AST ALGORITHMS OF ASYMPTOTIC ANALYSIS OF NETWORKS WITH UNRELIABE EDGES

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A problem of a reliability in networks with unreliable elements naturally origin in technical applications [1]. But a direct calculation of the reliability demands a number of operations which increases geometrically dependently on a number of edges. So it is necessary to use approximate methods and particularly asymptotic one. In [2] a reliability asymptotic is calculated in analogous asymptotic suggestions on the network edges. Main parameters in these asymptotic are a shortest way length and a maximal flow in a network. In this paper different partial classes of networks are considered and effective algorithms of their parameters calculations are suggested. These networks are networks originated by dynamic systems, networks with integer-valued lengths of edges, superposition of networks and bridge schemes.

1. Preliminaries

Define the graph Γ with the finite nodes set U and the set W of edges $w=(u,v)$. The graph Γ may contain cycles or not, its edges may be oriented or not. Denote by $\mathcal{R}(u)$ the set of all ways R of the graph Γ, which connect the nodes u_0, u, and assume that $\mathcal{R}(u)\neq\varnothing$, $u\in U$. Suppose that $\Gamma(u)$ is the sub-graph of the graph Γ, which consists of the ways $R\in\mathcal{R}(u)$. Consider the sets

$$\mathcal{A}(u)=\left\{A\subset U : u_0\in A, u\notin A\right\},\ L=L(A)=\left\{(u,u') : u\in A, u'\notin A\right\}$$

And the set $\mathcal{L}(u)=\left\{L(A), A\in\mathcal{A}(u)\right\}$ of all sections of the sub-graph $\Gamma(u)$.

Characterize each edge $w\in W$ of the graph Γ by the logic number $\alpha(w)=I$ (the edge w works), where $I(B)$ is the indicator function of the event B. Denote

$$\beta(u)=\bigvee_{R\in\mathcal{R}(u)}\ \bigwedge_{w\in R}\alpha(w)$$

the characteristic of the nodes u_0, u connectivity in the graph Γ. Suppose that $\alpha(w)$, $w\in W$, are independent random variables, $P(\alpha(w)=1)=p_w(h)$, $q_w(h)=1-p_w(h)$,
where h is small parameter: $h\to 0$. In [2] the following statements are proved.

Theorem 1. *Suppose that* $p_w(h)\sim\exp\left(-h^{-c(w)}\right)$, $h\to 0$, *where* $c(w)>0$, $w\in W$. *Then*

$$-\ln P\big(\beta(u)=1\big)\sim h^{-D(u)},\quad D(u)=\min_{R\in\mathcal{R}(u)}\max_{w\in R}c(w)\ . \tag{1}$$

Theorem 2. *Suppