$$\sigma(4) = \sqrt{N(4) - [M(4)]^2} \cong 0.4628,$$

- the mean values of the cover lifetimes in the reliability particular states

$$\overline{M}(1) = M(1) - M(2) \cong 4.5634 - 3.2268 = 1.3366,$$

$$\overline{M}(2) = M(2) - M(3) \cong 3.2268 - 2.0408 = 1.1860,$$

$$\overline{M}(3) = M(3) - M(4) \cong 2.0408 - 1.4431 = 0.5977,$$

$$\overline{M}(4) = M(4) \cong 1.4431.$$

Using the values given in these *Tables 5* and the formula (9) we find the approximate value of the moment when the system risk function exceeds an exemplary permitted level $\delta = 0.05$, namely

$$\tau = \boldsymbol{r}^{-1}(0.05) \cong 1.58.$$

5. Conclusion

Two recurrent formulae for multi-state reliability functions, a general one for non-homogeneous and its simplified form for homogeneous multi-state consecutive "k out of n: F" systems composed of ageing components have been proposed. The formulae for multi-state reliability function of a homogeneous multi-state consecutive "k out of n: F" system has been applied to reliability evaluation of the steel cover composed of ageing components. The considered steel cover was a five-state ageing consecutive "2 out of 24: F" system composed of components with Weibull reliability functions. On the basis of the recurrent formula for steel cover multi-state reliability function the approximate values of its vector components have been calculated and presented in tables and illustrated graphically. On the basis of these vales the mean values and standard deviations of the steel cover lifetimes in the reliability state subsets and the mean values of the steel cover lifetimes in particular reliability states have been estimated. Moreover, the cover risk function and the moment when the risk function exceeds the permitted risk level have been determined.
The input structural and reliability data of the considered steel cover have been assumed arbitrarily and therefore the obtained its reliability characteristics evaluations should be only treated as an illustration of the possibilities of the proposed methods and solutions. The proposed methods and solutions and the software are general and they may be applied to any multi-state consecutive "k out of n: F" system of ageing components.

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Some remarks on mean time between failures of repairable systems

Keywords

age replacement, increasing failure rate on average, mean time between failures, mean time between failures or repairs

Abstract

This paper describes a simple technique for approximating the mean time between failures (MTBF) of a system that has periodic maintenance at regular intervals. We propose an approximation of MTBF for vide range of systems than IFRA class of distributions.

1. Introduction

One important research area in reliability engineering is studying various maintenance policies. Maintenance can be classified by two major categories: corrective and preventive. Corrective maintenance is any maintenance that occurs when the system is failed. Some authors refer to corrective maintenance as repair and we will use this approach in this paper. Preventive maintenance is any maintenance that occurs when system is not failed.

A common measure used to describe the reliability characteristics of a repairable system is mean time between failures (MTBF). In most repairable systems, preventive maintenance used to reduce system failure frequency and hence increase the MTBF. It is easy that MTBF is the mean time to a repair service or an age replacement.

The MTBF for a system that has periodic maintenance at a time t can be described by [1], [10]:

$$MTBF = \frac{\int_0^t R(s)ds}{F(t)}. \qquad (1)$$

In paper [1], [10] an approximation to MTBF is provided. Moreover in [10] is proved that

$$\frac{tR(t)}{F(t)} \le MTBF \le \frac{t}{F(t)},$$

and for increasing failure rate on average (IFRA) class of distributions in [1] the following relation by proposed

$$-\frac{t}{ln(R(t)} \le MTBF \le \frac{t}{F(t)}.$$

In this paper, we propose an approximation to MTBF for wide class than IFRA. The equality (1) can be a basis to introduction a new class of ageing distributions (see [7], [8]).

Let us assumption the following notation

$$M(t) = \frac{1}{MTBF}.$$

The case when *M(t)* is monotonic was considered by Barlow and Campo [2], Marshall and Proschan [9], Klefsjö [5] and Knopik [7], [8].

Definition 1. The lifetime *T* belongs to the class (mean time to failure or replacement) MTFR, if the function *M (t)* is non-decreasing for $t \in \{t : F(t) > 0\}$.

It has been shown in Barlow [2] and Klefsjö [5] that

$$\text{IFR} \subset \text{MTFR} \subset \text{NBUE},$$

where IFR is increasing failure rate class, NBUE is new better than used in expectation class.

For absolutely continuous in [2] and for any random variable in [8] it has been proved, that

$$\text{IFRA} \subset \text{MTFR},$$

where IFRA is increasing failure rate on average class of distributions. Preservation of life distribution classes under reliability operations has been studied in [7], [8]. The class MTFR is closed under the operations:

(a) formation of a parallel system for absolutely continuous random variables,
(b) formation of series system with identically distributed and absolutely continuous random variables,
(c) weak convergence of distributions,
(d) convolution.

In this paper we propose new approximation and bounds, applicable for a wide range of systems. It is MTFR class of distributions, such that MTBF is non-increasing. The class MTFR contains distributions with unimodal failure rate function. We analyze special case of distribution with MTBF non-increasing as mixture of an exponential distribution and Rayleigh's distributions. This distribution has unimodal failure rate function. However, it is not easy to obtain distribution from mixtures with unimodal failure rate function [12]. The mixtures of two increasing linear failures rate functions were studied in [4]. In [4] showed that a mixture of two distributions with increasing linear failure rate functions does not give distributions with unimodal failure rate function.

In section 2, we introduce the proposed model of mixture and we estimate their parameters for an example of lifetime data [11]. In section 3 we propose a simple approximation of to MTBF for lifetime distributions with MTBF non-increasing.

2. Mixture of distributions

We consider a mixture of two lifetimes X_1 and X_2 with densities $f_1(t)$, $f_2(t)$, reliability functions $R_1(t)$, $R_2(t)$, failure rate functions $r_1(t)$, $r_2(t)$ and weights p and $q=1-p$, where $0 < p < 1$. The mixed density is then written as

$$f(t) = pf_1(t) + (1-p) f_2(t)$$

and the mixed reliability function is

$$R(t) = pR_1(t) + (1-p)R_2(t).$$

The failure rate function of the mixture can be written as the mixture

$$r(t) = \omega(t)\, r_1(t) + [1 - \omega(t)]\, r_2(t),$$

where $\omega(t) = pR_1(t)/R(t)$.

Moreover, from [3], we have under some mild conditions, that

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \min\{r_1(t), r_2(t)\}.$$

In the following propositions, we give some properties for the mixture failure rate function.

Proposition 1. For the first derivative of $\omega(t)$, we have

$$\omega'(t) = \omega(t)[1 - \omega(t)][r_2(t) - r_1(t)].$$

Proposition 2. For the first derivative of $r(t)$, we have

$$r'(t) = [1 - \omega(t)](-\omega(t)(r_2(t) - r_1(t))^2 + r'_2(t) + \omega(t) r'_1(t).$$

Proposition 3. If X_1 is exponentially distributed with parameter λ, then

$$r'(t) = [1 - \omega(t)](-\omega(t)(r_2(t) - \lambda)^2 + r'_2(t)).$$

We suppose that $r_2(t) = \alpha t$, where $\alpha > 0$. Consequently, the reliability function of X_2 is the reliability function of Rayleigh's distribution of the form

$$R_2(t) = \exp\{-\frac{\alpha}{2} t^2\} \quad \text{for } t \geq 0.$$

Proposition 4. If $p\lambda^2 \leq \alpha$, then r (t) is unimodal.

Proof. By *Proposition 1*, we conclude that ω (t) is decreasing for $t \in (0, t_1)$, where $t_1 = \lambda/\alpha$ and is increasing for $t \in (t_1, \infty)$. Hence, if $t < t_1$, then $\omega(t)(r_2(t) - \lambda)^2$ is decreasing from $p\lambda^2 < \alpha$ to 0, and if $t > t_1$ then $\omega(t)(r_2(t) - \lambda)^2$ is increasing from 0 to ∞. Thus the equation $\omega(t)(r_2(t) - \lambda)^2 = \lambda$ has only one solution t_2 and $r(t)$ is increasing for $t < t_2$ and decreasing for $t > t_2$.

Proposition 5. If $p\lambda^2 > \alpha$, then there exist t_3, and t_4, $t_3 < t_1 < t_4$, such that $r(t)$ decreases in $(0, t_3)$, increases in (t_3, t_4) and decreases in (t_4, ∞).

Proof. Let $h(t) = \alpha - \omega(t)(r_2(t) - \lambda)^2$. It is easy to find that $h(0) = \alpha - p\lambda^2 < 0$, $h(t_1) = \alpha$, $h(\infty) = -\infty$.
The function $h(t)$ is increasing from $h(0) < 0$ to $h(t_1) = \alpha > 0$, and is decreasing from $h(t_1) = \alpha > 0$ to $h(\infty) = -\infty$.
Thus, there exist t_3 and t_4, $0 < t_3 < t_1 < t_4$ such that $r(t)$ decreases on $(0, t_3)$, increases in (t_3, t_4) and decreases in (t_4, ∞). This completes the proof.

Example 1. In this example we consider a real life time data from the [11]. We estimate the parameters p, α, λ of the model with reliability function

$$R(t) = p \exp(-\lambda t) + (1-p) \exp(-0{,}5\alpha t^2) \text{ for } x \geq 0.$$

By maximizing the logarithm of likelihood function for grouped data, we calculate $p = 0.643316$, $\alpha = 0.001284$, $\lambda = 0.0288$. For these values of parameters, we prove Pearson's test of fit and compute $\chi^2 = 0.68$. By *Proposition 4*, we conclude that $r(t)$ is unimodal.

3. Bounds and Approximation

In this section we cover some of the well known bounds and approximations to the MTBF.
By the inequality

$$\int_0^t R(s)ds \le min\{t, ET\},$$

where ET is the mean value of T, we obtain the upper bound for MTBF:

$$MTBF_U = \frac{1}{F(t)} min\{t, ET\}.$$

In [1] for MTBF proposed is the following average approximation

$$MTBF_A = \frac{t}{2}\frac{1+R(t)}{F(t)}.$$

Proposition 6. If $f(t)$ is unimodal, then these exist t_1 such that $MTBF_A$ is a lower bound of MTBF for $t \in <0, t_1)$ and it is an upper bound of MTBF for $t \in (t_1, \infty)$

Proof. We consider the difference

$$g(t) = \frac{\int_0^t R(s)ds}{F(t)} - \frac{t}{2}\frac{R(t)+1}{F(t)}$$

and

$$g_1(t) = \int_0^t R(t)dt - \frac{1}{2}t(R(t)+1).$$

It is easy to find that $g_1(0) = 0$, $g_1(+\infty) = -\infty$. The first derivative of $g'_1(t)$ is

$$g'_1(t) = \frac{1}{2}[tf(t) - F(t)].$$

If $f(t)$ is decreasing then $g'_1(t) < 0$ and $MTBF_A$ is a lower bound for MTBF.

If $f(t)$ is unimodal then exists t_m and t_1 such that $f'(t_m) = 0$, $t_m < t_1$ and $g(t) \ge 0$ for $t \in (0, t_1)$, $g(t) \le 0$ for $t \in (t_1, \infty)$.

Proposition 7. If $T \in MTFR$, then

$$MTBF \ge \frac{1}{r(t)} \quad \text{for } t > 0.$$

Proof. By Definition 1, if $M(t)$ is non-decreasing, then we have

$$[M(t)]' = \frac{f(t)\int_0^t R(s)ds - F(t)R(t)}{(\int_0^t R(s)ds)^2} \ge 0$$

and

$$MTBF \ge MTBF_L = \frac{1}{r(t)} \quad \text{for } t \in \{t: r(t) > 0\}.$$

Proposition 8. If the lifetime T has unimodal failure rate function r (t), then $T \in MTFR$ if and only if $r(\infty)\, ET - 1 \ge 0$.

Proof. Let

$$h(t) = r(t)\int_0^t R(s)ds - F(t).$$

It is easy to show that $h(0) = 0$ and $h(\infty) = r(\infty)\, ET - 1$.
The first derivative of $h(t)$ is

$$h'(t) = r'(t)\int_0^t R(s)ds.$$

If r (t) is increasing, then $h(t)$ is increasing and if r (t) is decreasing, then $h(t)$ is decreasing. This completes the proof.

Example 2. Consider the system with failure rate function proposed in Example 1. The exact and approximate results for MTBF are shown in Table 1 for varying R (t) with the corresponding t. The results show that the average approximation $MTBF_A$ is greater than MTBF. For this data, we compute ET = 34.81 and $\lambda ET - 1 \ge 0$ and by Proposition 8 we obtain that $T \in MTFR$.

Table 1. The values of the exact and approximate MTBF of lifetime data

R(t)	t	MTBF	$MTBF_U$	$MTBF_L$	$MTBF_A$
0.99999	0.00054	53.97	53.97	53.97	53.97
0.9999	0.0054	53.97	53.97	53.97	53.97
0.999	0.05398	53.95	53.98	53.93	53.95
0,99	0.54031	53.76	54.03	53.55	53.76
0.9	5.43971	51.67	54.40	49.22	51.68
0.8	10.9261	49.15	54.63	44.05	49.17
0.7	16.4696	46.62	54.90	39.17	46.66
0.6	22.1619	44.21	55.40	34.90	44.32
0.5	29.1751	41.98	56.35	34.81	42.26
0.4	34.8007	39.94	58.00	34.81	40.60
0.3	42.5854	38.11	49.73	34.81	39.54
0.2	52.8174	36.50	43.51	34.81	39.61
0.1	70.2655	35.23	38.68	34.81	42.94

4. Conclusion

In this paper we show that, from a practical point view, the unimodal failure rate model can be obtained from a mixture of two common IFR models. This model is flexibility. Practical relevance and applicability have been demonstrated using well known data. In this paper a simple approximation of the MTBF of systems subjected to periodic maintenance has been proposed as well.

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Reliability modelling of complex systems - Part 1

Keywords

reliability, large system, asymptotic approach, limit reliability function

Abstract

The paper is concerned with the application of limit reliability functions to the reliability evaluation of large systems. Two-state large non-repaired systems composed of independent components are considered. The asymptotic approach to the system reliability investigation and the system limit reliability function are defined. Two-state homogeneous series, parallel and series-parallel systems are defined and their exact reliability functions are determined. The classes of limit reliability functions of these systems are presented. The results of the investigation concerned with domains of attraction for the limit reliability functions of the considered systems and the investigation concerned with the reliability of large hierarchical systems as well are discussed in the paper. The paper contains exemplary applications of the presented facts to the reliability evaluation of large technical systems.

1. Introduction

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and their complicated operating processes. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet large series systems, for instance, in piping transportation of water, gas, oil and various chemical substances. Large systems of these kinds are also used in electrical energy distribution. A city bus transportation system composed of a number of communication lines each serviced by one bus may be a model series system, if we treat it as not failed, when all its lines are able to transport passengers. If the communication lines have at their disposal several buses we may consider it as either a parallel-series system or an "*m* out of *n*" system. The simplest example of a parallel system or an "*m* out of *n*" system may be an electrical cable composed of a number of wires, which are its basic components, whereas the transmitting electrical network may be either a parallel-series system or an "*m* out of *n*"-series system. Large systems of these types are also used in telecommunication, in rope transportation and in transport using belt conveyers and elevators. Rope transportation systems like port elevators and ship-rope elevators used in shipyards during ship docking are model examples of series-parallel and parallel-series systems.

In the case of large systems, the determination of the exact reliability functions of the systems leads us to complicated formulae that are often useless for reliability practitioners. One of the important techniques in this situation is the asymptotic approach to system reliability evaluation. In this approach, instead of the preliminary complex formula for the system reliability function, after assuming that the number of system components tends to infinity and finding the limit reliability of the system, we obtain its simplified form.

The mathematical methods used in the asymptotic approach to the system reliability analysis of large systems are based on limit theorems on order statistics distributions, considered in very wide literature, for instance in [4]-[5], [7], [12]. These theorems have generated the investigation concerned with limit reliability functions of the systems composed of two-state components. The main and fundamental results on this subject that determine the three-element classes

of limit reliability functions for homogeneous series systems and for homogeneous parallel systems have been established by Gniedenko in [6]. These results are also presented, sometimes with different proofs, for instance in subsequent works [1], [8]. The generalizations of these results for homogeneous "*m* out of *n*" systems have been formulated and proved by Smirnow in [13], where the seven-element class of possible limit reliability functions for these systems has been fixed. As it has been done for homogeneous series and parallel systems classes of limit reliability functions have been fixed by Chernoff and Teicher in [2] for homogeneous series-parallel and parallel-series systems. Their results were concerned with so-called "quadratic" systems only. They have fixed limit reliability functions for the homogeneous series-parallel systems with the number of series subsystems equal to the number of components in these subsystems, and for the homogeneous parallel-series systems with the number of parallel subsystems equal to the number of components in these subsystems. Kolowrocki has generalized their results for non-"quadratic" and non-homogeneous series-parallel and parallel-series systems in [8]. These all results may also be found for instance in [9].

The results concerned with the asymptotic approach to system reliability analysis have become the basis for the investigation concerned with domains of attraction ([9], [11]) for the limit reliability functions of the considered systems and the investigation concerned with the reliability of large hierarchical systems as well ([3], [9]). Domains of attraction for limit reliability functions of two-state systems are introduced. They are understood as the conditions that the reliability functions of the particular components of the system have to satisfy in order that the system limit reliability function is one of the limit reliability functions from the previously fixed class for this system. Exemplary theorems concerned with domains of attraction for limit reliability functions of homogeneous series systems are presented here and the application of one of them is illustrated. Hierarchical series-parallel and parallel-series systems of any order are defined, their reliability functions are determined and limit theorems on their reliability functions are applied to reliability evaluation of exemplary hierarchical systems of order two.

All the results so far described have been obtained under the linear normalization of the system lifetimes. The paper contains the results described above and comments on their newest generalizations recently presented in [9].

2. Reliability of two-state systems

We assume that

E_i, $i = 1,2,...,n$, $n \in N$,

are two-state components of the system having reliability functions

$$R_i(t) = P(T_i > t),\ t \in (-\infty, \infty),$$

where

T_i, $i = 1,2,...,n$,

are independent random variables representing the lifetimes of components E_i with distribution functions

$$F_i(t) = P(T_i \le t),\ t \in (-\infty, \infty).$$

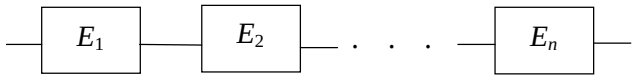
The simplest two-state reliability structures are series and parallel systems. We define these systems first.

Definition 1. We call a two-state system series if its lifetime T is given by

$$T = \min_{1 \le i \le n} \{T_i\}.$$

The scheme of a series system is given in *Figure 1*.

Figure 1. The scheme of a series system



Definition 1 means that the series system is not failed if and only if all its components are not failed, and therefore its reliability function is given by

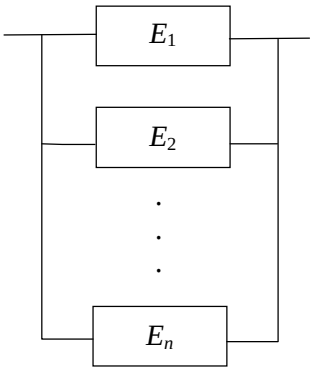
$$\overline{\boldsymbol{R}}_n(t) = \prod_{i=1}^{n} R_i(t),\ t \in (-\infty, \infty). \qquad (1)$$

Definition 2. We call a two-state system parallel if its lifetime T is given by

$$T = \max_{1 \le i \le n} \{T_i\}.$$

The scheme of a parallel system is given in *Figure 2*.

Figure 2. The scheme of a parallel system



Definition 2 means that the parallel system is failed if and only if all its components are failed and therefore its reliability function is given by

$$\boldsymbol{R}_n(t) = 1 - \prod_{i=1}^{n} F_i(t), \; t \in (-\infty, \infty). \tag{2}$$

Another basic, a bit more complex, two-state reliability structure is a series-parallel system. To define it, we assume that

$$E_{ij},\ i = 1,2,...,k_n,\ j = 1,2,...,l_i,\ k_n, l_1, l_2,..., l_{k_n} \in N,$$

are two-state components of the system having reliability functions

$$R_{ij}(t) = P(T_{ij} > t),\ t \in (-\infty, \infty),$$

where

$$T_{ij},\ i = 1,2,...,k_n,\ j = 1,2,...,l_i,$$

are independent random variables representing the lifetimes of components E_{ij} with distribution functions

$$F_{ij}(t) = P(T_{ij} \le t),\ t \in (-\infty, \infty).$$

Definition 3. We call a two-state system series-parallel if its lifetime T is given by

$$T = \max_{1 \le i \le k_n} \{\min_{1 \le j \le l_i} \{T_{ij}\}\}.$$

By joining the formulae (1) and (2) for the reliability functions of two-state series and parallel systems it is easy to conclude that the reliability function of the two-state series-parallel system is given by

$$\boldsymbol{R}_{k_n, l_1, l_2, ..., l_{k_n}}(t) = 1 - \prod_{i=1}^{k_n} [1 - \prod_{j=1}^{l_i} R_{ij}(t)], \; t \in (-\infty, \infty), \tag{3}$$

where k_n is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Definition 4. We call a two-state series-parallel system regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n,\ l_n \in N,$$

i.e. if the numbers of components in its series subsystems are equal.

The scheme of a regular series-parallel system is given in *Figure 3*.

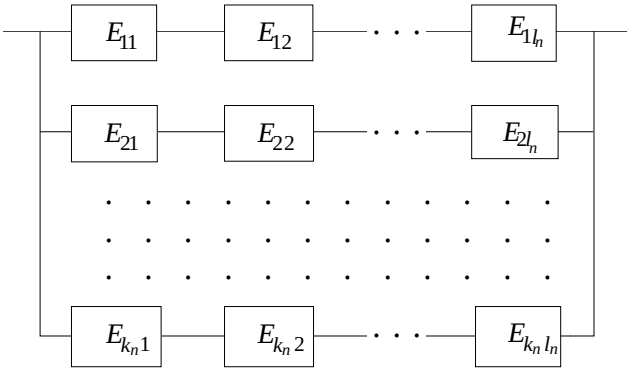


Figure 3. The scheme of a regular series-parallel system

Definition 5. We call a two-state system homogeneous if its component lifetimes have an identical distribution function $F(t)$, i.e. if its components have the same reliability function

$$R(t) = 1 - F(t),\ t \in (-\infty, \infty).$$

The above definition and equations (1)-(3) result in the simplified formulae for the reliability functions of the homogeneous systems stated in the following corollary.

Corollary 1. The reliability function of the homogeneous two-state system is given by

- for a series system

$$\overline{\boldsymbol{R}}_n(t) = [R(t)]^n,\ t \in (-\infty, \infty), \tag{4}$$

- for a parallel system

$$\boldsymbol{R}_n(t) = 1 - [F(t)]^n,\ t \in (-\infty, \infty), \tag{5}$$

- for a regular series-parallel system

$$\boldsymbol{R}_{k_n, l_n}(t) = 1 - [1 - [R(t)]^{l_n}]^{k_n},\ t \in (-\infty, \infty). \tag{6}$$

3. Asymptotic approach to system reliability

The asymptotic approach to the reliability of two-state systems depends on the investigation of limit distributions of a standardized random variable

$$(T - b_n)/a_n,$$

where T is the lifetime of a system and $a_n > 0$ and $b_n \in (-\infty, \infty)$ are suitably chosen numbers called normalizing constants.
Since

$$P((T - b_n)/a_n > t) = P(T > a_n t + b_n) = \boldsymbol{R}_n(a_n t + b_n),$$

where $\boldsymbol{R}_n(t)$ is a reliability function of a system composed of n components, then the following definition becomes natural.

Definition 6. We call a reliability function $\Re(t)$ the limit reliability function of a system having a reliability function $\boldsymbol{R}_n(t)$ if there exist normalizing constants $a_n > 0$, $b_n \in (-\infty, \infty)$ such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \Re(t) \text{ for } t \in C_{\Re},$$

where $C_{\Re}$ is the set of continuity points of $\Re(t)$.

Thus, if the asymptotic reliability function $\Re(t)$ of a system is known, then for sufficiently large n, the approximate formula

$$\boldsymbol{R}_n(t) \cong \Re((t - b_n)/a_n), \; t \in (-\infty, \infty). \qquad (7)$$

may be used instead of the system exact reliability function $\boldsymbol{R}_n(t)$.

3.1. Reliability of large two-state series systems

The investigations of limit reliability functions of homogeneous two-state series systems are based on the following auxiliary theorem.

Lemma 1. If
(i) $\overline{\Re}(t) = \exp[-\overline{V}(t)]$ is a non-degenerate reliability function,
(ii) $\overline{\boldsymbol{R}}_n(t)$ is the reliability function of a homogeneous two-state series system defined by (4),
(iii) $a_n > 0$, $b_n \in (-\infty, \infty)$,
then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_n(a_n t + b_n) = \overline{\Re}(t) \text{ for } t \in C_{\overline{\Re}}$$

if and only if

$$\lim_{n\to\infty} nF(a_n t + b_n) = \overline{V}(t) \text{ for } t \in C_{\overline{V}}$$

Proof. The proof may be found in [1], [6], [8].

Lemma 1 is an essential tool in finding limit reliability functions of two-state series systems. It also is the basis for fixing the class of all possible limit reliability functions of these systems. This class is determined by the following theorem.

Theorem 1. The only non-degenerate limit reliability functions of the homogeneous two-state series system are:

$$\overline{\Re}_1(t) = \exp[-(-t)^{-\alpha}] \text{ for } t < 0,$$

$$\overline{\Re}_1(t) = 0 \text{ for } t \geq 0, \; \alpha > 0;$$

$$\overline{\Re}_2(t) = 1 \text{ for } t < 0,$$

$$\overline{\Re}_2(t) = \exp[-t^{\alpha}] \text{ for } t \geq 0, \; \alpha > 0;$$

$$\overline{\Re}_3(t) = \exp[-\exp[t]] \text{ for } t \in (-\infty, \infty).$$

Proof. The proof may be found in [1], [6], [8].

3.2. Reliability of large two-state parallel systems

The class of limit reliability functions for homogeneous two-state parallel systems may be determined on the basis of the following auxiliary theorem.

Lemma 2. If
(i) $\Re(t) = 1 - \exp[-V(t)]$ is a non-degenerate reliability function,
(ii) $\boldsymbol{R}_n(t)$ is the reliability function of a homogeneous two-state parallel system defined by (5),
(iii) $a_n > 0$, $b_n \in (-\infty, \infty)$,
then

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \Re(t) \text{ for } t \in C_{\Re},$$

if and only if

$$\lim_{n\to\infty} nR(a_n t + b_n) = V(t) \text{ for } t \in C_V.$$

Proof. The proof may be found in [1], [6], [8].

By applying *Lemma 2* it is possible to fix the class of limit reliability functions for homogeneous two-state

parallel systems. However, it is easier to obtain this result using the duality property of parallel and series systems expressed in the relationship

$$\boldsymbol{R}_n(t) = 1 - \overline{\boldsymbol{R}}_n(-t) \text{ for } t \in (-\infty, \infty),$$

that results in the following lemma, [1], [6], [8]-[9].

Lemma 3. If $\overline{\Re}(t)$ is the limit reliability function of a homogeneous two-state series system with reliability functions of particular components $\overline{R}(t)$, then

$$\Re(t) = 1 - \overline{\Re}(-t) \text{ for } t \in C_{\overline{\Re}}$$

is the limit reliability function of a homogeneous two-state parallel system with reliability functions of particular components

$$R(t) = 1 - \overline{R}(-t) \text{ for } t \in C_{\overline{R}}.$$

At the same time, if $(a_{n,} b_n)$ is a pair of normalizing constants in the first case, then $(a_{n,} - b_n)$ is such a pair in the second case.
The application of *Lemma 3* and *Theorem 1* yields the following result.

Theorem 2. The only non-degenerate limit reliability functions of the homogeneous parallel system are:

$\Re_1(t) = 1$ for $t \le 0$,

$\Re_1(t) = 1 - \exp[-t^{-\alpha}]$ for $t > 0$, $\alpha > 0$;

$\Re_2(t) = 1 - \exp[-(-t)^{\alpha}]$ for $t < 0$,

$\Re_2(t) = 0$ for $t \ge 0$, $\alpha > 0$;

$\Re_3(t) = 1 - \exp[-\exp[-t]]$ for $t \in (-\infty, \infty)$.

Proof. The proof may be found in [1], [6], [8].

3.3. Reliability evaluation of large two-state series-parallel systems

The proofs of the theorems on limit reliability functions for homogeneous regular series-parallel systems and methods of finding such functions for individual systems are based on the following essential lemmas.
Lemma 4. If
(i) $k_n \to \infty$,
(ii) $\Re(t) = 1 - \exp[-V(t)]$ is a non-degenerate reliability function,
(iii) $\boldsymbol{R}_{k_n, l_n}(t)$ is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),
(iv) $a_n > 0$, $b_n \in (-\infty, \infty)$,
then

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n, l_n}(a_n t + b_n) = \Re(t) \text{ for } t \in C_{\Re}$$

if and only if

$$\lim_{n\to\infty} k_n [R(a_n t + b_n)]^{l_n} = V(t) \text{ for } t \in C_V .$$

Proof. The proof may be found in [8].

Lemma 5. If
(i) $k_n \to k$, $k > 0$, $l_n \to \infty$,
(ii) $\Re(t)$ is a non-degenerate reliability function,
(iii) $\boldsymbol{R}_{k_n, l_n}(t)$ is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),
(iv) $a_n > 0$, $b_n \in (-\infty, \infty)$,
then

$$\lim_{n\to\infty} \boldsymbol{R}_{k_n, l_n}(a_n t + b_n) = \Re(t) \text{ for } t \in C_{\Re},$$

if and only if

$$\lim_{n\to\infty} [R(a_n t + b_n)]^{l_n} = \Re_0(t) \text{ for } t \in C_{\Re_0},$$

where $\Re_0(t)$ is a non-degenerate reliability function and moreover

$$\Re(t) = 1 - [1 - \Re_0(t)]^k \text{ for } t \in (-\infty, \infty).$$

Proof. The proof may be found in [8].

The types of limit reliability functions of a series-parallel system depend on the system shape [7], i.e. on the relationships between the number k_n of its series subsystems linked in parallel and the number l_n of components in its series subsystems. The results based on *Lemma 4* and *Lemma 5* may be formulated in the form of the following theorem.

Theorem 3. The only non-degenerate limit reliability functions of the homogeneous regular two-state series-parallel system are:
Case 1. $k_n = n$, $|l_n - c \log n| >> s$, $s > 0$, $c > 0$.

$\Re_1(t) = 1$ for $t \le 0$,

$\Re_1(t) = 1 - \exp[-t^{-\alpha}]$ for $t > 0$, $\alpha > 0$;

$\Re_2(t) = 1 - \exp[-(-t)^{\alpha}]$ for $t < 0$,

$\Re_2(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_3(t) = 1 - \exp[-\exp[-t]]$ for $t \in (-\infty,\infty)$;

Case 2. $k_n = n$, $l_n - c \log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$.

$\Re_4(t) = 1$ for $t < 0$,

$\Re_4(t) = 1 - \exp[-\exp[-t^{\alpha} - s/c]]$ for $t \geq 0$, $\alpha > 0$;

$\Re_5(t) = 1 - \exp[-\exp[(-t)^{\alpha} - s/c]]$ for $t < 0$,

$\Re_5(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_6(t) = 1 - \exp[-\exp[\beta(-t)^{\alpha} - s/c]]$ for $t < 0$,

$\Re_6(t) = 1 - \exp[-\exp[-t^{\alpha} - s/c]]$ for $t \geq 0$, $\alpha > 0$, $\beta > 0$;

$\Re_7(t) = 1$ for $t < t_1$,

$\Re_7(t) = 1 - \exp[-\exp[-s/c]]$ for $t_1 \leq t < t_2$,

$\Re_7(t) = 0$ for $t \geq t_2$, $t_1 < t_2$;

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$.

$\Re_8(t) = 1 - [1 - \exp[-(-t)^{-\alpha}]]^k$ for $t < 0$,

$\Re_8(t) = 0$ for $t \geq 0$, $\alpha > 0$;

$\Re_9(t) = 1$ for $t < 0$,

$\Re_9(t) = 1 - [1 - \exp[-t^{\alpha}]]^k$ for $t \geq 0$, $\alpha > 0$;

$\Re_{10}(t) = 1 - [1 - \exp[-\exp t]]^k$ for $t \in (-\infty,\infty)$.

Proof. The proof may be found in [8].

Using the duality property of parallel-series and series-parallel systems similar to this given in *Lemma 3* for parallel and series systems it is possible to prove that the only limit reliability functions of the homogeneous regular two-state parallel-series system are

$$\overline{\Re}_i(t) = 1 - \Re_i(-t) \quad \text{for } t \in C_{\Re_i}, \quad i = 1,2,...,10.$$

Applying *Lemma 2*, it is possible to prove the following fact ([9]).

Corollary 2. If components of the homogeneous two-state parallel system have Weibull reliability functions

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \geq 0, \alpha > 0, \beta > 0$$

and

$$a_n = b_n/(\alpha \log n), \quad b_n = (\log n/\beta)^{1/\alpha},$$

then

$$\Re_3(t) = 1 - \exp[-\exp[-t]], \quad t \in (-\infty,\infty),$$

is its limit reliability function.

Example 1 (*a steel rope, durability*). Let us consider a steel rope composed of 36 strands used in ship rope elevator and assume that it is not failed if at least one of its strands is not broken. Under this assumption we may consider the rope as a homogeneous parallel system composed of $n = 36$ basic components. Further, assuming that the strands have Weibull reliability functions with parameters

$$\alpha = 2, \quad \beta = (7.07)^{-6},$$

by (5), the rope's exact reliability function takes the form

$$\boldsymbol{R}_{36}(t) = 1 - [1 - \exp[-(7.07)^{-6} t^2]]^{36} \text{ for } t \geq 0.$$

Thus, according to *Corollary 2*, assuming

$$a_n = (7.07)^3/(2\sqrt{\log 36}), \quad b_n = (7.07)^3 \sqrt{\log 36}$$

and applying (7), we arrive at the approximate formula for the rope reliability function of the form

$$\boldsymbol{R}_{36}(t) \cong \Re_3((t - b_n)/a_n)$$

$$= 1 - \exp[-\exp[-0.01071t + 7.167]]$$

for $t \in (-\infty,\infty)$.

The mean value of the rope lifetime T and its standard deviation, in months, calculated on the basis of the above approximate result and according to the formulae

$$E[T] = Ca_n + b_n, \quad \sigma = \pi a_n / \sqrt{6},$$

where $C \cong 0.5772$ is Euler's constant, respectively are:

$$E[T] \cong 723, \quad \sigma \cong 120.$$

The values of the exact and approximate reliability functions of the rope are presented in *Table 1* and graphically in *Figure 4*. The differences between them are not large, which means that the mistakes in replacing the exact rope reliability function by its approximate form are practically not significant.

Table 1. The values of the exact and approximate reliability functions of the steel rope

t	$\boldsymbol{R}_{36}(t)$	$\Re_3\left(\frac{t-b_n}{a_n}\right)$	$\Delta = \boldsymbol{R}_{36} - \Re_3$
0	1.000	1.000	0.000
400	1.000	1.000	0.000
500	0.995	0.988	–0.003
550	0.965	0.972	–0.007
600	0.874	0.877	–0.003
650	0.712	0.707	0.005
700	0.513	0.513	0.000
750	0.330	0.344	–0.014
800	0.193	0.218	–0.025
900	0.053	0.081	**–0.028**
1000	0.012	0.029	–0.017
1100	0.002	0.010	–0.008
1200	0.000	0.003	–0.003

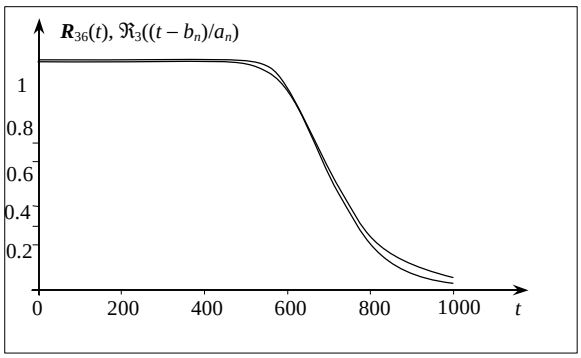


Figure 4. The graphs of the exact and approximate reliability functions of the steel rope

4. Domains of attraction for system limit reliability functions

The problem of domains of attraction for the limit reliability functions of two-state systems solved completely in [11] we will illustrate partly for two-state series homogeneous systems only. From *Theorem 1* it follows that the class of limit reliability functions for a homogeneous series system is composed of three functions, $\overline{\Re}_{ü}(t)$, $i = 1,2,3$. Now we will determine domains of attraction $D_{\overline{\Re}_{ü}}$ for these fixed functions, i.e. we will determine the conditions which the reliability functions $R(t)$ of the particular components of the homogeneous series system have to satisfy in order that the system limit reliability function is one of the reliability functions $\overline{\Re}_{ü}(t)$, $i = 1,2,3$.

Proposition 1. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\Re}_1}$$

if and only if

$$\lim_{r\to-\infty}\frac{1-R(r)}{1-R(rt)} = t^{\acute{o}} \quad \text{for } t>0.$$

Proposition 2. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\Re}_2}$$

if and only if

(i) $\exists\, y \in (-\infty,\infty)\ R(y) = 1$ and $R(y+\varepsilon) < 1$ for $\varepsilon > 0$,

(ii) $\lim_{r\to 0^+}\frac{1-R(rt+y)}{1-R(r+y)} = t^{\acute{o}}$ for $t>0$.

Proposition 3. If $R(t)$ is a reliability function of the homogeneous series system components, then

$$R(t) \in D_{\overline{\Re}_3}$$

if and only if

$$\lim_{n\to\infty} n[1-R(a_n t + b_n)] = e^t \quad \text{for } t \in (-\infty,\infty)$$

with

$$b_n = \inf\{t : R(t+0) \le 1-\frac{1}{n} \le R(t-0)\},$$

$$a_n = \inf\{t : R(t(1+0)+b_n)$$

$$\le 1-\frac{e}{n} \le R(t(1-0)+b_n)\}.$$

Example 2. If components of the homogeneous series system have reliability functions

$$R(t)=\begin{cases}1, & t<0\\ 1-t, & 0\le t<1\\ 0, & t\ge 1,\end{cases}$$

then

$$R(t)\in D_{\overline{\Re}_2}.$$

The results of the analysis on domains of attraction for limit reliability functions of two-state systems may automatically be transmitted to multi-state systems. To do this, it is sufficient to apply theorems about two-state systems such as the ones presented here to each vector co-ordinate of the multi-state reliability functions ([9], [14]).

5. Reliability of large hierarchical systems

Prior to defining the hierarchical systems of any order we once again consider a series-parallel system like a system presented in *Figure 3*. This system here is called a series-parallel system of order 1.
It is made up of components

$$E_{i_1 j_1},\ i_1=1,2,...,k_n,\ \ j_1=1,2,...,l_{i_1},$$

with the lifetimes respectively

$$T_{i_1 j_1},\ i_1=1,2,...,k_n,\ \ j_1=1,2,...,l_{i_1}.$$

Its lifetime is given by

$$T=\max_{1\le i_1\le k_n}\{\min_{1\le j_1\le l_{i_1}}\{T_{i_1 j_1}\}\}. \qquad (8)$$

Now we assume that each component

$$E_{i_1 j_1},\ i_1=1,2,...,k_n,\ \ j_1=1,2,...,l_{i_1},$$

of the series-parallel system of order 1 is a subsystem composed of components

$$E_{i_1 j_1 i_2 j_2},\ i_2=1,2,...,k_n^{(i_1 j_1)},\ j_2=1,2,...,l_{i_2}^{(i_1 j_1)},$$

and has a series-parallel structure.
This means that each subsystem lifetime $T_{i_1 j_1}$ is given by

$$T_{i_1 j_1}=\max_{1\le i_2\le k_n^{(i_1 j_1)}}\{\min_{1\le j_2\le l_{i_2}^{(i_1 j_1)}}\{T_{i_1 j_1 i_2 j_2}\}\}, \qquad (9)$$

$$i_1=1,2,...,k_n,\ j_1=1,2,...,l_{i_1},$$

where

$$T_{i_1 j_1 i_2 j_2},\ i_2=1,2,...,k_n^{(i_1 j_1)},\ j_2=1,2,...,l_{i_2}^{(i_1 j_1)},$$

are the lifetimes of the subsystem components $E_{i_1 j_1 i_2 j_2}$.
The system defined this way is called a hierarchical series-parallel system of order 2. Its lifetime, from (8) and (9), is given by the formula

$$T=\max_{1\le i_1\le k_n}\{\min_{1\le j_1\le l_{i_1}}[\max_{1\le i_2\le k_n^{(i_1 j_1)}}(\min_{1\le j_2\le l_{i_2}^{(i_1 j_1)}}T_{i_1 j_1 i_2 j_2})]\},$$

where k_n is the number of series systems linked in parallel and composed of series-parallel subsystems $E_{i_1 j_1}$, l_{i_1} are the numbers of series-parallel subsystems $E_{i_1 j_1}$ in these series systems, $k_n^{(i_1 j_1)}$ are the numbers of series systems in the series-parallel subsystems $E_{i_1 j_1}$ linked in parallel, and $l_{i_2}^{(i_1 j_1)}$ are the numbers of components in these series systems of the series-parallel subsystems $E_{i_1 j_1}$.
In an analogous way it is possible to define two-state parallel-series systems of order 2.
Generally, in order to define hierarchical series-parallel and parallel-series systems of any order r, $r\ge 1$, we assume that

$$E_{i_1 j_1 ... i_r j_r},$$

where

$$i_1=1,2,...,k_n,\ j_1=1,2,...,l_{i_1},\ i_2=1,2,...,k_n^{(i_1 j_1)},$$
$$j_2=1,2,...,l_{i_2}^{(i_1 j_1)},\ ...,\ i_r=1,2,...,k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})},$$
$$j_r=1,2,...,l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})}$$

and

$$k_n,\ l_{i_1},\ k_n^{(i_1 j_1)},\ l_{i_2}^{(i_1 j_1)},\ ...,\ k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})},$$
$$l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})}\in N,$$

are two-state components having reliability functions

$$R_{i_1 j_1 ... i_r j_r}(t)=P(T_{i_1 j_1 ... i_r j_r}>t),\ t\in(-\infty,\infty),$$

and random variables

$$T_{i_1 j_1 ... i_r j_r},$$

where

$$i_1=1,2,...,k_n,\ j_1=1,2,...,l_{i_1},\ i_2=1,2,...,k_n^{(i_1 j_1)},$$

$j_2 = 1,2,..., l_{i_2}^{(i_1 j_1)}$, ..., $i_r = 1,2,..., k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})}$,
$j_r = 1,2,..., l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})}$,

are independent random variables with distribution functions

$$F_{i_1 j_1 ... i_r j_r}(t) = P(T_{i_1 j_1 ... i_r j_r} \le t),\ t \in (-\infty, \infty),$$

representing the lifetimes of the components $E_{i_1 j_1 ... i_r j_r}$.

Definition 7. A two-state system is called a series-parallel system of order *r* if its lifetime *T* is given by

$$T = \max_{1 \le i_1 \le k_n} \{ \min_{1 \le j_1 \le l_{i_1}} \{ \max_{1 \le i_2 \le k_n^{(i_1 j_1)}} \{ \min_{1 \le j_2 \le l_{i_2}^{(i_1 j_1)}}$$

$$... \max_{1 \le i_r \le k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})}} (\min_{1 \le j_r \le l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})}} T_{i_1 j_1 ... i_r j_r})]...\}\}\},$$

where k_n, $k_n^{(i_1 j_1)}$, ..., $k_n^{(i_1 j_1 i_2 j_2 ... i_{r-1} j_{r-1})}$ are the numbers of suitable series systems of the system composed of series-parallel subsystems and linked in parallel, l_{i_1}, $l_{i_2}^{(i_1 j_1)}$, ..., $l_{i_{r-1}}^{(i_1 j_1 i_2 j_2 ... i_{r-2} j_{r-2})}$ are the numbers of suitable series-parallel subsystems in these series systems, and $l_{i_r}^{(i_1 j_1 i_2 j_2 K i_{r-1} j_{r-1})}$ are the numbers of components in the series systems of the series-parallel subsystems.

Definition 8. A two-state series-parallel system of order *r* is called homogeneous if its component lifetimes $T_{i_1 j_1 ... i_r j_r}$ have an identical distribution function

$$F(t) = P(T_{i_1 j_1 ... i_r j_r} \le t),\ t \in (-\infty, \infty),$$

where

$i_1 = 1,2,..., k_n$, $j_1 = 1,2,..., l_{i_1}$, $i_2 = 1,2,..., k_n^{(i_1 j_1)}$,
$j_2 = 1,2,..., l_{i_2}^{(i_1 j_1)}$, ..., $i_r = 1,2,..., k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})}$,
$j_r = 1,2,..., l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})}$,

i.e. if its components $E_{i_1 j_1 ... i_r j_r}$ have the same reliability function

$R(t) = 1 - F(t),\ t \in (-\infty, \infty)$.

Definition 9. A two-state series-parallel system of order *r* is called regular if

$$l_{i_1} = l_{i_2}^{(i_1 j_1)} = ... = l_{i_r}^{(i_1 j_1 ... i_{r-1} j_{r-1})} = l_n$$

and

$$k_n^{(i_1 j_1)} = ... = k_n^{(i_1 j_1 ... i_{r-1} j_{r-1})} = k_n$$

where k_n is the number of series systems in the series-parallel subsystems and l_n are the numbers of series-parallel subsystems or respectively the numbers of components in these series systems.

Using mathematical induction it is possible to prove that the reliability function of the homogeneous and regular two-state hierarchical series-parallel system of order *r* is given by

$$\boldsymbol{R}_{k,k_n,l_n}(t) = 1 - [1 - [\boldsymbol{R}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \text{ for } k = 2,3,...,r$$

and

$$\mathrm{R}_{1,k_n,l_n}(t) = 1 - [1 - [R(t)]^{l_n}]^{k_n},\ \mathrm{t} \in (-\infty, \infty),$$

where k_n and l_n are defined in *Definition 9.*

Corollary 3. If components of the homogeneous and regular two-state hierarchical series-parallel system of order *r* have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \ge 0,\ \lambda > 0,$$

then its reliability function is given by

$$\mathrm{R}_{k,k_n,l_n}(t) = 1 - [1 - [\mathrm{R}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \text{ for } t \ge 0$$

for $k = 2,3,...,r$ and

$$\boldsymbol{R}_{1,k_n,l_n}(t) = 1 - [1 - \exp[-\lambda l_n t]]^{k_n} \text{ for } t \ge 0.$$

Theorem 4. If

(i) $\Re(t) = 1 - \exp[-V(t)],\ t \in (-\infty, \infty)$, is a non-degenerate reliability function,

(ii) $\lim_{n \to \infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0$ for $r \ge 1$,

(iii) $\lim_{n \to \infty} k_n^{l_n^{r-1}+...+1} [R(a_n t + b_n)]^{l_n^r} = V(t)$ for $t \in C_V$, $r \ge 1,\ t \in (-\infty, \infty)$,

then

$$\lim_{n \to \infty} \boldsymbol{R}_{r,k_n,l_n}(a_n t + b_n) = \Re(t) \text{ for } t \in C_{\Re},\ r \ge 1,$$
$t \in (-\infty, \infty)$.

Proposition 4. If components of the homogeneous and regular two-state hierarchical series-parallel system of order r have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0, \ \lambda > 0,$$

$$\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0 \quad \text{for } r \geq 1,$$

and

$$a_n = \frac{1}{\lambda l_n^r}, \ b_n = \frac{1}{\lambda}\left(\frac{1}{l_n} + \frac{1}{l_n^2} + \ldots + \frac{1}{l_n^r}\right)\log k_n,$$

then

$$\Re_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty), \qquad (10)$$

is its limit reliability function.

Example 3. A hierarchical regular series-parallel homogeneous system of order $r = 2$ is such that $k_n = 200, \ l_n = 3$. The system components have identical exponential reliability functions with the failure rate $\lambda = 0.01$.
Under these assumptions its exact reliability function, according to *Corollary 3*, is given by

$$R_{2,200,3}(t) = 1 - [1 - [\,1 - [1 - \exp[-0.01 \cdot 3t]]^{200}\,]^3]^{200}$$

for $t \geq 0$.

Next applying *Proposition 4* with normalising constants

$$a_n = \frac{1}{0.01 \cdot 9} = 11.1,$$

$$b_n = \frac{1}{0.01}\left(\frac{1}{3} + \frac{1}{9}\right)\log 200 = 235.5,$$

we conclude that the system limit reliability function is given by

$$\Re_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty),$$

and from (7), the following approximate formula is valid

$$\boldsymbol{R}_{2,200,3}(t) \cong \Re_3(0.09t - 21.2)$$

$$= 1 - \exp[-\exp[-0.09t + 21.2]] \text{ for } t \in (-\infty, \infty).$$

Definition 10. A two-state system is called a parallel-series system of order r if its lifetime T is given by

$$T = \min_{1\leq i_1 \leq k_n}\{\max_{1\leq j_1 \leq l_{i_1}}\{\min_{1\leq i_2 \leq k_n^{(i_1 j_1)}}\{\max_{1\leq j_2 \leq l_{i_2}^{(i_1 j_1)}}$$

$$\ldots[\min_{1\leq i_r \leq k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}}(\max_{1\leq j_r \leq l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}} T_{i_1 j_1 \ldots i_r j_r})]\ldots\}\}\},$$

where k_n, $k_n^{(i_1 j_1)}$, …, $k_n^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of suitable parallel systems of the system composed of parallel-series subsystems and linked in series, l_{i_1}, $l_{i_2}^{(i_1 j_1)}$, ..., $l_{i_{r-1}}^{(i_1 j_1 i_2 j_2 \ldots i_{r-2} j_{r-2})}$ are the numbers of suitable parallel-series subsystems in these parallel systems, and $l_{i_r}^{(i_1 j_1 i_2 j_2 \ldots i_{r-1} j_{r-1})}$ are the numbers of components in the parallel systems of the parallel-series subsystems.

Definition 11. A two-state parallel-series system of order r is called homogeneous if its component lifetimes $T_{i_1 j_1 \ldots i_r j_r}$ have an identical distribution function

$$F(t) = P(T_{i_1 j_1 \ldots i_r j_r} \leq t),$$

where

$$i_1 = 1,2,\ldots,k_n, \ j_1 = 1,2,\ldots,l_{i_1}, \ i_2 = 1,2,\ldots,k_n^{(i_1 j_1)},$$
$$j_2 = 1,2,\ldots,l_{i_2}^{(i_1 j_1)},\ldots,$$
$$i_r = 1,2,\ldots,k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})}, \ j_r = 1,2,\ldots,l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})},$$

i.e. if its components $E_{i_1 j_1 \ldots i_r j_r}$ have the same reliability function

$$R(t) = 1 - F(t), \ t \in (-\infty, \infty).$$

Definition 12. A two-state parallel-series system of order r is called regular if

$$l_{i_1} = l_{i_2}^{(i_1 j_1)} = \ldots = l_{i_r}^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = l_n$$

and

$$k_n^{(i_1 j_1)} = \ldots = k_n^{(i_1 j_1 \ldots i_{r-1} j_{r-1})} = k_n$$

where k_n is the number of parallel systems in the parallel-series subsystems and l_n are the numbers of parallel-series subsystems or, respectively, the numbers of components in these parallel systems.

Applying mathematical induction it is possible to prove that the reliability function of the homogeneous and regular two-state hierarchical parallel-series system of order *r* is given by

$$\overline{\boldsymbol{R}}_{k,k_n,l_n}(t) = [1-[1-\overline{\boldsymbol{R}}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \quad \text{for } k = 2,3,...,r$$

and

$$\overline{\boldsymbol{R}}_{1,k_n,l_n}(t) = [1-[F(t)]^{l_n}]^{k_n}, \; t \in (-\infty,\infty),$$

where k_n and l_n are defined in *Definition 12*.

Corollary 4. If components of the homogeneous and regular two-state hierarchical parallel-series system of order *r* have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0, \; \lambda > 0,$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,k_n,l_n}(t) = [1-[1-\overline{\boldsymbol{R}}_{k-1,k_n,l_n}(t)]^{l_n}]^{k_n} \quad \text{for } k = 2,3,...,r$$

and

$$\overline{\boldsymbol{R}}_{1,k_n,l_n}(t) = [1-[1-\exp[-\lambda t]]^{l_n}]^{k_n} \quad \text{for } t \geq 0.$$

*Theorem 5***.** If

(i) $\overline{\Re}(t) = \exp[-\overline{V}(t)], \; t \in (-\infty,\infty),$ is a non-degenerate reliability function,

(ii) $\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0$ for $r \geq 1$,

(iii) $\lim_{n\to\infty} k_n^{l_n^{r-1}+...+1} [F(a_n t + b_n)]^{l_n^r} = \overline{V}(t)$ for $t \in C_V$, $r \geq 1, \; t \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_{r,k_n,l_n}(a_n t + b_n) = \overline{\Re}(t) \quad \text{for} \quad t \in C_{\overline{\Re}}, \quad r \geq 1, \; t \in (-\infty,\infty).$$

Proposition 5. If components of the homogeneous and regular two-state hierarchical parallel-series system of order *r* have an exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0, \; \lambda > 0,$$

$$\lim_{n\to\infty} l_n^{r-1} k_n^{-\frac{1}{l_n}} = 0 \quad \text{for } r \geq 1, \; \lim_{n\to\infty} l_n = l, \; l \in N,$$

and

$$a_n = \frac{1}{\lambda} \frac{1}{k_n^{\frac{1}{l_n}+...+\frac{1}{l_n^r}}}, \; b_n = 0,$$

then

$$\overline{\Re}_2(t) = \exp[-t^{l^r}] \text{ for } t \geq 0 \qquad (11)$$

is its limit reliability function.

Example 3. We consider a hierarchical regular parallel-series homogeneous system of order $r = 2$ such that $k_n = 200, \; l_n = 3,$ whose components have identical exponential reliability functions with the failure rate $\lambda = 0.01$.
Its exact reliability function, according to *Corollary 4*, is given by

$$\overline{\boldsymbol{R}}_{2,200,3}(t) = [1-[1-[\,1-[1-\exp[-0.01t]]^3]^{200}\,]^3]^{200}$$

for $t \geq 0$.

Next applying *Proposition 5* with normalising constants

$$a_n = \frac{1}{0.01} \cdot \frac{1}{200^{1/3+1/9}} = 9.4912, \; b_n = 0,$$

we conclude that

$$\overline{\Re}_2(t) = \exp[-t^9] \text{ for } t \geq 0$$

is the system limit reliability function, and from (7), the following approximate formula is valid

$$\overline{\boldsymbol{R}}_{2,200,3}(t) \cong \overline{\Re}_2(0.1054t) = \exp[-(0.1054t)^9]$$

for $t \geq 0$.

6. Conclusion

Generalizations of the results on limit reliability functions of two-state homogeneous systems for these and other systems in case they are non-homogeneous, are mostly given in [8] and [9]. These results allow us to evaluate reliability characteristics of homogeneous and non-homogeneous series-parallel and parallel-series systems with regular reliability structures, i.e. systems composed of subsystems having the same numbers of components. However, this fact does not restrict the completeness of the performed analysis, since by conventional joining of a suitable number of components which do not fail, in series sub-systems of the non-regular series-parallel systems, leads us to the regular non-homogeneous series-parallel systems.

Similarly, conventional joining of a suitable number of failed components in parallel subsystems of the non-regular parallel-series systems we get the regular non-homogeneous parallel-series systems. Thus the problem has been analyzed exhaustively.

The results concerned with the asymptotic approach to system reliability analysis, in a natural way, have led to investigation of the speed of convergence of the system reliability function sequences to their limit reliability functions ([9]). These results have also initiated the investigations of limit reliability functions of "*m* out of *n*"-series, series-"*m* out of *n*" systems, the investigations on the problems of the system reliability improvement and on the reliability of systems with varying in time their structures and their components reliability described briefly in [9] and presented in Part 2 ([10]) of this paper.

More general and practically important complex systems composed of multi-state and degrading in time components are considered in wide literature, for instance in [14]. An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is described in [9] for large multi-state systems with degrading components. The most important results regarding generalizations of the results on limit reliability functions of two-state systems dependent on transferring them to series, parallel, "*m* out of *n*", series-parallel and parallel-series multi-state systems with degrading components are given in [9]. Some practical applications of the asymptotic approach to the reliability evaluation of various technical systems are contained in [9] as well.

The proposed method offers enough simplified formulae to allow significant simplifying of large systems' reliability evaluating and optimizing calculations.

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Reliability modelling of complex systems - Part 2

Keywords

reliability, large system, asymptotic approach, limit reliability function

Abstract

Series-"*m* out of *n*" systems and "*m* out of *n*"-series systems are defined and exemplary theorems on their limit reliability functions are presented and applied to the reliability evaluation of a piping transportation system and a rope elevator. Applications of the asymptotic approach in large series systems reliability improvement are also presented. The paper is completed by showing the possibility of applying the asymptotic approach to the reliability analysis of large systems placed in their variable operation processes. In this scope, the asymptotic approach to reliability evaluation for a port grain transportation system related to its operation process is performed.

1. Reliability of large series-"*m* out of *n*" systems

Definition 1. A two-state system is called a series-"*m* out of k_n" system if its lifetime T is given by

$$T = T_{(k_n-m+1)}, \quad m = 1,2,...,k_n,$$

where $T_{(k_n-m+1)}$ is the mth maximal order statistic in the set of random variables

$$T_i = \min_{1\le j\le l_i}\{T_{ij}\}, \quad i = 1,2,...,k_n.$$

The above definition means that the series-"*m* out of k_n" system is composed of k_n series subsystems and it is not failed if and only if at least m out of its k_n series subsystems are not failed.
The series-"*m* out of k_n" system is a series-parallel for $m = 1$ and it becomes a series system for $m = k_n$.
The reliability function of the two-state series-"*m* out of k_n" system is given either by

$$\boldsymbol{R}^{(m)}_{k_n,l_1,l_2,...,l_{k_n}}(t) = 1$$

$$- \sum_{\substack{r_1,r_2,...,r_{k_n}=0 \\ r_1+r_2+...+r_{k_n}\le m-1}}^{1} \prod_{i=1}^{k_n}[\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[1-\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i},$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_1,l_2,...,l_{k_n}}(t)$$

$$= \sum_{\substack{r_1,r_2,...,r_{k_n}=0 \\ r_1+r_2+...+r_{k_n}\le \overline{m}}}^{1} \prod_{i=1}^{k_n}[1-\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i},$$

for $t \in (-\infty,\infty)$, where $\overline{m} = k_n - m$.

Definition 2. The series-"*m* out of k_n" system is called regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n, \quad l_n \in N.$$

Definition.3. The series-"*m* out of k_n" system is called homogeneous if its component lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t),\ t \in (-\infty,\infty),\ i = 1,2,...,k_n,\ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t),\ t \in (-\infty,\infty).$$

From the above definitions it follows that the reliability function of the homogeneous and regular series-"*m* out of k_n" system is given either by

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t)=1-\sum_{i=0}^{m-1}\binom{k_n}{i}[R^{l_n}(t)]^i[1-R^{l_n}(t)]^{k_n-i}$$

for $t\in(-\infty,\infty)$ or by

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t)=\sum_{i=0}^{\overline{m}}\binom{k_n}{i}[1-R^{l_n}(t)]^i[R^{l_n}(t)]^{k_n-i}$$

for $t\in(-\infty,\infty)$, $\overline{m}=k_n-m$, where k_n is the number of series subsystems in the "m out of k_n" system and l_n is the number of components of the series subsystems.

Corollary 1. If components of the homogeneous and regular two-state series-"m out of k_n" system have Weibull reliability function

$$R(t)=\exp[-\beta t^{\alpha}] \text{ for } t\geq 0,\ \alpha>0,\ \beta>0,$$

then its reliability function is given either by

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t)$$

$$=1-\sum_{i=0}^{m-1}\binom{k_n}{i}[\exp[-il_n\beta t^{\alpha}]][1-\exp[-l_n\beta t^{\alpha}]]^{k_n-i}$$

for $t\geq 0$ or by

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t)$$

$$=\sum_{i=0}^{\overline{m}}\binom{k_n}{i}[1-\exp[-l_n\beta t^{\alpha}]]^i[\exp[-(k_n-i)l_n\beta t^{\alpha}]] \quad (2)$$

for $t\geq 0$, $\overline{m}=k_n-m$.

Proposition 1. If components of the two-state homogeneous and regular series-"m out of k_n" system have Weibull reliability function

$$R(t)=\exp[-\beta t^{\alpha}] \text{ for } t\geq 0,\ \alpha>0,\ \beta>0,$$

and

$$\lim_{n\to\infty}k_n=k,\ k>0,\ 0<m\leq k,\ \lim_{n\to\infty}l_n=\infty,$$

$$a_n=(\beta l_n)^{-\frac{1}{\alpha}},\quad b_n=0,$$

then

$$\Re_9^{(2)}(t)$$

$$=1-\sum_{i=0}^{m-1}\binom{k}{i}\exp[-it^{\alpha}][1-\exp[-t^{\alpha}]]^{k-i} \text{ for } t\geq 0$$

is its limit reliability function, i.e., for $t\geq 0$, we have

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t)\cong\Re_9^{(2)}\left(\frac{t-b_n}{a_n}\right)$$

$$=1-\sum_{i=0}^{m-1}\binom{k}{i}\exp[-i\beta l_n t^{\alpha}][1-\exp[-\beta l_n t^{\alpha}]]^{k-i}. \quad (3)$$

Example 1. The piping transportation system is set up to receive from ships, store and send by carriages or cars oil products such as petrol, driving oil and fuel oil. Three terminal parts A, B and C fulfil these purposes. They are linked by the piping transportation systems. The unloading of tankers is performed at the pier. The pier is connected to terminal part *A* through the transportation subsystem S_1 built of two piping lines. In part *A* there is a supporting station fortifying tankers' pumps and making possible further transport of oil by means of subsystem S_2 to terminal part *B*. Subsystem S_2 is built of two piping lines. Terminal part *B* is connected to terminal part *C* by subsystem S_3. Subsystem S_3 is built of three piping lines. Terminal part *C* is set up for loading the rail cisterns with oil products and for the wagon carrying these to the railway station.

We will analyse the reliability of the subsystem S_3 only. This subsystem consists of $k_n = 3$ identical piping lines, each composed of $l_n = 360$ steel pipe segments. In each of lines there are pipe segments with Weibull reliability function

$$R(t) = \exp[-0.0000000008t^4] \text{ for } t\geq 0.$$

We suppose that the system is good if at least 2 of its piping lines are not failed. Thus, according to *Definitions 2-3*, it may be considered as a homogeneous and regular series-"2 out of 3" system, and according to *Proposition 1*, assuming

$$a_n=\frac{1}{(\beta l_n)^{1/\alpha}}=\frac{1}{(0.000000288)^{1/4}},\ b_n=0,$$

and using (3), its reliability function is given by

$$\boldsymbol{R}_{3,360}^{(2)}(t)\cong\Re_9^{(2)}\left(\frac{t}{a_n}\right)$$

$$=\sum_{i=0}^{1}\binom{3}{i}\exp[-i\cdot 0.000000288t^4]$$

$$\cdot[1-\exp[-0.000000288t^4]]^{3-i} \text{ for } t\geq 0.$$

2. Reliability of large "*m* out of *n*"-series systems

Definition 4. A two-state system is called an "m_i out of l_i"-series system if its lifetime T is given by

$$T = \min_{1\le i\le k_n} T_{(l_i-m_i+1)}, \ m_i = 1,2,...,l_i,$$

where $T_{(l_i-m_i+1)}$ is the m_ith maximal order statistic in the set of random variables

$$T_{i1}, T_{i2}, ..., T_{il_i}, \quad i = 1,2,...,k_n.$$

The above definition means that the "m_i out of l_i"-series system is composed of k_n subsystems that are "m_i out of l_i" systems and it is not failed if all its "m_i out of l_i" subsystems are not failed.
The "m_i out of l_i"-series system is a parallel-series system if $m_1 = m_2 = \ldots = m_{k_n} = 1$ and it becomes a series system if $m_i = l_i$ for all $i = 1,2, \ldots, k_n$.

The reliability function of the two-state "m_i out of l_i"-series system is given either by

$$\overline{\boldsymbol{R}^{(m_1,m_2,...,m_{k_n})}_{k_n,l_1,l_2,...,l_{k_n}}}(t)$$

$$= \prod_{i=1}^{k_n}[1 - \sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ r_1+r_2+...+r_{l_i}\le m_i-1}}^{1} [\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}]$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\overline{\boldsymbol{R}}^{(\overline{m}_1,\overline{m}_2,...,\overline{m}_{k_n})}_{k_n,l_1,l_2,...,l_{k_n}}}(t)$$

$$= \prod_{i=1}^{k_n}[\sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ r_1+r_2+...+r_{l_i}\le \overline{m}_i}}^{1} [1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{r_i}[\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}]$$

for $t \in (-\infty,\infty)$, where $\overline{m}_i = l_i - m_i, \ i = 1,2,...,k_n$.

Definition 5. The two-state "m_i out of l_i"-series system is called homogeneous if its component lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t), \ t \in (-\infty,\infty), \ i = 1,2,...,k_n, \ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t), \ t \in (-\infty,\infty).$$

Definition 6. The "m_i out of l_i"-series system is called regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n$$

and

$$m_1 = m_2 = \ldots = m_{k_n} = m, \text{ where } l_n, m \in N, \quad m \le l_n.$$

The reliability function of the two-state homogeneous and regular „m out of l_n"-series system is given either by

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t) = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [R(t)]^i [1 - R(t)]^{l_n - i}]^{k_n}$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}}(t) = [\sum_{i=0}^{\overline{m}} \binom{l_n}{i} [1 - R(t)]^i [R(t)]^{l_n - i}]^{k_n}$$

for $t \in (-\infty,\infty)$, $\overline{m} = l_n - m$ where k_n is the number of "m out of l_n" subsystems linked in series and l_n is the number of components in the "m out of l_n" subsystems.

Corollary 2. If the components of the two-state homogeneous and regular "m out of l_n"-series system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0, \ \alpha > 0, \ \beta > 0,$$

then its reliability function is given either by

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t)$$

$$= [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} \exp[-i\beta t^{\alpha}][1 - \exp[-\beta t^{\alpha}]]^{l_n - i}]^{k_n} \qquad (4)$$

for $t \ge 0$ or by

$$\overline{\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}}(t) = [\sum_{i=0}^{\overline{m}} \binom{l_n}{i} [1 - \exp[-\beta t^{\alpha}]]^i \exp[-(l_n - i)\beta t^{\alpha}]]^{k_n}$$

for $t \ge 0$, $\overline{m} = l_n - m$.

Proposition 2. If components of the two-state homogeneous and regular "m out of l_n"-series system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0, \ \alpha > 0, \ \beta > 0,$$

and

$$\lim_{n\to\infty} k_n = k,\ k>0,\ \ 0<m\le k,\ \ \lim_{n\to\infty} l_n = \infty,$$

$$a_n = \frac{b_n}{\alpha \log n},\ \ b_n = [\frac{\log n}{\beta}]^{\frac{1}{\alpha}}, \tag{5}$$

then

$$[\overline{\Re}_3^{(0)}(t)]^k = [1-\exp[-\exp[-t]]\sum_{i=0}^{m-1}\frac{\exp[-it]}{i!}]^k$$

for $t\in(-\infty,\infty)$, is its limit reliability function, i.e.

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t) \cong [\overline{\Re}_3^{(0)}(\frac{t-b_n}{a_n})]^k$$

$$=[1-\exp[-\exp[-\frac{t-b_n}{a_n}]]\sum_{i=0}^{m-1}\frac{\exp[-i\frac{t-b_n}{a_n}]}{i!}]^k \tag{6}$$

for $t\in(-\infty,\infty)$, where a_n and b_n are defined by (5).

Example 2. Let us consider the ship-rope transportation system (elevator). The elevator is used to dock and undock ships coming in to shipyards for repairs. The elevator is composed of a steel platform-carriage placed in its syncline (hutch). The platform is moved vertically with 10 rope hoisting winches fed by separate electric motors. During ship docking the platform, with the ship settled in special supporting carriages on the platform, is raised to the wharf level (upper position). During undocking, the operation is reversed. While the ship is moving into or out of the syncline and while stopped in the upper position the platform is held on hooks and the loads in the ropes are relieved.

In our further analysis we will discuss the reliability of the rope system only. The system under consideration is in order if all its ropes do not fail. Thus we may assume that it is a series system composed of 10 components (ropes). Each of the ropes is composed of 22 strands. Thus, considering the strands as basic components of the system and assuming that each of the ropes is not failed if at least $m=5$ out of its strands are not failed, according to *Definitions 5-6*, we conclude that the rope elevator is the two-state homogeneous and regular „5 out of 22"-series system. It is composed of k_n = 10 series-linked "5 out of 22" subsystems (ropes) with l_n = 22 components (strands). Assuming additionally that strands have Weibull reliability functions with parameters $\alpha=2,\ \beta=0.05,$ i.e.

$$R(t)=\exp[-0.05t^2] \text{ for } t\ge 0,$$

from (4), we conclude that the elevator reliability function is given by

$$\overline{\boldsymbol{R}_{10,22}^{(5)}}(t) =$$

$$[1-\sum_{i=0}^{4}(\tbinom{22}{i})\exp[-i0.05t^2][1-\exp[-0.05t^2]]^{22-i}]^{10}$$

for $t\ge 0$.

Next, applying *Proposition 2* with

$$a_n = \frac{7.8626}{2\log 22}\cong 1.2718,\ \ b_n = [\frac{\log 22}{0.05}]^{\frac{1}{2}}\cong 7.8626,$$

and (6) we get the following approximate formula for the elevator reliability function

$$\overline{\boldsymbol{R}_{10,22}^{(5)}}(t) \cong\ [\overline{\Re}_3^{(0)}(0.7863t-6.1821)]^{10}$$

$$=[1-\exp[-\exp[-0.7863t+6.1821]]$$

$$\cdot\sum_{i=0}^{4}\frac{\exp[-0.7863it+6.1821i]}{i!}]^{10},\ t\in(-\infty,\infty).$$

3. Asymptotic approach to systems reliability improvement

We consider the homogeneous series system illustrated in *Figure 1*.

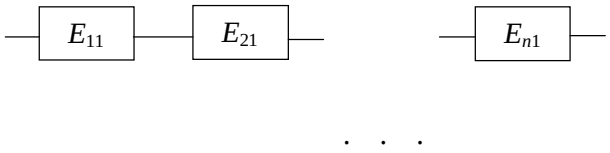


Figure 1. The scheme of a series system

It is composed of n components $E_{i1},\ i=1,2,...,n,$ having lifetimes $T_{i1},\ i=1,2,...,n,$ and exponential reliability functions

$$R(t)=\exp[-\lambda t] \text{ for } t\ge 0,\ \lambda>0.$$

Its lifetime and its reliability function respectively are given by

$$T^{(0)} = \min_{1\le i\le n}\{T_{i1}\},$$

$$\boldsymbol{R}_n(t)=[R(t)]^n=\exp[-\lambda nt],\ \ t\ge 0.$$

In order to improve of the reliability of this series system the following exemplary methods can be used:

– replacing the system components by the improved components with reduced failure rates by a factor ρ, $0<\rho<1$,

– a warm duplication (a single reservation) of system

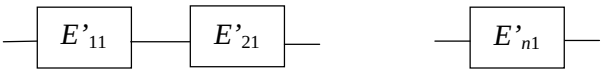
components,
– a cold duplication of system components,
– a mixed duplication of system components,
– a hot system duplication,
– a cold system duplication.

It is supposed here that the reserve components are identical to the basic ones.

The results of these methods of system reliability improvement are briefly presented below, giving the system schemes, lifetimes and reliability functions.

. . .

Case 1. Replacing the system components by the improved components E'_{i1} $i=1,2,...,n$, with reduced failure rates by a factor ρ, $0 < \rho < 1$, having lifetimes T'_{i1}, $i=1,2,...,n$, and exponential reliability functions



$$R(\rho t)=\exp[-\rho\lambda t] \text{ for } t\geq 0,\ \lambda>0.$$

Figure 2. The scheme of a series system with improved components

$$T^{(1)}=\min_{1\leq i\leq n}\{T'_{i1}\},$$

$$\boldsymbol{R}_n^{(1)}(t)=[R(\rho t)]^n=\exp[-\rho\lambda nt],\ t\geq 0.$$

Case 2. A hot reservation of the system components

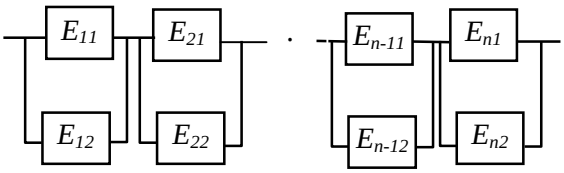


Figure 3. The scheme of a series system with components having hot reservation

$$T^{(2)}=\min_{1\leq i\leq n}\{\max_{1\leq j\leq 2}\{T_{ij}\}\},$$

$$\boldsymbol{R}_n^{(2)}(t)=[1-[F(t)]^2]^n=[1-[1-\exp[-\lambda t]]^2]^n,\ t\geq 0.$$

Case 3. A cold reservation of the system components

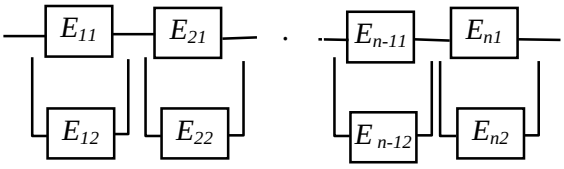


Figure 4. The scheme of a series system with components having cold reservation

$$T^{(3)}=\min_{1\leq i\leq n}\{\sum_{j=1}^{2}T_{ij}\},$$

$$\boldsymbol{R}_n^{(3)}(t)=[1-[F(t)]*[F(t)]]^n=[1+\lambda t]^n\exp[-n\lambda t],\ t\geq 0.$$

Case 4. A mixed reservation of the system components

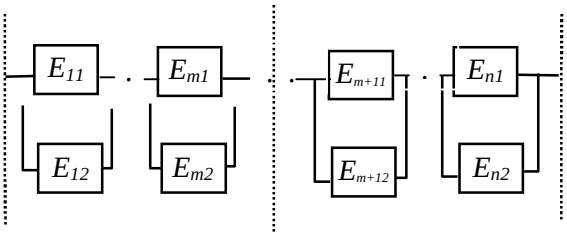


Figure 5. The scheme of a series system with components having mixed reservation

$$T^{(4)}=\min\{\min_{1\leq i\leq m}\{\sum_{j=1}^{2}T_{ij}\},\ \min_{m+1\leq i\leq n}\{\max_{1\leq j\leq 2}\{T_{ij}\}\}\},$$

$$\boldsymbol{R}_n^{(4)}(t)=[1-[F(t)]*[F(t)]]^m\,[1-R^2(t)]^{n-m}$$

$$=[1+\lambda t]^m\exp[-\lambda nt][2-\exp[-\lambda t]]^{n-m},\ t\geq 0.$$

Case 5. A hot system reservation

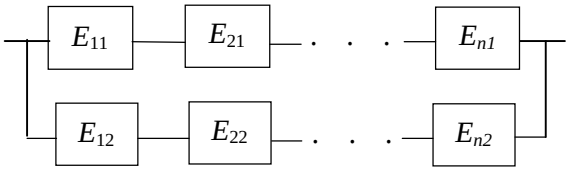


Figure 6. The scheme of a series system with hot reservation

$$T^{(5)}=\max_{1\leq j\leq 2}\{\min_{1\leq j\leq n}\{T_{ij}\}\},$$

$$\boldsymbol{R}_n^{(5)}(t)=1-[1-[R(t)]^n]^2=1-[1-\exp[-n\lambda t]]^2,\ t\geq 0.$$

Case 6. A cold system reservation

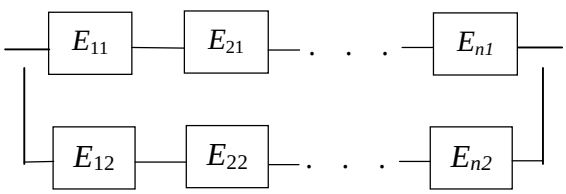


Figure 7. The scheme of a series system with cold reservation

$$T^{(6)} = \sum_{j=1}^{2} \min_{1\le i\le n}\{T_{ij}\},$$

$$\boldsymbol{R}_n^{(6)}(t) = 1-[1-[R(t)]^n]*[1-[R(t)]^n]$$

$$= [1+n\lambda t]\exp[-n\lambda t], \quad t \ge 0.$$

The difficulty arises when selecting the right method of improvement of reliability for a large system. This problem may be simplified and approximately solved by the application of the asymptotic approach. Comparisons of the limit reliability functions of the systems with different types of reserve and such systems with improved components allow us to find the value of the components' decreasing failure rate factor ρ, which gives rise to an equivalent effect on the system reliability improvement. Similar results are obtained under comparison of the system lifetime mean values. As an example we will present the asymptotic approach to the above methods of improving reliability for homogeneous two-state series systems.

Proposition 3.Case 1. If

$$a_n = 1/\lambda\rho n,\ b_n = 0,$$

then

$$\Re^{(1)}(t) = \exp[-t] \text{ for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with reduced failure rates of its components, i.e.

$$\boldsymbol{R}_n^{(1)}(t) = \Re^{(1)}(\lambda\rho nt) = \exp[-\lambda\rho nt] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(1)} = E[T^{(1)}] = \frac{1}{\lambda\rho n}.$$

Case 2. If

$$a_n = 1/\lambda\sqrt{n},\ b_n = 0,$$

then

$$\Re^{(2)}(t) = \exp[-t^2] \text{ for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with hot reservation of its components, i.e.

$$\boldsymbol{R}_n^{(2)}(t) \cong \Re^{(2)}(\lambda\sqrt{nt}) = \exp[-\lambda^2 nt^2] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(2)} = E[T^{(2)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda\sqrt{n}}.$$

Case 3. If

$$a_n = \sqrt{2}/\lambda\sqrt{n},\ b_n = 0,$$

then

$$\Re^{(3)}(t) = \exp[-t^2] \text{ for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with cold reservation of its components, i.e.

$$\boldsymbol{R}_n^{(3)}(t) \cong \Re^{(3)}(\lambda\sqrt{\frac{nt}{2}}) = \exp[-\frac{1}{2}\lambda^2 nt^2] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(3)} = E[T^{(3)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda}\sqrt{\frac{2}{n}}.$$

Case 4. If

$$a_n = \frac{1}{\lambda}\sqrt{\frac{2}{2n-m}},\ b_n = 0,$$

then

$$\Re^{(4)}(t) = \exp[-t^2] \text{ for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with mixed reservation of its components, i.e.

$$\boldsymbol{R}_n^{(4)}(t) \cong \Re^{(4)}(\lambda\sqrt{\frac{2n-m}{2}})$$

$$= \exp[-\frac{2n-m}{2}\lambda^2 t^2] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(4)} = E[T^{(4)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda}\sqrt{\frac{2}{2n-m}}.$$

Case 5. If

$$a_n = \frac{1}{\lambda n},\ b_n = 0,$$

then

$$\Re^{(5)}(t) = 1-[1-\exp[-t]]^2 \text{ for } t \geq 0,$$

is the limit reliability function of the homogeneous exponential series system with hot reservation, i.e.

$$\boldsymbol{R}_n^{(5)}(t) = \Re^{(5)}(\lambda nt) = 1-[1-\exp[-\lambda nt]]^2 \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(5)} = E[T^{(5)}] = \frac{3}{2\lambda n}.$$

Case 6. If

$$a_n = \frac{1}{\lambda n},\ b_n = 0,$$

then

$$\Re^{(6)}(t) = [1+t]\exp[-t] \text{ for } t \geq 0,$$

is the limit reliability function of the homogeneous exponential series system with cold reservation, i.e.

$$\boldsymbol{R}_n^{(6)}(t) = \Re^{(6)}(\lambda nt) = [1+\lambda nt]\exp[-\lambda nt] \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(6)} = E[T^{(6)}] = \frac{2}{\lambda n}.$$

Corollary 3. Comparison of the system reliability functions

$$\Re^{(i)}(t) = \Re^{(1)}(t),\ i = 2,3,\ldots,6,$$

results respectively in the following values of the factor ρ :

$$\rho = \rho(t) = \lambda t \text{ for } i = 2,$$

$$\rho = \rho(t) = \frac{1}{2}\lambda t \text{ for } i = 3,$$

$$\rho = \rho(t) = \frac{2n-m}{2n}\lambda t \text{ for } i = 4,$$

$$\rho = \rho(t) = 1-\log[2-\exp[-\lambda nt]] \text{ for } i = 5,$$

$$\rho = \rho(t) = 1-\frac{1}{\lambda nt}\log[1+\lambda nt] \text{ for } i = 6,$$

while comparison of the system lifetimes

$$\boldsymbol{T}^{(i)}(t) = \boldsymbol{T}^{(1)}(t),\ i = 2,3,\ldots,6$$

results respectively in the following values of the factor ρ :

$$\rho = \frac{1}{\Gamma(\frac{3}{2})\sqrt{n}} \text{ for } i = 2,$$

$$\rho = \frac{1}{\Gamma(\frac{3}{2})\sqrt{2n}} \text{ for } i = 3,$$

$$\rho = \frac{1}{\Gamma(\frac{3}{2})n\sqrt{\frac{2}{2n-m}}} \text{ for } i = 4,$$

$$\rho = \frac{2}{3} \text{ for } i = 5,$$

$$\rho = \frac{1}{2} \text{ for } i = 6.$$

Example 3. We consider a simplified bus service company composed of 81 communication lines. We suppose that there is one bus operating on each communication line and that all buses are of the same type with the exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0,\ \lambda > 0.$$

Additionally we assume that this communication system is working when all its buses are not failed, i.e. it is failed when any of the buses are failed. The failure rate of the buses evaluated on statistical data coming from the operational process of bus service company transportation system is assumed to be equal to 0.0049 h^{-1}.

Under these assumptions the considered transportation system is a homogeneous series system made up of components with a reliability function

$$R(t) = \exp[-0.0049t] \text{ for } t \geq 0.$$

Here we will use four sensible methods from those considered for system reliability improvement.

Namely, we apply the four previously considered cases.

Case 1. Replacing the system components by the improved components with reduced failure rates by a factor ρ.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot 81\rho} = \frac{1}{0.397\rho}, \; b_{81} = 0,$$

we conclude that

$$\Re^{(1)}(t) = \exp[-t] \quad \text{for } t \geq 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(1)}(t) = \Re^{(1)}(0.397\rho t) = \exp[-0.397\rho t] \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(1)} = E[T^{(1)}] = \frac{1}{0.397\rho} \text{ h.}$$

Case 2. Improving the reliability of the system by a single hot reservation of its components.

This means that each of 81 communication lines has at its disposal two identical buses it can use and its task is performed if at least one of the buses is not failed.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot \sqrt{81}} = \frac{1}{0.0441}, \; b_{81} = 0,$$

we conclude that

$$\Re^{(2)}(t) = \exp[-t^2], \; t \geq 0.$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_{81}^{(2)}(t) \cong \Re^{(2)}(0.0441t) \cong \exp[-0.0019t^2], \; t \geq 0,$$

and

$$\boldsymbol{T}^{(2)} = E[T^{(2)}] \cong \Gamma(\frac{3}{2}) \frac{1}{0.0049\sqrt{81}} \cong 20.10 \; h.$$

Case 4. Improving the reliability of the system by a single mixed reservation of its components.

This means that each of 81 communication lines has at its disposal two identical buses. There are $m = 50$ communication lines with small traffic which are using one bus permanently and after its failure it is replaced by the second bus (a cold reservation) and $n - m = 81 - 50 = 31$ communication lines with large traffic which are using two buses permanently (a hot reservation).

Applying *Proposition 3* with normalising constants

$$a_n = \frac{1}{0.0049}\sqrt{\frac{2}{112}} = \frac{1}{0.0367}, \; b_n = 0,$$

we conclude that

$$\Re^{(4)}(t) = \exp[-t^2] \quad \text{for } t \geq 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(4)}(t) \cong \Re^{(4)}(0.0367t) = \exp[-0.00135t^2] \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(4)} = E[T^{(4)}] \cong \Gamma(\frac{3}{2}) \frac{1}{0.0049}\sqrt{\frac{2}{112}} \cong 24.15 \; h.$$

Case 5. Improving the reliability of the system by a single hot reservation.

This means that the transportation system is composed of two independent companies, each of them operating on the same 81 communication lines and having at their disposal one identical bus for use on each line.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot 81} = \frac{1}{0.397}, \; b_{81} = 0,$$

we conclude that

$$\Re^{(5)}(t) = 1 - [1 - \exp[-t]]^2 \quad \text{for } t \geq 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(5)}(t) = \Re^{(5)}(0.397t)$$

$$= 1 - [1 - \exp[-0.397t]]^2 \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(5)} = E[T^{(5)}] = \frac{3}{2 \cdot 0.0049 \cdot 81} \cong 3.78 \text{ h.}$$

Comparing the system reliability functions for considered cases of improvement, from *Corollary 3*, results in the following values of the factor ρ:

$$\rho = \rho(t) = 0.0049t \quad \text{for } i = 2,$$

$$\rho = 0.0340t \quad \text{for } i = 4,$$

$$\rho = \rho(t) = 1 - \log[2 - \exp[-0.397t]] \quad \text{for } i = 5,$$

while comparison of the system lifetimes results respectively in:

$$\rho = 0.1254 \quad \text{for } i = 2,$$

$$\rho = 0.1043 \quad \text{for } i = 4,$$

$$\rho = 0.6667 \quad \text{for } i = 5.$$

Methods of system reliability improvement presented here supply practitioners with simple mathematical tools, which can be used in everyday practice. The methods may be useful not only in the operation processes of real technical objects but also in designing new operation processes and especially in optimising these processes. Only the case of series systems made up of components having exponential reliability functions with single reservations of their components and subsystems is considered. It seems to be possible to extend these results to systems that have more complicated reliability structures, and made up of components with different from the exponential reliability functions.

4. Reliability of large systems in their operation processes

This section proposes an approach to the solution of the practically very important problem of linking systems' reliability and their operation processes. To connect the interactions between the systems' operation processes and their reliability structures that are changing in time a semi-markov model ([1]) of the system operation processes is applied. This approach gives a tool that is practically important and not difficult for everyday use for evaluating reliability of systems with changing reliability structures during their operation processes. Application of the proposed methods is illustrated here in the reliability evaluation of the port grain transportation system.

We assume that the system during its operation process is taking different operation states. We denote by $Z(t)$, $t \in <0, \infty>$, the system operation process that may assume v different operation states from the set

$$Z = \{z_1, z_2, \ldots, z_v\}.$$

In practice a convenient assumption is that $Z(t)$ is a semi-markov process ([1]) with its conditional sojourn times θ_{bl} at the operation state z_b when its next operation state is z_l, $b, l = 1,2,...,v, \ b \neq l$. In this case this process may be described by:

- the vector of probabilities of the initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of its transitions between the states $[p_{bl}]_{vxv}$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the sojourn times θ_{bl}, $b \neq l$, where

$$H_{bl}(t) = P(\theta_{bl} < t) \text{ for } b, l = 1,2,...,v, \ b \neq l,$$

and

$$H_{bb}(t) = 0 \text{ for } b = 1,2,...,v.$$

Under these assumptions, the lifetime θ_{bl} mean values are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t), \ b, l = 1,2,...,v, \ b \neq l. \quad (7)$$

The unconditional distribution functions of the sojourn times θ_b of the process $Z(t)$ at the states z_b, $b = 1,2,...,v$, are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \ b = 1,2,...,v.$$

The mean values $E[\theta_b]$ of the unconditional sojourn times θ_b are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}, \ b = 1,2,...,v, \quad (8)$$

where M_{bl} are defined by (7).

Limit values of the transient probabilities at the states

$$p_b(t) = P(Z(t) = z_b), \ t \in <0, \infty), \ b = 1,2,...,v,$$

are given by ([1])

$$p_b = \lim_{t \to \infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \ b = 1,2,...,v, \quad (9)$$

where the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases}$$

We consider a series-parallel system and we assume that the changes of its operation process $Z(t)$ states

have an influence on the system components E_{ij} reliability and on the system reliability structure as well. Thus, we denote ([13]) the conditional reliability function of the system component E_{ij} while the system is at the operational state z_b, $b=1,2,...,v$, by

$$[R^{(i,j)}(t)]^{(b)} = P(T_{ij}^{(b)} \geq t / Z(t) = z_b),$$

for $t \in <0,\infty)$, $b=1,2,...,\nu$, and the conditional reliability function of the non-homogeneous regular series-parallel system while the system is at the operational state z_b, $b=1,2,...,\nu$, by

$$[\boldsymbol{R}_{k_n,l_n}(t)]^{(b)} = P(T^{(b)} \geq t / Z(t) = z_b)$$

$$= 1 - \prod_{i=1}^{a}[1 - [[R^{(i)}(t)]^{(b)}]^{l_n}]^{q_i k_n} \qquad (10)$$

for $t \in <0,\infty)$ and

$$[R^{(i)}(t)]^{(b)} = \prod_{j=1}^{e_i}[[R^{(i,j)}(t)]^{(b)}]^{p_{ij}},\ i = 1,2,...,a. \qquad (11)$$

The reliability function $[R^{(i,j)}(t)]^{(b)}$ is the conditional probability that the component E_{ij} lifetime $T_{ij}^{(b)}$ in the is not less than t, while the process $Z(t)$ is at the operation state z_b. Similarly, the reliability function $[\boldsymbol{R}_{k_n,l_n}(t)]^{(b)}$ is the conditional probability that the series-parallel system lifetime $T^{(b)}$ is not less than t, while the process $Z(t)$ is at the operation state z_b. In the case when the system operation time is large enough, the unconditional reliability function of the series-parallel system is given by

$$\boldsymbol{R}_{k_n,l_n}(t) = P(T > t) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{R}_{k_n,l_n}(t)]^{(b)} \qquad (12)$$

for $t \geq 0$ and T is the unconditional lifetime of the series-parallel system.

The mean values and variances of the series-parallel system lifetimes are

$$M \cong \sum_{b=1}^{\nu} p_b M_b, \qquad (13)$$

where

$$M_b = \int_0^{\infty} [\boldsymbol{R}_{k_n,l_n}(t)]^{(b)}\, dt, \qquad (14)$$

and

$$D[T^{(b)}] = 2\int_0^{\infty} t[\boldsymbol{R}_{k_n l_n}(t)]^{(b)}\, dt - [M_b]^2, \qquad (15)$$

for $b = 1,2,...,\nu$.

Example 5. We analyse the reliability of one of the subsystems of the port grain elevator. The considered system is composed of four two-state non-homogeneous series-parallel transportation subsystems assigned to handle and clearing of exported and imported grain. One of the basic elevator functions is loading railway trucks with grain.

In loading the railway trucks with grain the following elevator transportation subsystems take part: S_1 – horizontal conveyors of the first type, S_2 – vertical bucket elevators, S_3 – horizontal conveyors of the second type, S_4 – worm conveyors.

We will analyze the reliability of the subsystem S_4 only.

Taking into account experts opinion in the operation process, $Z(t)$, $t \geq 0$ of the considered transportation subsystem we distinguish the following as its three operation states:

an operation state z_1 – the system operation with the largest efficiency when all components of the subsystem S_4 are used,

an operation state z_2 – the system operation with less efficiency system when the first and second conveyors of subsystem S_4 are used,

an operation state z_3 – the system operation with least efficiency when the first conveyor of subsystem S_4 is used.

On the basis of data coming from experts, the probabilities of transitions between the subsystem S_4 operation states are given by

$$[p_{bl}] = \begin{bmatrix} 0 & 0.357 & 0.643 \\ 0.8 & 0 & 0.2 \\ 0.385 & 0.615 & 0 \end{bmatrix},$$

and their mean values, from (8), are

$$M_1 = E[\theta_1] = 0.357 \cdot 0.36 + 0.643 \cdot 0.2 \cong 0.257,$$

$$M_2 = E[\theta_2] = 0.8 \cdot 0.05 + 0.2 \cdot 0.2 \cong 0.08,$$

$$M_3 = E[\theta_3] = 0.385 \cdot 0.08 + 0.615 \cdot 0.05 \cong 0.062.$$

Since from the system of equations

$$\begin{cases} [\pi_1,\pi_2,\pi_3] = [\pi_1,\pi_2,\pi_3] \begin{bmatrix} 0 & 0.357 & 0.643 \\ 0.8 & 0 & 0.2 \\ 0.385 & 0.615 & 0 \end{bmatrix}, \\ \pi_1+\pi_2+\pi_3 = 1 \end{cases}$$

we get

$$\pi_1 = 0.374,\ \pi_2 = 0.321,\ \pi_3 = 0.305,$$

then the limit values of the transient probabilities $p_b(t)$ at the operation states z_b, according to (9), are given by

$$p_1 = 0.684,\ \ p_2 = 0.183,\ \ p_3 = 0.133. \qquad (16)$$

The subsystem S_4 consists of three chain conveyors. Two of these are composed of 162 components and the remaining one is composed of 242 components. Thus it is a non-regular series-parallel system. In order to make it a regular system we conventionally complete two first conveyors having 162 components with 80 components that do not fail. After this supplement subsystem S_4 consists of $k_n = 3$ conveyors, each composed of $l_n = 242$ components. In two of them there are:

- two driving wheels with reliability functions

$$R^{(1,1)}(t) = \exp[-0.0798t],$$

- 160 links with reliability functions

$$R^{(1,2)}(t) = \exp[-0.124t],$$

- 80 components with "reliability functions"
$R^{(1,3)}(t) = \exp[-\lambda_1(1)\,t]$, where $\lambda_1(1) = 0$.

The third conveyer is composed of:

- two driving wheels with reliability functions

$$R^{(2,1)}(t) = \exp[-0.167t]$$

- 240 links with reliability functions

$$R^{(2,2)}(t) = \exp[-0.208t].$$

At the operation state z_1 the subsystem S_4 becomes a non-homogeneous regular series-parallel system with parameters

$k_n = 3$, $l_n = 242$, $a = 2$, $q_1 = 2/3$, $q_2 = 1/3$,

$e_1 = 3$, $e_2 = 2$,

$p_{11} = 2/242,\ p_{12} = 160/242,\ p_{13} = 80/242,$

$p_{21} = 2/242,\ p_{22} = 240/242,$

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{3,242}(t)]^{(1)}$$

$$= 1 - [1 - \exp[-19.9892t]]^2[1 - \exp[-50.2628t]]$$

$$= 2\exp[-19.9892t] - 2\exp[-70.252t]$$

$$+ \exp[-50.2628t] + \exp[-90.2412t]$$

$$- \exp[-39.9784t] \text{ for } t \ge 0. \qquad (17)$$

According to (14)-(15), the subsystem lifetime mean value and the standard deviation are

$$M_1 \cong 0.078,\ \ \sigma_1 \cong 0.054. \qquad (18)$$

At the operation state z_2 the subsystem S_4 becomes a non-homogeneous regular series-parallel system with parameters

$k_n = 2$, $l_n = 162$, $a = 1$, $q_1 = 1$, $e_1 = 2$,

$p_{11} = 2/162,\ p_{12} = 160/162.$

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{2,162}(t)]^{(2)} = 1 - [1 - \exp[-20.007t]]^2$$

$$= 2\exp[-20.007t] - \exp[-40.014t] \text{ for } t \ge 0. \qquad (19)$$

According to (14)-(15), the subsystem lifetime mean value and the standard deviation are

$$M_2 \cong 0.075,\ \ \sigma_2 \cong 0.056. \qquad (20)$$

At the operation state z_3 the subsystem S_4 becomes a non-homogeneous regular series-parallel (series) system with parameters

$k_n = 1$, $l_n = 162$, $q_1 = 1$, $e_1 = 3$,

$p_{11} = 2/162,\ p_{12} = 160/162,$

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{1,162}(t)]^{(3)} = \exp[-19.999t] \text{ for } t \geq 0. \quad (21)$$

According to (14)-(15), the system lifetime mean value and the standard deviation are

$$M_3 \cong 0.050, \ \sigma_3 \cong 0.050. \quad (22)$$

Finally, considering (12), the subsystem S_4 unconditional reliability is given by

$$\boldsymbol{R}(t) \cong 0.684 \cdot [\boldsymbol{R}_{3,242}(t)]^{(1)} + 0.183 \cdot [\boldsymbol{R}_{2,162}(t)]^{(2)} + 0.133 \cdot [\boldsymbol{R}_{1,162}(t)]^{(3)}, \quad (23)$$

where $[\boldsymbol{R}_{3,242}(t)]^{(1)}$, $[\boldsymbol{R}_{2,162}(t)]^{(2)}$, $[\overline{\boldsymbol{R}}_{1,162}(t)]^{(3)}$, are given by (17), (19), (21).
Hence, applying (16) and (18), (20), (22), we get the mean values and standard deviations of the subsystem unconditional lifetimes given by

$$M \cong 0.684 \cdot 0.078 + 0.183 \cdot 0.075 + 0.133 \cdot 0.050 \cong 0.074, \quad (24)$$

$$\sigma(1) \cong 0.054. \quad (25)$$

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Transformed conditional probabilities in the analysis of stochastic sequences

Keywords

safety margin, stochastic sequence, conditional probability, system safety

Abstract

The need to use unsophisticated probability-based approaches and models in the structural safety analysis of the structures subjected to annual extreme service, snow and wind actions is discussed. Statistical parameters of single and coincident two extreme variable actions and their effects are analysed. Monotone and decreasing random sequences of safety margins of not deteriorating and deteriorating members are treated, respectively, as ordinary and generalized geometric distributions representing highly-correlated series systems. An analytical analysis of the failure or survival probabilities of members and their systems is based on the concepts of transformed conditional probabilities of safety margin sequences whose statistically dependent cuts coincide with extreme loading situations of structures. The probability-based design of members exposed to coincident extreme actions is illustrated by a numerical example.

1. Introduction

The stochastic systems and their subsystems consist of some particular members representing the only possible failure mode. To particular members belong cross and oblique sections of tension, compression, flexural and torsional structures. The structural members (beams, slabs, columns, walls) of buildings consist of two or three design particular members and may be treated as auto systems representing multicriteria failure modes. An overloading of members during severe service and climate actions may provoke a failure of structures. Therefore, the requirements of design codes should be satisfied at all sections along structural members.

Structural failures and collapses in buildings and construction works can be caused not only by irresponsibility and gross human errors of designers, builders or erectors but also by some conditionalities of recommendations and directions presented in design codes and standards. A possibility to ensure objectively the safety degree of structures subjected to extreme service loads, wind gust and snow pressures or wave surfs is hardly translated into reality using the traditional deterministic design methods of partial safety factors in Europe or load and resistance factors in the USA.

It is understandable that probabilistic design approaches are inevitable for the calibration of partial factors. However, it should be more expedient to analyse the structural safety of particular members and their systems by probability-based methods. Regardless of efforts to improve and modify deterministic design approaches, it is inconceivable to fix a real reliability index of structures a failure domain of which changes with time. The time-dependent safety assessment and prediction of deteriorating members and systems using unsophisticated methods is a significant concern of researchers.

Despite of fairly developed up-to-date concepts of reliability, hazard and risk theories, including the general principles on reliability for structures [6], [7], [15], it is difficult to apply probability-based approaches in structural safety analysis. These approaches may be acceptable to designers and building engineers only under the indispensable condition that the safety performance of members and their systems may be considered in a simple and easy perceptible manner. In other words, probabilistic methods may be implanted into structural design practice only using unsophisticated mathematical models helping us to assess all uncertainties due to the features of resistances and action effects of structures.

This paper deals with probability-based safety analysis of deteriorating and not deteriorating members and their systems under extreme gravity and lateral (horizontal) actions using unsophisticated but fairly exact design models.

2. Time dependent safety margin

According to probability-based approaches (design level III), the time-dependent safety margin as the performance of deteriorating particular members may be presented as follows:

$$Z(t) = g[\mathbf{и}, \mathbf{X}(t)] = \theta_R R(t) - \theta_g S_g - \theta_{q_1} S_{q_1}(t) -$$

$$- \theta_{q_2} S_{q_2}(t) - \theta_w S_w(t) \ , \qquad (1)$$

where $\mathbf{и}$ is the vector of additional variables characterizing uncertainties of models which give the values of resistance R, permanent S_g, sustained S_{q_1} and extraordinary S_{q_2} service and extreme wind S_w action effects of members (Figure 1, a). This vector may represent also the uncertainties of probability distributions of basic variables.

According to Rosowsky and Ellingwood [11], the annual extreme sum of sustained and extraordinary occupancy live action effects $S_q(t) = S_{q_1}(t) + S_{q_2}(t)$ can be modelled as an intermittent process and described by a Type 1 (Gumbel) distribution with the coefficient of variation $дS_q = 0.58$, characteristic S_{qk} and mean $S_{qm} = 0.47 S_{qk}$ values. Latter on Ellingwood and Tekie [4] recommended modelling extreme values of this sum during a 50 years period by a Type 1 distribution with the coefficient of variation $дS_q = 0.25$ and mean value $S_{qm} = S_{qk}$.

It is proposed to model the annual extreme climate (wind and snow) action effects by Gumbel distribution law with the mean values equal to $S_{wm} = S_{wk}/(1 + k_{0.98} дS_w)$ and $S_{sm} = S_{sk}/(1 + k_{0.98} дS_s)$ [3, 6, 7, 13, 15]. According to meteorological data, the strong wind conditions are characterized by a small wind extreme velocity variation, i.e. $дv \approx 0.1$. On the contrary, a large variation is characteristic of strong snow loading. Therefore, the coefficients of variation of wind and snow loads depending on the feature of a geographical area are equal to $дw = 0.2 - 0.4$ and $дs = 0.3 - 0.7$.

Probability distributions of material properties are close to a Gaussian distribution [3], [6], [9], [12]. Therefore, a normal distribution or a log-normal distribution may be convenient in resistance analysis models [5], [6], [7]. The permanent action effect S_g can be described by a normal distribution law [4], [5], [6], [10], [12]. Thus, for the sake of design simplifications, it is expedient to present the expression (1) in the form:

$$Z(t) = R_c(t) - S(t), \qquad (2)$$

where the component process

$$R_c(t) = \theta_R R(t) - \theta_g S_g, \qquad (3)$$

may be considered as the conventional resistance of members which may be modelled by a normal distribution;

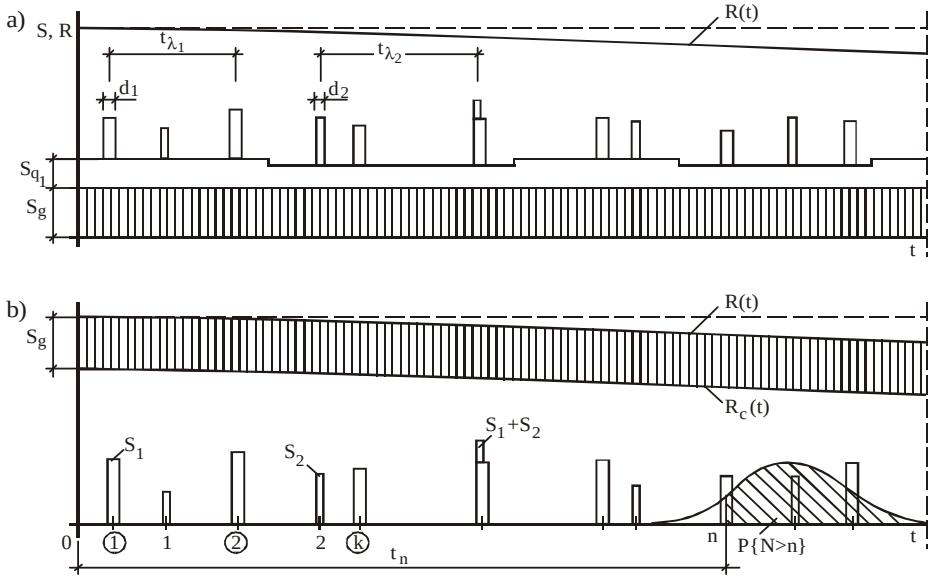

Figure 1. Real (a) and conventional (b) models for safety analysis of particular members (sections) of deteriorating structures

$$S(t) = \theta_q S_q(t) + [\theta_w S_w(t)], \qquad (4) \qquad S(t) = \theta_s S_s(t) + [\theta_w S_w(t)], \qquad (5)$$

are the joint processes of two annual extreme action effects when floor and roof structures, respectively, are under consideration. The components in square brackets belonging to the wind action effect are used in design analysis of wind-resistant members and systems. The action effect $S_s(t)$ in Equation (5) is caused by extreme snow loads.

3. Safety margin sequences with independent cuts

The data presented in Section 2 allow us to model extreme service and climate action effects as intermittent rectangular pulse renewal processes. These time-variant intermittent action effects belong to persistent design situations in spite of the short period of extreme events being much shorter than the design working life of structures. When variable action effects may be treated as rectangular pulse processes, the time-dependent safety margin (2) may be expressed as the finite rank random sequence and written as:

$$Z_k = R_{ck} - S_k,\ k = 1,\ 2,\ 3,\ ...,\ n-1,\ n. \tag{6}$$

There

$$R_{ck} = \theta_R R_k - \theta_g S_g, \tag{7}$$

$$S_k = \theta_q S_{qk} + \theta_w S_{wk} \text{ or } S_k = \theta_s S_{sk} + \theta_w S_{wk}, \tag{8}$$

are the components of this non-stationary sequence; $n = \lambda t_n$ is the number of sequence cuts as critical events (situations) during design working life t_n of members (*Figure 1, b*), where $\lambda = 1/t_\lambda$ is a mean renewal rate of these events per unit time when their return period is t_λ.

Usually the components R_{ck} and S_k are stochastically independent. The instantaneous survival probability of a member at *k*-th extreme situation (assuming that it was safe at the situations 1, 2, …, $k-1$) is:

$$\boldsymbol{P}_{sk} = \boldsymbol{P}\{R_{ck} > S_k\} = \int_0^\infty f_{R_{ck}}(x) F_{S_k}(x)\,dx, \tag{9}$$

where $f_{R_{ck}}(x)$ and $F_{S_k}(x)$ are the density and distribution functions of a conventional resistance R_{ck} by (7) and an extreme action effect S_k by (8). In this case, the instantaneous failure probability of members may be presented as:

$$\boldsymbol{P}_{fk} = (1 - \boldsymbol{P}_{sk}) \prod_{i=1}^{k-1} \boldsymbol{P}_{si}. \tag{10}$$

Thus, the random sequence of safety margins may be treated as a geometric distribution with ranked instantaneous survival probabilities of members $\boldsymbol{P}_{f1} < \boldsymbol{P}_{f2} < ... < \boldsymbol{P}_{fk} < ... \boldsymbol{P}_{f,n-1} < \boldsymbol{P}_{fn}$ calculated by Equation (10).

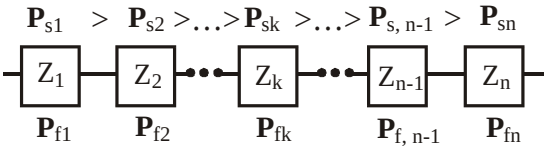

Figure 2. The scheme of series systems

Failure probabilities of structures should always be defined for some reference period t_n or as a number of extreme events n during this period. The scheme of series systems representing the safety margin sequences is given in *Figure 2*. When the cuts of rank random sequences are statistically independent, the cumulative distribution function and similarly a failure probability of members during their service life $[0, t_n]$ with n extreme situations may be presented as follows:

$$\boldsymbol{P}_f = F_N(n) = \boldsymbol{P}\{N \le n\} = \sum_{k=1}^{n} \left[(1 - \boldsymbol{P}_{sk}) \prod_{i=1}^{k-1} \boldsymbol{P}_{si} \right]$$

$$= 1 - \boldsymbol{P}_{s1} + (1 - \boldsymbol{P}_{s2}) \boldsymbol{P}_{s1} + ... + (1 - \boldsymbol{P}_{sk}) \prod_{i=1}^{k-1} \boldsymbol{P}_{si}$$

$$... + (1 - \boldsymbol{P}_{sn}) \prod_{i=1}^{n-1} \boldsymbol{P}_{si} = 1 - \prod_{k=1}^{n} \boldsymbol{P}_{sk}. \tag{11}$$

When the resistance $R(t)$ is a time-invariant function and treated as a stationary process, the instantaneous survival probability $\boldsymbol{P}_{sk}$ by (9) is characterized by the same value for all cuts of the monotone sequence. In this case, Equation (11) becomes a cumulative distribution function of an ordinary geometric distribution as follows:

$$\boldsymbol{P}_f = F_N(n) = \boldsymbol{P}\{N \le n\} = 1 - (1 - \boldsymbol{P}_{fk})^n. \tag{12}$$

The failure probability of members may be approximated by Equations (11) and (12) only for situations in which a variance of the action effect $\mathbf{y}^2 S$ is much larger than the value $\mathbf{y}^2 R_c$ for their conventional resistance by (7).

4. Safety margin sequences with dependent cuts

In design practice, only recurrent extreme action effects caused by extraordinary service and climate loads may be treated as stochastically independent variables. Usually, random sequence cuts of the safety margin (6) are dependent. The value of a coefficient of

autocorrelation ρ_{kl} of sequence cuts depends on uncertainties of material properties and dimensions of members. This coefficient may be defined as:

$$\rho_{kl}=\rho(Z_k,Z_l)=Cov(Z_k,Z_l)/(\mathbf{y}Z_k\times\mathbf{y}Z_l), \qquad (13)$$

where $Cov(Z_k,Z_l)$ and $\mathbf{y}Z_k,\mathbf{y}Z_l$ are an autocovariance and standard deviations of the random safety margins Z_k and Z_l.

The finite random sequence of member safety margins may be treated as a series stochastic system. The survival probability of highly correlated series systems consisting of two dependent elements can be expressed as follows:

$$\boldsymbol{P}\{Z_1>0 \text{ f } Z_2>0\}=\boldsymbol{P}_{s1}\times\boldsymbol{P}\{Z_2>0|Z_1>0\}$$

$$=\boldsymbol{P}_{s1}\times\boldsymbol{P}_{s2}+\rho^{a}(\boldsymbol{P}_{s2}-\boldsymbol{P}_1\times\boldsymbol{P}_2), \qquad (14)$$

where $a\approx 4.5/(1-0.98\rho_{12})$ is the bond index of survival probabilities of second-order series systems. The data calculated by (14) and computed by the complex numerical integration method presented by Ahammed and Melchers [1] are very close. Thus, a conditional probability $\boldsymbol{P}\{Z_2>0|Z_1>0\}$ may be transformed to a probability $\boldsymbol{P}_{s2}\left[1+\rho^{a}\left(\frac{1}{\boldsymbol{P}_{s1}}-1\right)\right]$.

Therefore, Equation (14) may be presented in the form:

$$\boldsymbol{P}\{Z_1>0 \text{ f } Z_2>0\}\approx\boldsymbol{P}_{s1}\times\boldsymbol{P}_{s2}$$

$$\times\left[1+\rho_{12}^{a}\left(\frac{1}{\boldsymbol{P}_{s1}}-1\right)\right] \qquad (15)$$

For not deteriorating structures, a member resistance is a time-invariant fixed random function the numerical values of which are random only at the beginning of a process. Therefore, the coefficient of correlation (13) of monotone sequence cuts may be expressed as:

$$\rho_{kl}=1/\left(1+\mathbf{y}^2S_k/\mathbf{y}^2R_c\right). \qquad (16)$$

When the monotone rank sequence of safety margins consists of n dependent elements, a failure probability of members is:

$$\boldsymbol{P}_f=\boldsymbol{P}\{N\le n\}=\boldsymbol{P}\left\{\mathop{\mathrm{r}}_{k=1}^{n}Z_k\le 0\right\}=1-\boldsymbol{P}\left\{\mathop{\mathrm{f}}_{k=1}^{n}Z_k>0\right\}$$

$$\approx 1-\boldsymbol{P}_{sk}^{n}\left[1+\rho_{kl}^{a}\left(\frac{1}{\boldsymbol{P}_{sk}}-1\right)\right]^{n-1}. \qquad (17)$$

When a ratio of variances $\mathbf{y}^2S_k/\mathbf{y}^2R_c>1$, the coefficient $\rho_{kl}^{a}\approx 0$ and the failure probability (17) becomes $\boldsymbol{P}_f=1-(1-\boldsymbol{P}_{fk})^{n}$ as it is expressed by Equation (12).

A long-term survival probability of not deteriorating members is:

$$\boldsymbol{P}_s=1-\boldsymbol{P}_f=\boldsymbol{P}_{sk}^{n}\left[1+\rho_{kl}^{a}\left(\frac{1}{\boldsymbol{P}_{sk}}-1\right)\right]^{n-1}. \qquad (18)$$

The decreasing rank sequence of safety margins of deteriorating members may be treated as a generalized geometric distribution. Similar to Equation (17), the failure probability of these members as series systems may be calculated by the formula:

$$\boldsymbol{P}_f=\boldsymbol{P}\{N\le n\}\approx 1-\prod_{k=1}^{n}\boldsymbol{P}_{sk}\left[1+\rho_{n\ldots1}^{a}\left(\frac{1}{\boldsymbol{P}_{s,n-1}}-1\right)\right]$$

$$\ldots\times\left[1+\rho_{k\ldots1}^{a}\left(\frac{1}{\boldsymbol{P}_{s,k-1}}-1\right)\right]$$

$$\ldots\times\left[1+\rho_{21}^{a}\left(\frac{1}{\boldsymbol{P}_{s1}}-1\right)\right], \qquad (19)$$

where the transformed rank coefficient of correlation is

$$\rho_{k\ldots1}=(\rho_{k,k-1}+\rho_{k,k-2}+\ldots+\rho_{k2}+\rho_{k1})/(k-1) \qquad (20)$$

The long-term survival probability of deteriorating members $\boldsymbol{P}_s=1-\boldsymbol{P}_f$, where the probability $\boldsymbol{P}_f$ is given in (19).

The presented method of transformed conditional probabilities may also be successfully used in the reliability analysis of random systems consisting of individual components and characterizing different failure modes of structures. In this case, it is expedient to base the structural safety analysis of systems on the ranked survival probabilities of their members as: $\boldsymbol{P}_{s1}>\boldsymbol{P}_{s2}>\ldots>\boldsymbol{P}_{sk}>\ldots>\boldsymbol{P}_{s,n-1}>\boldsymbol{P}_{sn}$ (*Figure 2*). A rank correlation matrix of systems is constructed taking into account this analysis rule.

4. The system of safety margin sequences

Due to the complexity of mathematical models, it is rather difficult to assess and predict a failure probability of structures subjected to two and more coincident recurrent and different by nature extraordinary actions. The methods based on the Markov-chain model and Turkstra's rule [14] may be quite unacceptable in a probabilistic analysis of not only deteriorating but also not deteriorating members and their systems. The Markov-chain model may be quite inaccurate for reliability analysis of members exposed to multiple combination of action-effect processes [12]. The Turkstra's rule may be assumed only in the case when the principal extreme load is strongly dominant [10].

Failure probabilities of members may be computed by modified numerical integration methods. It is suggested to use the theoretical expression of the cumulative distribution function of the maximum intensity of two load processes [10], the load overlap method [12] and the improved upper bounding techniques [13]. It leads to sufficiently accurate values but it is hard to realize these recommendations in engineering practice.

The need to simplify a reliability analysis of deteriorating structures is especially urgent. In any analysis case, it must be taken into account that a member failure caused by two statistically independent extreme action effects may occur not only in the case of their coincidence but also when the value of one out of two effects is extreme. Therefore, three finite random sequences of safety margins should be considered:

$$M_{1k} = R_{ck} - S_{1k}, k = 1,\ 2,\ ...,\ n_1, \tag{21}$$

$$M_{2k} = R_{ck} - S_{2k},\ k = 1,\ 2,\ ...,\ n_2, \tag{22}$$

$$M_{3k} = R_{ck} - S_{3k},\ k = 1,\ 2,\ ...,\ n_3. \tag{23}$$

There $S_{3k} = S_{1k} + S_{2k}$ is the joint action effect, the recurrence number of which during the period of time $[0, t_n]$ may be calculated by the equation:

$$n_3 = t_n(d_1 + d_2)\ \lambda_1\lambda_2, \tag{24}$$

where d_1, d_2 and λ_1, λ_2 are durations and renewal rates of extreme actions [8].

Mostly, the duration d_q of annual extreme gravity service loads is from 1 to 3 days. The durations of annual extreme snow and wind loads, respectively, are: $d_s = 14$-28 days and $d_w = 8$-12 hours. The renewal rates of these actions are: $\lambda_q = \lambda_s = \lambda_w = 1$/year. Therefore, for 50 years reference period, the recurrence numbers of extreme actions are: $n_{qw} = 0.2$-0.5 and $n_{sw} = 2$-4.

When probability distributions of random variables X and Y obey a Gumbel distribution law, the bivariate density function of the random variable $Z = X + Y$ may be presented in the form:

$$f_z(z) = \int_{-\infty}^{\infty} f_x(z - y, X_m - 0.45\mathbf{y}X) \times f_y(y, Y_m - 0.45\mathbf{y}Y)dy, \tag{25}$$

where X_m, Y_m and $\mathbf{y}X$, $\mathbf{y}Y$ are means and standard deviations of these variables.

Taking into account that $\mathbf{y}^2Z = \mathbf{y}^2X + \mathbf{y}^2Y$ is the variance of bivariate probability distribution, the joint density function may be expressed as:

$$f_Z(z) \approx f_Z(z, a_z), \tag{26}$$

$$a_z = X_m + Y_m - 0.19(\mathbf{y}X + \mathbf{y}Y) - 0.45\left(\mathbf{y}^2X + \mathbf{y}^2Y\right)^{1/2}.$$

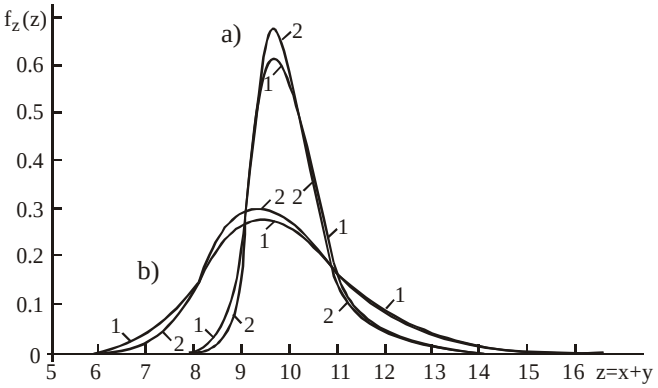

Figure 3. Bivariate density functions calculated by Equations (25) – 1 and (26) – 2: the coefficients of correlation $_{Д}X = {}_{Д}Y = 0.10$ (a) and 0.224 (b)

The probability density curves of joint extreme variable $Z = X + Y$ are given in Figure 3. It is not difficult to ascertain that the difference between the values computed by Equations (25) and (26) is fairly small. Besides, the upper tails of both density curves coincides. Therefore, in design practice it is expedient to use the conventional bivariate distribution function of two independent extreme action effects with the mean $S_{3k,m} = S_{1k,m} + S_{2k,m}$ and the variance $\mathbf{y}^2S_{3k} = \mathbf{y}^2S_{1k} + \mathbf{y}^2S_{2k}$.

6. Numerical example

The knee-joints of not deteriorating concrete frames of reliability class RC2 are under exposure of shear forces during 50 years period (*Figure 4*). The shear resistance of knee-joints is expressed as: $R = 0.068bhf_c$. The

characteristic, design and mean values of the concrete compressive strength and shear resistance of knee-joints are:

$$f_{ck} = 30 \text{ MPa},\ f_{cd} = 20 \text{ MPa},\ f_{cm} = 38 \text{ MPa};$$

$$R_k = 306 \text{ kN},\ R_d = 204 \text{ kN},\ R_m = 387.6 \text{ kN}.$$

The variance of shear resistance of knee-joints is:

$$\mathbf{y}^2 R = (0.128 \times 387.6)^2 = 2461.4 \ \text{(kN)}^2.$$

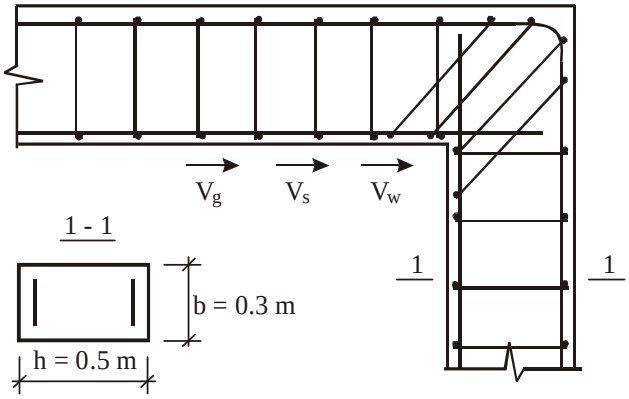

Figure 4. The knee-joint of concrete frames

The characteristic and design values of shear forces caused by permanent, snow and wind loads are:

$$V_{gk} = 77.72 \text{ kN},$$

$$V_{sk} = V_{wk} = 38.86 \text{ kN};$$

$$V_{gd} = 77.72 \times 1.35 = 104.92 \text{ kN},$$

$$V_{sd} = 38.86 \times 1.5 = 58.29 \text{ kN},$$

$$V_{wd} = 38.86 \times 0.7 \times 1.5 = 40.8 \text{ kN}.$$

Thus, the joint design shear force

$$V_d = V_{gd} + V_{sd} + V_{wd} = 204 \text{ kN} = R_d .$$

Therefore, according to deterministic calculation data, the frame knee-joints are reliable.
The coefficients of variation, means and variances of these extreme shear forces are:

$$\text{д}V_g = 0.1,$$

$$V_{gm} = V_{gk} = 77.72 \text{ kN},$$

$$\mathbf{y}^2 V_g = 60.4 \ \text{(kN)}^2;$$

$$\text{д}V_s = 0.6,$$

$$V_{sm} = V_{sk} / (1 + k_{0.98} \text{д}V_s) = 15.21 \text{ kN},$$

$$\mathbf{y}^2 V_s = 83.25 \ \text{(kN)}^2;$$

$$\text{д}V_w = 0.3,$$

$$V_{wm} = V_{wk} / (1 + k_{0.98} \text{д}V_w) = 21.86 \text{ kN};$$

$$\mathbf{y}^2 V_w = 43.0 \ \text{(kN)}^2.$$

The parameters of additional variables are:

$$\theta_{Rm} = 1.0,$$

$$\text{д}\theta_R = 0.1;$$

$$\theta_{Vm} = 1.0,$$

$$\text{д}\theta_g = \text{д}\theta_s = \text{д}\theta_w = 0.1,$$

$$\text{д}\theta_{sw} = 0.15.$$

Thus, the variances of revised shear forces are:

$$\mathbf{y}^2 (\theta_g V_g) = 120.8 \ \text{(kN)}^2,$$

$$\mathbf{y}^2 (\theta_s V_s) = 85.56 \ \text{(kN)}^2,$$

$$\mathbf{y}^2 (\theta_w V_w) = 47.8 \ \text{(kN)}^2,$$

$$\mathbf{y}^2 (\theta_{sw} V_{sw}) = 157.17 \ \text{(kN)}^2.$$

The parameters of conventional shear resistance (3) are:

$$R_{cm} = 387.6 - 77.72 = 309.9 \text{ kN},$$

$$\mathbf{y}^2 R_c = 1.0 \times 2461.4 + 387.6^2 \times 0.01 + 120.8 = 4084.6 \ \text{(kN)}^2.$$

According to (16), the coefficients of autocorrelation of the safety margins $Z_w = R_c - V_w$, $Z_s = R_c - V_s$ and $Z_{sw} = R_c - V_s - V_w$ of considered knee-joints are:

$$\rho_{w,kl} = 0.9884,$$

$\rho_{s,kl} = 0.9795,$

$\rho_{sw,kl} = 0.9629.$

The recurrence number of joint action effect $V_s + V_w$ calculated by Equation (24) is:

$$n_3 = 50\ [21/365 + 12/(24 \times 3.65)]\ 1 \times 1 = 2.945.$$

According to (9), the instantaneous survival probabilities of members are:

$\boldsymbol{P}_{sk,w} = 0.99999617,$

$\boldsymbol{P}_{sk,s} = 0.99999728,$

$\boldsymbol{P}_{sk,sw} = 0.9999837.$

Therefore, according to (18), the partial long-term survival probabilities of analysed knee-joints are:

$\boldsymbol{P}_{sw} = 0.9999717,$

$\boldsymbol{P}_{ss} = 0.9999710,$

$\boldsymbol{P}_{s,sw} = 0.9999747.$

According to (13), the coefficients of cross-correlation of safety margins are:

$\rho_{sw} = 0.9839,$

$\rho_{w,sw} = 0.9871,$

$\rho_{s,sw} = 0.9914.$

From Equation (19), the total survival probability of knee-joints is:

$$\boldsymbol{P} = 0.9999747 \times 0.9999717 \times 0.9999710 \times \left[1 + 0.98767^{11.84}\left(\frac{1}{0.9999717} - 1\right)\right] \times \left[1 + 0.9871^{11.73}\left(\frac{1}{0.9999747} - 1\right)\right] = 0.9999635.$$

It corresponds to the reliability index $\beta = 3.97 > \beta_{\min} (= 3.8)$ [5].

Despite high-correlated cuts of the safety margin sequences Z_w, Z_s and Z_{sw} of knee-joints, considerable differences among their instantaneous and long-term survival probabilities are corroborated.

The reliability verification of knee-joints of concrete frames by the deterministic partial factor method and probability-based approaches practically gave the same results.

7. Conclusion

When the system may be subjected to annual extreme service and climate actions, it is expedient to express its member performance processes by finite random sequences of safety margins, the dependent cuts of which coincide with the extreme loading situations of structures. Therefore, the generalized geometric distribution as the decreasing stochastic sequence may be successfully used in failure or survival probability analysis of highly correlated series systems. It leads to considerable perfections of probability-based analysis of deteriorating structures subjected to recurrent single and coincident actions as intermittent rectangular pulse renewal processes. A Gumbel distribution law may be used not only for joint sustained and extraordinary variable service loads but also for the sum of annual extreme action effects.

For the sake of simplifications of probabilistic time-dependent safety analysis of members, it is recommended to use design models with their conventional resistances and correlated sequence cuts of safety margins representing a variety of load combinations. The presented unsophisticated probability-based approaches and models may stimulate engineers having minimum appropriate skills to use full probabilistic methods in their engineering practice more courageously and effectively. It should be one more remedy in the struggle against deterministic approaches in the structural design.

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On asymptotic approach to reliability improvement of multi-state systems with components quantitative and qualitative redundancy: series and parallel systems

Keywords

reliability improvement, limit reliability functions

Abstract

The paper is composed of two parts, in this part after introducing the multi-state and the asymptotic approaches to system reliability evaluation the multi-state homogeneous series and parallel systems with reserve components are defined and their multi-state limit reliability functions are determined. In order to improve of the reliability of these systems the following methods are used: (i) a warm duplication of components, (ii) a cold duplication of components, (iii) a mixed duplication of components, (iv) improving the reliability of components by reducing their failure rate. Next, the effects of the systems' reliability different improvements are compared.

1. Introduction

Most real systems are very complex and it is difficult to analyze and to improve their reliability. Large numbers of components and subsystems and their operating complexity cause that the evaluation of their reliability is complicated. As a rule these are series systems, parallel systems or "*m* out of *n*" systems composed of a large number of components. One of the important techniques for reliability evaluation of large systems is the asymptotic approach. The mathematical methods are based on the limit theorems of order statistics distributions considered in a wide literature. These theorems generated investigations on limit reliability functions for systems with two-state components. Next, more general systems with multi-state components began to be considered. The asymptotic approach is also very useful in reliability improvement of large multi-state systems because of simplifying the calculation.

2. Multi-state and asymptotic approach

In multi-state reliability analysis presented in this paper it is supposed that:
- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the state set $\{0,1,...,z\}$,
- the state indices are ordered, the state 0 is the worst and the state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of the components E_i in the state subset $\{u,u+1,...,z\}$ while they were in the state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
- the system state degrades with time t without repair,
- $e_i(t)$ is a component E_i state at the time t, $t > 0$,
- $s(t)$ is a system state at the moment t, $t > 0$.

Definition 2.1. A vector

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1),..., R_i(t,z)],$$
$$t \in (-\infty,\infty),\ i = 1,2,...,n,$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t),$$
$$t \in (-\infty,\infty),\ u = 0,1,...,z,$$

is the probability that the component E_i is in the state subset $\{u,u+1,...,z\}$ at the time t, $t \in (-\infty,\infty)$ while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Definition 2.2. A vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],$$
$$t \in (-\infty,\infty),$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t),$$
$$t \in (-\infty,\infty),\ u = 0,1,...,z,$$

is the probability that the system is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in (-\infty,\infty)$ while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a system.

In the asymptotic approach to system reliability analysis we are interested in limit distributions of a standardized random variable

$$(T(u) - b_n(u))/a_n(u),\ u = 1,2,...,z,$$

where $T(u)$ is the lifetime of the system in the state subset $\{u,u+1,...,z\}$ and $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$, are some suitably chosen numbers, called normalizing constants.

Since

$$P((T(u) - b_n(u))/a_n(u) > t)$$

$$= P(T(u) > a_n(u)t + b_n(u))$$

$$= \boldsymbol{R}_n(a_n(u)t + b_n(u), u),\quad u = 1,2,...,z,$$

where

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state reliability function of the system, then we assume the following definition.

Definition 2.3. A vector

$$\Re(t,\cdot) = [1, \Re(t,1),..., \Re(t,z)],\ t \in (-\infty,\infty),$$

is called the limit multi-state reliability function of the system if there exist normalizing constants $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$ such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n(u)t + b_n(u), u) = \Re(t,u),$$
$$t \in C_{\Re(u)},\ u = 1,2,...,z,$$

where $C_{\Re(u)}$ is the set of continuity points of $\Re(t,u)$.

The knowledge of the system limit reliability function allow us, for sufficiently large n, to apply the following approximate formula

$$R_n(t,\cdot) = \Re((t - b_n(u))/a_n(u),\cdot),\ t \in (-\infty,\infty). \qquad (1)$$

3. System reliability improvement

3.1. Reliability improvement of a multi-state series system

Definition 3.1. A multi-state system is called a series system if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \min_{1\leq i\leq n}\{T_i(u)\},\ u = 1,2,...,z.$$

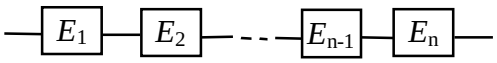

Figure 1. The scheme of a homogeneous series system

Definition 3.2. A multi-state series system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets $\{u,u+1,...,z\}$ have an identical distribution function

$$F_i(t,u) = F(t,u),\ u = 1,2,...,z,\ t \in (-\infty,\infty),\ i = 1,2,...,n,$$

The reliability function of the homogeneous multi-state series system is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1),..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = [R(t,u)]^n,\ t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Definition 3.3. A multi-state series system is called a system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = \min_{1\leq i\leq n}\{\max_{1\leq j\leq 2}\{T_{ij}(u)\}\},\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state series system with a hot reserve of its components is given by

$$\boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,\cdot) = [1,\ \boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,1),...,\ \boldsymbol{I}\overline{\boldsymbol{R}}_n^{(1)}(t,z)],$$

where

$$\overline{IR}_n^{(1)}(t,u) = [1-(F(t,u))^2]^n, t \in (-\infty,\infty). \quad (2)$$

Lemma 3.1. If

(i) $\overline{I\Re}^{(1)}(t,u) = \exp[-\overline{V}(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\overline{IR}_n^{(1)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state series system whit a hot reserve of its components defined by (2),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \overline{IR}_n^{(1)}(a_n(u)t + b_n(u)) = \overline{I\Re}^{(1)}(t,u), t \in C_{\overline{\Re}},$$
$u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u))]^2 = \overline{V}(t,u), t \in C_{\overline{V}},$$
$u = 1,2,...,z$.

Proposition 3.1. If components of the homogeneous multi-state series system with a hot reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \frac{1}{\lambda(u)}$, $b_n(u) = 0$, $u = 1,2,...,z$,

then

$$\overline{I\Re}^{(1)}(t,u) = 1, t < 0,$$

$$\overline{I\Re}^{(1)}(t,u) = \exp[-t^2], t \geq 0, u = 1,2,...,z,$$

is its limit reliability function.

The proof of Proposition 3.1 is given in [9].

Corollary 3.1. The reliability function of exponential series system whit a hot reserve of its components is given by

$$\overline{IR}_n^{(1)}(t,u) = 1, t < 0,$$

$$\overline{IR}_n^{(1)}(t,u) \cong \exp[-\lambda^2(u) n t^2], t \geq 0, u = 1,2,...,z. \quad (3)$$

Definition 3.4. A multi-state series system is called a system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(2)}(u) = \min_{1\leq i\leq n}\{\sum_{j=1}^{2} T_{ij}(u)\}, \quad u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components. The reliability function of the homogeneous multi-state series system with cold reserve of its components is given by

$$\overline{IR}_n^{(2)}(t,\cdot) = [1, \overline{IR}_n^{(2)}(t,1),..., \overline{IR}_n^{(2)}(t,z)],$$

where

$$\overline{IR}_n^{(2)}(t,u) = [1-F(t,u)*F(t,u)]^n, t \in (-\infty,\infty), \quad (4)$$
$u = 1,2,...,z$.

Lemma 3.2. If

(i) $\overline{I\Re}^{(2)}(t,u) = \exp[-\overline{V}(t,u)]$ $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\overline{IR}_n^{(2)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state series system whit a cold reserve of its components defined by (4),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \overline{IR}_n^{(2)}(a_n(u)t + b_n(u)) = \overline{I\Re}^{(2)}(t,u), t \in C_{\overline{I\Re}},$$
$u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= \overline{V}(t,u), t \in C_{\overline{V}}, \quad u = 1,2,...,z.$$

Proposition 3.2. If components of the homogeneous multi-state series system with a cold reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \frac{\sqrt{2}}{\lambda(u)\sqrt{n}}$, $b_n(u) = 0$, $u = 1,2,...,z$,

then

$$\overline{I\Re}^{(2)}(t,u) = 1, t < 0,$$

$$\overline{I\Re}^{(2)}(t,u) = \exp[-t^2], t \geq 0, u = 1,2,...,z,$$

is its limit reliability function.

The proof of Proposition 3.2 is given in [9].

Corollary 3.2. The reliability function of exponential series system whit a cold reserve of its components is given by

$$\boldsymbol{I\overline{R}}_{n}^{(2)}(t,u)=1,\ t<0,$$

$$\boldsymbol{I\overline{R}}_{n}^{(2)}(t,u)\cong\exp[-\lambda^{2}(u)nt^{2}/2],\ t\geq 0, \quad (5)$$
u =1,2,...,z.

Definition 3.5. A multi-state series system is called a system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u)=\min\{\min_{1\leq i\leq s_1 n}\{\max_{1\leq j\leq 2}\{T_{ij}(u)\}\},\ \min_{s_1 n+1\leq i\leq n}\{\sum_{j=1}^{2}T_{ij}(u)\}\},$$
u = 1,2,...,z,

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1+ s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state series system with a mixed reserve of its components is given by

$$\boldsymbol{I\overline{R}}_{n}^{(3)}(t,\cdot)=[1,\boldsymbol{I\overline{R}}_{n}^{(3)}(t,1),...,\boldsymbol{I\overline{R}}_{n}^{(3)}(t,z)],$$

where

$$\boldsymbol{I\overline{R}}_{n}^{(3)}(t,u)=[1-(F(t,u))^{2}]^{s_1 n}[1-F(t,u)*F(t,u)]^{s_2 n}, \quad (6)$$
$t\in(-\infty,\infty)$, u = 1,2,...,z.

Lemma 3.3. If

(i) $I\overline{\Re}^{(3)}(t,u) = \exp[-\overline{V}(t,u)]$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}_{n}^{(3)}(t,u)$, $t\in(-\infty,\infty)$, u = 1,2,...,z, is the reliability function of non-degenerate multi-state series system whit a mixed reserve of its components defined by (6),

(iii) $a_n(u) > 0$, $b_n(u)\in(-\infty,\infty)$, u = 1,2,...,z,

then

$$\lim_{n\to\infty}\boldsymbol{I\overline{R}}_{n}^{(3)}(a_n(u)t+b_n(u))=I\overline{\Re}^{(3)}(t,u),\ t\in C_{I\overline{\Re}},$$
u = 1,2,...,z,

if and only if

$$\lim_{n\to\infty}\ 2ns_1[F(a_n(u)t+b_n(u))+ns_2[F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]=\overline{V}(t,u),$$
$t\in C_{\overline{V}}$, u = 1,2,...,z.

Proposition 3.3. If components of the homogeneous multi-state series system with a mixed reserve of its components have multi-state exponential reliability functions

and $a_n(u)=\dfrac{1}{\lambda(u)\sqrt{n}}$, $b_n(u)=0$, u = 1,2,...,z,

then

$I\overline{\Re}(t,u)$ = 1 for t < 0,

$I\overline{\Re}(t,u)$ = exp[-(2s_1+s_2) t^2/2] , $t\geq 0$, u = 1,2,…,z,

is its limit reliability function.

Corollary 3.3. The reliability function of exponential series system whit mixed reserve of its components is given by

$$\boldsymbol{I\overline{R}}_{n}^{(3)}(t,u)=1,\ t<0,$$

$$\boldsymbol{I\overline{R}}_{n}^{(3)}(t,u)\cong\exp[-\lambda^{2}(u)n(2s_1+s_2)t^{2}/2],\ t\geq 0, \quad (7)$$
u = 1,2,...,z.

Proposition 3.4. If components of the homogeneous multi-state series system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u)\in$ <0,1>, u = 1,2,...,z, i.e.

$$\tilde{R}(t,u)=1,\ t<0,$$

$$\tilde{R}(t,u)=\exp[-\lambda(u)\rho(u)t],\ t\geq 0,\ \lambda(u)>0,$$
u = 1,2,…,z,

and $a_n(u)=\dfrac{1}{\lambda(u)\rho(u)n}$, $b_n(u)$ = 0, u = 1,2,...,z,

then

$I\overline{\Re}^{(4)}(t,u)$ = 1, t < 0,

$I\overline{\Re}^{(4)}(t,u)$ = exp[-t], $t\geq 0$, u = 1,2,...,z,

is its limit reliability function.

Corollary 3.4. The reliability function of exponential series system whit improved reliability functions of its components is given by

$\overline{\boldsymbol{IR}}_n^{(4)}(t,u) = 1$, t < 0,

$\overline{\boldsymbol{IR}}_n^{(4)}(t,u) = \exp[-\lambda(u)n\rho(u)t]$, $t \geq 0$, $u = 1,2,...,z$. (8)

3.2. Reliability improvement of a multi-state parallel system

Definition 3.6. A multi-state system is called a parallel system if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \max_{1\leq i\leq n}\{T_i(u)\},\ u = 1,2,...,z.$$

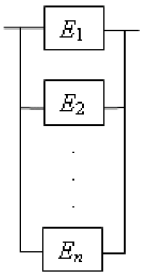

Figure 2. The scheme of a homogeneous parallel system

Definition 3.7. A multi-state parallel system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets $\{u,u+1,...,z\}$ have an identical distribution function

$F_i(t,u) = F(t,u)$, $u = 1,2,...,z$, $t \in (-\infty,\infty)$, $i = 1,2,...,n$.

The reliability function of the homogeneous multi-state parallel system is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1-[F(t,u)]^n,\ t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Definition 3.8. A multi-state parallel system is called a system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = \max_{1\leq i\leq n}\{\max_{1\leq j\leq 2}\{T_{ij}(u)\}\},\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state parallel system with a hot reserve of its components is given by

$$\boldsymbol{IR}_n^{(1)}(t,\cdot) = [1,\ \boldsymbol{IR}_n^{(1)}(t,1),...,\ \boldsymbol{IR}_n^{(1)}(t,z)],$$

where

$$\boldsymbol{IR}_n^{(1)}(t,u) = 1-[(F(t,u))^2]^n,\ t \in (-\infty,\infty),\quad (9)$$
$u = 1,2,...,z.$

Lemma 3.4. If

(i) $I\Re^{(1)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(1)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state parallel system whit a hot reserve of its components defined by (9),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(1)}(a_n(u)t + b_n(u)) = I\Re(t,u),\ t \in C_{I\Re},$$
$u = 1,2,...,z,$

if and only if

$$\lim_{n\to\infty} 2n[R(a_n(u)t + b_n(u))] = V(t,u),\ t \in C_V,$$
$u = 1,2,...,z.$

Proposition 3.5. If components of the homogeneous multi-state parallel system with a hot reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)}$, $b_n(u) = \dfrac{\log 2n}{\lambda(u)}$, $u = 1,2,...,z$,

then

$$I\Re^{(1)}(t) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),\ u = 1,2,...,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{t + \log 2n}{\lambda(u)} \to \infty \text{ as } n \to \infty$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Therefore

$$V(t,u) = \lim_{n\to\infty} 2nR(a_n(u)t + b_n(u))$$

$$= \lim_{n\to\infty} 2n\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} 2n\exp[-t - \log 2n]$$

$$= \exp[-t], t \in (-\infty, \infty),$$

which by Lemma 3.4 completes the proof.

Corollary 3.5. The reliability function of exponential parallel system whit a hot reserve of its components is given by

$$\boldsymbol{IR}_n^{(1)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)t + \log 2n], \quad (10)$$
$$t \in (-\infty, \infty),\ u = 1,2,...,z.$$

Definition 3.9. A multi-state parallel system is called a system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(2)}(u) = \max_{1 \le i \le n}\{\sum_{j=1}^{2} T_{ij}(u)\},\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state parallel system with a cold reserve of its components is given by

$$\boldsymbol{IR}_n^{(2)}(t,\cdot) = [1,\ \boldsymbol{IR}_n^{(2)}(t,1),..., \boldsymbol{IR}_n^{(2)}(t,z)],$$

where

$$\boldsymbol{IR}_n^{(2)}(t,u) = 1 - [F(t,u) * F(t,u)]^n, \quad (11)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Lemma 3.5. If

(i) $I\Re^{(2)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(2)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, is the reliability function of non-degenerate multi-state parallel system whit a cold reserve of its components defined by (11),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(2)}(a_n(u)t + b_n(u)) = I\Re^{(2)}(t,u),$$
$$t \in C_{\Re},\ u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= V(t,u),\ t \in C_V,\ u = 1,2,...,z.$$

Proposition 3.6. If components of the homogeneous multi-state parallel system with a cold reserve of its components have multi-state exponential reliability functions

and $a_n(u) = \dfrac{1}{\lambda(u)}$, $\dfrac{\exp[\lambda b_n(u)]}{\lambda(u) b_n(u)} = n$, $u = 1,2,...,z$,

then

$$\boldsymbol{I}\Re^{(2)}(t,u) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),$$
$$u = 1,2,...,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have

$a_n(u)t + b_n(u) \to \infty$ as $n \to \infty$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Therefore

$$V(t,u)$$

$$= \lim_{n\to\infty} n[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[(1 + \lambda(u)(a_n(u)t + b_n(u))) \cdot$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[(1 + t + \lambda(u)b_n(u)) \exp[-(t + \lambda(u)b_n(u))]$$

$$= \lim_{n\to\infty} n[\frac{1+t}{\exp[\lambda(u)b_n(u)]} + \frac{\lambda(u)b_n(u)}{\exp[\lambda(u)b_n(u)]}]\exp[-t]$$

$$= \exp[-t] \text{ for } t \in (-\infty,\infty),$$

which by Lemma 3.5 completes the proof.

Corollary 3.6. The reliability function of exponential parallel system whit a cold reserve of its components is given by

$$\boldsymbol{IR}_n^{(2)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)t + \lambda(u)b_n(u)], \quad (12)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Definition 3.10. A multi-state parallel system is called a system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u) = \max\{\max_{1\le i\le s_1 n}\{\max_{1\le j\le 2}\{T_{ij}(u)\}\},$$

$$\max_{s_1 n+1\le i\le n}\{\sum_{j=1}^{2} T_{ij}(u)\}\},$$

$u = 1,2,...,z,$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1 + s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state parallel system with mixed reserve of its components is given by

$$\boldsymbol{IR}_n^{(3)}(t,\cdot) = [1,\ \boldsymbol{IR}_n^{(3)}(t,1),...,\ \boldsymbol{IR}_n^{(3)}(t,z)\ ],$$

where

$$\boldsymbol{IR}_n^{(3)}(t,u) = 1-[(F(t,u))^2]^{s_1 n}[F(t,u)*F(t,u)]^{s_2 n}, \quad (13)$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z.$

Lemma 3.6. If

(i) $I\Re^{(3)}(t,u) = \exp[-V(t,u)]$, $u = 1,2,...,z,$ is non-degenerate reliability function,

(ii) $\boldsymbol{IR}_n^{(3)}(t,u)$, $t \in (-\infty,\infty)$, $u = 1,2,...,z,$ is the reliability function of non-degenerate multi-state parallel system with a mixed reserve of its components defined by (13),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z,$

then

$$\lim_{n\to\infty} \boldsymbol{IR}_n^{(3)}(a_n(u)t + b_n(u)) = I\Re^{(3)}(t,u), t \in C_{\Re},$$

$u = 1,...,z,$

if and only if

$$\lim_{n\to\infty} 2ns_1[R(a_n(u)t + b_n(u))]$$

$$+ ns_2[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= \overline{V}(t,u),\ t \in C_V,\ u = 1,2,...,z.$$

Proposition 3.7. If components of the homogeneous multi-state parallel system with a mixed reserve of its components have multi-state exponential reliability functions and

$$a_n(u) = \frac{1}{\lambda(u)b_n(u) - 1},\quad \frac{\exp[\lambda(u)b_n(u)]}{\lambda(u)b_n(u)} = s_2 n,$$

then

$$\boldsymbol{I\Re}^{(3)}(t,u) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),\ u = 1,\ldots,z,$$

is its limit reliability function.

Proof: Since for all fixed u, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty,\ t \in (-\infty,\infty),$$

and

$$\frac{1}{\lambda(u)b_n(u) - 1} \to 0 \text{ as } n \to \infty,\ t \in (-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[2s_1 \exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$+ s_2[(1 + \lambda(u)(a_n(u)t + b_n(u)))$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]]]$$

$$= \lim_{n\to\infty} n\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$s_2\lambda(u)(a_n(u)t + b_n(u))$$

$$[1 + \frac{s_2 + 2s_1}{s_2\lambda(u)(a_n(u)t + b_n(u))}]$$

$$= \lim_{n\to\infty} \exp[-\lambda(u)a_n(u)t]$$

$$\exp[-\lambda(u)b_n(u) + \log ns_2\lambda(u)b_n(u)$$

$$[1 + +o(1)]][1 + o(1)] = \exp[-t],$$

$t \in (-\infty,\infty)$, $u = 1,2,\ldots,z,$

which by Lemma 3.6 completes the proof.

Corollary 3.7. The reliability function of parallel system whit a mixed reserve of its components is given by

$$\boldsymbol{IR_n}^{(3)}(t,u)$$

$$\cong 1 - \exp[-\exp[-\frac{\lambda(u)b_n(u) - 1}{b_n(u)}t + (\lambda(u)b_n(u) - 1)],$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z.$ (14)

Proposition 3.8. If components of the homogeneous multi-state parallel system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u) \in <0,1>$, $u = 1,2,...,z,$ and

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)},\ b_n(u) = \frac{\log n}{\lambda(u)\rho(u)},\ u = 1,2,...,z,$$

then

$$I\Re^{(4)}(t,u) = 1 - \exp[-\exp[-t]],\ t \in (-\infty,\infty),$$
$$u = 1,2,...,z,$$

is its limit reliability function.

Corollary 3.8. The reliability function of exponential parallel system whit improved reliability functions of its components is given by

$$\boldsymbol{IR}_n^{(4)}(t,u) \cong 1 - \exp[-\exp[-\lambda(u)\rho(u)t + \log n]], \quad (15)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z.$$

4. Comparison of reliability improvement effects

The comparisons of the limit reliability functions of the systems with different kinds of reserve and such systems with improved components allow us to find the value of the components decreasing failure rate factor $\rho(u)$, which warrants an equivalent effect of the system reliability improvement.

4.1 Series system

The comparisons of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equations

$$\boldsymbol{I}\overline{\Re}^{(4)}\left((t - b_n(u))/a_n(u)\right)$$

$$= I\overline{\Re}^{(k)}\left((t - b_n(u))/a_n(u)\right),\ u = 1,2,...,z, \quad (16)$$
$$k = 1,2,3.$$

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state series system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (16) are respectively given by

$$k = 1\ \ \rho(u) = \rho(t,u) = \lambda(u)t,\quad u = 1,2,...,z,$$

$$k = 2\ \ \rho(u) = \rho(t,u) = \frac{\lambda(u)t}{2},\quad u = 1,2,...,z,$$

$$k = 3\ \ \rho(u) = \rho(t,u) = \frac{2s_1 + s_2}{2}\lambda(u)t,\ u = 1,2,...,z.$$

4.2. Parallel system

The comparisons of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equation

$$\boldsymbol{I}\Re^{(4)}\left((t - b_n(u))/a_n(u)\right)$$

$$= \boldsymbol{I}\Re^{(k)}\left((t - b_n(u))/a_n(u)\right),\ u = 1,2,...,z, \quad (17)$$
$$k = 1,2,3.$$

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state parallel system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (17) are respectively given by

$$k = 1\ \ \rho(u) = \rho(t,u) = 1 - \frac{\log 2}{\lambda(u)t},\quad u = 1,2,...,z,$$

$$k = 2\ \ \rho(u) = \rho(t,u) = 1 - \frac{\lambda(u)b_n(u) - \log n}{\lambda(u)t},$$
$$u = 1,2,...,z,$$

$$k = 3$$
$$\rho(u) = \rho(t,u) = \frac{\lambda(u)b_n(u) - 1}{\lambda(u)}$$

$$- \frac{\lambda(u)b_n(u) - 1 - \log n}{\lambda(u)t},$$
$$u = 1,2,...,z.$$

5. Conclusion

Proposed in the paper application of the limit multi-state reliability functions for reliability of large systems evaluation and improvement simplifies calculations. The methods may be useful not only in the technical objects operation processes but also in their new processes designing, especially in their optimization. The case of series, parallel and "*m* out of *n*" (in part 2) systems composed of components having exponential reliability functions with double reserve of their components is considered only. It seems to be possible to extend the results to the systems having other much complicated reliability structures and components with different from the exponential reliability function. Further, it seems to be reasonable to elaborate a computer programs supporting calculations and accelerating decision making, addressed to reliability practitioners.

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On asymptotic approach to reliability improvement of multi-state systems with components quantitative and qualitative redundancy: „*m* out of *n*" systems

Keywords

reliability improvement, limit reliability functions

Abstract

The paper is composed of two parts, in this part the multi-state homogeneous „*m* out of *n*" systems with reserve components are defined and their multi-state limit reliability functions are determined. In order to improve of the reliability of these systems the following methods are used: (i) a warm duplication of components, (ii) a cold duplication of components, (iii) a mixed duplication of components, (iv) improving the reliability of components by reducing their failure rate. Next, the effects of the systems' reliability different improvements are compared.

1. Introduction

Presented paper is continuation of a work about reliability improvement of large system. In the first part of this work are defined the component's and system's multi-state reliability functions and next the asymptotic approach are brought forward. There are presented results concerned with improvement of large series and parallel systems, their multi-state limit reliability functions in case when the systems have reserve components and in case when the reliability of components is improved by reducing their failure rate. As the main result are found the forms of reducing their failure rate factor for both kinds of large systems.

2. Reliability improvement of a multi-state „*m* out of *n*" system

Definition 2.1. A multi-state system is called an „*m* out of *n*" system if its lifetime in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = T_{(n-m+1)}(u),\ m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m th* maximal order statistics in the sequence of the component lifetimes $T_1(u), T_2(u),..., T_n(u)$.

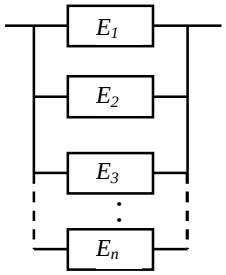

Figure 1. The scheme of a homogeneous „m out of n" system

The above definition means that the multi-state „*m* out of *n*" system is in the state subset $\{u,u+1,...,z\}$ if and only if at least *m* out of *n* its components is in this state subset and it is a multi-state parallel system if $m = 1$ and it is a multi-state series system if $m = n$.

Definition 2.2. A multi-state „*m* out of *n*" system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets have an identical distribution function

$$F_i(t,u) = F(t,u),\ u = 1,2,...,z,\ t \in (-\infty,\infty),\ i = 1,2,...,n.$$

The reliability function of the homogeneous multi-state „*m* out of *n*" system is given either by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1,\boldsymbol{R}_n^{(m)}(t,1),...,\boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \binom{n}{i} [R(t,u)]^i [F(t,u)]^{n-i},$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z,$$

or by

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\mathbf{R}}_n^{(\overline{m})}(t,u) = \sum_{i=0}^{\overline{m}} \binom{n}{i} [F(t,u)]^i [R(t,u)]^{n-i},\ t \in (-\infty,\infty),$$
$$\overline{m} = n - m,\ u = 1,2,...,z.$$

Definition 2.3. A multi-state system is called an „*m* out of *n*" system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = T_{(n-m+1)}(u),\ m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \max_{1 \le j \le 2} \{T_{ij}(u)\},\ i = 1,2,..,n,\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a hot reserve of its components is given either by

$$\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,\cdot) = [1, \boldsymbol{IR}^{(1)}{}_n^{(m)}(1,z), ..., \boldsymbol{IR}^{(1)}{}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \binom{n}{i} [1 - (F(t,u))^2]^i [F(t,u)]^{2(n-i)}, \quad (1)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z,$$

or by

$$\overline{\boldsymbol{IR}}^{(1)}{}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{IR}}^{(1)}{}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{IR}}^{(1)}{}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{IR}}^{(1)}{}_n^{(\overline{m})}(t,u)$$
$$= \sum_{i=0}^{\overline{m}} \binom{n}{i} [(F(t,u))]^{2i} [1 - (F(t,u))^2]^{(n-i)}, \quad (2)$$
$$\overline{m} = n - m,\ t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Lemma 2.1.
case 1: If

(i) $\mathrm{I}\Re^{(1)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{[V(t,u)]^i}{i!} \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (16),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(1)}{}_n^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(1)(m)}(t,u),$$
$$t \in C_{\mathrm{I}\Re},\ u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[1 - F^2(a_n(u)t + b_n(u))] = V(t,u),\ t \in C_V,$$
$$u = 1,2,...z,$$

case 2: If

(i) $\mathrm{I}\Re^{(1)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}} dx$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (16),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, as $n \to \infty$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(1)}{}_n^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(1)(\mu)}(t,u),$$
$$t \in C_{\boldsymbol{I}\Re},\ u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} \frac{(n+1)[1 - F^2(a_n(u)t + b_n(u))] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = v(t,u),$$
$$u = 1,2,...,z.$$

case 3: If

(i) $\mathrm{I}\overline{\Re}^{(1)(\overline{m})}(t,u) = \sum_{i=0}^{m} \frac{[\overline{V}(t,u)]^i}{i!} \exp[-\overline{V}(t,u)]$,

$\overline{m} = n - m$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{I}\overline{\boldsymbol{R}}^{(1)}{}_{n}^{(\overline{m})}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (17),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $n - m = \overline{m}$ = constant $(m/n \to 1 \; as \; n \to \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I}\overline{\boldsymbol{R}}^{(1)}{}_{n}^{(\overline{m})}(a_n(u)t + b_n(u)) = \mathrm{I}\overline{\Re}^{(1)(\overline{m})}(t,u),$$

$t \in C_{\mathrm{I}\overline{\Re}}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)]^2 = \overline{V}(t,u), t \in C_{\overline{V}},$$

$u = 1,2,...,z$.

Proposition 2.1. If components of the homogeneous multi-state „*m* out of *n*" system with a hot reserve of its components have multi-state exponential reliability functions

and

case 1 *m* = constant,

$$a_n(u) = \frac{1}{\lambda(u)}, \; b_n(u) = \frac{1}{\lambda(u)} \log 2n, \; u = 1,2,...,z,$$

then

$$\boldsymbol{I}\Re^{(1)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]],$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 2 $m/n \to \mu$, $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{\sqrt{\mu}}{\lambda(u)2\sqrt{n+1}}, \; b_n(u) = \frac{1}{\lambda(u)}\sqrt{1-\mu},$$

$u = 1,2,...,z$,

then

$$\boldsymbol{I}\Re^{(1)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx,$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 3 $n - m = \overline{m}$ = constant, $(m/n \to 1, \; n \to \infty)$,

$$a_n(u) = \frac{1}{\sqrt{n}\lambda(u)}, \; b_n(u) = 0, \; u = 1,2,...,z,$$

then

$$\mathrm{I}\overline{\Re}^{(1)(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\mathrm{I}\overline{\Re}^{(1)(\overline{m})}(t,u) = \sum_{i=0}^{n-m} \frac{t^{2i}}{i!} \exp[-t^2], \; t \geq 0, \; u = 1,2,...,z,$$

is its limit reliability function.

Proof:

case 1: Since for all fixed *u*, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty.$$

Therefore

$$\begin{aligned} V(t,u) &= \lim_{n\to\infty} n[1 - F^2(a_n(u)t + b_n(u))] \\ &= \lim_{n\to\infty} n[2\exp[-\lambda(u)(a_n(u)t + b_n(u))] \\ &\quad - \exp[-2\lambda(u)(a_n(u)t + b_n(u))]] \\ &= \lim_{n\to\infty} 2n\exp[-\lambda(u)(a_n(u)t + b_n(u))] \\ &\quad [1 - \frac{1}{2}\exp[-\lambda(u)(a_n(u)t + b_n(u))]] \\ &= \lim_{n\to\infty} \exp[-t][2n\exp[-\lambda(u)b_n(u)] \\ &\quad - n\exp[-t]\exp[-2\lambda(u)b_n(u)]] \\ &= \lim_{n\to\infty} \exp[-t][2n\frac{1}{n} - n\frac{1}{n^2}\exp[-t]] \\ &= \exp[-t], \; t \in (-\infty,\infty), \; u = 1,2,...,z, \end{aligned}$$

which by *case* 1 in Lemma 2.1 completes the proof.

case 2: Since for all fixed *u*, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty,$$

moreover

$$1 - F^2(a_n(u)t + b_n(u))$$

$$= 2\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t + b_n(u))]$$

$$= 2[1-\lambda(u)(a_n(u)t + b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2]$$

$$-[1-2\lambda(u)(a_n(u)t + b_n(u))$$

$$+\frac{1}{2}4\lambda^2(u)(a_n(u)t + b_n(u))^2] + o(\frac{1}{(n+1)})$$

$$= 1-\lambda^2(u)(a_n(u)t + b_n(u))^2 + o(\frac{1}{(n+1)}),$$

next

$$\nu(t,u) = \lim_{n\to\infty}\frac{(n+1)[1-F^2(a_n(u)t + b_n(u))]]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty}\frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{\sqrt{n+1}})]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty}\frac{-\sqrt{\mu(1-\mu)}t + o(1)}{\sqrt{\mu(1-\mu)}} = -t,\ t\in(-\infty,\infty),$$

u = 1,2,...,z,

which by *case* 2 in Lemma 2.1 completes the proof.

case 3: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{t}{\lambda(u)\sqrt{n}} < 0 \ \text{ for } t < 0$$

and

$$a_n(u)t + b_n(u) = \frac{t}{\lambda(u)\sqrt{n}} \geq 0 \ \text{ for } t \geq 0,$$

then

$$F^2(a_n(u)t + b_n(u)) = 0,\ t < 0$$

and

$$F^2(a_n(u)t + b_n(u))$$

$$= [1-\exp[-\lambda(u)(a_n(u)t + b_n(u))]]^2$$

$$= [1-\exp[-\frac{t}{\sqrt{n}}]]^2,\ t \geq 0.$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[F(a_n(u)t + b_n(u))]^2 = 0,\ t < 0,$$

u = 1,2,...,z,

and

$$V(t,u) = \lim_{n\to\infty} n[F(a_n(u)t + b_n(u))]^2$$

$$= \lim_{n\to\infty} n[1-\exp[-\frac{t}{\sqrt{n}}]]^2$$

$$= \lim_{n\to\infty} n[\frac{t}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})]^2 = t^2,\ t \geq 0,$$

u = 1,2,...,z,

which by *case* 3 in Lemma 2.1 completes the proof.

Corollary 2.1. The reliability function of exponential „m out of n" system with a hot reserve of its components is given by

case 1

$$IR_n^{(1)(m)}(t,u) \cong 1-\sum_{i=0}^{m-1}\frac{\exp[-i(\lambda(u)t - \log 2n)]}{i!}$$

$$\exp[-\exp[-\lambda(u)t + \log 2n]], \qquad (3)$$

$t\in(-\infty,\infty)$, u = 1,2,...,z.

case 2

$$\boldsymbol{IR}_n^{(1)(\mu)}(t,u) \cong 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Im} e^{-\frac{x^2}{2}}dx \qquad (4)$$

where

$$\Im = \frac{2\lambda(u)\sqrt{n+1}}{\sqrt{\mu}}t - \frac{2\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}},\ t\in(-\infty,\infty), \quad (5)$$

u = 1,2,...,z.

case 3

$$\boldsymbol{IR}_n^{(1)(\overline{m})}(t,u) = 1,\ t < 0,$$

$$\boldsymbol{IR}^{(1)}{}_{n}^{(\overline{m})}(t,u) \cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\sqrt{nt}]^{2i}}{i!} \exp[-\lambda^2(u)nt^2], \quad (6)$$

$t \geq 0$, $u = 1,2,...,z$.

Definition 2.4. A multi-state system is called an „*m* out of *n*" system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(2)}(u) = T_{(n-m+1)}(u), m = 1,2,...,n, u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \sum_{j=1}^{2} T_{ij}(u), i = 1,2,..,n, u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a cold reserve of its components is given either by

$$\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,\cdot) = [1, \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,1),..., \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$$

$$= 1 - \sum_{i=0}^{m-1} \binom{n}{i} [1 - F(t,u) * F(t,u)]^i$$

$$= \cdot [F(t,u) * F(,u)]^{n-i} \quad (7)$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

or by

$$\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,\cdot) = [1, \boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,1),..., \boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,z)],$$

where

$$\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,u)$$

$$= \sum_{i=0}^{\overline{m}} \binom{n}{i} [F(t,u) * F(t,u)]^i [1 - F(t,u) * F(t,u)]^{n-i}, \quad (8)$$

$t \in (-\infty,\infty)$, $\overline{m} = n - m$, $u = 1,2,...,z$.

Lemma 2.2.
case 1: If

(i) $\mathrm{I}\Re^{(2)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{[V(t,u)]^i}{i!} \exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (24),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(1)(m)}(t,u), t \in C_{I\Re},$$

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= V(t,u), t \in C_V, u = 1,2,...z,$$

case 2: If

(i) $\mathrm{I}\Re^{(2)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}} dx$, $u = 1,2,...,z$, is non-degenerate reliability function,

(i) $\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (24),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, as $n \to \infty$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(2)(\mu)}(t,u),$$

$t \in C_{I\Re}$,

if and only if

$$\lim_{n\to\infty} \frac{(n+1)[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \nu(t,u), u = 1,2,...,z.$$

case 3: If

(i) $\mathrm{I}\overline{\Re}^{(2)(\overline{m})}(t,u) = \sum_{i=0}^{m} \frac{[\overline{V}(t,u)]^i}{i!} \exp[-\overline{V}(t,u)]$,

$\overline{m} = n - m$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}^{(2)(\overline{m})}_{n}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (25),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $n - m = \overline{m}$ = constant $(m/n \to 1 \text{ as } n \to \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}}^{(2)(\overline{m})}_{n}(a_n(u)t + b_n(u)) = I\overline{\Re}^{(2)(\overline{m})}(t,u),$$

$t \in C_{I\overline{\Re}}$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u) * F(a_n(u)t + b_n(u)] = \overline{V}(t,u),$$

$t \in C_{\overline{V}}$, $u = 1,2,...,z$.

Proposition 3.2. If components of the homogeneous multi-state „*m* out of *n*" system with a cold reserve of its components have multi-state exponential reliability functions
and
case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)}, \quad \frac{\exp[\lambda(u)b_n(u)]}{\lambda(u)b_n(u)} = n, \quad u = 1,2,...,z,$$

then

$$I\Re^{(2)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]],$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 2 $m/n \to \mu$ $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{\sqrt{\mu}}{\lambda(u)\sqrt{n+1}}, \quad b_n(u) = \frac{\sqrt{1-\mu}}{\lambda(u)}, \quad u = 1,2,...,z,$$

then

$$I\Re^{(2)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx, \; t \in (-\infty,\infty),$$

$u = 1,2,...,z$,

case 3 $n - m = \overline{m}$ = constant $(m/n \to 1, \; n \to \infty)$,

$$a_n(u) = \frac{\sqrt{2}}{\sqrt{n}\lambda(u)}, \; b_n(u) = 0, \; u = 1,2,...,z,$$

then

$$I\overline{\Re}^{(2)(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\boldsymbol{I\overline{\Re}}^{(2)(\overline{m})}(t,u) = \sum_{i=0}^{n-m} \frac{t^{2i}}{i!} \exp[-t^2], \; t \geq 0, \; u = 1,2,...,z,$$

is its limit reliability function.

Proof:
case 1: Since for all fixed *u*, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty, \; t \in (-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[\frac{1+t}{\exp[\lambda(u)b_n(u)]}$$

$$+ \frac{\lambda(u)b_n(u)}{\exp[\lambda(u)b_n(u)]}]\exp[-t]$$

$$= \exp[-t], \; t \in (-\infty,\infty), \; u = 1,2,...z,$$

which by *case* 1 in Lemma 2.2 completes the proof.

case 2: Since for all fixed *u*, we have

$$a_n(u)t + b_n(u) = \frac{\sqrt{\mu}}{\lambda(u)\sqrt{n+1}}t + \frac{1}{\lambda(u)}\sqrt{1-\mu}$$

$$\to \frac{1}{\lambda(u)}\sqrt{1-\mu} > 0 \text{ as } n \to \infty$$

and

$$1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$

$$= [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$[1 - \lambda(u)(a_n(u)t + b_n(u)) +$$

$$\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2 - o(\frac{1}{n+1})]$$

$$=1-\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2$$

$$-o(\frac{1}{n+1})],\ t\in(-\infty,\infty).$$

Therefore

$$v(t,u)$$

$$=\lim_{n\to\infty}\frac{(n+1)[1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$=\lim_{n\to\infty}\frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t+\mu-o(\frac{1}{n+1})]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$=\lim_{n\to\infty}\frac{-\sqrt{\mu(1-\mu)}t+o(1)}{\sqrt{\mu(1-\mu)}}=-t,\ t\in(-\infty,\infty),$$

which by *case* 2 in Lemma 2.2 completes the proof.

case 3: Since for all fixed *u*, we have

$$a_n(u)t+b_n(u)=\frac{\sqrt{2}t}{\lambda(u)\sqrt{n}}<0 \text{ for } t<0$$

and

$$a_n(u)t+b_n(u)=\frac{t\sqrt{2}}{\lambda(u)\sqrt{n}}\geq 0 \text{ for } t\geq 0,$$

then

$$F(a_n(u)t+b_n(u))=0 \text{ for } t<0$$

and

$$F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))$$

$$=[1-[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$=1-(1+\frac{t\sqrt{2}}{\sqrt{n}})\exp[-\frac{t\sqrt{2}}{\sqrt{n}}]$$

$$=\frac{t^2}{n}+o(\frac{1}{n}),\ t\geq 0.$$

Therefore

$$v(t,u)=\lim_{n\to\infty}n[F(a_n(u)t+b_n(u))$$

$$*F(a_n(u)t+b_n(u))]$$

$$=\lim_{n\to\infty}n[1-(1+\frac{t\sqrt{2}}{\sqrt{n}})\exp[-\frac{t\sqrt{2}}{\sqrt{n}}]]$$

$$=\lim_{n\to\infty}n[\frac{t^2}{n}+o(\frac{1}{n})]=t^2,\ t\geq 0,\ u=1,2,...z,$$

which by *case* 3 in Lemma 2.2 completes the proof.

Corollary 2.2. The reliability function of exponential „*m* out of *n*" system whit a cold reserve of its components is given by

case 1

$$\boldsymbol{IR}^{(2)}{}_n^{(m)}(t,u)\cong 1-\sum_{i=0}^{m-1}\frac{\exp[-i\lambda(u)(t-b_n(u))]}{i!}$$

$$\exp[-\exp[-\lambda(u)t+\lambda(u)b_n(u)], \qquad (9)$$

$t\in(-\infty,\infty),\ u=1,2,...,z.$

case 2

$$\boldsymbol{IR}^{(2)}{}_n^{(\mu)}(t,u)\cong 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Im}e^{-\frac{x^2}{2}}dx \qquad (10)$$

where

$$\Im=\frac{\lambda(u)\sqrt{n+1}}{\sqrt{\mu}}t-\frac{\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}},\ t\in(-\infty,\infty). \qquad (11)$$

case 3

$$\boldsymbol{IR}^{(2)}{}_n^{(\overline{m})}(t,u)=1,\ t<0,$$

$$\boldsymbol{IR}^{(2)}{}_n^{(\overline{m})}(t,u)]$$

$$\cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\sqrt{nt}/\sqrt{2}]^{2i}}{i!} \exp[-\lambda^2(u)nt^2/2], \qquad (12)$$

$t \geq 0$, $u = 1,2,...,z$.

Definition 2.5. A multi-state series system is called an „*m* out of *n*" system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u) = T_{(n-m+1)}(u), m = 1,2,...,n, u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \{\max_{1\leq i\leq s_1 n}\{\max_{1\leq j\leq 2}\{T_{ij}(u)\}\}, \max_{s_1 n+1\leq i\leq n}\{\sum_{j=1}^{2} T_{ij}(u)\}\},$$
$$i = 1,2,..,n., u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1 + s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a mixed reserve of its components is given either by

$$\boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t,\cdot) = [1, \boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t,1),..., \boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t) = 1 - \sum_{i=0}^{m-1}\binom{n}{i}[1-(F(t,u))^2]^{s_1 i}$$
$$[1-F(t,u)*F(t,u)]^{s_2 i}[F(t,u)]^{2(n-i)s_1}$$
$$[F(t,u)*F(t,u)]^{(n-i)s_2}, \qquad (13)$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

or by

$$\overline{\boldsymbol{IR}}^{(3)}{}_{n}^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{IR}}^{(3)}{}_{n}^{(\overline{m})}(t,1),..., \overline{\boldsymbol{IR}}^{(3)}{}_{n}^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{IR}}^{(3)}{}_{n}^{(\overline{m})}(t) = \sum_{i=0}^{\overline{m}}\binom{n}{i}[F(t,u)]^{2s_1 i}[F(t,u)*F(t,u)]^{s_2 i}$$
$$[1-(F(t,u))^2]^{(n-i)s_1}[1-F(t,u)*F(t,u)]^{(n-i)s_2}, \qquad (14)$$
$t \in (-\infty,\infty)$, $\overline{m} = n-m$, $u = 1,2,...,z$.

Lemma 2.3.

case 1: If

(i) $\mathrm{I}\Re^{(3)(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\frac{[V(t,u)]^i}{i!}\exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system whit a mixed reserve of its components defined by (30),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(3)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(3)(m)}(t,u),$$
$t \in C_{I\Re}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[s_1[1-[F(a_n(u)t+b_n(u))]^2]$$
$$+ s_2[1 - F(a_n(u)t+b_n(u)) * F(a_n(u)t+b_n(u))]]$$

$$= V(t,u), \ t \in C_V, \ u = 1,2,...z,$$

case 2: If

(i) $\mathrm{I}\Re^{(3)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}}dx$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(3)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system whit a mixed reserve of its components defined by (30),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, przy $n \to \infty$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(3)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \mathrm{I}\Re^{(3)(\mu)}(t,u),$$
$t \in C_{I\Re}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} \frac{(n+1)[s_1[1-[F(a_n(u)t+b_n(u))]^2]}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$\frac{s_2[1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$=\nu(t,u),\ u=1,2,...,z.$$

case 3: If

(i) $\mathrm{I}\overline{\Re}^{(3)(\overline{m})}(t,u) = \sum_{i=0}^{m}\frac{[\overline{V}(t,u)]^i}{i!}\exp[-\overline{V}(t,u)]$, $\overline{m}=n-m$, u = 1,2,...,z, is non-degenerat reliability function,

(ii) $\boldsymbol{I\overline{R}}^{(3)}{}_{n}^{(\overline{m})}(t,u)$ is the reliability function of non-degenerate multi-state „m out of n" system with a mixed reserve of its components defined by (31),

(iii) $a_n(u) > 0$, $b_n(u)\in(-\infty,\infty)$, u = 1,2,...,z,

(iv) $n - m = \overline{m}$ = constant $(m/n\to 1\ as\ n\to\infty)$,

then

$$\lim_{n\to\infty}\boldsymbol{I\overline{R}}^{(3)}{}_{n}^{(\overline{m})}(a_n(u)t+b_n(u)) = \mathrm{I}\overline{\Re}^{(3)(\overline{m})}(t,u),$$

$t\in C_{\mathrm{I}\overline{\Re}}$, u = 1,2,...,z,

if and only if

$$\lim_{n\to\infty} n[s_1[F(a_n(u)t+b_n(u))]^2$$

$$+ s_2[F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]]$$

$$=\overline{V}(t,u)$$

$t\in C_{\overline{V}}$, u = 1,2,...,z.

Proposition 2.3. If components of the homogeneous multi-state „m out of n" system with a mixed reserve of its components have multi-state exponential reliability functions
and
case 1 m = constant,

$$a_n(u)=\frac{1}{\lambda(u)},\ \frac{\exp[\lambda(u)b_n(u)]}{2s_1+s_2\lambda(u)b_n(u)}=n,\ u=1,2,...,z,$$

then

$$\mathrm{I}\Re^{(3)(m)}(t,u) = 1-\sum_{i=0}^{m-1}\frac{\exp[-it]}{i!}\exp[-\exp[-t]],$$

$t\in(-\infty,\infty)$, u = 1,2,...,z,

case 2 $m/n\to\mu$ $0<\mu<1$, $n\to\infty$,

$$a_n(u)=\frac{\sqrt{\mu/2}}{\lambda(u)\sqrt{(2s_1+s_2)(n+1)}},$$

$$b_n(u)=\frac{1}{\lambda(u)}\sqrt{\frac{2(1-\mu)}{2s_1+s_2}},\ u=1,2,...,z,$$

then

$$\mathrm{I}\Re^{(3)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-\frac{x^2}{2}}dx,$$

$t\in(-\infty,\infty)$, u = 1,2,...,z,

case 3 $n - m = \overline{m}$ = constant ($m/n\to 1$, $n\to\infty$),

$$a_n(u)=\frac{\sqrt{2}}{\lambda(u)\sqrt{(2s_1+s_2)n}},\ b_n(u)=0,\ u=1,2,...,z,$$

then

$$\mathrm{I}\overline{\Re}^{(3)(\overline{m})}(t,u)=1,\ t<0,$$

$$\mathrm{I}\overline{\Re}^{(3)(\overline{m})}(t,u) = \sum_{i=0}^{n-m}\frac{t^{2i}}{i!}\exp[-t^2],\ t\ge 0,\ u=1,2,...,z,$$

is its limit reliability function.

Proof:
case 1: Since for all fixed u, we have

$$a_n(u)t+b_n(u)\to\infty \text{ as } n\to\infty \text{ for } t\in(-\infty,\infty),$$

and

$$1-[F(a_n(u)t+b_n(u))]^2$$

$$=2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t+b_n(u))],\ t\in(-\infty,\infty),$$

$$1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))$$

$$=[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))],\ t\in(-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[s_1[1-[F(a_n(u)t+b_n(u))]^2]$$

$$+ s_2[1 - F(a_n(u)t + b_n(u))$$

$$* F(a_n(u)t + b_n(u))]]$$

$$= \lim_{n\to\infty}[ns_1[2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$
$$-\exp[-2\lambda(u)(a_n(u)t+b_n(u))]]$$

$$+ ns_2[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$= \lim_{n\to\infty}[ns_1 2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$

$$[1-1/2\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$+ ns_2\lambda(u)b_n(u)[1+\frac{1+\lambda(u)a_n(u)t}{\lambda(u)b_n(u)}]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$= \lim_{n\to\infty}\exp[-t][ns_1 2\exp[-\lambda(u)b_n(u)]$$

$$[1-\frac{1}{2}\exp[-t]\exp[-\lambda(u)b_n(u)]]$$

$$+ ns_2\lambda(u)b_n(u)[1+\frac{1+t}{\lambda(u)b_n(u)}]$$

$$\exp[-\lambda(u)b_n(u)]]$$

$$= \lim_{n\to\infty}\exp[-t][ns_1 2\exp[-\lambda(u)b_n(u)]$$

$$- ns_1\exp[-t]\exp[-2\lambda(u)b_n(u)]$$

$$+ ns_2\lambda(u)b_n(u)\exp[-\lambda(u)b_n(u)]$$

$$+ ns_2[1+t]\exp[-\lambda(u)b_n(u)]]$$

$$= \lim_{n\to\infty}\exp[-t]$$

$$[1-\frac{s_1}{n(2s_1+\lambda(u)b_n(u)s_2)^2}\exp[-t]$$

$$+\frac{s_2}{(2s_1+\lambda(u)b_n(u)s_2)}(1+t)]$$

$$= \exp[-t],\ t\in(-\infty,\infty),\ u = 1,2,...,z,$$

which by *case* 1 in Lemma 2.3 completes the proof.

case 2: Since for all fixed *u*, we have

$$a_n(u)t+b_n(u) \to \frac{1}{\lambda(u)}\sqrt{\frac{2(1-\mu)}{2s_1+s_2}} > 0 \text{ as } n\to\infty,$$
$$t\in(-\infty,\infty).$$

and

$$1 - [F(a_n(u)t+b_n(u))]^2$$

$$= 2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t+b_n(u))]$$

$$= 2[1-\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2]$$

$$-[1-2\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}4\lambda^2(u)(a_n(u)t+b_n(u))^2]+o(\frac{1}{n+1})$$

$$=1-\lambda^2(u)(a_n(u)t+b_n(u))^2 + o(\frac{1}{n+1}),$$
$$t\in(-\infty,\infty),$$

$$1 - F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))$$

$$=[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$[1-\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2 - o(\frac{1}{n+1})]$$

$$=1-\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2 - o(\frac{1}{n+1})],$$
$$t\in(-\infty,\infty).$$

Therefore

$$s_1[1-[F(a_n(u)t+b_n(u))]^2]$$

$$+ s_2[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]$$

$$= s_1[1 - \lambda^2(u)(a_n(u)t + b_n(u))^2]$$

$$+ s_2[1 - \frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2] + o(\frac{1}{n+1})$$

$$= 1 - \frac{2s_1 + s_2}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2 + o(\frac{1}{n+1})$$

$$= -\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{n+1}), \quad t \in (-\infty, \infty),$$

next

$$\nu(t,u) = \lim_{n\to\infty} \frac{(n+1)[s_1[1 - [F(a_n(u)t + b_n(u))]^2]}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$+ \frac{s_2[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{n+1})] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{-\sqrt{\mu(1-\mu)}t + o(1)}{\sqrt{\mu(1-\mu)}} = -t, \quad t \in (-\infty, \infty),$$

$u = 1,2,..., z,$

which by *case* 2 in Lemma 2.3 completes the proof.

case 3: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{\sqrt{2}t}{\lambda(u)\sqrt{(2s_1 + s_2)n}} < 0, \quad t < 0$$

and

$$a_n(u)t + b_n(u) = \frac{t\sqrt{2}}{\lambda(u)\sqrt{(2s_1 + s_2)n}} \geq 0, \quad t \geq 0,$$

then

$$F(a_n(u)t + b_n(u)) = 0, \quad t < 0,$$

and for $t \geq 0$

$$[F(a_n(u)t + b_n(u))]^2$$

$$= [1 - \exp[-\lambda(u)(a_n(u)t + b_n(u))]]^2,$$

$$F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$

$$= [1 - [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]].$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[s_1[F(a_n(u)t + b_n(u))]^2$$

$$+ s_2[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]]$$

$$= \lim_{n\to\infty}[ns_1[1 - \exp[-\lambda(u)(a_n(u)t + b_n(u))]]^2$$

$$+ ns_2[1 - [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]]]$$

$$= \lim_{n\to\infty}[ns_1[\lambda(u)(a_n(u)t + b_n(u))]^2$$

$$+ ns_2[\lambda^2(u)(a_n(u)t + b_n(u))^2$$

$$- \frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2]]$$

$$= \lim_{n\to\infty} n[\frac{2s_1 + s_2}{2}(\lambda(u)a_n(u)t)^2]$$

$$= \lim_{n\to\infty} n[\frac{t^2}{n}] = t^2, \quad t \geq 0, \; t \in (-\infty, \infty), \; u = 1,2,...,z,$$

which by *case* 3 in Lemma 2.3 completes the proof.

Corollary 2.3. The reliability function of exponential „*m* out of *n*" system with a mixed reserve of its components is given by

case 1

$$\boldsymbol{IR}^{(3)}{}_n^{(m)}(t,u) \cong 1 - \sum_{i=0}^{m-1} \frac{\exp[-i\lambda(u)(t - b_n(u))]}{i!}$$

$$\exp[-\exp[-\lambda(u)t + \lambda(u)b_n(u)], \qquad (15)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 2

$$\boldsymbol{IR}^{(3)}{}_{n}^{(\mu)}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Im} e^{-\frac{x^2}{2}} dx, \qquad (16)$$

where

$$\Im = \frac{\lambda(u)\sqrt{(2s_1 + s_2)(n+1)2}}{\sqrt{\mu}} t - \frac{2\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}}, \quad (17)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 3

$$\boldsymbol{IR}^{(3)}{}_{n}^{(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\boldsymbol{IR}^{(3)}{}_{n}^{(\overline{m})}(t,u) \cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\sqrt{(2s_1 + s_2)nt}/\sqrt{2}]^{2i}}{i!}$$

$$\exp[-\lambda^2(u)(2s_1 + s_2)nt^2/2], \qquad (18)$$

$t \geq 0$ $u = 1,2,...,z$.

Proposition 2.3. If components of the homogeneous multi-state „*m* out of *n*" system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u) \in <0,1>$, $u = 1,2,...,z$,
and
case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)}, \; b_n(u) = \frac{\log n}{\lambda(u)\rho(u)}, \; u = 1,2,...,z,$$

then

$$\mathrm{I}\Re^{(4)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]],$$

$t \in (-\infty, \infty)$,

case 2 $m/n \to \mu$, $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)\sqrt{n+1}} \sqrt{\frac{1-\mu}{\mu}}, \; b_n(u) = \frac{\log(1/\mu)}{\lambda(u)\rho(u)},$$

$u = 1,2,...,z$,

then

$$\mathrm{I}\Re^{(4)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx, \; t \in (-\infty, \infty),$$

case 3 $n - m = \overline{m}$ constant ($m/n \to 1$, $n \to \infty$),

$$a_n(u) = \frac{1}{n\lambda(u)\rho(u)}, \; b_n(u) = 0, \; u = 1,2,...,z,$$

then

$$\mathrm{I}\overline{\Re}^{(4)(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\mathrm{I}\overline{\Re}^{(4)(\overline{m})}(t,u) = \sum_{i=0}^{n-m} \frac{t^i}{i!} \exp[-t], \; t \geq 0,$$

is its limit reliability function.

Corollary 2.3. The reliability function of exponential „*m* out of *n*" system with improved reliability functions of its components is given by
case 1

$$\boldsymbol{IR}^{(4)}{}_{n}^{(m)}(t,u) \cong 1 - \sum_{i=0}^{m-1} \frac{\exp[-i\lambda(u)\rho(u)t - \log n)]}{i!}$$

$$\exp[-\exp[-\lambda(u)\rho(u)t + \log n], \qquad (19)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 2

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\mu)}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Im} e^{-\frac{x^2}{2}} dx, \qquad (20)$$

where

$$\Im = \frac{\lambda(u)\rho(u)\sqrt{(n+1)\mu}}{\sqrt{1-\mu}} t - \frac{\sqrt{n+1}\sqrt{\mu}}{\sqrt{1-\mu}} \log(1/\mu), \quad (21)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 3

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\overline{m})}(t,u)$$

$$\cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\rho(u)nt]^i}{i!} \exp[-\lambda(u)\rho(u)nt], \qquad (22)$$

$t \geq 0$, $u = 1,2,...,z$.

3. Comparison of reliability improvement effects

The comparisons of the limit reliability functions of the systems with different kinds of reserve and such systems with improved components allow us to find the value of the components decreasing failure rate factor $\rho(u)$, which warrants an equivalent effect of the system reliability improvement.

3.1. The "*m* out of *n*" system

The comparison of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equations

$$I\Re^{(4)(m)}\left((t-b_n(u))/a_n(u)\right)$$

$$= I\Re^{(k)(m)}\left((t-b_n(u))/a_n(u)\right), \qquad (23)$$

$u = 1,2,...,z$, $k = 1,2,3$.

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state „*m* out of *n*" system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (23) are respectively given by

k = 1

case 1 $\rho(u) = \rho(t,u) = 1 - \dfrac{\ln 2}{\lambda(u)t}$, $u = 1,2,...,z$,

case 2 $\rho(u) = \dfrac{2\sqrt{1-\mu}}{\mu} - \dfrac{2(1-\mu)+\mu\log\mu}{\lambda(u)\mu t}$,
$u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \lambda(u)t$, $u = 1,2,...,z$,

k = 2

case 1 $\rho(u) = 1 - \dfrac{\lambda(u)b_n(u) - \log n}{\lambda(u)t}$

$$= 1 - \frac{\log \lambda(u)b_n(u)}{\lambda(u)t},\ u = 1,2,...,z,$$

case 2 $\rho(u) = \dfrac{\sqrt{1-\mu}}{\mu} - \dfrac{1-\mu+\mu\log\mu}{\lambda(u)\mu t}$, $u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \dfrac{\lambda(u)t}{2}$, $u = 1,2,...,z$,

k = 3

case 1 $\rho(u) = 1 - \dfrac{\lambda(u)b_n(u) - \log n}{\lambda(u)t}$

$$= 1 - \frac{\log(2s_1 + s_2\lambda(u)b_n(u))}{\lambda(u)t},\ u = 1,2,...,z,$$

case 2

$$\rho(u) = \frac{\sqrt{2(2s_1+s_2)}\sqrt{1-\mu}}{\mu} - \frac{2(1-\mu)+\mu\log\mu}{\lambda(u)\mu t},$$
$u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \dfrac{(2s_1+s_2)\lambda(u)t}{2}$, $u = 1,2,...,z$.

4. Conclusion

Proposed in the paper application of the limit multi-state reliability functions for reliability of large systems evaluation and improvement simplifies calculations. The methods may be useful not only in the technical objects operation processes but also in their new processes designing, especially in their optimization. The case of series, parallel (in part 1) and „*m* out of *n*" systems composed of components having exponential reliability functions with double reserve of their components is considered only. It seems to be possible to extend the results to the systems having other much complicated reliability structures and components with different from the exponential reliability function. Further, it seems to be reasonable to elaborate a computer programs supporting calculations and accelerating decision making, addressed to reliability practitioners.

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Fundamentals of the Dempster-Shafer theory and its applications to system safety and reliability modelling

Keywords

Dempster-Shafer Theory, evidence, uncertainty, expert assessment, Event Tress Analysis, Fault Tree Analysis

Abstract

The Dempster-Shafter Theory is well-known for its usefulness to express uncertain judgments of experts. This contribution shows how to apply the calculus to safety and reliability modelling, especially to expert judgement; Failure Modes, Effects, and Criticality Analysis; Event Tree Analysis, Fault Tree Analysis, and Reliability Centred Maintenance. Including a tutorial introduction to the Dempster-Shafer Theory, the differences between the Probability and the Dempster-Shafer Theory are discussed widely.

1. Introduction

In the middle of the 1960s Arthur P. Dempster developed a theory [1], [2], [3] that includes a kind of "upper and lower probabilities". Later, it turned out that this approach is very useful to express uncertain judgments of experts.

About ten years later the work of Dempster was extended, refined, recast, and published by Glenn Shafer [18] as a "Mathematical Theory of Evidence". Shafer e.g. rebuilt the mathematical theory around the Dempster concept and introduced degrees of belief instead of lower probabilities. The Theory of Evidence was also denoted as the *Dempster-Shafer Theory* (DST) or the *Dempster-Shafer Evidential Theory*.

The more the *Dempster-Shafer Theory* was further developed, the more the evidence measures of DST departed from being probabilities, e.g. Klir & Folger [11] revised DST in that sense. As stated by [4] and [20], the advantage of DST is that it allows coping with absence of preference, due to limitations of the available information, which results in indeterminacy.

2. Fundamentals

The DST became known to the safety and reliability community in the early 1990s, refer e.g. Guth [7]. The reliability-oriented approach to DST as presented here is based on a *scenario* that contains the system with all hypotheses, pieces of evidence and data sources.

The *hypotheses* represent all the possible states (e.g. faults) of the system considered. It is required that all hypotheses are elements (singletons) of the *frame of discernment*, which is given by the finite universal set Ω. The set of all subsets of Ω is its power set 2^{Ω}. A subset of those 2^{Ω} sets may consist of a single hypothesis or of a conjunction of hypotheses. Moreover, it is required that all hypotheses are unique, not overlapping and mutually exclusive.

In this context *pieces of evidence* are symptoms or events (e.g. failures) that occurred or may occur within a system. One piece of evidence is related to a single hypothesis or a set of hypotheses. It is not allowed that different pieces of evidence lead to the same hypothesis or set of hypotheses.

The qualitative relation between a piece of evidence and a hypothesis corresponds to a cause-consequence chain: A piece of evidence implies a hypothesis or a set of hypotheses, respectively. The strength of an evidence-hypothesis assignment, and thereby the strength of this implication, is quantified by a statement of a data source.

Data sources are persons, organisations, or any other entities that provide information for a scenario. In safety and reliability engineering, data sources are usually the results of empirical studies or they are experts, who give subjective quantifiable statements. As required by O'Neill [13], data sources have to be representative (e.g. studies) or as free from bias as possible (e.g. experts).

Some misunderstandings in interpretations concerning the plot of hypotheses have to be cleared up. From an objective point of view, which might e.g. be lo-

cated outside the system (e.g. observer), exactly one single hypothesis is true; from a subjective point of view of a data source (e.g. expert or operator), it might be uncertain which hypothesis fits best to reality. Therefore, DST makes it possible to model several

- single pieces of evidence within single hypothesis relations or
- single pieces of evidence within multi hypotheses relations

as uncertain assessments of a system in which exactly one hypothesis is objectively true. Both points of view, the objective and the subjective, have to be distinguished clearly. The DST calculus describes the subjective viewpoint as an assessment for an unknown objective fact.

By means of a data source, a mapping

$$m\colon 2^{\Omega} \rightarrow [0,\ 1] \tag{1}$$

assigns an evidential weight to a set $\boldsymbol{A} \subseteq \Omega$, which contains a single hypothesis or a set of hypotheses. This is the most significant difference to the Probability Theory: The DST mapping distinguishes clearly between the evidence measures and probabilities with mapping $\Omega \rightarrow [0,\ 1]$. Each $\boldsymbol{A}$ that holds $m(\boldsymbol{A}) > 0$ is called a focal element. The function m is called a *basic assignment* and fulfils

$$\sum_{\boldsymbol{A}\subseteq\Omega} m(\boldsymbol{A}) = 1\ . \tag{2}$$

This equation means that all statements of a single data source have to be normalised, just to ensure that the evidence presented by each data source is equal in weight, e.g. no data source is more important than another one. – For the "sake of simplicity" (Klir & Folger [11]), it is assumed that

$$m(\varnothing) = 0\ ; \tag{3}$$

however, this property requires an appropriate choice of the universal set Ω. That means, the set Ω has to be complete and contain all possible hypotheses of the scenario considered.

In some publications m is called the *basic probability assignment*, refer Shafer [18], which misleads to the assumption $m(\boldsymbol{A})$ might be a probability. Further denotations are the *basic belief assignment* [19], the *belief structure* [21], [4] or the *mass assignment function* [14].

A clear distinction has to be made between probabilities and basic belief assignment: probability distribution functions are defined on Ω and basic assignment functions on the power set 2^{Ω}. In addition, m has three further properties, which distinguishes it from being a probability function, refer Klir & Folger [11]: It is not required

- that $m(\Omega) = 1$,
- that $m(\boldsymbol{A}) \leq m(\boldsymbol{B})$ if $\boldsymbol{A} \subset \boldsymbol{B}$, or
- that there is a relationship between $m(\boldsymbol{A})$ and $m(\neg\boldsymbol{A})$.

Therefore, it seems to be useful to avoid the terms *probability* and *belief* (which is defined next) in the denotation of m.

By applying the *basic assignment function*, several *evidential functions* can be created. A *belief measure* is given by the function $bel\colon 2^{\Omega} \rightarrow [0,1]$. There is

$$bel(\boldsymbol{A}) = \sum_{B\subseteq A;B\neq\phi} m(\boldsymbol{B})\,. \tag{4}$$

The counterpart of *bel* is the plausibility measure $pl\colon 2^{\Omega} \rightarrow [0,1]$ with

$$pl(\boldsymbol{A}) = \sum_{B\cap A\neq\phi} m(\boldsymbol{B})\,. \tag{5}$$

The measure $pl(\boldsymbol{A})$ shall not be understood as the complement of $bel(\boldsymbol{A})$. Only

$$\{\boldsymbol{A} \subseteq \Omega \mid m(\boldsymbol{A}) > 0\} \neq \varnothing \rightarrow bel(\boldsymbol{A}) \leq pl(\boldsymbol{A}) \tag{6}$$

has to be fulfilled. In addition to *bel* and *pl*, a third evidential function can be defined. Shafer [18] introduced the *commonality measure* with $cmn\colon 2^{\Omega} \rightarrow [0,1]$ and

$$cmn(\boldsymbol{A}) = \sum_{B\supseteq A} m(\boldsymbol{B})\,. \tag{7}$$

Figure 1 shows a graphical representation of the above-defined measures *belief* and *plausibility*. The difference $pl(\boldsymbol{A}) - bel(\boldsymbol{A})$ describes the evidential interval range, which represents the uncertainty concerning the set $\boldsymbol{A}$.

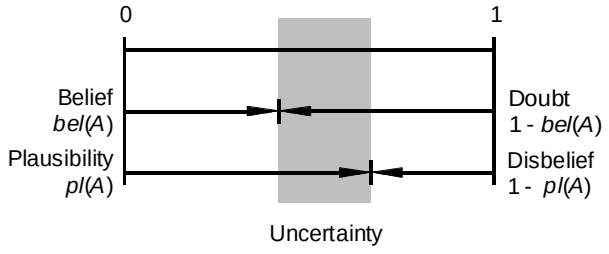

Figure 1. Measures of belief and plausibility and its complements for a given bel(**A**) < pl(**A**). The evidential or the uncertainty interval, respectively, is shaded grey.

The complements to the measures *belief* and *plausibility* are *doubt* and *disbelief*, respectively. Although Shafer ([18], page 43) defines *doubt* as a complement

to the measure plausibility, it seems to make more sense to distinguish between *doubt* and *disbelief* in the way given above because DST does not require a causal relationship between a hypothesis and its negation. As Flack [6] emphasises, lack of *belief* does not imply *disbelief*. The *disbelief* of set $\underline{\underline{\boldsymbol{A}}}$ is the belief in the complement. There is

$$bel(\neg \boldsymbol{A}) = 1 - pl(\boldsymbol{A}) \; , \; pl(\neg \boldsymbol{A}) = 1 - bel(\boldsymbol{A}) \tag{8}$$

with

$$bel(\neg \boldsymbol{A}) \leq pl(\neg \boldsymbol{A}) \; . \tag{9}$$

The difference $pl(\boldsymbol{A}) - bel(\boldsymbol{A})$ describes the *uncertainty* concerning the hypothesis $\boldsymbol{A}$ represented by the evidential interval, see *Figure 1*.

2.1. Interpretations on Evidence Measures

Some helpful and interesting interpretations of the evidence measures are given in the literature and cited here.

Basic Assignment

- The measure $m(\boldsymbol{A})$ assigns an evidential weight to the set $\boldsymbol{A}$, refer Flack [6].
- The measure $m(\boldsymbol{A})$ is the degree of evidence that the element in question belongs exactly to the set $\boldsymbol{A}$, refer Klir & Folger [11].
- The measure $m(\boldsymbol{A})$ is the degree of evidence supporting the claim that a specific element of Ω belongs to the set $\boldsymbol{A}$, but not to any special subset of $\boldsymbol{A}$, refer Klir & Folger [11].
- The quantity $m(\boldsymbol{A})$ is the degree of belief that the above specified claim is warranted, refer Klir & Folger [11].

Belief

- The measure $bel(\boldsymbol{A})$ is the degree of evidence that the element in question belongs to the set $\boldsymbol{A}$ as well as to the various special subsets of $\boldsymbol{A}$, refer Klir & Folger [11].
- The measure $bel(\boldsymbol{A})$ can be interpreted as the total amount of justified support given to $\boldsymbol{A}$, refer Denoeux [4].
- The measure $bel(\boldsymbol{A})$ is the degree of evidence supporting the claim that a specific element of Ω belongs to the set $\boldsymbol{A}$, but not to any special subset of $\boldsymbol{A}$, refer Klir & Folger [11].

Plausibility

- The quantity $pl(\boldsymbol{A})$ is the degree of evidence that the element in question belongs to the set $\boldsymbol{A}$ or to any of its subsets [or to any set that overlaps with **A**], refer Klir & Folger [11].
- The quantity $pl(\boldsymbol{A})$ can be interpreted as the maximum amount of specific support that could be given to $\boldsymbol{A}$, if justified by additional information, refer Smets [19].

2.2. Bayesian Statistical Modelling versus Dempster-Shafer Theory

Flack [6] describes the differences between the *Bayesian statistical modelling* and the *Dempster-Shafer Theory* as a difference in concepts. A Bayesian model describes a Boolean type of phenomena, which either exist or do not exist. Little belief in the existence of a phenomenon implies a strong belief in its non-existence. This implication does not necessarily hold for DST. Here, no causal relationship is required between both, belief in existence and belief in non-existence. For example, a statement concerning the failure probability of an item also implies a statement about its counterpart, the reliability of the same item. DST does not require this sub-proposition; and that adds new aspects and possibilities to reliability modelling.

As stated by Ferson et al. [5], the Dempster-Shafer Theory has been widely studied in computer science and artificial intelligence, but has never achieved complete acceptance among probabilists and traditional statisticians. (By this, the question arises if any other theory than the Probability Theory would ever be accepted by probabilists or traditional statisticians.)

However, there are still some disadvantages of the Probability Theory for the DST, which should also be stated here. There are three undesired main properties as listed by [4]:

- *Lack of introspection or assessment strategies*: The main criticisms of the Bayesian statistical modelling is its unreasonable requirement for precision. But the necessity to assign precise numbers in DST applications to each subset $\boldsymbol{A} \subseteq \Omega$ by the basic assignment m is constraining in the same way. Precise degrees of the desired measures may exist, but it is perhaps too difficult to determine them with the necessary precision.
- *Instability*: Underlying beliefs may be unstable. Estimated beliefs may be influenced by the conditions of its estimation.
- *Ambiguity*: Ambiguous or imprecise judgement could not be expressed by the evidence measures.

Additional statements to disadvantages of the Dempster-Shafer Theory are:

- DST lists all the hypotheses into the frame of discernment Ω, which resembles the fault space in the Bayesian Theory. However, given k hypotheses, it can consist of up to 2^k elements, representing all possible subsets of Ω. This leads to a similar problem encountered in the Bayesian Theory, except that it is worse because human experts have to es-

timate a larger number of belief values than after the Bayesian theory [12].

- Another caveat of the applicability of DST is that it does not offer a procedure for implementation of a diagnostic system [10].

2.3. Dempster-Shafer Rule of Combination

Dempster [2], [3] followed by Shafer [18] suggested a rule of combination which allows that the basic assignments are combined. There is

$$m(\boldsymbol{Z}) = \frac{\sum_{\boldsymbol{A}\cap\boldsymbol{B}=\boldsymbol{Z}\neq\phi} m(\boldsymbol{A})\cdot m(\boldsymbol{B})}{1-\sum_{\boldsymbol{A}\cap\boldsymbol{B}=\phi} m(\boldsymbol{A})\cdot m(\boldsymbol{B})}, \tag{10}$$

with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Z} \subseteq \Omega$. Verbally: the numerator represents the accumulated evidence for the sets $\boldsymbol{A}$ and $\boldsymbol{B}$, which supports the hypothesis $\boldsymbol{Z}$, and the denominator sum quantifies the amount of conflict between the two sets. Depending on the application, the denominator of

$$m(\boldsymbol{Z}) = \frac{\sum_{\boldsymbol{A}\cap\boldsymbol{B}=\boldsymbol{Z}\neq\phi} m(\boldsymbol{A})\cdot m(\boldsymbol{B})}{\sum_{\boldsymbol{A}\cap\boldsymbol{B}\neq\phi} m(\boldsymbol{A})\cdot m(\boldsymbol{B})}, \tag{11}$$

is easier to apply.

3. Illustration

The following illustration is strictly step-wise structured and gives an easy-to-understand introduction to the calculus of the Dempster-Shafer Theory.

The given scenario discusses a typical situation in a power plant. The operators at the control panel detect serious changes of the system properties. Some failures are detectable; however, their consequences or the system fault, respectively, can neither be determined exactly nor interpreted certainly. (This situation is widely discussed e.g. in the ATHEANA Report [14] and by Hollnagel [8].) To avoid an error forcing context, pieces of evidence are collected, hypotheses are postulated, and conclusions are made on this basis. Therefore, a Dempster-Shafer approach is applied to support the operators in decision making.

Step ∈ – *Creating the Scenario*

The scenario consists of a power plant (the system considered), two operators (data sources, denoted by the index *l*), the failures detected (pieces of evidence), and the system fault states (set of hypotheses). As described above, the pieces of evidence correspond to *failures* or *causes* and the hypotheses to *faults* or *consequences*.

The faults can be determined to at most three precisely defined hypotheses represented by the set Ω with

$$\Omega = \{h_1, h_2, h_3\} . \tag{12}$$

With that, the frame of discernment of this context is given. It should be noted that Ω is postulated by the operators based on their subjective points of view, assuming that Ω is complete. The corresponding power set of Ω is

$$2^{\Omega} = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \Omega\} . \tag{13}$$

The first operator mainly states that the faults h_1 or h_2 are the reason for the problems. For example, the failure ev_3 might have occurred and resulted in the consequences h_1 or h_2. The assignments of the second operator are slightly different. Here, focus is on the faults h_1 or h_3. Both operators give their statements to the four pieces of evidence found. The complete survey of the qualitative failure-fault(s) assignments is given in *Table 1*.

Table 1. Qualitative failure-fault(s) assignments given by the operators involved

		Failure	Fault(s)
Operator	1st	ev_1	h_1
		ev_2	h_2
		ev_3	h_1, h_2
		ev_4	h_1, h_2, h_3
	2nd	ev_1	h_1
		ev_2	h_3
		ev_3	h_1, h_3
		ev_4	h_1, h_2, h_3

Please note that contrary to the *Fault Tree Analysis* (FTA), DST does not allow that more than one failure lead to the same fault (hypothesis). However, different failures may have different set of consequences, which may contain the same hypotheses as elements, e.g. ev_1, ev_3, and ev_4 in *Table 1* lead to hypotheses, which all contain h_1 as a fault. Again, DST allows modelling several

- *single*-failure-*single*-fault relations and
- *single*-failure-*multi*-fault relations

as an uncertain assessment of a system, which can take exactly one state at a time. Generally, DST emphasizes more on the hypotheses (faults) than on the pieces of evidence (failures), which are of minor interest in the next steps.

As described in Section 0, the system is exactly in one state of Ω. In other words, exactly one hypothesis

of $\{h_1, h_2, h_3\}$ is true for the given scenario and situation if the system would be observed from an objective point of view. Subjectively, the operators are not sure, in which state the system actually is.

Step • – *Quantification of Statements*

At this step, both operators quantify their statements as given in Table 2. The set of hypotheses $\boldsymbol{A}$ is assigned to the first operator, $\boldsymbol{B}$ to the second operator. For example, the second operator claims that the consequences h_1 or h_3 may have occurred with a basic assignment of 0.4. (Formulating this sentence verbally, it is rather difficult to avoid that the tongue mentions "probability". Again, basic assignments are not probabilities, see equation (1).) Non-specified statements are assigned by 0 and are not focal elements.

The subjective quantifications of the operators are based on their system experiences and mostly on their "engineering feelings". Certainly, these quantifications are imprecise.

Table 2. Quantitative statements given by the operators involved (outer columns). The inner column contains all subsets of the power set 2^{Ω}.

1st operator	2^{Ω}	2nd operator
$m(\boldsymbol{A}_1) = 0.2$	$\{\boldsymbol{h}_1\}$	$m(\boldsymbol{B}_1) = 0.2$
$m(\boldsymbol{A}_2) = 0.1$	$\{\boldsymbol{h}_2\}$	$m(\boldsymbol{B}_2) = 0$
$m(\boldsymbol{A}_3) = 0$	$\{\boldsymbol{h}_3\}$	$m(\boldsymbol{B}_3) = 0.2$
$m(\boldsymbol{A}_4) = 0.6$	$\{\boldsymbol{h}_1 \cup \boldsymbol{h}_2\}$	$m(\boldsymbol{B}_4) = 0$
$m(\boldsymbol{A}_5) = 0$	$\{\boldsymbol{h}_1 \cup \boldsymbol{h}_3\}$	$m(\boldsymbol{B}_5) = 0.4$
$m(\boldsymbol{A}_6) = 0$	$\{\boldsymbol{h}_2 \cup \boldsymbol{h}_3\}$	$m(\boldsymbol{B}_6) = 0$
$m(\boldsymbol{A}_7) = 0.1$	$\{\boldsymbol{h}_1 \cup \boldsymbol{h}_2 \cup \boldsymbol{h}_3\}$	$m(\boldsymbol{B}_7) = 0.2$

Based on the basic assignments given by both operators, the *belief* and *doubt*, *commonality*, *plausibility* and *disbelief* measures can be calculated. For example, the *belief* in the set of hypotheses $\{h_1 \cup h_2\}$ is the sum of its own basic assignment with those of all of its subsets

$$\{h_1\}, \{h_2\}, \{h_1 \cup h_2\} \subseteq \{h_1 \cup h_2\}, \quad (14)$$

see equation (4). For the fourth statement of the first operator there is

$$bel(\boldsymbol{A}_4) = m(\boldsymbol{A}_1) + m(\boldsymbol{A}_2) + m(\boldsymbol{A}_4) = 0.9, \quad (15)$$

with the corresponding *doubt* measure

$$1 - bel(\boldsymbol{A}_4) = 0.1 . \quad (16)$$

The *commonality* takes every statement into account, which includes the discussed statement completely. There is for $\boldsymbol{A}_4$

$$\{h_1 \cup h_2\}, \{h_1 \cup h_2 \cup h_3\} \supseteq \{h_1 \cup h_2\}, \quad (17)$$

$$cmn(\boldsymbol{A}_4) = m(\boldsymbol{A}_4) + m(\boldsymbol{A}_7) = 0.7 . \quad (18)$$

The *plausibility* includes basic assignments of all statements which have got at least one hypothesis with those of the discussed statement in common. Concerning $\boldsymbol{A}_4$, there is

$$\{h_1\}, \{h_2\}, \{h_1 \cup h_2\}, \{h_1 \cup h_3\}, \{h_2 \cup h_3\},$$

$$\{h_1 \cup h_2 \cup h_3\} \cap \{h_1 \cup h_2\} \neq \varnothing, \quad (19)$$

which results in the *plausibility*

$$pl(\boldsymbol{A}_4) = m(\boldsymbol{A}_1) + m(\boldsymbol{A}_2) + m(\boldsymbol{A}_4) + m(\boldsymbol{A}_5) + m(\boldsymbol{A}_6) + m(\boldsymbol{A}_7) = 1 \quad (20)$$

and finally in no *disbelief* at all

$$1 - pl(\boldsymbol{A}_4) = 0 . \quad (21)$$

Table 3 shows the results for *belief* and *plausibility* of all statements ($k = 1,\ldots, 7$).

Table 3. This table corresponds to *Table 2* and shows the values of basic assignments (bold typing), belief, and plausibility for each statement and operator (first left side, second right side).

$m(A_k)$	$bel(A_k)$	$pl(A_k)$	2^{Ω}	$m(B_k)$	$bel(B_k)$	$pl(B_k)$
0.2	0.2	0.9	$\{\boldsymbol{h}_1\}$	**0.2**	0.2	0.8
0.1	0.1	0.8	$\{\boldsymbol{h}_2\}$	**0**	0	0.2
0	0	0.1	$\{\boldsymbol{h}_3\}$	**0.2**	0.2	0.8
0.6	0.9	1	$\{\boldsymbol{h}_1 \cup \boldsymbol{h}_2\}$	**0**	0.2	0.8
0	0.2	0.9	$\{\boldsymbol{h}_1 \cup \boldsymbol{h}_3\}$	**0.4**	0.8	1
0	0.1	0.8	$\{\boldsymbol{h}_2 \cup \boldsymbol{h}_3\}$	**0**	0.2	0.8
0.1	1	1	Ω	**0.2**	1	1

Step Ž – *Combining Hypotheses*

The third step combines each hypothesis or set of hypotheses, respectively, from one data source (operator) with one from the other source and builds the cut set of both, see *Table 4*. Depending on quantifications given at Step •, some of the cut sets may not be focal elements before or thereafter. Actually, this step was fit in for illustrative purpose and can be combined with Step •.

Table 4. The Combination Table contains the full plot of hypotheses cut sets of $\boldsymbol{A}$ and $\boldsymbol{B}$

∩	A_1	A_2	A_3	A_4	A_5	A_6	A_7
B_1	h_1	∅	∅	h_1	h_1	∅	h_1
B_2	∅	h_2	∅	h_2	∅	h_2	h_2
B_3	∅	∅	h_3	∅	h_3	h_3	h_3
B_4	h_1	h_2	∅	$h_1 \cup h_2$	h_1	h_2	$h_1 \cup h_2$
B_5	h_1	∅	h_3	h_1	$h_1 \cup h_3$	h_3	$h_1 \cup h_3$
B_6	∅	h_2	h_3	h_2	h_3	$h_2 \cup h_3$	$h_2 \cup h_3$
B_7	h_1	h_2	h_3	$h_1 \cup h_2$	$h_1 \cup h_3$	$h_2 \cup h_3$	Ω

Step • – *Reducing the Combination Table*

To avoid mathematical effort, those columns and rows of the Combination *Table 4* were dropped, which are related to non-focal elements (non-specified statements with $m(\boldsymbol{A}_k) = 0$, $m(\boldsymbol{B}_k) = 0$). In this context, columns $\boldsymbol{A}_3$, $\boldsymbol{A}_5$, $\boldsymbol{A}_6$ and rows $\boldsymbol{B}_2$, $\boldsymbol{B}_4$, $\boldsymbol{B}_6$ are not applicable. *Table 5* shows the reduced plot containing the combinations of focal elements exclusively.

Table 5. The reduced Combination Table

∩	A_1	A_2	A_4	A_7
B_1	h_1	∅	h_1	h_1
B_3	∅	∅	∅	h_3
B_5	h_1	∅	h_1	$h_1 \cup h_3$
B_7	h_1	h_2	$h_1 \cup h_2$	Ω

Step • – *Calculating Products and Sums of Combined Basic Assignments*

At this step, products of the related basic assignments are calculated from the non-empty sets. Products of basic assignments corresponding to the same cut set have to be added. For $\{h_1\}$ yields

$$\boldsymbol{Z}_1 = \boldsymbol{A}_1 \cap \boldsymbol{B}_1 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_1) = m(\boldsymbol{A}_1) \cdot m(\boldsymbol{B}_1) = 0.04\,, \tag{22}$$

$$\boldsymbol{Z}_2 = \boldsymbol{A}_1 \cap \boldsymbol{B}_5 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_2) = m(\boldsymbol{A}_1) \cdot m(\boldsymbol{B}_5) = 0.08\,, \tag{23}$$

$$\boldsymbol{Z}_3 = \boldsymbol{A}_1 \cap \boldsymbol{B}_7 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_3) = m(\boldsymbol{A}_1) \cdot m(\boldsymbol{B}_7) = 0.04\,, \tag{24}$$

$$\boldsymbol{Z}_4 = \boldsymbol{A}_4 \cap \boldsymbol{B}_1 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_4) = m(\boldsymbol{A}_4) \cdot m(\boldsymbol{B}_1) = 0.12\,, \tag{25}$$

$$\boldsymbol{Z}_5 = \boldsymbol{A}_4 \cap \boldsymbol{B}_5 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_5) = m(\boldsymbol{A}_4) \cdot m(\boldsymbol{B}_5) = 0.24\,, \tag{26}$$

$$\boldsymbol{Z}_6 = \boldsymbol{A}_7 \cap \boldsymbol{B}_1 = \{h_1\}$$

$$\Rightarrow m(\boldsymbol{Z}_6) = m(\boldsymbol{A}_7) \cdot m(\boldsymbol{B}_1) = 0.02 \tag{27}$$

with the sum

$$\sum_{k=1}^{6} m(\boldsymbol{Z}_k) = 0.54\,. \tag{28}$$

Hypotheses h_2 or h_3 are supported by

$$\boldsymbol{Z}_7 = \boldsymbol{A}_2 \cap \boldsymbol{B}_7 = \{h_2\}$$

$$\Rightarrow m(\boldsymbol{Z}_7) = m(\boldsymbol{A}_2) \cdot m(\boldsymbol{B}_7) = 0.02\,, \tag{29}$$

$$\boldsymbol{Z}_8 = \boldsymbol{A}_7 \cap \boldsymbol{B}_3 = \{h_3\}$$

$$\Rightarrow m(\boldsymbol{Z}_8) = m(\boldsymbol{A}_7) \cdot m(\boldsymbol{B}_3) = 0.02\,. \tag{30}$$

There is for the sets $\{h_1 \cup h_2\}$, $\{h_1 \cup h_3\}$, and $\{h_1 \cup h_2 \cup h_3\}$:

$$\boldsymbol{Z}_9 = \boldsymbol{A}_4 \cap \boldsymbol{B}_7 = \{h_1 \cup h_2\}$$

$$\Rightarrow m(\boldsymbol{Z}_9) = m(\boldsymbol{A}_4) \cdot m(\boldsymbol{B}_7) = 0.12\,, \tag{31}$$

$$\boldsymbol{Z}_{10} = \boldsymbol{A}_7 \cap \boldsymbol{B}_5 = \{h_1 \cup h_3\}$$

$$\Rightarrow m(\boldsymbol{Z}_{10}) = m(\boldsymbol{A}_7) \cdot m(\boldsymbol{B}_5) = 0.04\,, \tag{32}$$

$$\boldsymbol{Z}_{11} = \boldsymbol{A}_7 \cap \boldsymbol{B}_7 = \{h_1 \cup h_2 \cup h_3\}$$

$$\Rightarrow m(\boldsymbol{Z}_{11}) = m(\boldsymbol{A}_7) \cdot m(\boldsymbol{B}_7) = 0.02\,. \tag{33}$$

To illustrate this formal procedure, the results are given in *Table 6*.

Table 6. This table corresponds directly to *Table 5* and represents the products (•) of basic assignments; û means *no focal element*

•	A_1	A_2	A_4	A_7
B_1	0.04	û	0.12	0.02
B_3	û	û	û	0.02
B_5	0.08	û	0.24	0.04
B_7	0.04	0.02	0.12	0.02

Step ‘ – *Combining Basic Assignments*

The sum over all combinations calculated in Step • ,

$$\sum_{k=1}^{11} m(\boldsymbol{Z}_k) = 0.76\,. \tag{34}$$

is identical with the denominator of equation (11). With that, the basic assignment of every hypothesis can be calculated. There is

$$m(\{h_1\}) = \frac{\sum_{k=1}^{6} m(\mathbf{Z}_k)}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.7105 \quad (35)$$

for the hypothesis h_1 and

$$m(\{h_2\}) = \frac{m(\mathbf{Z}_7)}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.0263 \quad (36)$$

$$m(\{h_3\}) = \frac{m(\mathbf{Z}_8)}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.0263 \quad (37)$$

$$m(\{h_1 \cup h_2\}) = \frac{m(\mathbf{Z}_9)}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.1579 \quad (38)$$

$$m(\{h_1 \cup h_3\}) = \frac{m(\mathbf{Z}_{10})}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.0526 \quad (39)$$

$$m(\{h_1 \cup h_2 \cup h_3\}) = \frac{m(\mathbf{Z}_{11})}{\sum_{k=1}^{11} m(\mathbf{Z}_k)} \approx 0.0263 \quad (40)$$

for all relevant sets of hypotheses.

Step ’ – Evidence Measures of Combined Hypotheses

The evidence measures of combined hypotheses are calculated according to Step Œ, *Table 7*. The set $\{h_1 \cup h_2\}$ does not occur because it vanished in *Table 5*.

Step “ – *Interpretation*

Starting at the same low value, h_3 takes roughly half the range of uncertainty that h_2 takes. However, both hypotheses alone should not be considered further due to the low values of belief and plausibility. With about ~0.24, the single hypothesis h_1 is assigned with a wide range of uncertainty. The combination of h_1 and h_3 covers a smaller range (~0.18) than h_1 alone, and it has a higher plausibility. Finally, a combination of h_1 and h_2 shows the smallest range of uncertainty (~0.08) with the same (highest) plausibility as in case of the combination of h_1 and h_3.

The conclusion is that a combination of h_1 and h_2 may be responsible for the serious changes of the system properties. Please note that a probabilistic approach would have blamed h_1 alone for being responsible. (And please consider the consequences.) This result clearly shows the differences between both theories, based on their different mappings $\Omega \rightarrow [0, 1]$ versus $2^{\Omega} \rightarrow [0, 1]$, see Section 0.

Table 7. The *basic assignments* (bold) and the resulting evidence measures *belief*, *commonality*, and *plausibility* are given; hypotheses are ranked by their belief measures, all values are rounded.

2^{Ω}	*m*	*bel*	*cmn*	*pl*
Ω	**0.0263**	1	0.0263	1
$\{h_1 \cup h_2\}$	**0.1579**	0.8947	0.1842	0.9737
$\{h_1 \cup h_3\}$	**0.0526**	0.7895	0.0789	0.9737
$\{h_1\}$	**0.7105**	0.7105	0.9474	0.9471
$\{h_2\}$	**0.0263**	0.0263	0.2105	0.2105
$\{h_3\}$	**0.0263**	0.0263	0.1053	0.1053

4. Applications to System Safety and Reliability Modelling

The introductory descriptions of the *Failure Modes, Effects, and Criticality Analysis*, the *Event Tree Analysis*, and the *Fault Tree Analysis* are taken from the “System Safety Analysis Handbook” written by Stephens & Talso [29] and published by the *System Safety Society*. The descriptions are shortened and slightly revised. The introductory description of the Reliability Centred Maintenance is taken from [23]. Additionally, the IEC standards are recommended for the application of all listed methods, refer to Section 0.

4.1. Failure Modes, Effects, and Criticality Analysis

As described by Stephens & Talso [29], the *Failure Modes, Effects, and Criticality Analysis* (FMECA) tabulates a list of items in a process along with all the possible failure modes for each item. The effect of each failure is evaluated and ranked according to a severity classification. An FMECA includes the following steps:

- Define the worksheet formats and ground rules.
- Give analysis assumptions.
- Identify the lowest indenture level of analysis.
- Code the system description.
- Give failure definitions and evaluations.

The usefulness of the FMECA as a design tool and in the decision making process depends on the effectiveness with which problem information is communicated for early design attention. Probably the most severe criticism of the FMECA has been its limited use for improvement of designs, as Stephens & Talso [29] claim. The main causes for this have been the

untimeliness and the isolated performance of the FMECA without adequate inputs to the design process. While the objective of an FMECA is to identify all modes of failure within a system design, its first purpose is the early identification of all critical failure probabilities so that they can be eliminated or minimised through design correction at the earliest possible time.

The Dempster-Shafer calculus, as described by the illustration in Section 0, can easily be applied to the FMECA. If more than one expert is involved in the quantitative assessments of the criticality and the occurrence of an item failure. The results are narrow or wide ranges of uncertainties, which require an interpretation similar to Step " Section 0.

However, it is an interesting question if institutions, which conduct system homologations, would accept these results.

4.2. Event Tree Analysis

As summarised by Stephens & Talso [29], the *Event Tree Analysis* (ETA) is an analytical tool that can be used to organise, characterise, and quantify potential failures in a methodical manner. An event tree models the sequence of events that results from a single initiating event. The ETA is a bottom-up analysis versus the top-down approach for the Fault Tree Analysis, see Section 0.

Conducting an ETA starts with selection of the initiating events, both the desired events and the ones not desired. Thereafter, their consequences are developed through consideration of component, module, and system failure-and-success alternatives, respectively. The identification of initiating events may be based on review of the system design and operation, the results of another safety analysis, or personal operating experience acquired with a similar system. Then the success and failure of the mitigating systems are postulated and continued through all alternate paths, considering each consequence as a new initiating event. The basic steps for construction of an event tree include the following:

- List all possible initiating events.
- Identify functional system responses.
- Identify support system responses.
- Group initiating events with all responses.
- Define failure sequences.
- Assign probabilities to each step in the event tree to arrive at total probability of occurrence for each failure sequence.

The method is universally applicable to all kinds of systems, with the limitation that all events must be anticipated to produce meaningful analytical results.

Among the methods presented in the "System Safety Analysis Handbook" by Stephens & Talso [29], the *Event Tree Analysis* is definitely one of the most exhaustive, if it is applied properly. Axiomatically, their use also consumes large quantities of resources. Their use, therefore, is well reserved for systems in which risks are regarded as high and well concealed.

As described by Stephens & Talso [29], probabilities are assigned to each step in the event tree. To apply evidence measures instead of probabilities, the following steps are conducted, which lead to the *Dempster-Shafer Event Tree Analysis DS*-ETA.

It is assumed that the considered event tree consists of bifurcations only; i.e., any symbol within an event tree has one input and two outputs (the event "failure" or the event "no failure"), see *Figure 2*.

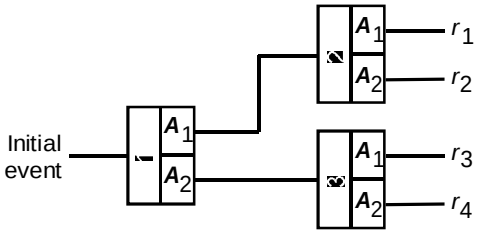

Figure 2. Event tree with three bifurcations resulting in four sequences, $\boldsymbol{A}_1$ denotes "failure" and $\boldsymbol{A}_2$ "no failure"

Every bifurcation $i = 1, \ldots, n$ of an event tree is considered separately and independently from the $n - 1$ other bifurcations. The assigned set of hypotheses contains $\Omega_i = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3\} \equiv$ {"failure", "no failure", "uncertain"}. Hence, any expert (data source) l has to give three values for the basic assignments $m_{l,i}(\boldsymbol{A}_k)$, $k = 1, 2, 3$ of any bifurcation i, representing his/her degree of belief that $\boldsymbol{A}_k$ may occur. Obviously, the uncertainty of an expert concerning an event $\boldsymbol{A}_3$ does not appear in the graphical representation. There is for the *Expert Assessment Matrix*

$$\boldsymbol{M}_l = \begin{bmatrix} m_{l,1}(\boldsymbol{A}_1) & m_{l,1}(\boldsymbol{A}_2) & m_{l,1}(\boldsymbol{A}_3) \\ \text{j} & \text{j} & \text{j} \\ m_{l,i}(\boldsymbol{A}_1) & m_{l,i}(\boldsymbol{A}_2) & m_{l,i}(\boldsymbol{A}_3) \\ \text{j} & \text{j} & \text{j} \\ m_{l,n}(\boldsymbol{A}_1) & m_{l,n}(\boldsymbol{A}_2) & m_{l,n}(\boldsymbol{A}_3) \end{bmatrix} \tag{41}$$

with $m_{l,i}(\boldsymbol{A}_1) + m_{l,i}(\boldsymbol{A}_2) + m_{l,i}(\boldsymbol{A}_3) = 1$.

The *Combination Matrix* $\boldsymbol{C}_i$ combines each hypothesis or set of hypotheses from two experts and builds the cut set of both. Depending on the quantifications given, some of the cut sets (here: the cut sets of "failure" and "no failure") may not be focal elements before or afterwards. To avoid mathematical effort, those columns and rows were dropped, which are related to empty cut sets or non-specified statements (non-focal elements with a 0% basic assignment). In

case of two experts and three possible answers, as presented here, the matrix

$$\boldsymbol{C}_i = \begin{bmatrix} m_{1,i}(\boldsymbol{A}_1)\cdot m_{2,i}(\boldsymbol{A}_1) & 0 & m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_1) \\ 0 & m_{1,i}(\boldsymbol{A}_2)\cdot m_{2,i}(\boldsymbol{A}_2) & m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_2) \\ m_{1,i}(\boldsymbol{A}_1)\cdot m_{2,i}(\boldsymbol{A}_3) & m_{1,i}(\boldsymbol{A}_2)\cdot m_{2,i}(\boldsymbol{A}_3) & m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_3) \end{bmatrix} \tag{42}$$

represents the combinations of focal elements exclusively assigned to the *i*-th event within the tree. The focal sum $\sigma(i)$ is the sum of all matrix elements $\boldsymbol{C}_i$. In this case, it is given by

$$\sigma(i) = 1 - \left(m_{1,i}(\boldsymbol{A}_1)\cdot m_{2,i}(\boldsymbol{A}_2) + m_{1,i}(\boldsymbol{A}_2)\cdot m_{2,i}(\boldsymbol{A}_1)\right), \tag{43}$$

considering the fact that "failure" ($\boldsymbol{A}_1$) and "no failure" ($\boldsymbol{A}_2$) are mutually exclusive. The combined basic assignments $m_i(\boldsymbol{A}_k)$ for a "failure", "no failure", and "uncertain" are

$$m_i(\boldsymbol{A}_1) = \frac{m_{1,i}(\boldsymbol{A}_1)\cdot m_{2,i}(\boldsymbol{A}_1) + m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_1) + m_{1,i}(\boldsymbol{A}_1)\cdot m_{2,i}(\boldsymbol{A}_3)}{\sigma(i)}, \tag{44}$$

$$m_i(\boldsymbol{A}_2) = \frac{m_{1,i}(\boldsymbol{A}_2)\cdot m_{2,i}(\boldsymbol{A}_2) + m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_2) + m_{1,i}(\boldsymbol{A}_2)\cdot m_{2,i}(\boldsymbol{A}_3)}{\sigma(i)}, \tag{45}$$

$$m_i(\boldsymbol{A}_3) = \frac{m_{1,i}(\boldsymbol{A}_3)\cdot m_{2,i}(\boldsymbol{A}_3)}{\sigma(i)} \ . \tag{46}$$

With $m_i(\boldsymbol{A}_k)$ as given above, the evidence measures for a "failure" and a "no failure" decision can be calculated by

$$bel_i(\boldsymbol{A}_1) = m_i(\boldsymbol{A}_1) \ ,$$

$$pl_i(\boldsymbol{A}_1) = m_i(\boldsymbol{A}_1) + m_i(\boldsymbol{A}_3) \ ; \tag{47}$$

$$bel_i(\boldsymbol{A}_2) = m_i(\boldsymbol{A}_2) \ ,$$

$$pl_i(\boldsymbol{A}_2) = m_i(\boldsymbol{A}_2) + m_i(\boldsymbol{A}_3) \ . \tag{48}$$

The next step of modelling applies the basic operations of *interval arithmetic* to the given event tree and assigns the evidential measures as input variables. The addition and multiplication operations are commutative, associative and sub-distributive. There is for $i \neq ii$,

$$[bel_i(\boldsymbol{A}_k),\ pl_i(\boldsymbol{A}_k)] + [bel_{ii}(\boldsymbol{A}_k),\ pl_{ii}(\boldsymbol{A}_k)]$$

$$= [bel_i(\boldsymbol{A}_k) + bel_{ii}(\boldsymbol{A}_k),\ pl_i(\boldsymbol{A}_k) + pl_{ii}(\boldsymbol{A}_k)] \ , \tag{49}$$

$$[bel_i(\boldsymbol{A}_k),\ pl_i(\boldsymbol{A}_k)] \cdot [bel_{ii}(\boldsymbol{A}_k),\ pl_{ii}(\boldsymbol{A}_k)]$$

$$= [\min[bel_i(\boldsymbol{A}_k)\cdot bel_{ii}(\boldsymbol{A}_k),\ bel_i(\boldsymbol{A}_k)\cdot pl_{ii}(\boldsymbol{A}_k),$$

$$pl_i(\boldsymbol{A}_k)\cdot bel_{ii}(\boldsymbol{A}_k),\ pl_i(\boldsymbol{A}_k)\cdot pl_{ii}(\boldsymbol{A}_k)],$$

$$\max[bel_i(\boldsymbol{A}_k)\cdot bel_{ii}(\boldsymbol{A}_k),\ bel_i(\boldsymbol{A}_k)\cdot pl_{ii}(\boldsymbol{A}_k),$$

$$pl_i(\boldsymbol{A}_k)\cdot bel_{ii}(\boldsymbol{A}_k),\ pl_i(\boldsymbol{A}_k)\cdot pl_{ii}(\boldsymbol{A}_k)],] \ . \tag{50}$$

Instead of subtraction, the complements of the evidence measures are applied; this yields

$$bel_i(\boldsymbol{A}_1) = 1 - pl_i(\boldsymbol{A}_2) \ , \tag{51}$$

$$pl_i(\boldsymbol{A}_1) = 1 - bel_i(\boldsymbol{A}_2) \ . \tag{52}$$

With the inputs and operations given, the evidence of a sequence r_j, see *Figure 2.*,can be calculated easily.

As described above, the basic operations of interval arithmetic are applied within the *DS*-ETA. Fortunately, the structure avoids the trouble that the sub-distributivity of subtraction operations may cause, e.g. as known from the *Fuzzy Fault Tree Analysis* (*f*-FTA), see [17].

4.3. Fault Tree Analysis

The *Fault Tree Analysis* (FTA) can model the failure of a single event or multiple failures which lead to a single system failure denoted as the *top event*, refer to Stephens & Talso [29]. However, the FTA is a top-down analysis versus the bottom-up approach for the Event Tree Analysis; i.e., the method identifies an undesirable top event and the contributing elements (down to the so-called *basic events*) that would pre-

cipitate it. The contributors are interconnected with the top event, using network paths through Boolean logic gates. The following basic steps are used to conduct a fault tree analysis:

- Define the top event of interest.
- Define the physical and analytical boundaries.
- Define the tree-top structure.
- Develop the path of failures for each branch to the logical initiating failure, represented by the basic event.

Figure 3 shows a typical fault tree with meshed basic events 1, 4, and 7.

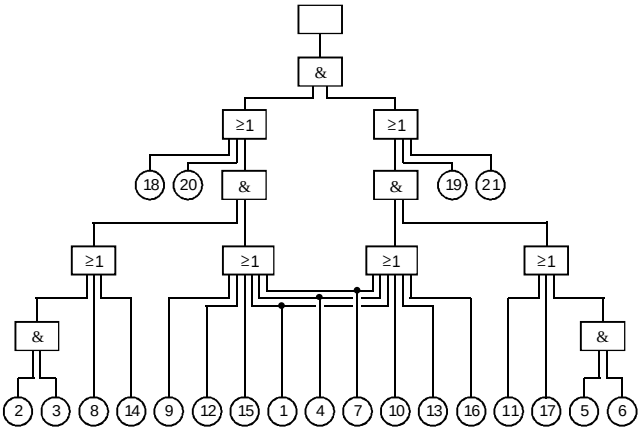

Figure 3. Typical fault tree with basic events, gates, and top event; the example represents one fault of a braking module as applied in railway vehicles

Once the fault tree has been developed to the desired degree of detail, the various paths can be evaluated to arrive at a probability of occurrence. Cut sets are combinations of components failure causing the top event. Minimal cut sets are the smallest combinations causing the top event. The method is universally applicable to systems of all kinds, with the following ground rules:

- The top events which are to be analysed, and their contributors, must be foreseen.
- Each of those top events must be analysed individually.
- The contributing factors have been adequately identified and explored in sufficient depth.

The FTA has got several strengths. The procedures are well defined and focus on failures. The top-down approach requires analysis completeness at each level before proceeding. It cannot guarantee identification of all failures, but the systematic approach enhances the likelihood of completeness. The FTA addresses effects of multiple failures by identifying interrelationships between components and identifying minimal failure combinations that cause the system to fail (minimal cut sets). The method addresses the effects of design, operation, and maintenance. The FTA can handle complex systems. It provides a graphical representation that helps to understand these complex operations and interrelationships between modules and components. Finally, FTA provides both qualitative and quantitative information.

The method is capable of producing numerical statements of the probability of occurrence of undesirable events, given probabilities of contributing factors. As Stephens & Talso ([29], page 136) claim, this capability leads to a common abuse: much effort can be expended in producing refined numerical statements of probability, based on contributing factors whose individual probabilities are hardly known and to which broad confidence limits should be attached. Applying the Dempster-Shafer Theory to FTA can help modelling uncertainties with less effort as shown by Guth [7].

Guth discusses $\Omega = \{h_1, h_2, h_3\} \equiv$ {"event occurs", "uncertain", "event does not occur"}. In the following, two events $\boldsymbol{A}$ and $\boldsymbol{B}$ are considered as inputs and $\boldsymbol{Z}$ as output of an And or Or gate, respectively. There is

$$m(\boldsymbol{A}_1) = bel(\boldsymbol{A}) \,, \tag{53}$$

$$m(\boldsymbol{A}_2) = pl(\boldsymbol{A}) - bel(\boldsymbol{A}) \,, \tag{54}$$

$$m(\boldsymbol{A}_3) = 1 - pl(\boldsymbol{A}) \,, \tag{55}$$

$$m(\boldsymbol{A}_1) + m(\boldsymbol{A}_2) + m(\boldsymbol{A}_3) = 1 \,; \tag{56}$$

the same holds for $\boldsymbol{B}$. Please note that the Guth approach to the *Dempster-Shafer Fault Tree Analysis* (*DS*-FTA) considers *events* where the *DS*-ETA considers *bifurcations*. *Table 8* shows the (underlying) combination of hypotheses with the different results given for the And and Or gate.

Table 8. Combination Table

And	$\boldsymbol{A}_1$	$\boldsymbol{A}_2$	$\boldsymbol{A}_3$
$\boldsymbol{B}_1$	h_1	h_2	h_3
$\boldsymbol{B}_2$	h_2	h_2	h_3
$\boldsymbol{B}_3$	h_3	h_3	h_3

Or	$\boldsymbol{A}_1$	$\boldsymbol{A}_2$	$\boldsymbol{A}_3$
$\boldsymbol{B}_1$	h_1	h_1	h_1
$\boldsymbol{B}_2$	h_1	h_2	h_2
$\boldsymbol{B}_3$	h_1	h_2	h_3

Following Step • in Section 0, the combined basic assignments are calculated. An And gate yields

$$m(\boldsymbol{Z}_1) = m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_1) \,, \tag{57}$$

$$m(\boldsymbol{Z}_2) = m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_2) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_1) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_2) \,, \tag{58}$$

$$m(\boldsymbol{Z}_3) = m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_1) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_2) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_3)$$

$$= m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_3)\,. \quad (59)$$

For an or gate

$$m(\boldsymbol{Z}_1) = m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_1) + m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_2) + m(\boldsymbol{A}_1)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_1) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_1)$$

$$= m(\boldsymbol{A}_1) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_1) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_1)\,, \quad (60)$$

$$m(\boldsymbol{Z}_2) = m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_2) + m(\boldsymbol{A}_2)\, m(\boldsymbol{B}_3) + m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_2)\,, \quad (61)$$

$$m(\boldsymbol{Z}_3) = m(\boldsymbol{A}_3)\, m(\boldsymbol{B}_3) \quad (62)$$

holds similarly. With that, both evidence measures $bel(\boldsymbol{Z})$ and $pl(\boldsymbol{Z})$ can now be calculated recursively to the equations (53) to (55).

Cheng [22] claims that a calculus based on interval arithmetic is more concise and efficient in operation than the calculus proposed by Guth [7] and presented above. However, contrary to an event tree structure, a fault tree structure may cause trouble with the sub-distributivity property of subtraction operations as known from the *Fuzzy Fault Tree Analysis*, see [17]. This applies especially if events are meshed within a fault tree, see Figure 3.

Some authors apply an m: $\Omega\times\Omega \rightarrow [0, 1]$ mapping instead of the well-known m: $2^{\Omega} \rightarrow [0, 1]$ mapping which mainly characterises the calculus of the Dempster-Shafer Theory. However, an $\Omega\times\Omega$ rather represents operations in interval arithmetic, where the lower bound and the upper bound are just labelled as *belief* and *plausibility*, respectively.

4.4. Reliability Centred Maintenance

Reliability Centred Maintenance (RCM), which was first introduced in the aircraft industry, has been used with considerable success in the last decades in many industrial branches. As described in [23], the RCM analysis starts with establishing an expert group and initiating the collection of important component and system data based on the system documentation. Then the system functions are broken down to the desired component level. All relevant component information should be collected in the form of a modified FMECA. The main modification of the FMECA consists of the inclusion of information facilitating the choice of the optimum maintenance strategy. This is generally performed by an RCM decision diagram. Many different decision diagrams are proposed to apply in an RCM analysis. To illustrate the approach, a diagram as given in *Figure 4* is discussed, see [28]. Note that the given diagram is not necessarily complete or applicable in every context given.

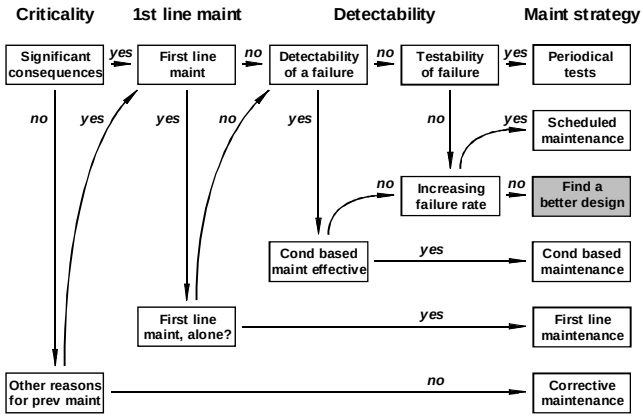

Figure 4. Example of an RCM decision diagram [23]

Supported by the diagram and the integrated questions, a choice for the best fitting maintenance strategy should be made. Finally, the maintenance program should be implemented, and feedback from operation experience and new data should be used to improve the program regularly.

The choice of the best maintenance strategy is the objective of applying RCM; however, reasoning may be difficult, due to questions without a definite answer. For example, the question whether or not a component is critical could not easily be answered with either "yes" or "no". A Dempster-Shafer based alternative makes the avoidance of crisp "yes" or "no" decisions possible and leads to weighted recommendations on which maintenance strategy to choose.

The application of the DST to RCM (denoted as *DS*-RCM [28]) follows the procedure of the *DS*-ETA as described in Section 0. The RCM decision diagram corresponds to the event tree, where the decisions are the counterparts of the events. Consequently, the decisions "yes" and "no" correspond to the events "failure" and "no failure". The *Expert Assessment Matrix* $\boldsymbol{M}_l$, the *Combination Matrix* $\boldsymbol{C}_i$, and the focal sum $\sigma(i)$ are defined and applied analogously. Additionally, an *RCM Decision Diagram Matrix* $\boldsymbol{D}$ with

$$\boldsymbol{D} = \begin{bmatrix} m_1(\boldsymbol{A}_1) & m_1(\boldsymbol{A}_2) & m_1(\boldsymbol{A}_3) \\ \text{j} & \text{j} & \text{j} \\ m_8(\boldsymbol{A}_1) & m_8(\boldsymbol{A}_2) & m_8(\boldsymbol{A}_3) \end{bmatrix} \quad (63)$$

is defined, which collects the combined basic assignment values (results) of each decision within the RCM diagram. Finally, the weighted recommendations on all maintenance strategies are listed by the *Recommendation Matrix*

$$\boldsymbol{R} = \begin{bmatrix} bel(r_1) & pl(r_1) \\ \text{j} & \text{j} \\ bel(r_6) & pl(r_6) \end{bmatrix}, \qquad (64)$$

which collects the values of the evidence measures *belief* and *plausibility* for every strategy.

The main advantages of the *DS*-RCM as against the qualitative RCM can be summarised as follows:

- Experts feel more comfortable giving degrees of belief instead of taking "yes" or "no" decisions. It might therefore be easier to obtain relevant data for the RCM analysis.
- The *DS*-RCM approach results in a profile of all possible maintenance strategies. Decision making based on this profile helps preventing "weak decisions" and may in any case be more comprehensive than relying on a single strategy.
- In some RCM studies it may be desirable to analyse modules and not separate components. In these cases *DS*-RCM is especially useful, since it does not force a single strategy.
- As shown in the discussion of the example case, the *DS*-RCM approach helps to reveal possible design problems and their causes.

A possible disadvantage of this approach is that some experts may find the evidential numbers more complicated than a simple "yes" or "no" decision. A short discussion about the nature of these numbers should therefore be given as an introduction to an RCM session.

5. Conclusions

The Dempster-Shafter Theory is well-known for its usefulness to express uncertain judgments of experts. It is shown in this contribution, how to apply the calculus to safety and reliability modelling. Approaches to expert judgement; Failure Modes, Effects, and Criticality Analysis; Event Tree Analysis; Fault Tree Analysis, and Reliability Centred Maintenance are discussed.

Generally, the Dempster-Shafter Theory adds a new flavour to safety and reliability modelling compared to probabilistic approaches. The illustration (Section 0) clearly shows the differences between the Probability and the Dempster-Shafer Theory, based on their different mappings $\Omega \rightarrow [0, 1]$ versus $2^{\Omega} \rightarrow [0, 1]$, see Section 0. Probability theory would identify a single hypothesis and DST a combination of two to be responsible for the serious changes of the system considered.

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7. Symbols

$\boldsymbol{A}$, $\boldsymbol{B}$	set of hypotheses
bel	belief measure
$\boldsymbol{C}$	Combination Matrix
cmn	commonality measure
$\boldsymbol{D}$	RCM Decision Diagram Matrix
h	single hypothesis
i, *n*	index, resp. number of an element
j	sequence index (ETA) or statement index (RCM)
k	set or statement index
l	data source index
$\boldsymbol{M}$	Expert Assessment Matrix
m	basic assignments
pl	plausibility measure
$\boldsymbol{R}$	Recommendation Matrix
r_j	sequence evidence (ETA) or evaluation value of the maintenance strategies (RCM)
$\sigma(i)$	focal sum
$\boldsymbol{Z}$	combined set of hypotheses

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Systems reliability analysis in variable operation conditions

Keywords

reliability function, semi-markov process, large multi-state system

Abstract

The semi-markov model of the system operation process is proposed and its selected parameters are defined. There are found reliability and risk characteristics of the multi-state series- "*m* out of *k*" system. Next, the joint model of the semi-markov system operation process and the considered multi-state system reliability and risk is constructed. The asymptotic approach to reliability and risk evaluation of this system in its operation process is proposed as well.

1. Introduction

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and complicated operating processes. This complexity very often causes evaluation of systems reliability to become difficult. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet these systems, for instance, in piping transportation of water, gas, oil and various chemical substances or in transport using belt conveyers and elevators.

Taking into account the importance of safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis. The assumption that the systems are composed of multi-state components with reliability state degrading in time without repair gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability characteristics is often very difficult to fix and to analyse. A convenient tool for solving this problem is semi-markov modelling of the systems operation processes which is proposed in the paper. In this model, the variability of system components reliability characteristics is pointed by introducing the components' conditional reliability functions determined by the system operation states. Therefore, the common usage of the multi-state system's limit reliability functions in their reliability evaluation and the semi-markov model for system's operation process modelling in order to construct the joint general system reliability model related to its operation process is proposed. On the basis of that joint model, in the case, when components have exponential reliability functions, unconditional multi-state limit reliability functions of the series- *m* out k_n system are determined.

2. System operation process

We assume that the system during its operation process has *v* different operation states. Thus, we can define $Z(t),\ t \in <0,+\infty>,$ as the process with discrete states from the set

$$Z = \{z_1, z_2, \ldots, z_v\}.$$

In practice a convenient assumption is that $Z(t)$ is a semi-markov process [1] with its conditional sojourn times θ_{bl} at the operation state z_b when its next operation state is z_l, $b, l = 1,2,...,v,\ b \neq l$. In this case this process may be described by:
- the vector of probabilities of the initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of its transitions between the states $[p_{bl}]_{vxv}$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the sojourn times θ_{bl}, $b \neq l$.

If the sojourn times θ_{bl}, $b,\ l = 1,2,...,v,\ b \neq l$, have Weibull distributions with parameters α_{bl}, β_{bl}, i.e., if for $b,l = 1,2,...,v,\ b \neq l$,

$$H_{bl}(t) = P(\theta_{bl} < t) = 1 - \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0,$$

then their mean values are determined by

$$M_{bl} = E[\theta_{bl}] = \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}), \quad (1)$$
$$b,l = 1,2,...,v,\ b \neq l.$$

The unconditional distribution functions of the process $Z(t)$ sojourn times θ_b at the operation states z_b, $b = 1,2,...,v,$ are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl}[1 - \exp[-\alpha_{bl} t^{\beta_{bl}} t]],$$

$$= 1 - \sum_{l=1}^{v} p_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0, \quad (2)$$
$$b = 1,2,...,v,$$

and, considering (1), their mean values are

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}$$

$$= \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}),\ b = 1,2,...,v, \quad (3)$$

and variances are

$$D_b = D[\theta_b] = E[(\theta_b)^2] - (M_b)^2, \quad (4)$$

where, according to (2),

$$E[(\theta_b)^2] = \int_0^{\infty} t^2 dH_b(t)$$

$$= \sum_{l=1}^{v} p_{bl} \int_0^{\infty} t^2 \alpha_{bl} \beta_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}] t^{\beta_{bl}-1} dt$$

$$= \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{2}{\beta_{bl}}} \Gamma(1 + \frac{2}{\beta_{bl}}),\ b = 1,2,...,v.$$

Limit values of the transient probabilities

$$p_b(t) = P(Z(t) = z_b),\ t \geq 0,\ b = 1,2,...,v,$$

at the operation states z_b are given by

$$p_b = \lim_{t \to \infty} p_b(t) = \pi_b M_b / \sum_{l=1}^{v} \pi_l M_l,\ b = 1,2,...,v, \quad (5)$$

where M_b are given by (3) and the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases}$$

3. Multi-state series- "*m* out of k_n" system

In the multi-state reliability analysis to define systems with degrading components we assume that all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \geq 1$, the reliability states are ordered, the state 0 is the worst and the state z is the best and the component and the system reliability states degrade with time t without repair. The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse ones. The way in which the components and system states change is illustrated in *Figure 1*.

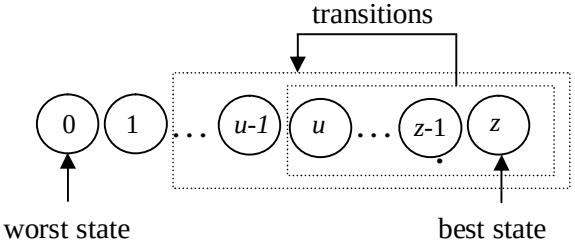

Figure 1. Illustration of states changing in system with ageing components

One of basic multi-state reliability structures with components degrading in time are series- "m out of k_n" systems.

To define them, we additionally assume that E_{ij}, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, $l_2,...,l_{k_n}$, $n \in N$, are components of a system, $T_{ij}(u)$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, $l_2,...,l_{k_n}$, $n \in N$, are independent random variables representing the lifetimes of components E_{ij} in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$, $e_{ij}(t)$ are components E_{ij} states at the moment t, $t \in <0,\infty)$, $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u,u+1,...,z\}$ while it was in the reliability state z at the moment $t = 0$ and $s(t)$ is the system reliability state at the moment t, $t \in <0,\infty)$.

Definition 1. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1),..., R_{ij}(t,z)], \ t \in <0,\infty),$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t)$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, is the probability that the component E_{ij} is in the reliability state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

Definition 2. A vector

$$\boldsymbol{R}^{(m)}_{k_n l_n}(t,\cdot) = [1, \boldsymbol{R}^{(m)}_{k_n l_n}(t,0), \boldsymbol{R}^{(m)}_{k_n l_n}(t,1),..., \boldsymbol{R}^{(m)}_{k_n l_n}(t,z)],$$

where

$$\boldsymbol{R}^{(m)}_{k_n l_n}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t)$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, is the probability that the system is in the reliability state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a system.

It is clear that from *Definition 1* and *Definition 2*, for $u = 0$, we have $R_{ij}(t,0) = 1$ and $\boldsymbol{R}^{(m)}_{k_n l_n}(t,0) = 1$.

Definition 3. A multi-state system is called series- "m out of k_n" if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = T_{(k_n-m+1)}(u), \ u = 1,2,...,z,$$

where $T_{(k_n-m+1)}(u)$ is m-th maximal statistics in the random variables set

$$T_i(u) = \min_{1 \leq j \leq l_i} \{T_{ij}(u)\}, \ i = 1,2,...,k_n, \ u = 1,2,...,z.$$

Definition 4. A multi-state series- "m out of k_n" system is called regular if $l_1 = l_2 = \ldots = l_{k_n} = l_n$, $l_n \in N$.

Definition 5. A multi-state series- "m out of k_n" system is called homogeneous if its component lifetimes $T_{ij}(u)$ have an identical distribution function, i.e.

$$F(t,u) = P(T_{ij}(u) \leq t), \ t \in <0,\infty), \ u = 1,2,...,z,$$
$$i = 1,2,...,k_n, \ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function, i.e.

$$R(t,u) = 1 - F(t,u), \ t \in <0,\infty), \ u = 1,2,...,z.$$

From the above definitions it follows that the reliability function of the homogeneous and regular series- "m out of k_n" system is given by [3]

$$\boldsymbol{R}^{(m)}_{k_n l_n}(t,\cdot) = [1, \boldsymbol{R}^{(m)}_{k_n l_n}(t,1), \boldsymbol{R}^{(m)}_{k_n l_n}(t,2), ..., \boldsymbol{R}^{(m)}_{k_n l_n}(t,z)], \quad (6)$$

where

$$\boldsymbol{R}^{(m)}_{k_n,l_n}(t,u) \quad (7)$$
$$= 1 - \sum_{i=0}^{m-1} \binom{k_n}{i} [R^{l_n}(t,u)]^i [1 - R^{l_n}(t,u)]^{k_n - i}$$
$$\text{for } t \in <0,\infty), \ u = 1,2,...,z,$$

or by

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n l_n}(t,\cdot) = [1, \overline{\boldsymbol{R}}^{(\overline{m})}_{k_n l_n}(t,1), \overline{\boldsymbol{R}}^{(\overline{m})}_{k_n l_n}(t,2), ..., \overline{\boldsymbol{R}}^{(\overline{m})}_{k_n l_n}(t,z)], \quad (8)$$

where

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t,u) = \sum_{i=0}^{\overline{m}} \binom{k_n}{i} [1 - R^{l_n}(t,u)]^i [R^{l_n}(t,u)]^{k_n - i} \quad (9)$$
$$\text{for } t \in <0,\infty), \ u = 1,2,...,z, \ \overline{m} = k_n - m,$$

where k_n is the number of series subsystems in the "*m* out of k_n" system and l_n is the number of components of the series subsystems.

Under these definitions, if $\boldsymbol{R}_{k_n l_n}^{(m)}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z$, or $\overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z$, then

$$M(u) = \int_0^\infty \boldsymbol{R}_{k_n l_n}^{(m)}(t,u)dt, \ u = 1,2,..., z, \tag{10}$$

or

$$M(u) = \int_0^\infty \overline{\boldsymbol{R}}_{k_n l_n}^{(m)}(t,u)dt, \ u = 1,2,..., z, \tag{11}$$

is the mean lifetime of the multi-state non-homogeneous regular series "*m* out of k_n" system in the reliability state subset $\{u, u+1,..., z\}$, and the variance is given by

$$D[T(u)] = 2\int_0^\infty t\boldsymbol{R}_{k_n l_n}^{(m)}(t,u)dt - E^2[T(u)], \tag{12}$$

or by

$$D[T(u)] = 2\int_0^\infty t\overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,u)dt - E^2[T(u)]. \tag{13}$$

The mean lifetime $\overline{M}(u)$, $u = 1,2,..., z$, of this system in the particular states can be determined from the following relationships

$$\overline{M}(u) = M(u) - M(u+1), \ u = 1,2,..., z-1,$$

$$\overline{M}(z) = M(z). \tag{14}$$

Definition 6. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t), \ t \in <0,\infty),$$

that the system is in the subset of states worse than the critical state *r*, $r \in \{1,...,z\}$ while it was in the reliability state *z* at the moment $t = 0$ is called a risk function of the multi-state homogeneous regular series "*m* out of k_n" system.

Considering *Definition 6* and *Definition 2*, we have

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}_{k_n l_n}^{(m)}(t,r), \ t \in <0,\infty), \tag{15}$$

and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{16}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

4. Multi-state series- "*m* out of k_n" system in its operation process

We assume that the changes of the process $Z(t)$ states have an influence on the system components E_{ij} reliability and the system reliability structure as well. Thus, we denote the conditional reliability function of the system component E_{ij} while the system is at the operational state z_b, $b = 1,2,...,v$, by

$$[R^{(i,j)}(t,\cdot)]^{(b)} = [1, [R^{(i,j)}(t,1)]^{(b)}, ..., [R^{(i,j)}(t,z)]^{(b)}],$$

where for $t \in <0,\infty)$, $u = 1,2,..., z$, $b = 1,2,...,v$,

$$[R^{(i,j)}(t,u)]^{(b)} = P(T_{ij}^{(b)}(u) > t \,|\, Z(t) = z_b)$$

and the conditional reliability function of the system while the system is at the operational state z_b, $b = 1,2,...,v$, by

$$[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,\cdot)]^{(b)} = [1, [\boldsymbol{R}_{k_n,l_n}^{(m)}(t,1)]^{(b)}, ..., [\boldsymbol{R}_{k_n,l_n}^{(m)}(t,z)]^{(b)}$$

for $t \in <0,\infty)$, $u = 1,2,..., z$, $b = 1,2,...,\nu$,
where according to (7), we have

$$[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)} = P(T^{(b)}(u) > t \,|\, Z(t) = z_b)$$

$$= 1 - \sum_{i=0}^{m-1} \binom{k_n}{i} [[R(t,u)]^{(b)}]^{l_n i} \cdot [1 - [[R(t,u)]^{(b)}]^{l_n}]^{k_n - i}$$

for $t \in <0,\infty)$, $u = 1,2,..., z$, $b = 1,2,...,\nu$,

or by

$$[\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,\cdot)]^{(b)} = [1, [\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,1)]^{(b)}, ..., [\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,z)]^{(b)}$$

for $t \in <0,\infty)$, $u = 1,2,..., z$, $b = 1,2,...,\nu$,
where according to (9), we have

$$[\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,u)]^{(b)} = P(T^{(b)}(u) > t \,|\, Z(t) = z_b)$$

$$= \sum_{i=0}^{\overline{m}} \binom{k_n}{i} [1-[[R(t,u)]^{(b)}]^{l_n}]^i$$

$$\cdot [[[R(t,u)]^{(b)}]^{l_n}]^{k_n-i} \text{ for } t \in <0,\infty),$$

$u = 1,2,..., z,\ b = 1,2,...,\nu$.

The reliability function $[R^{(i,j)}(t,u)]^{(b)}$ is the conditional probability that the component E_{ij} lifetime $T_{ij}^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. Similarly, the reliability function $[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)}$ or $[\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. In the case when the system operation time θ is large enough, the unconditional reliability function of the system

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,\cdot) = [1,\ \boldsymbol{R}_{k_n,l_n}^{(m)}(t,1),\ ...,\ \boldsymbol{R}_{k_n,l_n}^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u) = P(T(u) > t) \text{ for } u = 1,2,...,z,$$

or

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,\cdot) = [1,\ \overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,1),,...,\ \overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,u) = P(T(u) > t) \text{ for } u = 1,2,...,z,$$

and $T(u)$ is the unconditional lifetime of the system in the reliability state subset $\{u, u+1,..., z\}$, is given by

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)}, \tag{17}$$

or

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,u) \cong \sum_{b=1}^{\nu} p_b [\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,u)]^{(b)} \tag{18}$$

for $t \geq 0$ and the mean values and variances of the system lifetimes in the reliability state subset $\{u, u+1,..., z\}$ are

$$M(u) \cong \sum_{b=1}^{\nu} p_b M_b(u) \text{ for } u = 1,2,..., z, \tag{19}$$

where

$$M_b(u) = \int_0^{\infty} [\boldsymbol{R}_{\boldsymbol{k_n},l_n}^{(m)}]^{(b)}(t,u)dt, \tag{20}$$

or

$$M_b(u) = \int_0^{\infty} [\overline{\boldsymbol{R}}_{\boldsymbol{k_n},l_n}^{(\overline{m})}]^{(b)}(t,u)dt, \tag{21}$$

and

$$D[T^{(b)}(u)] = 2\int_0^{\infty} t[\boldsymbol{R}_{k_n l_n}^{(m)}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)], \tag{22}$$

or

$$D[T^{(b)}(u)] = 2\int_0^{\infty} t[\overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)] \tag{23}$$

for $b = 1,2,...,\nu$, $t \geq 0$, and p_b are given by (4).
The mean values of the system lifetimes in the particular reliability states u, by (14), are

$$\overline{M}(u) = M(u) - M(u+1),\ u = 1,2,..., z-1,$$

$$\overline{M}(z) = M(z). \tag{24}$$

5. Large multi-state series- "*m* out of k_n" system in its operation process

Definition 7. A reliability function

$$\Re(t,\cdot) = [1, \Re(t,1),..., \Re(t,z)],\ t \in (-\infty,\infty),$$

where

$$\Re(t,u) = \sum_{b=1}^{\nu} p_b \Re^{(b)}(t,u),$$

is called a limit reliability function of a multi-state homogeneous regular series- "*m* out of k_n" system in its operation process with reliability function

$$\boldsymbol{R}_{k_n l_n}^{(m)}(t,\cdot) = [1,\ \boldsymbol{R}_{k_n l_n}^{(m)}(t,1),...,\ \boldsymbol{R}_{k_n l_n}^{(m)}(t,z)],$$

or

$$\overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,\ \cdot) = [1,\ \overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,1),...,\ \overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,z)\ ],$$

where $\boldsymbol{R}_{k_n l_n}^{(m)}(t,u),\ \overline{\boldsymbol{R}}_{k_n l_n}^{(\overline{m})}(t,u),\ u=1,2,...,z,$ are given by (17) and (18) if there exist normalising constants

$$a_n^{(b)}(u)>0,\ b_n^{(b)}(u)\in(-\infty,\infty),\ b=1,2,...,v,$$
$$u=1,2,...,z,$$

such that for $t\in C_{\Re^{(b)}(u)}$, $u=1,2,...,z,\ b=1,2,...,v,$

$$\lim_{n\to\infty}[\boldsymbol{R}_{k_n,l_n}^{(m)}(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}=\Re^{(b)}(t,u),$$

or

$$\lim_{n\to\infty}[\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}=\Re^{(b)}(t,u).$$

Hence, the following approximate formulae are valid

$$\boldsymbol{R}_{\boldsymbol{k_n} l_n}^{(m)}(t,u)\cong\sum_{b=1}^{v}p^b\Re^{(b)}(\frac{t-b_n^{(b)}}{a_n^{(b)}},u),\qquad(25)$$
$$u=1,2,...,z,$$

or

$$\overline{\boldsymbol{R}}_{\boldsymbol{k_n} l_n}^{(\overline{m})}(t,u)\cong\sum_{b=1}^{v}p^b\Re^{(b)}(\frac{t-b_n^{(b)}}{a_n^{(b)}},u),\qquad(26)$$
$$u=1,2,...,z.$$

The following auxiliary theorem is proved in [7].

Lemma 1. If

(i) $\lim_{n\to\infty}k_n=\infty$, m = constant

$(m/k_n\to 0$ and $k_n\to\infty)$,

(ii) $\Re^{(m)}(t,u)$

$$=1-\sum_{b=1}^{v}p_b\sum_{i=0}^{m-1}\exp[-V^{(b)}(t,u)]\frac{[V^{(b)}(t,u)]^i}{i!}$$

is a non-degenerate reliability function,

(iii) $\boldsymbol{R}_{\boldsymbol{k_n},l_n}^{(m)}(t,\ \cdot)=[1,\ \boldsymbol{R}_{k_n,l_n}^{(m)}(t,1),...,\boldsymbol{R}_{k_n,l_n}^{(m)}(t,z)],$

$t\in(-\infty,\infty),$

where

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t)\cong\sum_{b=1}^{v}p_b[\boldsymbol{R}_{k_n,l_n}^{m}(t)]^{(b)}$$

is the reliability function of a homogeneous regular multi-state series- "m out of k_n" system, where

$$[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)}$$

$$=1-\sum_{i=0}^{m-1}\binom{k_n}{i}[R^{(b)}(t,u)]^{l_n i}[1-[R^{(b)}(t,u)]^{l_n}]^{k_n-i}\ ,$$

$t\in(-\infty,\infty),\ u=1,2,...,z$,

is its reliability function at the operational state z_b, then

$$\Re^{(m)}(t,\cdot)=[1,\ \Re^{(m)}(t,1),...,\Re^{(m)}(t,z)],$$
$$t\in(-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if [7]

$$\lim_{n\to\infty}k_n[R^{(b)}(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{l_n}$$

$$=V^{(b)}(t,u),\ t\in C_{V^{(b)}(u)},\qquad(27)$$
$$u=1,2,...,z,\ b=1,2,...,v.$$

Proposition 1. If components of the multi-state homogeneous, regular series- "m out of k_n" system at the operational state z_b

(i) have exponential reliability functions,

$$R^{(b)}(t,u)=1 \text{ for } t<0,$$
$$R^{(b)}(t,u)=\exp[-\lambda^{(b)}(u)t]\ \text{ for }\ t\ge 0,\qquad(28)$$
$$u=1,2,...,z,\ b=1,2,...,v,$$

(ii) $m=$ constant, $k_n=n,\ l_n>0,$

(iii) $a_n^{(b)}(u)=\dfrac{1}{\lambda^{(b)}(u)l_n},\ b_n^{(b)}=\dfrac{1}{\lambda^{(b)}(u)l_n}\log n,$

$u=1,2,...,z,\ b=1,2,...,v,$

then

$$\Re_3^{(m)}(t,\cdot)=[1,\ \Re_3^{(m)}(t,1),...,\Re_3^{(m)}(t,z)],\qquad(29)$$
$$t\in(-\infty,\infty),$$

Where

$$\Re_3^{(m)}(t,u)=1-\sum_{b=1}^{v}p_b\sum_{i=0}^{m-1}\exp[-\exp(-t)]\frac{\exp[-it]}{i!}\qquad(30)$$

for $t\in(-\infty,\infty)$, $u=1,2,...,z,$ is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u) \cong 1-\sum_{b=1}^{v} p_b \sum_{i=0}^{m-1}\exp[-\exp(-\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)})]$$

$$\cdot\frac{\exp[-i\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}]}{i!}$$

$$\cong 1-\sum_{b=1}^{v} p_b \sum_{i=0}^{m-1}\exp[-\exp(-\lambda^{(b)}(u)l_n t-\log n)]$$

$$\cdot\frac{\exp[-i\lambda^{(b)}(u)l_n t-i\log n]}{i!} \qquad (31)$$

for $t\in(-\infty,\infty)$, $u=1,2,...,z$.

Proof. For *n* large enough we have

$$a_n^{(b)}(u)t+b_n^{(b)}(u)=\frac{t+\log n}{\lambda^{(b)}(u)l_n}\geq 0 \text{ for } t\in(-\infty,\infty)$$
$$u=1,2,...,z,\ b=1,2,...,v.$$

Therefore, according to (28) for *n* large enough, we obtain

$$R^{(b)}(a_n^{(b)}(u)t+b_n^{(b)}(u),u)$$

$$=\exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t+b_n^{(b)}(u))]$$

$$=\exp[\frac{-t-\log n}{l_n}] \text{ for } t\in(-\infty,\infty),\ u=1,2,...,z,$$
$$b=1,2,...,v.$$

Hence, considering (27), it appears that

$$[V(t,u)]^{(b)}=\lim_{n\to\infty}k_n[R^{(b)}(a_n^{(b)}(u)t+b_n^{(b)}(u))]^{l_n}$$

$$=\lim_{n\to\infty} n\exp[l_n\frac{-t-\log n}{l_n}]=\exp[-t]$$

for $t\in(-\infty,\infty)$, $u=1,2,...,z$, $b=1,2,...,v$,
which means that according to *Lemma 1* the limit reliability function of that system is given by (29)-(30).

The next auxiliary theorem is proved in [7].

Lemma 2. If

(i) $\frac{m}{k_n}\to\eta$, $0<\eta<1$ for $n\to\infty$,

$$\frac{m}{k_n}-\eta=o(\frac{1}{\sqrt{k_n}}),$$

(ii) $\tilde{\Re}^{(\eta)}(t,u)=1-\frac{1}{\sqrt{2\pi}}\sum_{b=1}^{v}p_b\int_{-\infty}^{-v^{(b)}(t,u)}\exp[-\frac{x^2}{2}]dx$,

is a non-degenerate reliability function, where $v^{(b)}(t,u)$ is a non-increasing function

(iii) $\boldsymbol{R}_{\boldsymbol{k_n},l_n}^{(m)}(t,\cdot)=[1,\ \boldsymbol{R}_{k_n,l_n}^{(m)}(t,1),...,\boldsymbol{R}_{k_n,l_n}^{(m)}(t,z)]$,
$t\in(-\infty,\infty)$, where

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)\cong\sum_{b=1}^{v}p_b[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)},\ t\in(-\infty,\infty),$$

is the reliability function of a homogeneous regular multi-state series- "*m* out of k_n" system, where

$$[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)}$$

$$=1-\sum_{i=0}^{m-1}\binom{k_n}{i}[R^{(b)}(t,u)]^{l_n i}[1-[R^{(b)}(t,u)]^{l_n}]^{k_n-i},$$
$$t\in(-\infty,\infty),\ u=1,2,...,z,\ b=1,2,...,v,$$

is its reliability function at the operational state z_b, then

$$\tilde{\Re}^{(\eta)}(t,\cdot)=[1,\ \tilde{\Re}^{(\eta)}(t,1),...,\tilde{\Re}^{(\eta)}(t,z)],$$
$$t\in(-\infty,\infty)$$

is the multi-state limit reliability function of that system if and only if [7]

$$\lim_{n\to\infty}\frac{\sqrt{k_n+1}[R^{l_n}(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}-\eta]}{\sqrt{\eta(1-\eta)}}$$

$$=v^{(b)}(t,u) \text{ for } t\in C_{v^{(b)}(u)},\ u=1,2,...,z, \qquad (32)$$
$$b=1,2,...,v.$$

Proposition 2. If components of the multi-state homogeneous, regular series- "*m* out of k_n" system at the operational state z_b

(i) have exponential reliability functions,
$R^{(b)}(t,u)=1$ for $t<0$,
$R^{(b)}(t,u)=\exp[-\lambda^{(b)}(u)t]$ for $t\geq 0$, (33)
$u=1,2,...,v$, $b=1,2,...,v$,

(ii) $\frac{m}{k_n}\to\eta$, $0<\eta<1$ for $n\to\infty$, $k_n=n$, $l_n>0$,

(iii) $a_n^{(b)}(u)=\frac{\sqrt{1-\eta}}{\lambda^{(b)}(u)l_n\sqrt{\eta n}}$, $b_n^{(b)}(u)=\frac{-\log\eta}{\lambda^{(b)}(u)l_n}$,

$u = 1,2,...,z,\ b = 1,2,...,v,$

then

$$\tilde{\Re}_{7}^{(\eta)}(t,\cdot) = [1,\ \tilde{\Re}_{7}^{(\eta)}(t,1),...,\tilde{\Re}_{7}^{(\eta)}(t,z)], \quad (34)$$
$$t \in (-\infty,\infty),$$

where

$$\tilde{\Re}_{7}^{(\eta)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \sum_{b=1}^{v} p_b \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx \quad (35)$$
for $t \in (-\infty,\infty),\ u = 1,2,...,z,$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}} \sum_{b=1}^{v} p_b \int_{-\infty}^{\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}} e^{-\frac{x^2}{2}} dx$$

$$\cong 1 - \frac{1}{\sqrt{2\pi}}$$

$$\cdot \sum_{b=1}^{v} p_b \int_{-\infty}^{\frac{\sqrt{\eta n}(\lambda^{(b)}(u) l_n t + \log n}{\sqrt{1-\eta}}} e^{-\frac{x^2}{2}} dx \quad (36)$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z.$

Proof. Since, for sufficiently large n, we have

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u) l_n}\left(\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t - \log\eta\right) > 0$$
for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$

then according to (33) for sufficiently large n, we obtain

$$R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u),$$

$$= \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$
$$= \exp[-\frac{1}{l_n}(\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t - \log\eta)] \text{ for } t \in (-\infty,\infty),$$
$u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (32), it appears that

$$v^{(b)}(t,u)$$

$$= \lim_{n\to\infty} \frac{\sqrt{k_n+1}[[R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{l_n} - \eta]}{\sqrt{\eta(1-\eta)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(\exp[-l_n(\frac{\sqrt{1-\eta}}{l_n\sqrt{\eta n}} t - \frac{\log\eta}{l_n})] - \eta)}{\sqrt{\eta(1-\eta)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(\exp[-\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t + \log\eta] - \eta)}{\sqrt{\eta(1-\eta)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(\eta(\exp[-\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t] - 1))}{\sqrt{\eta(1-\eta)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(\eta(1 - \frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t + o(\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t) - 1))}{\sqrt{\eta(1-\eta)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(-\frac{\sqrt{\eta(1-\eta)}}{\sqrt{n}} t + \eta \cdot o(\frac{\sqrt{1-\eta}}{\sqrt{\eta n}} t))}{\sqrt{\eta(1-\eta)}}$$

$= -t$ for $t \in (-\infty,\infty)$, $b = 1,2,...,v,$

which means that according *Lemma 2* the limit reliability function of that system is given by (34)-(35).

The next auxiliary theorem is proved in [7].

Lemma 3. If

(i) $k_n \to \infty$, $\frac{m}{k_n} \to 1$, $(k_n - m) \to \overline{m}$ = constant
for $n \to \infty$,

(ii) $\overline{\Re}^{(\overline{m})}(t,u) = \sum_{b=1}^{v} p_b \sum_{i=0}^{\overline{m}} \exp[-\overline{V}^{(b)}(t,u)] \frac{[\overline{V}^{(b)}(t,u)]^i}{i!}$
is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}}_{\boldsymbol{k_n},l_n}^{(\overline{m})}(t,\cdot) = [1,\ \overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,1),...,\overline{\boldsymbol{R}}_{k_n,l_n}^{(\overline{m})}(t,z)],$

$t \in (-\infty, \infty)$, where

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t,u) \cong \sum_{b=1}^{v} p_b [\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t,u)]^{(b)}, \ t \in (-\infty,\infty),$$

is the reliability function of a homogeneous regular multi-state series- "*m* out of k_n" system, where

$$[\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t)]^{(b)}$$

$$= \sum_{i=0}^{k_n-m} \binom{k_n}{i} [1-[R^{(b)}(t)]^{l_n}]^i [R^{(b)}(t)]^{l_n(k_n-i)},$$

$$t \in (-\infty,\infty), \ \ u = 1,2,...,z\,, b = 1,2,...,v,$$

is its reliability function at the operational state z_b, then

$$\overline{\Re}^{(\overline{m})}(t,\cdot) = [1, \overline{\Re}^{(\overline{m})}(t,1),...,\overline{\Re}^{(\overline{m})}(t,z)],$$
$$t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if [7]

$$\lim_{n\to\infty} k_n l_n F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)$$

$$= \overline{V}^{(b)}(t,u) \ \text{ for } t \in C_{\overline{V}^{(b)}(u)} \tag{37}$$
$$u = 1,2,...,z, \ \ b = 1,2,...,v.$$

Proposition 3. If components of the multi-state homogeneous, regular series- "*m* out of k_n" system at the operational state z_b

(i) have exponential reliability functions,

$$R^{(b)}(t,u) = 1 \text{ for } t < 0,$$
$$R^{(b)}(t,u) = \exp[-\lambda^{(b)}(u)t] \ \text{ for } t \ge 0, \tag{38}$$
$$u = 1,2,...,z, \ b = 1,2,...,v,$$

(ii) $k_n \to \infty$, $\lim_{n\to\infty} k_n - m = \overline{m} = \text{constant}$,

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u) l_n k_n}$, $b_n^{(b)}(u) = 0$,

$$u = 1,2,...,z, \ b = 1,2,...,v,$$

then

$$\overline{\Re}_2^{(\overline{m})}(t,\cdot) = [1, \overline{\Re}_2^{(\overline{m})}(t,1),...,\overline{\Re}_2^{(\overline{m})}(t,z)], \tag{39}$$
$$t \in (-\infty,\infty),$$

where

$$\overline{\Re}_2^{(\overline{m})}(t,u) = \begin{cases} 1, & t < 0, \\ \sum_{b=1}^{v} p_b \sum_{i=0}^{\overline{m}} \exp[-t]\dfrac{t^i}{i!}, & t \ge 0, \end{cases} \tag{40}$$

is the multi-state limit reliability function of that system , i.e. for *n* large enough we have

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t,u)$$

$$\cong \begin{cases} 1, & t < 0, \\ \sum_{b=1}^{v} p_b \sum_{i=0}^{\overline{m}} \exp[-\dfrac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)}] \\ \cdot \dfrac{[\dfrac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)}]^i}{i!}, & t \ge 0, \end{cases}$$

$$\cong \begin{cases} 1, & t < 0, \\ \sum_{b=1}^{v} p_b \sum_{i=0}^{\overline{m}} \exp[t\lambda^{(b)}(u) l_n k_n \\ \cdot \dfrac{[t\lambda^{(b)}(u) l_n k_n]^i}{i!}, & t \ge 0. \end{cases} \tag{41}$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u) l_n k_n} < 0 \ \text{ for } \ t < 0,$$
$$u = 1,2,...,z, \ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u) l_n k_n} \ge 0 \ \text{ for } \ t \ge 0,$$
$$u = 1,2,...,z, \ b = 1,2,...,v,$$

therefore, according to (38), we obtain

$$F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u) = 0 \ \text{ for } \ t < 0,$$
$$u = 1,2,...,z, \ b = 1,2,...,v,$$

and

$$F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)$$
$$= 1 - \exp[-\frac{t}{k_n l_n}] \ \text{ for } \ t \ge 0, \ u = 1,2,...,z,$$
$$b = 1,2,...,v.$$

Hence, considering (37), it appears that

$$\overline{V}^{(b)}(t,u)$$

$$= \lim_{n\to\infty} k_n l_n F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u) = 0 \text{ for } t < 0,$$
$$u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$\overline{V}^{(b)}(t,u) = \lim_{n\to\infty} k_n l_n F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)$$

$$= \lim_{n\to\infty} k_n l_n \left(1 - \exp[-\frac{t}{k_n l_n}]\right)$$

$$= \lim_{n\to\infty} k_n l_n \left(1 - 1 + \frac{t}{k_n l_n} - o(\frac{t}{k_n l_n})\right)$$

$$= t \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according *Lemma 3* the limit reliability function of that system is given by (39)-(40).

The next auxiliary theorem is proved in [7].

Lemma 4. If

(i) $\lim_{n\to\infty} k_n = k,\ k > 0,\ 0 < m \leq k,\ \lim_{n\to\infty} l_n = \infty$,

(ii) $\Re(t,u) = \sum_{b=1}^{v} p_b \Re^{(b)}(t,u)$ is a non-degenerate reliability function,

(iii) $\boldsymbol{R}_{\boldsymbol{k_n},l_n}^{(m)}(t,\cdot) = [1,\ \boldsymbol{R}_{k_n,l_n}^{(m)}(t,1),...,\boldsymbol{R}_{k_n,l_n}^{(m)}(t,z)]$, $t \in (-\infty,\infty)$, where

$$\boldsymbol{R}_{k_n,l_n}^{(m)}(t) \cong \sum_{b=1}^{v} p_b [\boldsymbol{R}_{k_n l_n}^{(m)}(t)]^{(b)}$$

is the reliability function of a homogeneous regular multi-state series- "*m* out of k_n" system, where

$$[\boldsymbol{R}_{k_n,l_n}^{(m)}(t,u)]^{(b)}$$

$$= 1 - \sum_{i=0}^{m-1} \binom{k_n}{i} [R^{(b)}(t,u)]^{l_n i} [1 - [R^{(b)}(t,u)]^{l_n}]^{k_n - i}$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

is its reliability function at the operational state z_b, then

$$\Re(t,\cdot) = [1,\ \Re(t,1),...,\Re(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if [7]

$$\lim_{n\to\infty} [R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{l_n} = \Re_0^{(b)}(t,u) \qquad (42)$$
$$\text{for } t \in C_{\Re_0^{(b)}(u)},\ u = 1,2,...,z,\ b = 1,2,...,v,$$

where $\Re_0^{(b)}(t,u)$, $u = 1,2,...,z$, is a non-degenerate reliability function and

$$\Re(t,u)$$

$$= 1 - \sum_{b=1}^{v} p_b \sum_{i=0}^{m-1} \binom{k}{i} [\Re_0^{(b)}(t,u)]^i [1 - \Re_0^{(b)}(t,u)]^{k-i} \qquad (43)$$
$$\text{for } t \in (-\infty,\infty),\ u = 1,2,...,z.$$

Proposition 4. If components of the multi-state homogeneous, regular series- "*m* out of k_n" system at the operational state z_b

(i) have exponential reliability functions,

$$R^{(b)}(t,u) = 1 \text{ for } t < 0,$$
$$R^{(b)}(t,u) = \exp[-\lambda^{(b)}(u)t] \text{ for } t \geq 0, \qquad (44)$$
$$u = 1,2,...,z,\ b = 1,2,...,v,$$

(ii) $k_n \to k,\ k > 0,\ l_n \to \infty,\ m = \text{const}$,

(iii) $a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u) l_n},\ b_n^{(b)}(u) = 0$,

$u = 1,2,...,z,\ b = 1,2,...,v,$

then

$$\Re_9^{(m)}(t,\cdot) = [1,\ \Re_9^{(m)}(t,1),...,\Re_9^{(m)}(t,z)], \qquad (45)$$
$$t \in (-\infty,\infty),$$

where

$$\Re_9^{(m)}(t,u)$$

$$\cong \begin{cases} 1, & t < 0, \\ 1 - \sum_{b=1}^{v} p_b \sum_{i=0}^{m-1} \binom{k}{i} \exp[-t]]^i & \\ \cdot [1 - \exp[-t]]^{k-i}, & t \geq 0, \end{cases} \qquad (46)$$

is the multi-state limit reliability function of that system , i.e. for *n* large enough we have

$$\Re_9^{(m)}(t,u)$$

$$\cong \begin{cases} 1, & t<0, \\ 1-\sum_{b=1}^{v} p_b \sum_{i=0}^{m-1} \binom{k}{i} [\exp[-\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}]]^i \\ \cdot [1-\exp[-\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}]]^{k-i}, & t \geq 0, \end{cases}$$

$$\cong \begin{cases} 1, & t<0, \\ 1-\sum_{b=1}^{v} p_b \sum_{i=0}^{m-1} \binom{k}{i} [\exp[-t\lambda^{(b)}(u) l_n]]^i \\ \cdot [1-\exp[-t\lambda^{(b)}(u) l_n]]^{k-i}, & t \geq 0. \end{cases} \quad (47)$$

Proof . Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u) l_n} < 0 \text{ for } t<0,$$
$$u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u) l_n} \geq 0 \text{ for } t \geq 0,$$
$$u = 1,2,...,z,\ b = 1,2,...,v,$$

therefore, according to (44), we obtain

$$[R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{l_n} = 1 \text{ for } t<0,$$
$$u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)] = \exp[-\frac{t}{l_n}] \text{ for } t \geq 0,$$
$$u = 1,2,...,z,\ b = 1,2,...,v.$$

Hence, according (42)-(43), it appears that

$$\Re_0^{(b)}(t,u) = \lim_{n\to\infty}[R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u]^{l_n} = 1$$
$$\text{for } t<0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$\Re_0^{(b)}(t,u)$$

$$= \lim_{n\to\infty}[R^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u]^{l_n}$$

$$= \lim_{n\to\infty}[\exp[-\frac{t}{l_n}]]^{l_n}$$

$$= \exp[-t] \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

which, by *Lemma 4*, completes the proof.

6. Conclusion

The purpose of this paper is to give the method of reliability analysis of selected multi-state systems in variable operation conditions. As an example a multi-state series-"*m* out of *k*" systems are analyzed. Their exact and limit reliability functions, in constant and in varying operation conditions, are determined. The paper proposes an approach to the solution of practically very important problem of linking the systems' reliability and their operation processes. To involve the interactions between the systems' operation processes and their varying in time reliability structures a semi-markov model of the systems' operation processes and the multi-state system reliability functions are applied. This approach gives practically important in everyday usage tool for reliability evaluation of the large systems with changing their reliability structures and components reliability characteristic during their operation processes. The results can be applied to the reliability evaluation of real technical systems.

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Reliability of complex system with one shot items

Keywords

reliability, complex system, one shot item

Abstract

This article deals with modelling and analysis of the reliability of complex systems that use one-shot items during their operation. It includes an analysis of the impact of the reliability of used one-shot items on the resulting reliability of the system as a whole. Practical application of theoretical knowledge is demonstrated on an example of a model of reliability of an aircraft gun that was used for optimization of the gun's design during its development and design. The analysed gun uses two types of one-shot items – rounds intended for conducting of fire and special pyrotechnic cartridges designed for re-charging a gun after a possible failure of the round.

1. Introduction

This contribution is supposed to contribute to a solution of dependability qualities of the complex (in this case) weapon system as an observed object. I would like to show one of the ways how to specify a value of single dependability measures of a set. The aim of our paper is to verify the suggested solution in relation to some functional elements which influence fulfillment of a required function in a very significant manner. [1], [3]

A weapon set is a complex mechatronics system which is designed and constructed for military purposes. We are talking about a barrel shooting gun – a fast shooting two-barrel cannon. It is going to be implemented in military air force in particular.

Generally speaking the set consists of mechanical parts, electric, power and manipulation parts, electronic parts and ammunition. For the purpose of use in our paper we are going to deal with isolated functional blocks and ammunition only. In this case we view the ammunition as recommended standardised rounds and pyrotechnic cartridges.

Single parts of the set can be described with qualitative and most importantly quantitative indices which present their quality. In my paper I am dealing especially with quality in terms of dependability characteristics. We are working first and foremost with probability values which characterize single indices, and which describe functional range and required functional abilities of the set. We focus on the part handling rounds and pyrotechnic cartridges which are crucial for this case. In order to continue our work it is necessary to define all terms and specify every function.

2. Essential terms and definitions

We are always talking about an object in terms of reliability analyses. The definition for object is the same as the used in IEC 60500 (191/50). Consequently we need to describe the basic object's measures [2].

Object's function:

The main function: The main function of the object is putting into effect a fire from a gun using standard ammunition.

The step function: Manipulation with ammunition, its charging, initiation, detection and indication of ammunition failure during initiation, initiation of backup system used for re-charging of a failed cartridge.

It is expected that the object will be able to work under different operating conditions especially in different temperature spectra, under the influence of varied static, kinetic and dynamic effects, in various zones of atmospheric and weather conditions.

In this case we will not take into account any of the operating conditions mentioned above. However, their influence might be important while considering successful mission completion.

One of the main terms we are going to develop is:

Mission: It is an ability to complete a regarded mission by an object in specified time, under given conditions and in a required quality.
In our contribution it is a case of cannon ability to put into effect a fire in a required amount – in a number of shot ammunition at a target in required time, and under given operating and environmental conditions.
As it follows from the definition of a mission it is a case of a set of various conditions which have to be fulfilled all at once in a way to satisfy us completely. Our object is supposed to be able to shoot a required amount of ammunition which has to hit the target with required accuracy (probability). We will not take into consideration circumstances relating to evaluation of shooting results, weapon aiming, internal and external ballistics, weather conditions and others. We will focus only on an ability of an object to shoot. [4]
As we have stated above we will deal with isolated function blocks only. We are presuming that these blocks act according to required and determined boundary conditions. In order to understand functional links fully we introduce our way of dividing an object.
We are talking about the following block:

- manipulation with ammunition, its charging, initiation, failure detection and indication during initiation, initiation of a backup system in order to recharge a failed cartridge, all mechanical parts, all electric and electronic parts, interface elements with a carrying device - Block A;
- ammunition – Block B;
- pyrotechnic cartridges – Block C.

3. Description of a process

The process as a whole can be described this way:
From a mathematical and technical point of view it is a fulfilling of requirements´ quee which gradually comes into the service place of a chamber. The requirements´ quee is a countable rounds´ chain where the rounds wait for their turn and are transported from the line where they wait in to a service place (fulfilment of a requirement) of a chamber and there they are initiated. After the initiation the requirement is fulfilled. An empty shell (one of the essential parts of a round) leaves a chamber taking a different way than a complete round. When the requirement is fulfilled, another system which is an integral part of a set detects process of fulfilling the requirement. The process is detected and indicated on the basis of interconnected reaction processes. In this case fulfilling the requirement is understood as a movement of a barrel breech going backwards. Both fulfilling the requirement and its detection are functionally connected with transport of another round waiting in a line to go into a chamber.
Let's presume that rounds are placed in an ammunition feed belt of an exactly defined length. A maximum number of rounds which could be placed in a belt is limited by the length then. The length is given either by construction limitations or by tactical and technical requirements for a weapon set. Let's presume that despite different lengths of an ammunition belt, this will be always filled with rounds from the beginning to the end. Let's also assume that the rounds are not non-standard and are designed for the set.
The process of fulfilling the requirement is monitored all the time by another system which is able to differentiate if it is fulfilled or not. The fulfilment itself means that a round is transported into a chamber, it is initiated, shot, and finally an empty shell leaves a chamber according to a required principle. If the process is completed in a required sequence, the system detects it as a right one.
Because of unreliability of rounds the whole system is designed in the way to be able to detect situations in which the requirement is not fulfilled in a demanded sequence and that is why it is detected as faulty.
Although a round is transported into a chamber and is initiated, it is not fired. A function which is essential for a round to leave a chamber is not provided either, and therefore another round waiting in line cannot be transported into a chamber. That is the reason why fulfilling of the requirement is not detected.
The system is designed and constructed in such a way that it is able to detect an event like this and takes appropriate countermeasures. A redundant system which has been partly described above is initiated. After a round is initiated and the other steps don't carry out (non-fire, non-movement of a barrel breech backwards, non-detection of fulfilling the requirement, non-leaving of a chamber by an empty shell, and non-transport of another round into a chamber) a system of pyrotechnic cartridges is initiated. It is functionally connected with all the system providing mission completion. A pyrotechnic cartridge is initiated and owing to this a failed round is supposed to leave a chamber. A failed functional link is established and another round waiting in line is transported into a chamber.
In order to restore the main function we use a certain number of backup pyrotechnic cartridges. Our task is to find out a minimum number which is essential for completing the mission successfully.

4. Mathematical model

To meet the needs of our requirements we are going to use a mathematical way which helps us to express successful completing the mission. We know that the number of rounds n in an ammunition belt is final. We also know that an event-failure of a round $\overline{B}$ (ammunition block – B) can occur with a probability p_n. All the requirements and specifications mentioned above will be used in further steps.

Because it is about a stream of rounds of a number n which wait in line to meet the requirement, and each of them has a potential quality p_n, a number of failed rounds has a binomial distribution (*Bi)* of a an event occurrence. The distribution is specified by the parameters n and p_n: $Bi(n,p_n)$. A number of occurrences X_n of an event $\overline{B}$ follows the distribution in Bernoulli's row n of independent experiments, and probability of event occurrence $P(\overline{B}) = p_n$. A number p_n is the same in every experiment. [5]; [6]

Because there is an occurrence of a number of events in an observed file we are talking about a counting distribution of an observed random variable. A random variable is in this case a number of failed rounds. A probability function of a binomial distribution can be put that way:

$$P(X_n = x) = \binom{n}{x} p_n{}^x (1-p_n)^{n-x}\ ;\ x \in \{0,1,2,...,n\}. \quad (1)$$

Qualities of binomial distribution like a mean value $E(X_n)$ and dispersion $D(X_n)$ are obtained by calculating the formula:

$$E(X_n) = n\ .\ p_n\ , \quad (2)$$

$$D(X_n) = n\ .\ p_n\ .\ (1- p_n). \quad (3)$$

A number of failed rounds follows a binomial distribution with parameters n – a number of rounds and p_n – failure occurrence probability of a round.

In order to specify a mean number of possible failures in an ammunition belt of a given length (there is a certain amount of rounds) we quantify the formula (2) and replace n by a real number of rounds in an ammunition belt.

On the basis of construction, technical and technical requirements we can have ammunition belts of different length at a given moment, and consequently we have a different number of rounds. Only a maximum number of rounds in an ammunition belt is considered in another calculation. The ammunition belt is supposed to be of a maximum length which is able to fit a loading device

In case a round fails initiation of a backup system for function restoration occurs according to a mechanism described above. It is a case of successive initiation of pyrotechnic cartridges (in a system of pyrotechnic cartridges) which are supposed to guarantee restoring of a required broken chain of function. A number of pyrotechnic cartridges in a backup system is m. Pyrotechnic cartridges have also a probability p_m of a failure occurrence which unable their initiation. Pyrotechnic cartridges too are placed in line waiting for meeting the requirement which results from their function. In case of a failure of the first pyrotechnic cartridge the next one is initiated up to the moment when either a function is restored or all pyrotechnic cartridges are used up.

On the basis of the facts mentioned above it is obvious that the process of fulfilling the requirements follows geometrical distribution (*Ge*). It means that the process of fulfilling the requirements repeats so often until it meets them in terms of reversion of all the process to an operational state. It is a case of an observed discreet random variable. Pyrotechnic cartridges also have failure rate p_m (failure probability) and there is a limited number of them. It means that a failure can occur up to *m-times*. A geometrical distribution $Ge(p_m)$ generally follows this outline.

We are going to assess the succession of independent attempts, and probability of an observed event occurrence equals the same number p_m in each attempt. The quantity X_m is a serial number of the first success which means that a required event occurs. The event here means a function of a block C, and a probability p_m means an event occurrence $\overline{C}$. Characteristics of the process are as follow. A probability function:

$$P(X_m=x) = p_m{}^{x-1}(1-p_m); x \in \{1,2,3,\ldots,m\}. \quad (4)$$

It is a special case of a geometrical distribution when a probability of an event occurrence (a pyrotechnic cartridge failure) does not depend on a number of previous unsuccessful attempts of a value 0. Characteristics of a geometrical distributions, for example mean value $E(X_m)$ (a mean number of pyrotechnic cartridges necessary for removing one failed round) and dispersion $D(X_m)$ are obtained by a calculation of a formula:

$$E(X_m) = \sum_{x=0}^{\infty} x.\Pr(X_m = x) = \frac{1}{1-p_m}. \quad (5)$$

While completing the mission during either training or a real deployment a few scenarios can occur, and the course of them depends on single functional blocks. To complete the mission M successfully single blocks are expected to be failure free as stated above. The function of the blocks mentioned above are designated as A, B, C, the opposite is $\overline{A}$; $\overline{B}$; $\overline{C}$. The relation can be expressed by using events this way:

$$M = A \cap (B \cup C). \quad (6)$$

Using probability expression we talk about probability of mission completion M. We can put it that way:

$$P(M) = P(A)\ .\ [P(B) + P(C) - P(B \cap C)]; \quad (7)$$

5. Description of scenarios

Description of the scenarios which can occur during completing or defaulting the mission relate only to an ammunition block and to a redundant mechatronics system with pyrotechnic cartridges.

The mission is completed. In the first case there can be a situation when all the ammunition of a certain amount which is placed in an ammunition belt is used up and a round failure occurs or it is used up and a round failure does not occur. In this case a backup system of pyrotechnic cartridges is able to reverse a system into an operational state. Using up can be single, successive in small bursts with breaks between different bursts, or it might be mass using one burst. Shooting is failure free or there is a round failure occurrence *n*. In case a round failure occurs, a system which restores a function of pyrotechnic cartridges is initiated. There are two scenarios too – a system restoring a pyrotechnic cartridges function is failure free, or a pyrotechnic cartridge fails. If a function of pyrotechnic cartridges is applied, it can remove a failure *m-times*. So a number of restorations of the function is the same as the number of available pyrotechnic cartridges. In order to complete the mission successfully we need a higher amount of pyrotechnic cartridges *m*, or in the worst case the number of pyrotechnic cartridges should be equal to a number of failures. Another alternative is the situation that a round fails and in this case a pyrotechnic cartridge fails too. A different pyrotechnic cartridge is initiated and it restores the function. This must satisfy the requirements that an amount of all round failures *n* is lower or at least equal to a number of operational (undamaged) pyrotechnic cartridges *m*. The mission is completed in all the cases mentioned above and when following a required level of readiness of a block A.

The mission is not completed. In the second case the shooting is carried out one at a time, in small bursts or in one burst, and during the shooting there will be *n* round failures. At the time the failure occurs a backup system for restoring the function will be initiated. Unlike the previous situation there will be *m* pyrotechnic cartridges´ failures and a total number of pyrotechnic cartridges´ failures equals at least a number of round failures, and is equal to a number of implemented pyrotechnic cartridges M at the most. It might happen in this case that restoring of the function does not take place and the mission is not completed at the same time because there are not enough implemented pyrotechnic cartridges.

The relation of transition among the states can be expressed by the theory of Markov chains.

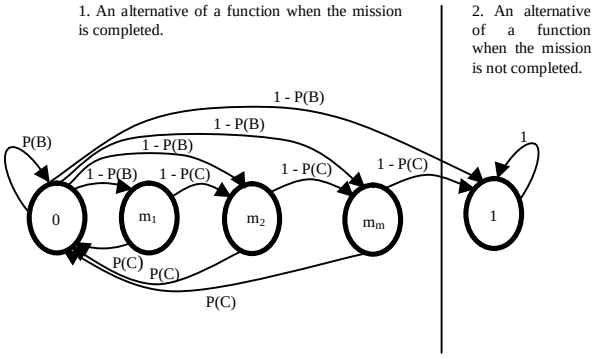

Figure 1. Description of transitions among the states

Characteristics of the states:

0 state: An initial state of an object until a round failure occurs with a probability function of a round P(B). It is also a state an object can get with a pyrotechnic cartridge probability P(C) in case a round failure occurs

$$P(\overline{B}) = 1 - P(B),$$

or

$$P(\mathrm{C}|\,\overline{B}) = \frac{P(C \cap \overline{B})}{P(\overline{B})}.$$

$m_1 \ldots m_m$ state: A state an object can get while completing the mission. Either a round failure occurs in probability $P(\overline{B}) = 1 - P(B)$, or there is a pyrotechnic cartridge failure in probability $P(\overline{C}) = 1 - P(C)$.

1 state: A state an object can get while completing the mission. It is so called an absorption state. Transition to the state is described as probability $P(\overline{C}) = 1 - P(C)$ of a failure of last pyrotechnic cartridge as long as an object was in a state „k_n“ before this state, or it can be described as probability of a round failure occurrence $P(\overline{B}) = 1 - P(B)$ as long as an object was in a state 0 before this state and all pyrotechnic cartridges are eliminated from the possibility to be used.

Transitions among different states as well as absolute probability might be put in the following formulae:

$$P(0) = P(B) + P(C_{k_1 0}) + P(C_{k_2 0}) + \ldots + P(C_{k_{1n} 0}), \qquad (8)$$

$$P(m_1) = 1 - P(B), \qquad (9)$$

$$P(m_m) = (1 - P(B)) + (1 - P(C)), \qquad (10)$$

$$P(1) = 1. \qquad (11)$$

We suggest the subsequent steps for all the scenarios mentioned above. Following the mathematical formula (1) it is possible to find out probability of a number of round failures´ occurrences in an ammunition belt of a length *n*. Following the equation (2) we can specify an expected mean value of a mean number of round failures in an ammunition belt of a given length.
The mean value result is recommended to be used for a maximum length of an ammunition belt (a maximum number of rounds) which could be implemented into a weapon set concerning construction as well as tactical and technical views. The result informs us of a minimum number of pyrotechnic cartridges which are to be applied for a successful completing the mission.
In this case there is a threat of a pyrotechnic cartridge failure which could cause a system failure (as far as a number of round failures is higher than a number of available pyrotechnic cartridges). In this case we would not complete the mission.
In order to assess dependability of a shooting function it is necessary to know a number of pyrotechnic cartridges and, depending on this, probability of completing the mission. To fulfil the requirements I suggest three steps:

1) To determine a required number of pyrotechnic cartridges;
2) To quantify generally probabilities of completing the mission;
3) To quantify exactly probabilities of completing the mission

Following the steps mentioned above we suggest this method.

Ad 1) To determine a required number of pyrotechnic cartridges

When we calculate a mean number of failed rounds $E(X_n)$ which is determined from a maximum number of rounds *n* in a ammunition belt (see above) and probability of a round failure occurrence p_n, see the formula (2), we get a minimum recommended number of pyrotechnic cartridges which are supposed to guarantee completing the mission in case a round fails.
The calculation would be successful in case a pyrotechnic cartridge failure does not occur. However, even a system of pyrotechnic cartridges concerning a failure occurrence depends on counting distribution of a discreet random variable which is specified in our case by a geometrical distribution. (Because the system is activated so long until the observed and required event occurs – in terms of repairing the failure.) We suggest calculating a mean number of pyrotechnic cartridges´ failures following the formula (5). For the calculation we will need only pyrotechnic cartridge failure probability p_m. On the basis of this calculation we get an average number of pyrotechnic cartridges required to repair a failure of one round.
In order to complete the mission a number of available (operational) pyrotechnic cartridges should be at least the same as a number of failed rounds. When we multiply the mean values we obtain a total number of pyrotechnic cartridges *M* which will guarantee completing the mission (even in the situation when besides failed rounds there are failed pyrotechnic cartridges too)

$$M = E(X_n) \,.\, E(X_m) = \frac{n.p_n}{1-p_m}. \qquad (12)$$

Logically a number of pyrotechnic cartridges which are essential for completing the mission successfully is continually proportioned to a number of rounds *n* and to probability of their failure p_n, and inversely proportioned to probability of pyrotechnic cartridge "success" $1\text{-}p_m$. The *Figure 2* shows a typical course of dependability $M\ (p_n;p_m)$, it means a invariant *M* which depends on variables p_n a p_m. This way might be the first of the alternatives how to solve the problem. It suggests a total number of pyrotechnic cartridges which are essential for completing the mission but it does not show the way how to quantify probability of mission completion.
While recording distribution parameters we are going to use an equivalent *m* standing for a value *M*.

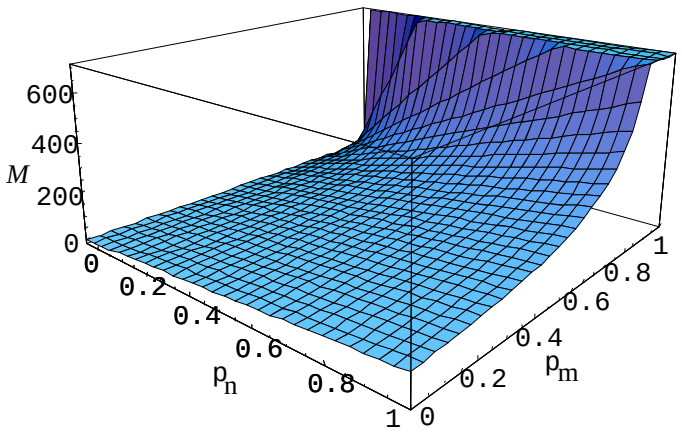

Figure 2. Course of dependability of a number of pyrotechnic cartridges *M* on variables p_n and p_m

Ad 2) To quantify generally probability of completing the mission

In this case we follow the solution which has been stated in the part Ad 1. We take into account that there is a number of pyrotechnic cartridges required for completing the mission. So, we determine an α fractile which provides an upper limit of a number of rounds which fail in probability α. After we specify β fractile

which provides an upper limit of pyrotechnic cartridges which fail in probability β.
While working with fractiles we follow the general information. 100% fractile of a random variable *X* is a number x_p, and a probability *p* where 0<*p*<1 is denoted by

$$P(X \le x_p) \ge p \tag{13}$$

and

$$\lim_{x \to x_p^-} P(x) \le p\,. \tag{14}$$

The fractile of an observed random variable we are working with is expressed by

$$p_n = \sum_{n=0}^{x_\alpha} P(X_\alpha = n). \tag{15}$$

We put it into words this way – occurrence probability *n* of a number of events is specified by a sum of probabilities for the occurrence of all events from *0* to *n*.
In our case we take into account that round failures´ distribution is binomial $B_i=(n;p_n)$ and a fractile determining an upper limit of a number of rounds which might fail in probability α will be designated as x_α. We put it that way

$$P(X_n \le x_\alpha) = \alpha\,. \tag{16}$$

We suppose that a general distribution of a pyrotechnic cartridge follows a binomial distribution too $Bi(m;p_m)$. A fractile providing an upper limit of a number of pyrotechnic cartridges which fail in probability β is denoted by y_β. Thus

$$P(Y_m \le y_\beta) = \beta\,. \tag{17}$$

The equation can be put in a different way as

$$Pr(m - Y_m \ge m - y_\beta) = \beta\,. \tag{18}$$

The following interpretation of a fractile y_β is useful for other steps – at least $m - y_\beta$ of pyrotechnic cartridges will be available with probability β.
As it was stated before we are supposed to know a total number of pyrotechnic cartridges *M* which are essential for completing the mission. The requirement is shown in the following equation:

$$(M - y_\beta) \ge x_\alpha\,. \tag{19}$$

The equation shows that a number of available pyrotechnic cartridges (we obtain it when we subtract failed pyrotechnic cartridges from a total amount of all applied pyrotechnic cartridges) will be at least the same (it would be better to have a higher number) as a number of failed rounds. If this assumption is fulfilled, we can expect that the mission will be completed in probability p_{mis}. Probability of completing the mission can be put that way

$$p_{mis} = \alpha\,.\,\beta\,. \tag{20}$$

The formula can be described like this – probability of completing the mission equals a multiplication of probabilities$\alpha;\beta \in (0;1)$ which provide us an upper limit of failed rounds and an upper limit of failed pyrotechnic cartridges for required levels of fractiles.
If the level of mission completion probability is known in advance, e.g. it is specified by technical requirements for a set, we can put it in the formula which is based on an assumption that the mission will be completed in case a number of available pyrotechnic cartridges is at least the same as rounds which are supposed to fail

$$x_\alpha \le m - y_\beta\,. \tag{21}$$

If it goes this way, the mission will be completed in probability expressed in the formula (20).
If we have the values α, *n*, β, p_{mis}, we may find a value *m (M)* using quantitative methods. At the end of my contribution there is an example of this solution.

Ad 3) To quantify exactly probabilities of completing the mission

In the last step we are going to examine how to quantify an exact value of mission completion probability p_{mis}. On the basis of the assumption described above we know that probability of completing the mission depends on reliability of two key blocks. It is an ammunition block (B) and a pyrotechnic cartridges´ block (C). Following the last two alternatives we might specify both a required total number of pyrotechnic cartridges which is essential to complete the mission (in case all conditions are met), and a general value of mission completion probability in case general conditions are followed. This solution might satisfy us under certain circumstances but it is not always like that. Therefore we suggest the last way how to quantify probability of completing the mission based on more exact method.
It is necessary to define indices and quantities which effect directly probability of completing the mission p_{mis}. These are a number of rounds *n*, probability of a

round failure occurrence p_n, a number of pyrotechnic cartridges m, and probability of a pyrotechnic cartridge failure occurrence p_m. A general function of mission completion probability and its variables is put that way:

$$p_{mis}(n,p_n,m,p_m). \tag{22}$$

Further steps follow well known assumptions. The function of a rounds´ failure takes form of a binomial distribution with parameters n and p_n – $Bi(n,p_n)$, and the rounds which may fail can be marked with k where $k \in \{0;1;2;\ldots..;n\}$. Moreover, we introduce functions of a pyrotechnic cartridges´ failure Y_k where $k \in \{0;1;2;\ldots..;m\}$. They show us possibility of a pyrotechnic cartridge failure while shooting as soon as it is necessary to remove a failed round. Let us assume that a sum of functions of a pyrotechnic cartridges´ failure will be lower than a number of available pyrotechnic cartridges used for removing a failed round. We put it in the following formula

$$\sum_{k=0}^{m} Y_k \le m \approx Y_0 + Y_1 + \ldots..Y_k \le m\,. \tag{23}$$

Following the assumption mentioned above we consider the case that the first available pyrotechnic cartridge follows geometrical distribution of a function of its activity $Ge(p_m)$ during the failure of the k-th round Y_k. The function p_m means probability of pyrotechnic cartridge failure occurrence. It can be described as

$$Y_k \sim Ge(p_m). \tag{24}$$

The equation showing probability of completing the mission is put that way

$$\begin{aligned} &P(n, p_n, m, p_m) \\ &= \sum_{k=0}^{n} P(X=k).P(Y_0 + Y_1 + \ldots. + Y_k \le m) \end{aligned} \tag{25}$$

where in case $k=0$ (it reflects a situation where there is no round failure) a function would be specified additionally provided that $P(Y_1+\ldots.Y_k \le m)=1$. And in order to solve a probability value of completing the mission we would use so called completing the formula taking advantage of forming functions. From a mathematical point of view this is much more demanding but it offers a very exact value expressing probability of completing the mission p_{mis} while using a variation of function factors. On its basis it is easy to prove a dependability of a total number of used pyrotechnic cartridges on a level of mission completion probability p_{mis}.

An example of a possible solution:

Given:
p_n = 0,000 1 - round failure probability;
n = 200 - maximum rounds´ number during one process;
p_m = 0,01 - pyrotechnic cartridge failure probability;
p_{mis} = 0,99 - probability of mission success.

Solution according to "Ad 1)": We are looking for a sufficient number of pyrotechnic cartridges used for removing a possible failure

$$M = \frac{n.\alpha}{1-\beta} = \frac{200.0{,}0001}{1-0{,}01} \cong 0{,}02\,.$$

The formula shows us that having at least one pyrotechnic cartridge is enough to complete the mission successfully. However, we cannot quantify probability for completing the mission.

Solution according to "Ad 2)": We are looking for a level of mission completion probability p_{mis} as well as a required number of pyrotechnic cartridges. We follow the values described above. The solution is put in the table.

Table 1. Results of example

α	x_α	$\beta = \frac{p_{mis}}{\alpha}$	m
0,991	1	0,998991	2
0,992	1	0,997984	2
0,993	1	0,996979	2
0,994	1	0,995976	2
0,995	1	0,994975	2
0,996	1	0,993976	2
0,997	1	0,992979	2
0,998	1	0,991984	2
0,999	1	0,990991	2

If we take into account this solution and starting marginal conditions, two pyrotechnic cartridges will be enough to complete the mission successfully in 0,99 probability.

6. Conclusion

This contribution is supposed to serve as one of the alternatives solving the problems connected with providing a function of an object whose function is redundant (backed up) because its failure is important to complete the mission. In order to solve the problem we chose the methods which are supposed to be the most suitable for it. Other ways are also likely to be used in order to reach the aim but it is not the intention of this contribution.

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Soft computing methods applied to condition monitoring and fault diagnosis for maintenance

Keywords

soft computing, artificial neural networks, fuzzy logic, genetic algorithms, condition monitoring, fault diagnosis, maintenance

Abstract

Malfunctions in equipment and components are often sources of reduced productivity and increased maintenance costs in various industrial applications. For this reason, machine condition monitoring is being pursued to recognize incipient faults in the strive towards optimising maintenance and productivity. In this respect, the following lecture notes provide the basic concepts underlying some methodologies of soft computing, namely neural networks, fuzzy logic systems and genetic algorithms, which offer great potential for application to condition monitoring and fault diagnosis for maintenance optimisation. The exposition is purposely kept on a somewhat intuitive basis: the interested reader can refer to the copious literature for further technical details.

1. Introduction

Managing an industrial plant entails evaluating and trading off the conflicting objectives of economic service and safe operation. The first scientific approaches to this management problem date back to the 1950's and 1960's and can be found in the review paper by McCall [1] and in the book from Barlow and Proschan [3]. As a result, various so-called periodic maintenance *optimisation models* were introduced in which both costs and benefits of *periodic maintenance* were quantified and an optimum compromise between the two was sought. Well-known models originating from this period are the so-called age and *block replacement* models.

From the practical point of view, at that time, preventive maintenance was strongly advocated as a means to reduce failures, for safety reasons, and unplanned downtime, for economic reasons. In many companies, large time-based preventive maintenance programs were set up.

Nowadays, modern production plants are expected to run continuously for extended hours. In this situation, unexpected downtime due to components and equipment failures has become more costly than ever before. The faults can degrade the quality of a product line or even cause the entire plant to function incorrectly, possibly resulting in downtime of the production system with consequent economic loss.

On the other hand, proper monitoring of the conditions of the components and systems can be highly cost effective in minimizing maintenance downtime by providing advanced warning and lead time to prepare the appropriate corrective actions upon an adequate fault diagnosis.

For this reason, condition monitoring has become a popular approach for predicting component failures using physical information on the actual state of the equipment. The possibility of monitoring the system state, continuously for operating systems or by tests and inspections for stand-by safety systems, allows a more dynamic preventive maintenance practice, called condition-based maintenance (CBM), in which the decision of maintaining the system is taken on the basis of the observed condition of the system. This, in principle, allows saving resources by preventively maintaining the system only when necessary. In many practical instances, this approach has proved more effective than the previous large preventive maintenance programs.

Analytical results for single-component deteriorating systems have been established under simplifying assumptions. Markov and semi-Markov models have been the preferred approach in modelling CBM [4], [12], [16]- [17], [21], [27] but other approaches, like counting processes [1], have also been proposed. The majority of the models appeared in the literature

assume that the system's degradation level can only be known through periodic inspection as typical in safety systems such as those employed in nuclear plants [2]-[3], [26]-[27]; Kopnov [16] considers the case in which the system is continuously monitored and Lam [17] considers both cases. Another common assumption is to consider that repairs/replacements always restore the system to a 'good-as-new' condition, which, in practice, may not be very realistic; Kopnov [16] has allowed also for partial recovery.

The dynamic CBM policies for single-component systems whose condition can only be known through inspection, developed in [10], [12] and [19], are all based on control-limit rules which define when to repair/replace a component and when to schedule the next inspection.

For the continuously inspected systems investigated by Kopnov [16], the two-level policies from the Inventory Theory have been adapted to the CBM problem of degrading systems. Semi-Markov processes are also considered; a death process is proposed for a unit subject to corrosion and a Markov chain is used for modelling fatigue crack growth.

A common feature of the models discussed is that the state of the system is described as a state of a Markov process and then the analysis proceeds to finding analytically the probabilities of the various states. However, if the system is made of several multi-state components the analysis becomes excessively complicated. Simulation tools are hence needed when treating more complex systems. Bérenguer et al. [[4] have extended the work of Grall et al. [10] by investigating two-component deteriorating systems using simulation. Their maintenance model takes into consideration economic dependence between components and again the state of the system is only known through periodic inspections. Barata et al. [2] developed a stochastic degradation model for repairable multi-component systems and embedded its simulation within a maintenance optimisation scheme. The condition of each component is known continuously. The novelty of the model stems from the fact that the component's failures can occur not only because of excessive degradation which leads to a critical state of the system, but also because of random shocks which suddenly fail the system and whose occurrence probability is degradation-dependent. While in some cases the system degradation level depends on the combination of many mechanisms and can only be known through inspection [4], [12], [12], [19], [26] there are other mechanisms such as fatigue and corrosion of structures in which deterministic laws are known and the uncertainty is on the value of the parameters that govern those laws.

Regarding the deterioration models themselves, Hontelez et al. [12] give several examples, all of deterministic nature, from the civil engineering field. Grall et al. [10] use a model in which the degradation level increases randomly according to an exponential distribution. Degradation models describing fatigue and corrosion of metal structures are described by Guedes Soares and Garbatov in [23], [24].

The success of condition monitoring and condition-based maintenance strongly relies on the capability of modelling the degradation processes and the corresponding plant dynamic responses under different configurations and conditions. However, the complexity and non-linearities of the involved processes are such that analytical modelling becomes burdensome, if at all feasible without resorting to unrealistic simplifying assumptions. For this reason, empirical modelling is becoming very popular since it does not require a detailed physical understanding of the processes nor knowledge of the material properties, geometry and other characteristics of the plant and its components and it does not resort to simplifying assumptions: the underlying dynamic model is identified by fitting plant operational data, with a procedure often referred to as 'learning' or 'training'.

Among the various techniques of empirical modelling, the so-called soft computing methods offer powerful algorithms for constructing non-linear models from operational data. As a fact, they are being used with increasing frequency as an alternative to traditional models in a variety of engineering applications including monitoring, prediction, diagnostics, control and safety.

The main soft computing methodologies are Neural Networks (NNs), Fuzzy Logic Systems (FLSs) and Genetic Algorithms (GAs). These methodologies are inspired by biology and natural behaviour and provide potentially powerful tools for effectively tackling difficult multivariate, non-linear problems, which often cannot be solved with ease by means of traditional analytical or numerical methods.

In the present lecture notes, we shall try to give a brief description of the concepts underlying the different methodologies and point out their main advantages and limitations. With this objective in mind, we shall refer our discussion to a multidimensional non-linear input/output mapping, for NNs and FLSs, or searching space, for GAs optimisation.

NNs and FLSs are capable of establishing the existing non-linear input/output relationships, which map the inputs of a system to its outputs. They reconstruct the complex non-linear relations by combining multiple simple functions. More precisely, through an analogy with the functioning of the human brain, NNs form the shape of the mapping of interest by appropriately combining a large number of sigmoid, radial or other

simple parameterised functions, which are adjusted (enlarged, shrunk, shifted, etc.) by means of appropriate parameters and synaptic weights [20], [21]. The great power of this technique lies in the fact that the adjustments can be made 'automatically' through a training phase based on available input/output data: this training phase allows to adjust the NN-model parameters so as to obtain the best interpolation of the multivariate, non-linear functional relation between input and output.

FLSs, on the contrary, partition the input/output spaces into several typically overlapping areas, whose shapes are established by assigned membership functions and whose mapping relationships are governed by distinct, simple IF-THEN rules [28], [13]. The great advantage of this method lies in the inherent capability of handling imprecise data and in the physical transparency and interpretability offered by this particular way of representing the underlying model relations.

Finally, if the input/output multidimensional space is seen as a searching space in which the inputs are the decision variables and the outputs are the performance indicators of the search problem, the GAs offer a powerful method for evaluating a best input solution with respect to the optimisation (minimization or maximization) of the performance indicators of interest [9], [11]. The main advantages of the method are that the search is performed by manipulation of a population of points, contrary to classical methods which proceed from a single solution point to another, and that the search is solely based on the evaluation of the performance indicators, with no need of other information, e.g. of derivative nature.

2. Artificial Neural Networks

Artificial neural networks (ANNs) are information processing systems composed of simple processing elements (nodes) linked by weighted connections. Their functioning is inspired by the biological neural networks.

A biological neuron consists of dendrites, a cell body and axons (*Figure 1a*). The connections between a dendrite and the axons of other neurons are called synapses. In correspondence of each synapse, electric pulses from other neurons are transformed into chemical information which is input to the cell body: if the sum of the inputs received by the neuron through all its synapses exceeds a given threshold, then it fires an electric pulse which activates the neuron function. The network of all these neurons makes up the most essential part of the human brain and its operation enables the incredible variety of human activities. In synthesis, the function of a biological neuron is 'simply' to output pulses, with the characteristics of a quasi-step switching function, according to a weighed combination of the multiple signals received from the other connected neurons. A second important function of the neuron is to appropriately modify the rate of transition through the different synapses to optimise the whole network.

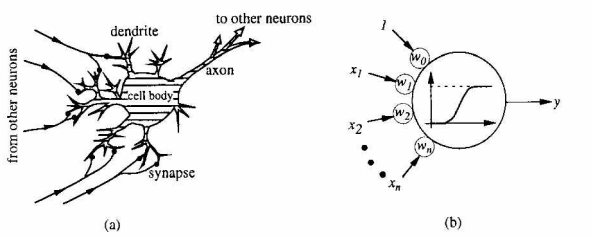

Figure 1. A biological neuron (a) and an artificial neuron model (b) [25]

An artificial neuron (node) aims at simulating the operation of a biological neuron: thus, it accepts multiple inputs $x_1, x_2, ..., x_m$, it weighs them by means of adaptive synaptic weights, $w_0, w_1, ..., w_m$, and it simulates the switching function characteristic of the input/output relation to provide the output (*Figure 1b*). Connecting several artificial neurons together one obtains an artificial neural network which, by construction, constitutes an information processing system composed of simple processing elements (nodes) linked by weighted synaptic connections [21]. The adaptation of the synaptic weights is realized through a training phase during which properly devised learning algorithms are used to change the synaptic weights of the network in an effort to optimise its mapping performance [20].

Here, we limit ourselves to briefly describing the most commonly used multi-layered feed-forward neural network which, in its simplest form, consists of three layers of processing elements: the input, the hidden and the output layers, with n_i, n_h and n_o nodes, respectively (*Figure 2*). The signal is processed forward from the input to the output layer. Each node collects the output values, weighted by the connection weights, from all the nodes of the preceding layer, processes this information through a sigmoid function

$$f(x) = \left(1 + e^{-x}\right)^{-1}$$

and then delivers the result towards all the nodes of the successive layer. Typically, both input and hidden layers are provided with an additional bias node, which serves as a threshold in the argument of the activation function and whose output always equals unity.

As for the determination of the connection weights, i.e. the model parameters, the learning technique most commonly employed is the so-called error back-propagation algorithm, which follows from the

general gradient-descent method [20]. In short, starting from random values of the synaptic weights the back-propagation algorithm performs the steepest descent in the weight space on a surface whose height at any point is equal to the error function; in practice, it consists of an iterative gradient algorithm designed to minimize the mean square error between the actual network output and the true value. A number n_p of sets (patterns) of input and associated outputs are repeatedly presented to the network and the values of the connection weights are modified so as to minimize the average squared output deviation error function, or Energy function, defined as:

$$E = \frac{1}{2n_p n_o} \sum_{n=1}^{n_p} \sum_{l=1}^{n_o} (\hat{y}_{nl} - y_{nl})^2 \qquad (1)$$

where y_{nl} and $\hat{y}_{nl}$ are the true output value of the *n*-th pattern and the corresponding network-computed output value at the *l*-th node, $l = 1, 2, ..., n_o$. Through this training procedure, the network is able to build an internal representation of the input/output mapping of the problem under investigation. The success of the training strongly depends on the normalization of the data and on the choice of the training parameters. Typically, each signal is transformed by an affined mapping in an interval such as (0.2, 0.8) or similar and the connection weights are initialised randomly within an interval such as (-0.3, 0.3) or similar.

After the training is completed, the final connection weights are kept fixed. New input patterns are presented to the network, which is capable of recalling the information stored in the connection weights during training to produce the corresponding output, coherent with the internal representation of the input/output mapping. Notice that the non-linearity of the sigmoid function of the processing elements allows the neural network to learn arbitrary non-linear mappings [6], [15]. Moreover, each node acts independently of all the others and its functioning relies only on the local information provided through the adjoining connections. In other words, the functioning of one node does not depend on the states of those other nodes to which it is not connected. This allows for efficient distributed representation and parallel processing, and for an intrinsic fault-tolerance and generalization capability.

These attributes render the artificial neural networks a powerful tool for signal processing, non-linear mappings and near-optimal solution to combinatorial optimisation problems.

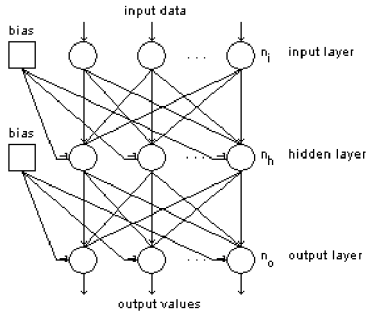

Figure 2. Scheme of a three-layered, feedforward neural network

2.1. Feedforward artificial neural networks for regression

In order to understand further the way of functioning of neural networks, let us consider an artificial feedforward neural network to be trained for performing the task of non-linear regression, i.e. estimating the underlying non-linear relationship existing between a multi-dimensional vector of input variables *x* and an output target *y*, assumed mono-dimensional for simplicity of illustration ($n_0 = 1$, in esq. (1)), based on a finite set of input/output data examples (the above mentioned patterns),

$$D \equiv \left\{ (x_n, y_n),\ n = 1, 2, ..., n_p \right\}.$$

It is assumed that the target *y* is related to the input vector *x* by an unknown deterministic function $\mu_y(x)$ corrupted by a *white noise* ε, viz.

$$y = \mu_y(x) + \varepsilon(x);\ \varepsilon(x) \quad N(0, \sigma_\varepsilon^2(x)) \qquad (2)$$

The objective of the regression task is to estimate $\mu_y(x)$ by means of a *regression function* $f(x; \hat{w})$, dependent on a set of parameters $\hat{w}$ to be properly determined on the basis of an available set of input/output patterns *D*.

A feedforward neural network provides a non-linear form of the function $f(x; \hat{w})$ for the regression task. As above explained, the parameters $\hat{w}$ are called network weights and are usually determined by a *training* procedure which aims at minimizing the *quadratic error function*

$$E = \frac{1}{2n_p} \sum_{n=1}^{n_p} (\hat{y}_n - y_n)^2 \qquad (3)$$

where for simplicity of notation the output node subscript *l* has been dropped since the case considered concerns a single output.

The network output corresponding to input x_n is a function of the weight values, $y_n = f(x;w)$. If the network architecture and training parameters are suitably chosen and the minimization done to determine the weights values is successful, the obtained function $f(x;w)$ gives a good estimate of the unknown, true regression function $\mu_y(x)$. Indeed, it is possible to show that in the ideal case of an infinite training data set and perfect minimization algorithm, a neural network trained to minimize the error function in (3) provides a function f which performs a mapping from the input x into the expected value of the target y conditioned on x, i.e. the true deterministic function $E[y|x] = \mu_y(x)$ [5]. In other words, the network averages over the noise on the data and discovers the underlying deterministic generator. Unfortunately, all the training sets are finite and there is no guarantee that the selected minimization algorithm can achieve the global minimum.

The quadratic error function in (3) can be motivated from the principle of maximum likelihood applied to the set of available training patterns $D \equiv \{(x_n, y_n),\ n = 1, 2, ..., n_p\}$. The likelihood of the observed data set D is defined as

$$L(w|D) = \prod_{n=1}^{n_p} p(x_n, y_n|w) = \prod_{n=1}^{n_p} p(y_n|x_n, w)\, p(x_n|w) \tag{4}$$

where it is assumed that each pattern (x_n, y_n) is drawn independently from the same distribution $P(x_n, y_n)$. The unknown weights w of the neural model $f(x;w)$ are determined by maximization of the likelihood $L(w|D)$ of observing the training set D [5]. Instead of maximizing the likelihood, it is computationally more convenient to minimize its negative logarithm,

$$L^*(w|D) = -\ln L(w|D) = -\sum_{n=1}^{n_p} \ln p(y_n|x_n, w) - \sum_{n=1}^{n_p} \ln p(x_n|w) \tag{5}$$

The distribution of the input values x_n, i.e. the second term in the rhs of (5), is independent of the parameters w of the neural model $f(x;w)$; thus, the parameters w can be found by minimization of the first term only, i.e. the following error function

$$E = -\sum_{n=1}^{n_p} \ln p(y_n|x_n, w) \tag{6}$$

Different forms of the conditional distribution $p(y|x,w)$ lead to different error functions. In particular, the assumption of a Gaussian distribution for the target as in (2) leads to a quadratic error function of the kind in (3) [20].

Indeed, from eq. (2) we have

$$\varepsilon(x) = y - \mu_y(x) \sim N\left(0, \sigma_\varepsilon^2(x)\right)$$

and using the regression function $y = f(x;w)$ to estimate $\mu_y(x)$, we get

$$p(y|x,w) = \frac{1}{\sqrt{2\pi}\sigma_\varepsilon(x)} e^{-\frac{1}{2}\frac{(y-y)^2}{\sigma_\varepsilon^2(x)}} \tag{7}$$

and the error function in (6) becomes

$$E = -\sum_{n=1}^{n_p} \ln\left[\frac{1}{\sqrt{2\pi}\sigma_\varepsilon(x_n)} e^{-\frac{1}{2}\frac{(y_n - y_n)^2}{\sigma_\varepsilon^2(x_n)}}\right]. \tag{8}$$

When the noise variance is independent of the input x, i.e. $\sigma_\varepsilon^2(x) = \sigma_\varepsilon^2$,

$$E = \frac{1}{2\sigma_\varepsilon^2}\sum_{n=1}^{n_p}(y_n - y_n)^2 + n_p \ln\sigma_\varepsilon + \frac{n_p}{2}\ln 2\pi, \tag{9}$$

and the error function reduces to the form (3) since the other terms do not depend on the weights w of the neural model.

Obviously, the quadratic error function in (3) can be used also for regression on targets, which are not Gaussian-distributed: in this case, the resulting regression function $f(x;w)$ cannot distinguish between the true distribution and any other with same mean and variance.

Finally, notice that the value of the error function (3) at the minimum gives a measure of the variance of the target data, averaged over the input.

2.2. Neural network uncertainty

In practical regression problems, there are two types of prediction that one may want to obtain in correspondence of a given input x: an estimate

$f(x;\mathbf{w})$ of the underlying deterministic function $\mu_y(x)$ and an estimate of the target value *y* itself, as given by eq.(2), with their corresponding measures of confidence. This requires that the various sources of uncertainty affecting the determination of the weights $\mathbf{w}$ be properly accounted for [7].

For what concerns the estimate $f(x;\mathbf{w})$ of $\mu_y(x)$, it must be considered that, from a probabilistic point of view, the data set $D \equiv \{(x_n, y_n),\ n = 1,2,...,n_p\}$ used for training the network is only one of an infinite number of possible data sets. This variability in the training data set is due to the variability in the sampling of the input vectors $x_n, n = 1,2,...,n_p$ and in the random fluctuation of the corresponding target output y_n. Each possible training set *D* can give rise to a different set of network weights $\mathbf{w}$. Correspondingly, there is a distribution of regression functions $f(x;\mathbf{w})$ with variance (with respect to the training set *D*):

$$E\left\{\left[f(x,\mathbf{w}) - E[f(x,\mathbf{w})]\right]^2\right\} \tag{10}$$

Since in practice a neural network structure is not a perfect algorithm, it systematically under/over estimates the correct result, i.e. the expected value $E[f(x,\mathbf{w})]$ is not equal to the true underlying deterministic function, $\mu_y(x)$, their difference being the so-called *bias*. Of course, the bias would be zero in the case of a perfect neural network.

To quantify the confidence in the estimate $f(x;\mathbf{w})$ of the true deterministic function $\mu_y(x)$, it is customary to refer to the *confidence intervals* of the error $f(x,\mathbf{w}) - \mu_y(x)$ whose variance with respect to all possible training data sets is:

$$E\{[f(x,\mathbf{w}) - \mu_y(x)]^2\}$$

$$= E\left\{\left[\begin{array}{l} f(x,\mathbf{w}) - E[f(x,\mathbf{w})] + \\ \\ + E[f(x,\mathbf{w})] - \mu_y(x) \end{array}\right]^2\right\}$$

$$= \left\{\begin{array}{l} \left[f(x,\mathbf{w}) - E[f(x,\mathbf{w})]\right]^2 \\ \\ + \left[E[f(x,\mathbf{w})] - \mu_y(x)\right]^2 \\ \\ + 2\left[f(x,\mathbf{w}) - E[f(x,\mathbf{w})]\right] \\ \\ \cdot \left[E[f(x,\mathbf{w})] - \mu_y(x)\right] \end{array}\right\}$$

$$= \left\{\left[f(x,\mathbf{w}) - E[f(x,\mathbf{w})]\right]^2\right\}$$

$$+ \left\{E[f(x,\mathbf{w})] - \mu_y(x)\right\}^2 \tag{11}$$

where the first term is the variance of the distribution of regression function values $f(x;\mathbf{w})$ and measures the extent to which the network regression function $f(x;\mathbf{w})$ differs from the ensemble average over different training data sets, whereas the second term is the square of the bias which measures the extent to which the average (over all possible training data sets) of the network regression function $f(x;\mathbf{w})$ differs from the true underlying deterministic function, $\mu_y(x)$.

2.3. Sources of uncertainty

A first source of uncertainty comes from a wrong choice of the network architecture. Indeed, in case of a network with too few nodes, i.e. too few parameters, a large bias occurs since the regression function $f(x;\mathbf{w})$ has insufficient flexibility to model the data adequately, which results in poor generalization properties of the network. On the other side, excessively increasing the flexibility of the model by introducing too many parameters increases the variance term because the network regression function tends to over-fit the training data. Thus, in both cases, the network performs poorly when fed with new input pattern in the generalization phase. A trade-off is, then, necessary. This trade-off is typically achieved by controlling the model complexity (i.e., the number of parameters) and the training procedure (by adding a regularization term in the error function or by early stopping of the training [5]) so as to achieve a good fit of the training data but with a reasonably smooth regression function which is not over-fit to the data.

An additional source of uncertainty in the network performance, due to uncertainty in the weights $\mathbf{w}$, arises from the minimization algorithm itself, which may get stuck in a local minimum of the error function. Furthermore, the training may be stopped prematurely, before reaching the minimum.

Besides the above uncertainties in the regression function $f(x;\mathbf{w})$ due to uncertainty in the weights $\mathbf{w}$, in practice there is also uncertainty in the input values *x* due to noise and uncertainty in the model structure $f(\cdot)$ itself.

For what concerns the prediction of the target value, *y*, it is clear that even in the ideal case of a regression function $f(x;\mathbf{w})$ equal to the true deterministic function, $\mu_y(x)$, the target *y* could not be predicted

with certainty due to the presence of the noise term $\varepsilon(x)$ in (eq. 2) which accounts for the intrinsic random fluctuations. To quantify the accuracy of the estimate of *y*, it is customary to refer to the *prediction intervals* of the deviation

$$y - f(x, \hat{w}) = \left[\mu_y(x) - f(x, \hat{w})\right] + \varepsilon \ .$$

The variance of such deviation is:

$$\begin{aligned} & E\left\{\left[f(x, \hat{w}) - y\right]^2\right\} \\ & = E\left\{\left[f(x, \hat{w}) - \mu_y(x)\right]^2\right\} + E\left\{\left[\varepsilon\right]^2\right\} \qquad (12) \\ & = E\left\{\left[f(x, \hat{w}) - \mu_y(x)\right]^2\right\} + \sigma_\varepsilon^2 \end{aligned}$$

Note that the first term is the variance of the distribution of the error $f(x, \hat{w}) - \mu_y(x)$ (eq. 11), so that the prediction intervals include the confidence intervals.

From the above said, it appears that artificial neural networks are *unstable* predictors: small changes in the training data may produce very different regression models and consequently different generalization performances on new, unseen data. For this reason, the generalization performance of a single artificial neural network, particularly if trained on small data sets, should be tested by means of a *k*-fold cross validation, where the available set of n_p input/output patterns is divided into *k* subsets of (approximately) equal size and the network is trained *k* times on a training set in which each time one of the *k* subsets is left out and used to verify the network generalization performance. If *k* equals the sample size, the procedure is called *leave-one-out* cross-validation [14].

3. Fuzzy logic system

Fuzzy logic systems are founded on the theory of fuzzy sets, which, in general, deals with vague information, where *vagueness* is defined as the uncertainty associated with linguistic or intuitive information. For example, the quality of an image may be judged as *bad*, *medium* or *good*. From this example, it appears that vagueness is related to immeasurable issues and involves situations in which the transitions among linguistic statements occur across boundaries, which are not sharp.

A few words on the concepts underlying fuzzy set theory seem then in order [28]. Let us consider a variable *x*, for example a measured output of a plant. Mutating from classical logic, the set *U* which contains all the possible values of *x* is usually called the universe of discourse (UOD) of *x* or the universal set. Suppose that the UOD *U* has been subdivided in a sequence of subsets $X_i \subset U$. In classical set theory, the X_i's are mutually exclusives so that a given value of *x* may belong to only one of them. These sets are called *crisp* and the membership of a crisp value of *x* to a set X_i is specified by the (rectangular) characteristic function χ_{X_i}, which is equal to unity or zero according to whether the value of *x* belongs or not to X_i.

In fuzzy set theory, the situation is quite different: the subsets X_i of the universe of discourse *U* of a linguistic variable *x* are not necessarily exclusive, so that a given crisp value of $x \in U$, may simultaneously belong to more than one of them with different degrees of membership. This feature clearly distinguishes fuzzy set theory from probability theory, which operates on crisp events. In fuzzy set theory, then, the subsets are not identified by sharp boundaries but by linguistic labels (*words*). For example we may consider the linguistic variable *temperature* defined in the universe of discourse $U = (0°, 40°)$ subdivided in the subsets $X_1 = (0°, 20°)$, $X_2 = (10°, 30°)$, $X_3 = (20°, 40°)$, labelled by the words *cold*, *warm* and *hot*, respectively. Clearly a given temperature value may belong to more than one set, e.g. 15° belongs to X_1 and X_2.

Fuzzy set theory aims at quantifying the meanings of the words attached by the analyst to the subsets X_i (such as cold, warm or hot in the above example) within the framework of set theory. To this aim, to each set X_i the analyst associates, for all values of $x \in U$, the membership function $\mu_{X_i}(x)$, which represents the degree to which he postulates that *x* belongs to X_i. As opposed to the characteristic functions of classical set theory, which are rectangular in shape and disjoint, the membership functions associated to fuzzy sets have subjective shapes and may overlap to describe a continuous transition from one set to another, thus providing for the possibility that a given value of $x \in U$ simultaneously belongs to several sets with different degrees of membership.

In summary, in the fuzzy context we deal with linguistic variables (e.g. *temperature*) whose arguments are words, also called fuzzy values (e.g. *negative*, *approximately zero*, *low*, *positive*, *high*). Each of these words refers to a subset of the universe of discourse and the degree of membership of the crisp values within the subset is analytically specified by the associated membership function.

Fuzzy logic systems build on the theory of fuzzy sets to realize a complex non-linear input/output relation as a synthesis of multiple simple input/output relations. This idea is similar to that of NNs. The difference is that in FLSs each simple input/output relation is embedded in a different IF-THEN rule with 'fuzzy', and not sharp, boundaries so that going from one rule to the other the system output gradually changes [28].
In general, the generic *j*-th fuzzy rule is made up of a number of antecedent and consequent linguistic statements, suitably related by fuzzy connections:

IF (x_1 is X_{1j}) AND (…) AND (x_m is X_{mj})
THEN (y_1 is Y_{1j}) AND (…) AND (y_k is Y_{kj})

The linguistic variables $x_p, p = 1, 2, ..., m$, are the antecedents, represented in terms of the fuzzy sets X_{pj} of the universe of discourse (range) X_p, with membership functions $\mu_{X_{pj}}(x_p)$. The linguistic variables $y_q, q = 1, 2, ..., k$, are the consequents, represented by the fuzzy sets Y_{qj} of the universe of discourse Y_q, with membership functions $\mu_{Y_{qj}}(y_q)$.
The connective operator AND links two fuzzy concepts and it is generally implemented by means of a *t-norm*, typically the minimum operator $\wedge$ or the algebraic product. Since one of the key features of FLS lies in allowing the overlapping of the rules, a given input vector (FACT) will typically activate more than one rule.
Another feature of FLSs is the ability to separate logic and fuzziness [25]. Conventional binary logic systems are unable to do so and thus their governing rules have to be modified when either the system logic or the variables fuzziness needs to be changed. On the contrary, FLSs modify their rules when the logic needs to be changed whereas they modify the supporting membership functions when fuzziness should be changed. To clarify this, consider the performance of an inverted pendulum controller [25]. Define as ω and ω' the angle that the pole forms on the right side with the vertical line and the associated angular velocity, respectively. Let two correct logic rules for the control of the pendulum be:

1) IF ω is positive big AND ω' is big, THEN move the car to the right quickly
2) IF ω is negative small AND ω' is small, THEN move the car to the left slowly

If the performance of the controller is unsatisfactory, one needs not change the fuzzy rules themselves, which are logically correct, but rather must only modify appropriately the definition of fuzziness in the linguistic terms *big*, *small*, *quickly*, *slowly*.
On the other hand, binary logic rules such as

3)IF $40° < \omega < 60°$ AND $50° s^{-1} < \omega' < 80° s^{-1}$, THEN move the car at $0.5 m/s$
4)IF $-20° < \omega < -10°$ AND $10° s^{-1} < \omega' < 20° s^{-1}$, THEN move the car at $-0.1 m/s$

must be modified whenever the logic of the system or the quantitative definitions of angle, angular velocity and car speed are changed.
To understand FLSs, we address on an intuitive basis the problem of controlling a plant [25]. The mathematical-based approach of classical and modern control theory stems on the observation of the system, the construction of its mathematical model and the design of a model-based controller. The focus is placed on the behaviour of the target system and on its mathematical representation.
On the contrary, fuzzy-logic control does not utilize the target system for modelling but it is based, in principle, on the linguistic control rules used by experienced and skilled operators. Although most skilled operators do not know the mathematical behaviour of the systems they are required to control, they can still perform successfully. For example, a skilled driver most likely ignores the mathematical equations underlying the physical behaviour of the car when turning to the right while driving up an unpaved hill and, yet, he or she can still handle the car safely and successfully. In this view, a fuzzy logic controller aims at reproducing the knowledge and experience supporting the control actions of skilled human operators using IF-THEN fuzzy rules.
Clearly the set of IF-THEN fuzzy rules constitutes the heart of the input/output mapping model provided by the FLS. When the experience of the skilled human operators is unavailable or insufficient, because of the complexity of the system, input/output data can be used to generate a set of fuzzy rules representative of the mapping from the input space into the output one. This phase of rule construction during which both the system input and output are known is often referred to with the term 'training', in analogy to the procedure for determining the weights of a neural network model illustrated in Section 2.

3.1. Establishing the antecedent part of a rule

The IF part of a rule is called *antecedent*. Establishing the antecedent parts of the rules of the FLSs is related to the partitioning of the multivariate input space. For simplicity, let us consider a two-dimensional input space (x_1, x_2) and a one-dimensional output space *y*.

Most of the times it is possible to assume that all input variables are independent and, thus, partition separately the input space in each direction (*Figure 3*). This assumption makes it easy not only to partition the input space but also to interpret the partitioned areas in linguistic terms. For example, the rule IF *temperature* is A_1 AND *humidity* is A_2, THEN …, is easy to understand because the variables of temperature and humidity are separated. The difference between 'crisp' and 'fuzzy' rule-based systems lies in the way the input space is partitioned (*Figure 3*). The idea behind FLSs is that in the real analog world, changes are not sudden and sharp but gradual in nature so that overlapping of rules domains should be allowed. The degree of overlapping is defined in terms of membership functions and the intrinsic gradual property allows for smooth control.

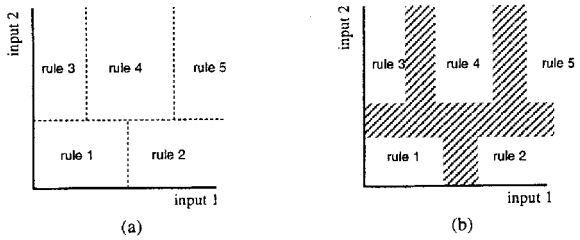

Figure 3. Rule partition of an input space (a) partition for crisp rules and (b) partition for fuzzy rules [25]

3.2. Establishing the consequent part of a rule

The THEN part of a rule is called *consequent*. In the control case, establishing the consequent parts of the rules of the FLSs must eventually lead to defining the control action value corresponding to each rule. In this respect, fuzzy models are classified into three types according to the form used for the consequent *y*:

Model type	Consequent expression	Characteristic
Mamdani	*Y is Y*	*Y* is a fuzzy set
Takagi Sugeno Kang (TSK)	$y = c_0+\Sigma_i a_i x_i$	a_i 's are constant and the x_i 's are the input variables
Simplified fuzzy	$y = c$	*c* is constant

In the Mamdani type FLSs the consequent is a fuzzy variable defined by a membership function. These systems are more difficult to compute than those whose consequents are numerically defined but they better describe the qualitative knowledge related to the consequent.
The consequents of the TSK models are expressed as a weighed linear combination of the input variables. It is also possible to use non-linear combination for better performance, but at the expense of the transparency of the rules.
The simplified fuzzy model has fuzzy rules whose consequents are constant values. Thus, it is a special case of both Mamdani and TSK types. Even if the output of each rule is a constant, the overall FLS output is non-linear because it contains the characteristics of the underlying model membership functions.

3.3. Inferring the output corresponding to a given input: fuzzy reasoning and aggregation

Now that the IF and THEN parts of the rules have been designed, the next step is to infer the output of the FLS resulting from a given *m*-dimensional crisp input vector $x^{(*)}$, also called FACT. This is done in two steps: 1) determination of the rules strengths; 2) aggregation of each rule output into the final FLS numerical output.
As mentioned, the connective operator AND links two fuzzy concepts in a rule and it is generally implemented by means of a *t-norm* applied to the membership functions of the rule's antecedents evaluated in correspondence of the crisp input vectors $x^{(*)}$ constituting the FACT. Typically, the minimum operator $\wedge$ or the algebraic product is employed. This gives the strength of the rule for the given FACT: a measure of how active that rule is for the given FACT or, in other words, how much the FACT is described by the antecedents of the rule. Considering a generic rule *l*, the strength $s_l(x^{(*)})$ is given, in the case of the algebraic product, by the product of its antecedent membership values in correspondence of the crisp inputs:

$$s_l(x^{(*)}) = \prod_{p=1}^{m} \mu_{X_{pl}}(x_p^{(*)}) \qquad (13)$$

where X_{pl} denotes the fuzzy set characterizing the *p*-th antecedent of the *l*-th rule.
In the case of the min operator:

$$s_l(x^{(*)}) = \min_{\mu}(\mu_{X_{pl}}(x_p^{(*)})) \qquad (14)$$

Obviously, if the *l*-th rule is not activated by $x^{(*)}$ then its strength is zero. This occurs if at least one of the elements of the crisp input vector, say $x_p^{(*)}$, does not belong to X_{pl}, the corresponding fuzzy set in the rule.
Let us now consider the aggregation step and denote by R_f the number of rules, which are activated (fired)

by the input $x^{(*)}$ and by y^l the consequent in the *l*-th activated rule, $l = 1, 2, ..., R_f$. The output *y* can be determined as a normalized weighed sum of the consequents y^l of the R_f active rules, the weights being the strengths of the rules,

$$y = \frac{\sum_{l=1}^{R_f} s_l(\overset{\circ}{x}^{(*)}) y^l}{\sum_{l=1}^{R_f} s_l(\overset{\circ}{x}^{(*)})} \tag{15}$$

Figure 4 shows an example of a simple TSK-type FLS with four fuzzy rules [25]. The first rule for example could be:

IF x_1 is small AND x_2 is small,
THEN $y = 3x_1 + 2x_2 - 4$

Corresponding to the input vector $(x_1, x_2) = (10, 0.5)$, the membership functions of the fuzzy sets constituting the antecedents of the first rule are readily evaluated as 0.8 and 0.3.
If the algebra product is used as the *t*-norm operator for the AND connection, then the rule strength is $s_1(10, 0.5) = 0.8 \cdot 0.3 = 0.24$. Similarly, the strengths of the second, third and fourth rules are $s_2(10, 0.5) = 0.8$, $s_3(10, 0.5) = 1.0$, $s_4(10, 0.5) = 0.3$, respectively. The output of each rule corresponding to the given input vector is $y^1 = 27$, $y^2 = 23.5$, $y^3 = -9$, $y^4 = -20.5$, respectively. Then, the final system output is

$$y = \frac{0.24 \cdot 27 + 0.8 \cdot 23.5 + 1 \cdot (-9) + 0.3 \cdot (-20.5)}{0.24 + 0.8 + 1.0 + 0.3}$$

$$\approx 4.33 \tag{16}$$

In the Mamdani type model, with fuzzy consequents, the output of the fuzzy inference engine consists of a fuzzy set $Y' \subseteq Y$ with compact support (η_1, η_2), whose membership function is $\mu_{Y'}(y)$. However, eventually we are interested in finding a crisp number y^* that represents the information encoded in the output fuzzy set Y'. This conversion, called *defuzzification*, may be done in several ways, the most commonly used being the *Centre of Area* (COA) method:

$$y^* = y_{COA} = \frac{\int_{\eta_1}^{\eta_2} y \cdot \mu_{Y'}(y) dy}{\int_{\eta_1}^{\eta_2} \mu_{Y'}(y) dy} \tag{17}$$

The crisp number y^* thereby obtained can be taken as the output resulting from the given input vector (FACT) $x^{(*)}$.

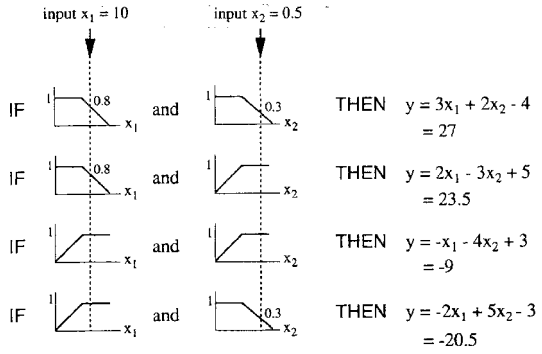

Figure 4. Example aggregation of TSK model [25]

3.4. Interpretation of fuzzy rules

The major difference between fuzzy systems and other non-linear approximates, such as neural networks, is the possibility of interpretation of the rules underlying the fuzzy model. This, however, does not automatically follow from the existence of a rule structure. Therefore, it is relevant to discuss the circumstances under which a fuzzy system is really interpretable and this depends on the application. Yet, some general guidelines can be given. The following factors may influence the interpretability of a fuzzy logic system [25]:

- *Number of rules*. If the number of rules is too large, the fuzzy system can be hardly interpreted. Especially for systems with many inputs, the number of rules often becomes overwhelmingly large if all antecedent combinations are realized.
- *Number of antecedents in the rule premise*. Rules with premises that have many, say more than three or four, antecedents are hard to interpret. In the human language, most rules include only very few antecedents even if the total number of inputs relevant for the problem is large.
- *Dimension of input fuzzy sets*. One-way to avoid, or at least, reduce the difficulties with high-dimensional input spaces and to decrease the number of rules is to resort to multi-dimensional input fuzzy sets. However, multidimensional input fuzzy sets with more than three inputs are certainly beyond human imagination and it is precisely the conjunction of one-dimensional

input fuzzy sets that make a fuzzy logic model interpretable.

- *Arrangement of fuzzy sets*. Fuzzy sets should be arranged in proper order over the universe of discourse so that, for example, very small is followed by *small*, followed by medium, large and so on. If the fuzzy model is developed on the basis of expert knowledge such ordering comes natural. However, if rule construction by training on input/output data is used to optimise the FLS, the ordering of the fuzzy sets might be lost if no precautions are taken. This typically leads to constraint optimisations in which for example the ordering of the input membership functions is constrained. This not only leads to an easier interpretation of the fuzzy system but, often, also provides an improved performance.

4. Genetic algorithms

Search or optimisation algorithms inspired on the biological laws of genetics are called evolutionary computing algorithms [8]. The main features of these algorithms are that the search is conducted i) using a population of multiple solution points or candidates, ii) using operations inspired by the evolution of species, such as breeding and genetic mutation, iii) based on probabilistic operations, iv) using only information on the objective or search function and not on its derivatives. Typical paradigms belonging to the class of evolutionary computing are genetic algorithms (GAs), evolution strategies (ESs), evolutionary programming (EP) and genetic programming (GP). In the following we shall focus on the more popular GAs.

As a first definition, it may be said that genetic algorithms are numerical search tools aiming at finding the global maximum (or minimum) of a given real objective function of one or more real variables, possibly subject to various linear or non linear constraints [11]. Genetic algorithms have proven to be very powerful search and optimisation tools especially when only little about the underlying structure in the data is known. They employ operations similar to those of natural genetics to guide their path through the search space. Essentially, they embed a survival of the fittest optimisation strategy within a structured, yet randomised, information exchange [9].

Since the GAs owe their name to the fact that their functioning is inspired by the rules of the natural selection, the adopted language contains many terms borrowed from biology, which need to be suitably redefined to fit the algorithmic context. Thus, when we say that the GA operates on a set of (artificial) chromosomes, these must be understood as strings of numbers, generally sequences of binary digits 0 and 1. If the objective function has many arguments, each string is partitioned in as many substrings of assigned lengths, one for each argument and, correspondingly, we say that each chromosome is analogously partitioned in (artificial) genes. The genes constitute the so-called genotype of the chromosome and the substrings, when decoded in real numbers called control factors, constitute its phenotype. When the objective function is evaluated in correspondence of the values of the control factors of a chromosome, its value is called the fitness of that chromosome. Thus each chromosome gives rise to a trial solution to the problem.

The GA search is performed by constructing a sequence of populations of chromosomes, the individuals of each population being the children of those of the previous population and the parents of those of the successive population. The initial population is generated by randomly sampling the bits of all the strings. At each step, the new population is then obtained by manipulating the strings of the old population in order to arrive at a new population hopefully characterized by an increased mean fitness. This sequence continues until a termination criterion is reached. As for the natural selection, the string manipulation consists in selecting and mating pairs of chromosomes in order to groom chromosomes of the next population. This is done by repeatedly performing on the strings the four fundamental operations of reproduction, crossover, replacement and mutation, all based on random sampling. These operations will be detailed below [18].

Finally, it is by now acknowledged that GAs take a more global view of the search space than many other optimisation methods. The main advantages are i) fast convergence to near global optimum, ii) superior global searching capability in complicated search spaces, iii) applicability even when gradient information is not readily achievable. The first two advantages are related to the population-based searching property *(Figure 5*). Indeed, while the gradient method determines the next searching point using the gradient information at the current searching point, the GA determines the next set of multiple search points using the evaluation of the objective function at the current multiple searching points. When only gradient information is used, the next searching point is strongly influenced by the local geometric information of the current searching point so that the search may remain trapped in a local minimum. On the contrary, the GA determines the next multiple searching points using the fitness values of the current searching points, which are spread throughout the searching space, and it can also resort to the additional mutation to escape from local minima.

The key disadvantage of a GA is that its convergence speed becomes slow near the global optimum.

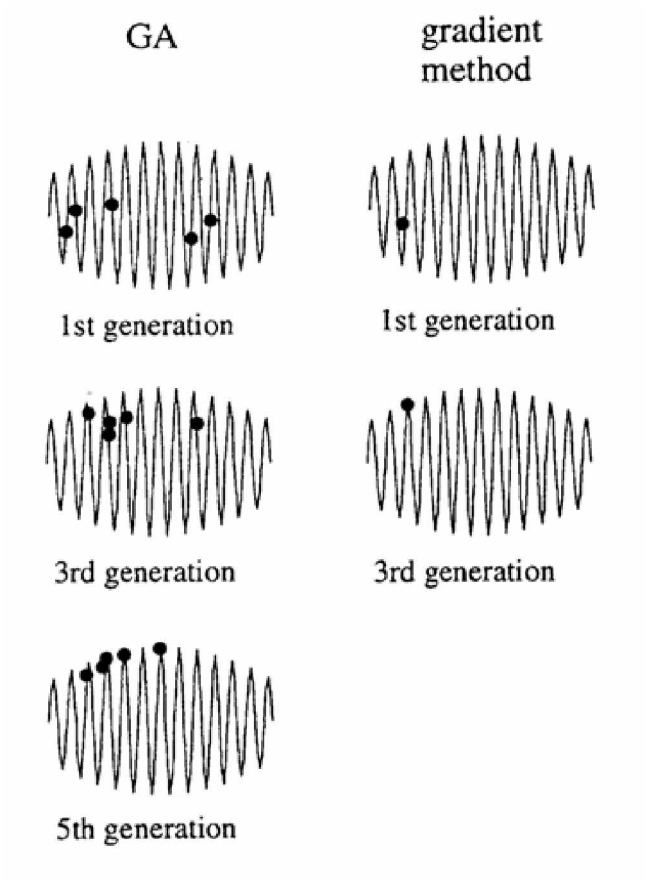

Figure 5. GA search and gradient-based search [25]

4.1. Definitions

It is important to be acquainted with the technical terms of GAs.

Individuals and Population: An individual is a chromosome, constituted by $n \geq 1$ genes and a population is a collection of individuals. To code/decode the i-th gene in a control factor, that is in an argument of the objective function, the user:

- defines the range (a_i, b_i) of the corresponding argument in the objective function;
- assigns the resolution of that independent variable by dividing the range (a_i, b_i) in 2^{n_i} intervals. A number n_i of bits is then assigned to the substring representative of the gene and the relation between a real value $x \in (a_i, b_i)$ and its binary counterpart β is

$$x = a_i + \beta \frac{b_i - a_i}{2^{n_i}} \qquad (18)$$

The values a_i, b_i, n_i are called the *phenotyping parameters* of the gene.

Figure 6 shows the constituents of a chromosome made up of three genes and the relation between the genotype and the external environment, i.e. the phenotype, constituted by three control factors, x_1, x_2, x_3, one for each gene. The passage from the genotype to the phenotype and vice versa is ruled by the phenotyping parameters of all genes, which perform the coding/decoding actions. Each individual is characterized by fitness, defined as the value of the objective function calculated in correspondence of the control factors pertaining to that individual. Thus a population is a collection of points in the solution space, i.e. in the space of *f*.

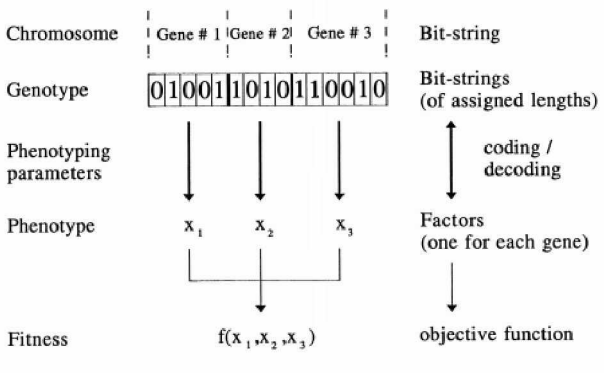

Figure 6. Components of an individual (a chromosome) and its fitness

An important feature of a population is its genetic diversity: if the population is too small, the scarcity of genetic diversity may result in a population dominated by almost equal chromosomes and then, after decoding the genes and evaluating the objective function, in the quick convergence towards an optimum which may well be a local one. At the other extreme, in too large populations, the overabundance of genetic diversity can lead to clustering of individuals around different local optima: then the mating of individuals belonging to different clusters can produce children (newborn strings) lacking the good genetic part of either of the parents. In addition, the manipulation of large populations may be excessively expensive in terms of computer time.

In most computer codes the population size is kept fixed at a value set by the user so as to suit the requirements of the model at hand. The individuals are left unordered, but an index is sorted according to their fitnesses. During the search, the fitnesses of the newborn individuals are computed and the fitness index is continuously updated.

4.2. Creation of the initial population

As said above, the initial population is generated by random sampling the bits of all the strings. This procedure corresponds to uniformly sampling each control factor within its range. The chromosome creation, while quite simple in principle, presents

some subtleties worth to mention: indeed it may happen that the admissible hypervolume of the control factors is only a small portion of that resulting from the Cartesian product of the ranges of the single variables, so that one must try to reduce the search space by resorting to some additional condition in terms of suitable physical criteria to be satisfied. This remark also applies to the *chromosome replacement*, below described.

4.3. The traditional breeding algorithm

The breeding algorithm is the way in which the $(n+1)$-th population is generated from the *n*-th previous one.

The first step of the breeding procedure is the generation of a temporary new population. Assume that the user has chosen a population of size N (generally an even number). The population reproduction is performed by resorting to the Standard Roulette Selection rule: to find the new population, the cumulative sum of the fitnesses of the individuals in the old population is computed and normalized to sum to unity. The new population is generated by random sampling individuals, one at a time with replacement, from this cumulative sum, which then plays the role of a cumulative distribution function (cdf) of a discrete random variable (the position of an individual in the population). By so doing, on the average, the individuals in the new population are present in proportion to their relative fitness in the old population. Since individuals with relatively larger fitness have more chance to be sampled, most probably the mean fitness of the new population is larger.

The second step of the breeding procedure, i.e. the *crossover*, is performed as indicated in *Figure 7*: after having generated the new (temporary) population as above said, *N/2* pairs of individuals, the parents, are sampled at random without replacement and irrespectively of their fitness, which has already been taken into account in the first step. In each pair, the corresponding genes are divided into two portions by inserting at random a separator in the same position in both genes (one-site crossover): finally, the first portions of the genes are exchanged. The two chromosomes so produced, the children, are thus a combination of the genetic features of their parents. A variation of this procedure consists in performing the crossover with an assigned probability p_c (generally rather high, say $p_c \geq 0.6$): a random number R is uniformly sampled in (0,1] and the crossover is performed only if $R < p_c$. Vice versa, if $R \geq p_c$, the two children are copies of the parents.

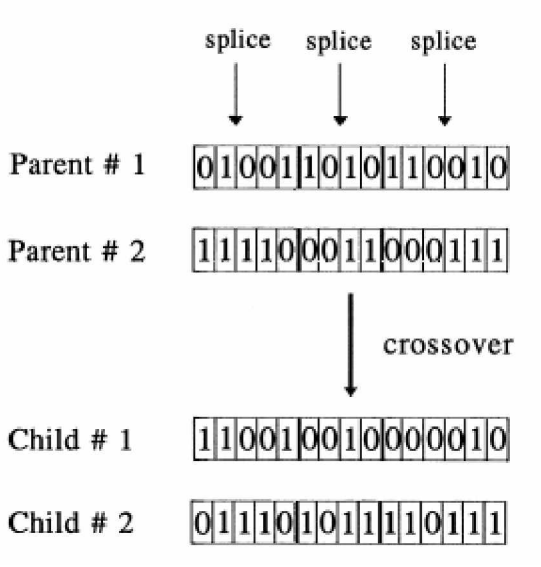

Figure 7. Crossover in a population with chromosomes constituted by three genes

The third step of the breeding procedure, performed after each generation of a pair of children, concerns the replacement in the new population of two among the four involved individuals. The simplest recipe, again inspired by natural selection, just consists in the children replacing the parents: children live, parents die. In this case, each individual breeds only once.

The fourth and last step of the breeding procedure eventually gives rise to the final $(n+1)$-th population by applying the mutation procedure to the (up to this time temporary) population obtained in the course of the preceding steps. The procedure concerns the mutation of some bits in the population, i.e. the change of some bits from their actual values to the opposite one ($0 \rightarrow 1$) and vice versa. The mutation is performed on the basis of an assigned mutation probability for a single bit (generally quite small, say 10^{-3}). The product of this probability by the total number of bits in the population gives the mean number μ of mutations. If $\mu < 1$ a single bit is mutated with probability μ. Those bits to be actually mutated are then located by randomly sampling their positions within the entire bit population.

The sequence of successive population generations is usually stopped according to one of the following criteria:

1. when the mean fitness of the individuals in the population increases above an assigned convergence value;
2. when the median fitness of the individuals in the population increases above an assigned convergence value;
3. when the fitness of the best individual in the population increases above an assigned

convergence value. This criterion guarantees that at least one individual is good enough;

4. when the fitness of the weakest individual in the population drops below an assigned convergence value. This criterion guarantees that the whole population is good enough;
5. when the assigned number of population generations is reached.

More sophisticated techniques of reproduction, crossover and replacement can be employed for a more effective search.

Furthermore, in general, the initial population sampled contains a majority of second-rate individuals together with few chromosomes, which, by chance, have moderately good fitnesses. Then, the selection rules are such that, in a few generations, almost all the moderately good chromosomes, which are actually mediocre individuals, are selected as parents and generate children of similar fitnesses; thus, almost all the second-rate individuals disappear, and most of the population gathers in a small region of the search space around one of the mediocre individuals. In this case, the genetic diversity is drastically reduced and the algorithm may achieve a premature convergence of the population fitness to a local maximum. In the course of the successive generations, the crossover procedure generates mediocre individuals, which are substituted in place of other mediocre individuals, so that the genetic selection may be seen as a random walk among mediocres. To obviate to this unpleasant premature convergence to mediocrity, a pre-treatment of the fitness function is often welcome. This is typically done by means of an affined transform of the fitnesses. Instead of applying the selection rules to the fitness $f(x)$ one works with its affined transform $f'(x)$, viz.,

$$f'(x) = a\, f(x) + b \tag{19}$$

where *a* and *b* are chosen so as to favour, at the beginning, the less fit individuals, thus maintaining genetic diversity and avoiding premature convergence [18].

5. Conclusion

These lecture notes have briefly sketched some of the concepts underlying the modern computational paradigms of neural networks, fuzzy logic systems and genetic algorithms, which are becoming of significant interest for application to condition monitoring and fault diagnostics for maintenance. Due to the limitation in the number of pages, only an intuitive and non-exhaustive treatment has been provided. Whereas some examples of practical application will be illustrated during the lecture, the interested reader is invited to refer to the specialized literature for further, in-depth details on the different techniques.

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Optimising a fuzzy fault classification tree by a single-objective genetic algorithm

Keywords

fault classification, decision tree, fuzzy logic, genetic algorithm

Abstract

In this paper a single-objective Genetic Algorithm is exploited to optimise a Fuzzy Decision Tree for fault classification. The optimisation procedure is presented with respect to an ancillary classification problem built with artificial data. Work is in progress for the application of the proposed approach to a real fault classification problem.

1. Introduction

In recent years, many efforts have been devoted to the development of automatic diagnostic techniques based on statistical or geometric methods, neural networks, expert systems, fuzzy and neuro-fuzzy approaches [22], [11], [15], [5], [7]. These techniques have proven to be very effective but often remain "black boxes" as to the interpretation of the physical relationships underpinning the fault classification.

In an effort to overcome this limitation, a systematic approach to fault classification has been introduced by the authors leading to a Fuzzy Decision Tree (FDT) [25], [26]. The main advantages of the proposed approach from the operator point of view are the transparency of the resulting classification model and its visualization in the form of a DT [14], [20].

The construction of the Decision Tree (DT) is pursued starting from the fuzzy rules of a Fuzzy Rule Base (FRB) derived from a clustering algorithm tailored to fault classification [25]. To do this, every Fuzzy Set (FS) representing a deviation of the monitored signals in the respective ranges of variability (Universes of Discourse, UODs, in Fuzzy Logic terminology) is associated to a symptom of a fault class and the FRB of the model is translated into a Symptom Table in which the relationships between fault classes and symptoms are explicitly laid out.

In practice, however, it is often difficult to attribute the detected symptoms to a given fault class, given that one fault may cause several symptoms and dually a symptom may describe more than one possible fault.

To solve this problem, the relationships between fault classes and symptoms contained in the Symptoms Table are systematically represented in a DT, which is then quantified by applying the rules of Fuzzy Logic.

The design of the DT entails the successive consideration of the symptoms. These can be considered in different orders, leading to different structures of the DT and thus different classification performances. Hence, a combinatorial optimisation problem arises with regards to the DT design.

In this paper, a single-objective genetic algorithm search is devised to find the sequence of symptoms leading to the optimal configuration of the DT, i.e. that which achieves the maximum classification performance.

The paper is organized as follows. Section 2 illustrates the procedure adopted for the construction of the DT and its fuzzy quantification. In Section 3, the results related to its application to an artificial case study regarding the classification of data randomly extracted from six different Gaussian distributions are reported. Section 4 presents the optimisation of the FDT by a single-objective genetic algorithm maximizing the percentage of correct classifications. A synthetic

discussion of the findings of the work is provided in the last Section.

2. From a Fuzzy Classification Model to a Fuzzy decision Tree

Let us consider an industrial system or plant whose "state of health" is monitored by a set of sensors, which collect the relevant parameters data at a given frequency. These data (also called "signals") provide a picture of the health state of the plant. A particular picture corresponds to the plant functioning in nominal conditions, with all the signals within their design envelope. Deviations from the nominal states are due to faults of different types (classes), which may occur to the components of the plant, leading to different "pictures" of the monitored signals.

When a generic fault of class Γ_j, $j=1,...,c$, occurs in the plant, corresponding representative symptoms are observed by the monitoring system, in terms of variations in the signal values. A symptom associated to the fault of class Γ_j is a deviation of a monitored signal from its reference value, outside of the allowed design envelope. The objective of fault identification is to build a system capable of recognizing the fault as of class Γ_j on the basis of the measured symptoms, i.e. the monitored signals.

In this work, we assume that the classification of the fault is performed by applying a previously built FRB (for example, a possible method for building an FRB from available pre-classified data is presented in [25]). Such FRB has one fuzzy rule for each fault class: the generic rule j associates the symptoms of the monitored signals (input data) to the fault class Γ_j. In the fuzzy rules, each one of the FSs of the antecedents describes a deviation of a monitored signal, i.e. a symptom, except those FSs representing the still nominal conditions of those monitored signals which are unaffected by the particular fault. Correspondingly, the generic FS X_{pj} associated to the p-th antecedent in rule j, $p=1,...,n$, $j=1,...,c$, represents a symptom for the class of faults Γ_j.

Notice that the relations between fault classes and symptoms (signals deviations) are not univocal: the faults of a given class may initiate several symptoms and in turn one symptom may be a legitimate representative of several possible fault classes.

On the other hand, an adequately designed monitoring system should be capable of associating to each fault class a unique set of symptoms (signal deviations). This leads to a Symptom Table such as the one reported in *Table 1*, where S_r, $r=1,...,s$, denotes the generic symptom.

The binary vector $\sigma_j=[I_{j1},I_{j2},...,I_{js}]$ represents the reference symptoms vector for fault class Γ_j, $j=1,...,c$. Each element I_{jr} is a binary value that corresponds to the presence or absence of symptom r when a fault of class Γ_j has occurred, $r=1,...,s$, $j=1,...,c$.

Table 1. Symptom Table: Reference relations between fault classes and symptoms [13]

Fault Class	Symptom Type					
	S_1	S_2	⋯	S_r	⋯	S_s
Γ_1	I_{11}	I_{12}	⋯	I_{1r}	⋯	I_{1s}
Γ_2	I_{21}	I_{22}	⋯	I_{2r}	⋯	I_{2s}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Γ_j	I_{j1}	I_{j2}	⋯	I_{jr}	⋯	I_{js}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Γ_c	I_{c1}	I_{c2}	⋯	I_{cr}	⋯	I_{cs}

During operation, the monitored signals could then be translated into an observation vector $\sigma'=(I_1',I_2',...,I_s')$, which carries the information on the presence or absence of the symptoms. As explained earlier, a symptom is present in the system if its representative measured signal has deviated from its nominal value beyond the design envelope. For example, a patient has the symptom "fever" if his or her monitored temperature rises to a "high" value, i.e. above 37°C.

However, in practice often the presence or absence of a symptom remains uncertain and ambiguous due to the complexity of the non-linear signal behaviours associated to the various faults, to the measurement errors of the monitoring sensors and to the imprecise and ambiguous definition of the signal deviation ranges and the associated linguistic labels [25]. To reflect this uncertainty, a fuzzy observation vector $\sigma_f'=(\mu_1',\mu_2',...,\mu_s')$ is associated to a pattern of

deviations of the monitored signals measured in correspondence of a given fault, where $\mu_r^{'}$, $r=1,...,s$, is the value of the membership of the FS corresponding to the symptom S_r and gives the degree with which it is present in the monitored situation being examined.
The fault identification problem is then to identify which fault class is occurring in the plant on the basis of the fuzzy observation vector $\sigma_f^{'}$. To tackle this problem a systematic procedure for constructing a DT is presented below [26].

2.1. Decision Tree

Based on the Symptom *Table 1*, a complete DT can be diagrammed by examining all s symptoms one by one [7]. Taking into consideration all possible combinations of symptoms, the DT will have 2^s branches given that each of the s symptoms can be either present or absent. On the other hand, only one combination of symptoms corresponds to a given fault: thus, only c of the 2^s tree branches correspond to a class while the remaining $2^s - c$ combinations of symptoms cannot be associated to a class.
For building a smaller, more transparent and easier to interpret DT, two main hypotheses are assumed [26], [13]:
- if a symptom is indicated as present in the measured observation vector $\sigma^{'}$, it is certainly present in the system;
- the presence of a single symptom characteristic of a fault suffices to conclude that the measured pattern of signals belongs to that fault class.
In this context, defining an "unwanted" symptom as a symptom that, although not present in the system, somehow is present by mistake in the observation vector and a "missing" symptom as a symptom that is not observed although it is present in the system [13], the first hypothesis can be called of "impossibility of unwanted symptoms" and the second of "possibility of missing symptoms".
The procedure for building the DT proceeds as follows:
1. A root node is placed at the top of the tree. This node refers to all possible fault classes identified for the system under analysis.
2. A symptom from the Symptom Table is associated to this node.
3. The root node is split into two branches: the left corresponding to the presence of the symptom, the right to the absence of the symptom.
4. The fault classes for which the symptom is present are associated to a node under the left branch. If only one fault class is found to contain the symptom, then the associated node is a terminal leaf of the branch and its identification is guaranteed by the fact that it has been assumed that a symptom that is absent in the system cannot be indicated as present (impossibility of unwanted symptoms hypothesis). The fault class associated to the identified leaf may be also associated to other leaves, at the end of other branches in the tree. This accounts for the possibility that a symptom is not indicated as present by the monitoring system although it actually is (possibility of missing symptoms hypothesis). If more than one fault class are associated to the node characterized by the identified symptom, a new symptom is searched in the Symptom Table and associated to the node in order to differentiate between the identified fault classes. To select the new differentiating symptom, the previous procedure is applied, starting from step 2.
5. The right branch from the root node is further developed by first adding a node associated to all possible fault classes. This node is then treated as a local root node to which the branching procedure is applied starting from step 2.
6. The tree development terminates when all symptoms have been considered and their associated branches developed down to the distinguishing leaves of the individual fault classes.
A path through the branches of the tree, from the root node to a leaf, identifies a crisp observation vector $\sigma^{'}$ of symptoms representative of the fault class associated to the corresponding leaf. As pointed out above, different paths may lead to different leaves associated to the same fault class, due to the possibility of missing symptoms.
In operation, the DT gives the correct diagnosis when the measured symptom vector matches completely with the reference symptom vector of a fault class; on the contrary, the diagnosis is conservative in case of a missing symptom, i.e. it is not necessary to have all the symptoms to diagnose the fault.
Finally, in case of unwanted symptoms, the classification is driven by the structure of the tree and the classification will be wrong if the first symptom considered is an unwanted symptom.
From the above it appears that an issue of adequate DT design arises with respect to the order with which the successive symptoms are considered for optimal classification performance.

2.2. Classification by the FDT

In the realistic case of ambiguity in the actual presence or absence of a symptom, in correspondence of a given pattern of signal deviations the degree of activation of

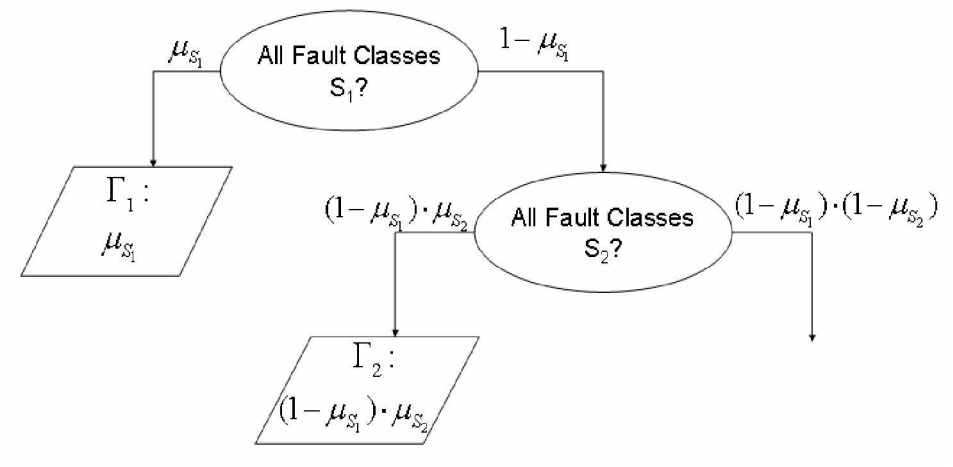

Figure 1. Propagation of fuzzy information through the DT

each symptom S_r, $r=1,...,s$, is computed from the MF of the corresponding FS. The DT then becomes a FDT and the possibility classification of a given pattern of measured signal deviations is performed by proceeding through all the branches of the tree and computing the MFs to each fault class, at the tree leaves.

The symptoms degrees of activation are then propagated through the DT according to the rules of FS theory. In particular, the logic operator of negation of a symptom S_r is implemented by $\left(1-\mu_{S_r}\right)$ in the right branch corresponding to the absence of the symptom whereas its complement μ_{S_r} is propagated along the left branch associated to its presence (*Figure 1*). The connection between two nodes of the tree represent a logic operator of intersection (*and*), implemented as the algebraic product of the membership values in this work.

Finally, since more than one terminal leaf can indicate the same class, the final membership to a given class is computed through the logic operation of union (*or*) of all the leaves associated to that class. The logic operator *or* is here implemented as the MFs sum limited to 1, accordingly to the rules of FS arithmetic.

As mentioned at the end of Section 2.1, different sequences of symptoms lead to different DTs and, implicitly, to different classification performances. For realistic problems, the number of possible sequences of symptoms for building the DT is combinatorial, so that a trial and error process for finding the optimal structure of the tree, i.e. that which allows obtaining the maximum classification performance, would not be practical. For example, the number of possible sequences of a group of 15 symptoms, would be approximately 10^{11} and to each sequence corresponds a different DT whose classification performance must be evaluated.

3. Application of the FDT to an artificial case study

The classification approach has been applied to the artificial four-dimensional, six-classes data set of *Figure 2*. The data have been obtained by random sampling from 6 different Gaussian distributions and can be assumed to represent the system response signals resulting from 6 different types of system faults.

The previously illustrated procedure for classifying the data into the six classes (Section 2) consists of two main steps: the first one is the building of the FRB, i.e. a set of transparent and accurate fuzzy rules and the second one is the construction and quantification of the corresponding FDT.

A fuzzy clustering – based method has been used for obtaining the FRB from available pre-classified data [25]. The resulting FRB is composed of $c=6$ rules, one for each class *(Table 2*).

To build the associated DT, first each antecedent of the rules in the FRB is associated to a symptom, resulting in 15 possible symptoms, indicated as S_i, $i=1,2,...,15$, in *Table 2*. This allows the translation of the FRB in the Symptom *Table 3*.

Then, by applying the steps 1.– 6. of the procedure for building the DT (Section 2.1) on the sequence of symptoms $\Sigma_0=[S_1;S_2;\ ...\ ;S_{15}]$, one obtains the DT reported in *Figure 3*.

The possibility quantification of the degree of membership to different classes can be performed as described in Section 2.2, i.e. propagating through the branches of the tree the degree of activation of each symptom according to the rules of Fuzzy Logic. The final assignment of an incoming pattern of signals to a class is conservatively realized as follows:

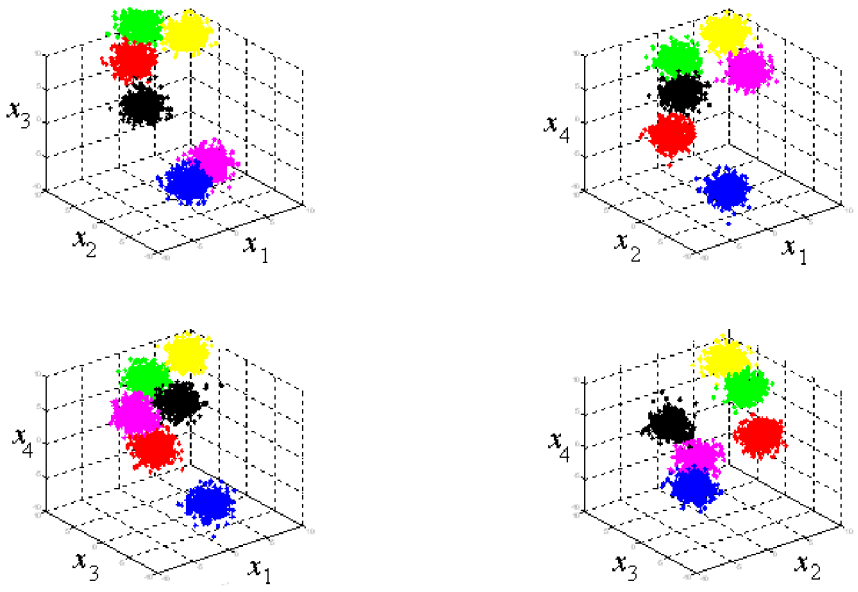

Figure 2. Four-dimensional data set comprised of six classes

Table 2. The Table of rules of the FRB

Rule		x_1	x_2	x_3	x_4		Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
1		Low S_1	Low S_4	Low S_9	Medium S_{12}		Yes	No	No	No	No	No
2		High S_2	Medium S_5	Medium S_{10}	High S_{13}		No	Yes	No	No	No	No
3	IF	High S_2	High S_6	Medium S_{10}	High S_{13}	THEN	No	No	Yes	No	No	No
4		High S_2	Low S_4	Medium S_{10}	Low S_{14}		No	No	No	Yes	No	No
5		High S_2	Higher S_7	Medium S_{10}	Medium S_{12}		No	No	No	No	Yes	No
6		Higher S_3	Highest S_8	High S_{11}	Higher S_{15}		No	No	No	No	No	Yes

Table 3. Symptom Table

	x_1 low	x_1 high	x_1 higher	x_2 low	x_2 medium	x_2 high	x_2 higher	x_2 highest	x_3 low	x_3 medium	x_3 high	x_4 medium	x_4 high	x_4 low	x_4 higher
	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
Γ_1	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0
Γ_2	0	1	0	0	1	0	0	0	0	1	0	0	1	0	0
Γ_3	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0
Γ_4	0	1	0	1	0	0	0	0	0	1	0	0	0	1	0
Γ_5	0	1	0	0	0	0	1	0	0	1	0	1	0	0	0
Γ_6	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1

- the pattern is declared assigned to a class (in possibility terms), if the membership grade to the respective class is larger than a confidence threshold γ (here chosen equal to 0.6);
- the pattern is declared 'atypical', if none of the membership grades is larger than γ ;
- the pattern is declared 'ambiguous', if more than one membership grade is larger than γ .

A test on a set of 600 data has resulted in only 40.67% correct classifications to the six fault classes, while 10.5% of the data are considered as atypical, 2.33% as ambiguous and 46.5% are assigned to the wrong class. The obtained performance is obviously unacceptable and motivates the search for an optimal or near-optimal sequence of symptoms upon which to build the DT. The objective of the optimisation algorithm is to find the sequence of symptoms that leads to the DT with the best classification performance in terms of percentage of correct classifications. The number of possible sequences of symptoms is 15! (¼10^{11}), in Section 4, this combinatorial optimisation problem is tackled by a single-objective genetic algorithm.

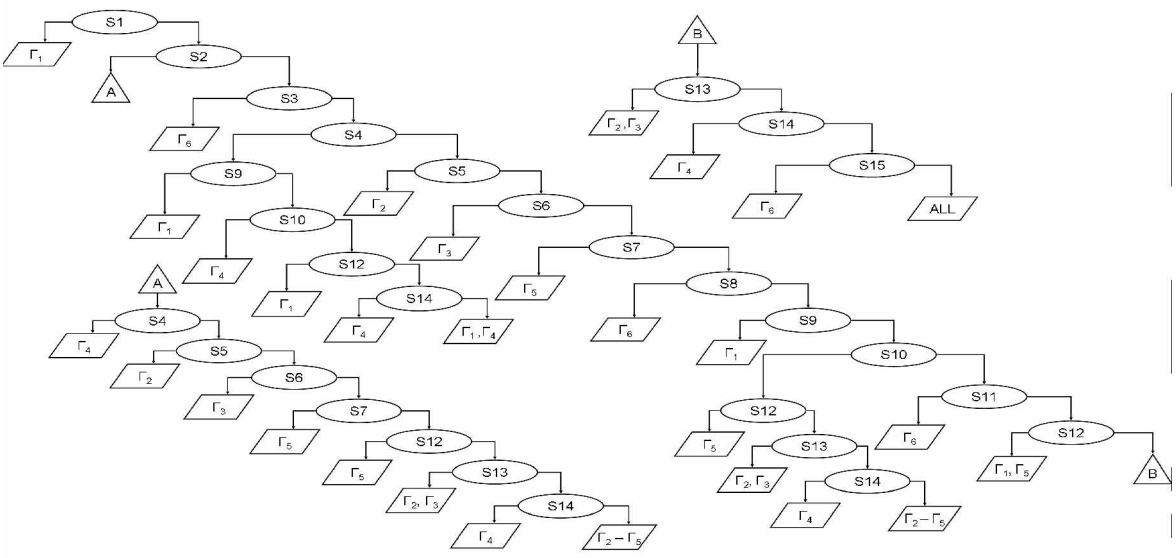

Figure 3. DT for classification, built with the ordered sequence of symptoms Σ_0

4. Genetic algorithm optimisation of the decision tree design

, a procedure based on a single-objective genetic In this Section algorithm (Appendix A) is carried out for determining the sequence of symptoms to which corresponds the FDT with the maximum classification performance. The genetic algorithm can be seen as performing a wrapper search [12] around the classification algorithm (*Figure 4*): the symptoms sequence selected during the search is evaluated using as criterion (fitness) the percentage of correct classified data achieved by the FDT itself.

The data and rules of the genetic algorithm search are given in *Table 4*. These parameters have been established through a systematic procedure of experimentation. The objective (fitness) function to be maximized is the percentage of correct data classifications; the decision variable is the symptoms sequence.

Table 4. GA run parameters

Number of chromosomes in the population	100
Number of generations (termination criterion)	50
Selection	Standard Roulette
Replacement	Children - Parents
Mutation probability	0.01
Crossover probability (one-site)	1

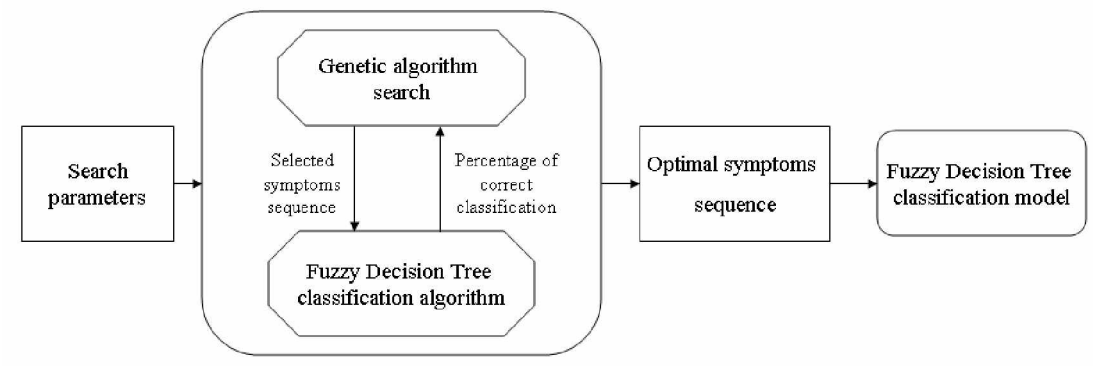

Figure 4. Single-objective genetic algorithm "wrapper" search

Each chromosome is made up by 15 genes, one gene for each symptom. The single gene can assume any integer value in [0,15] that encodes the "swap" position of the symptom along the sequence. An

example of a chromosome coding a particular sequence is given in *Figure 5*. To decode the chromosome in its corresponding symptom sequence, a 15 – steps procedure is performed, one for each gene. At the generic step $i = 1, 2, ..., 15$, the ordered sequence Σ_{i-1} and the value k contained in the i-th gene are considered: the symptom in the i-th position of Σ_{i-1} is swapped with the symptom in the k-th position of the sequence. For example in the first step of Figure 5, the value 7 in gene 1 means that the symptom S_1 is placed in position 7 of the sequence and simultaneously the symptom that occupied position 7 is swapped to position 1. This operation is carried out until the 15th gene of the chromosome is worked out, leading to the final sequence:

$$\Sigma_{15} = [S_3; S_{11}; S_5; S_{12}; S_6; S_8; S_7; S_2; S_1; S_{10}; S_{13}; S_9; S_{14}; S_4; S_{15}]$$

Note that this original chromosome random design leads to a coherent symptom sequence, i.e. without repetition of symptoms, thus avoiding computationally burdensome chromosome coherence checking a posterior.

The optimal sequence found at convergence of the genetic algorithm is:

$$\Sigma_1 = [S_4; S_6; S_7; S_3; S_{12}; S_{10}; S_{13}; S_{15}; S_5; S_1; S_{14}; S_{11}; S_8; S_9; S_2]$$

The FDT built following this sequence ends into 46 leaves and achieves a classification performance of 91.34%, while 5.33% of the data are considered as atypical, 0.33% as ambiguous and only 3% are assigned to the wrong class.

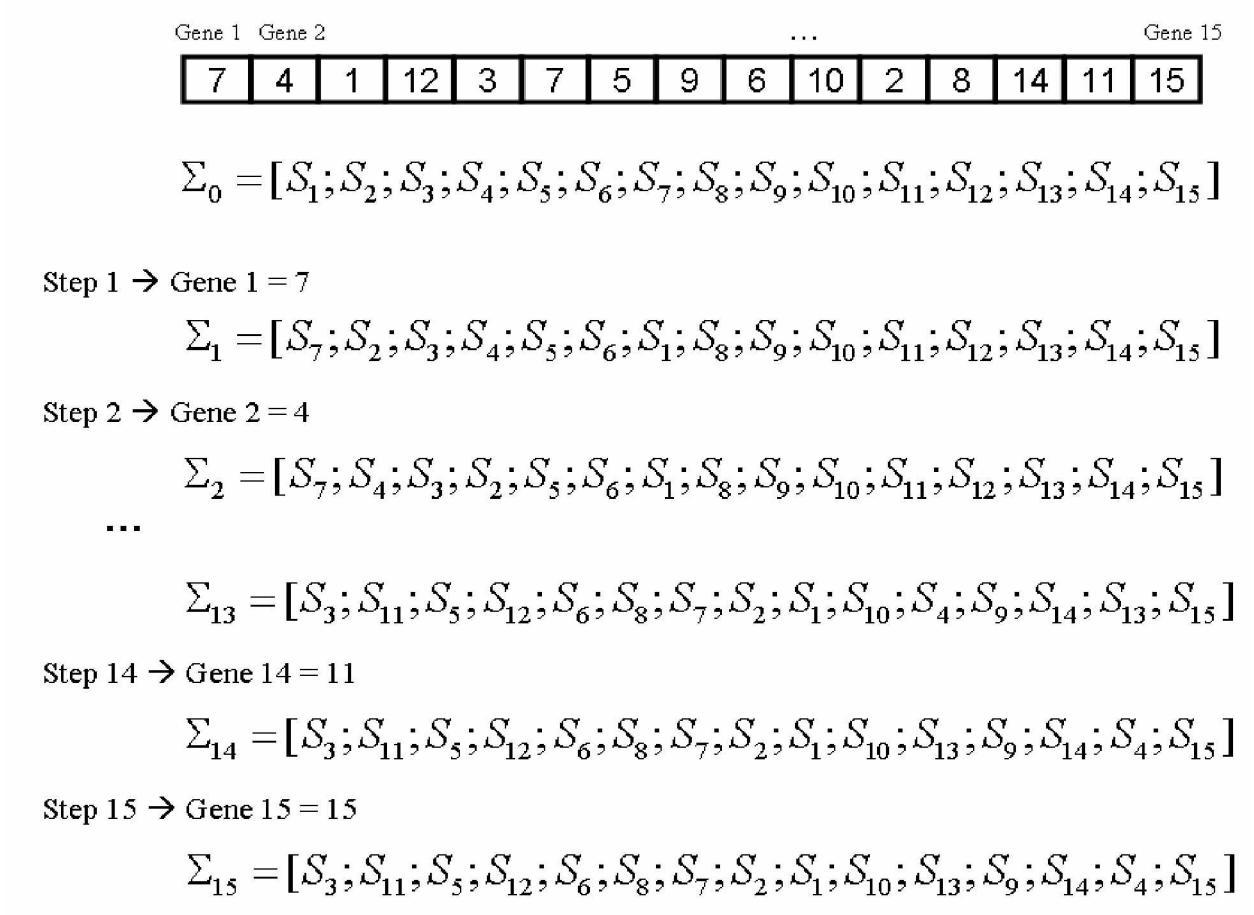

Figure 5. Example of a chromosome and the corresponding sequence

5. Conclusion

In realistic applications, fault classification is usually based on ambiguous information, which can be effectively handled within a fuzzy logic framework. A Fuzzy Decision Tree can then be built to logically structure the uncertain information available. To each fault class corresponds a classification rule, with mono-dimensional Fuzzy Sets representing the characteristic symptoms for the corresponding fault class.

The classification performance by the resulting FDT is dependent on the order in which the symptoms are considered in the building procedure of the DT. This leads to an optimisation problem with respect to the construction of the tree. In this work, this problem has been tackled by means of a single-objective genetic algorithm in which the different sequences of symptoms are coded into the chromosomes of the genetic population by an original procedure, which guarantees coherence, i.e. no repetition of symptoms in the sequence.

The genetic algorithm-based optimisation procedure developed has been successfully applied to a test case regarding the development of Fuzzy Decision Trees for the classification of artificial data. Undergoing research concerns the application of the developed classification procedure to a real diagnostic problem.

Appendix A: A brief recall of Genetic Algorithms

In the following, only a concise snapshot is provided on the basics of genetic algorithms optimisation. For an extensive and detailed presentation of this computational paradigm, the interested reader is invited to consult the available copious literature [18], [17], [3], [6], [9], [4], [2].

Genetic Algorithms (GAs) are optimisation methods aiming at finding the global optimum of a set of real objective functions, $F \equiv \{f(\cdot)\}$, of one or more decision variables, $U \equiv \{u\}$, possibly subject to various linear or non linear constraints. Their main properties are that the search is conducted i) using a population of multiple solution points or candidates, ii) using operations inspired by the evolution of species, such as breeding and genetic mutation, iii) using probabilistic operations, iv) using only information on the objective or search function and not on its derivatives [16].

GAs owe their name to their operational similarities with the biological and behavioural phenomena of living beings. After the pioneering theoretical work by John Holland [10], in the last decade a flourishing literature has been devoted to their application to real problems. The basics of the method may be found in Goldberg [8]; some applications in various contexts are included in Chambers [1].

The terminology adopted in GAs contains many terms borrowed from biology, suitably redefined to fit the algorithmic context. Thus, GAs operate on a set of (artificial) chromosomes, which are strings of numbers, generally sequences of binary digits 0 and 1. If the objective function of the optimisation has many arguments (typically called control factors or decision variables), each string is partitioned in as many substrings of assigned lengths, one for each argument and, correspondingly, we say that each chromosome is partitioned in (artificial) genes. The genes constitute the so-called genotype of the chromosome and the substrings, when decoded in real numbers, constitute its phenotype. When the objective functions are evaluated in correspondence of a set of values of the control factors of a chromosome, its values are called the fitness of that chromosome. Thus, each chromosome gives rise to a trial solution to the problem at hand in terms of a set of values of its control factors.

The GA search is performed by constructing a sequence of populations of chromosomes, the individuals of each population being the children of those of the previous population and the parents of those of the successive population. The initial population is generated by randomly sampling the bits of all the strings. At each step, the new population is then obtained by manipulating the strings of the old population in order to arrive at a new population hopefully characterized by increased mean fitness. This sequence continues until a termination criterion is reached. As for the natural selection, the string manipulation consists in selecting and mating pairs of chromosomes in order to groom chromosomes of the next population. This is done by repeatedly performing on the strings the four fundamental operations of reproduction, crossover, replacement and mutation, all based on random sampling: the parents' selection step determines the individuals which participate in the reproduction phase; reproduction itself allows the exchange of already existing genes whereas mutation introduces new genetic material; the substitution defines the individuals for the next population. This way of proceeding enables to efficiently arrive at optimal or near-optimal solutions.

With regards to their performance, it is acknowledged that GAs takes a more global view of the search space than many other optimisation methods. The main advantages are i) fast convergence to near global optimum, ii) superior global searching capability in complicated search spaces, iii) applicability even when gradient information is not readily achievable.

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ISSN 1932-2321

http://www.gnedenko-forum.org/Journal/index.htm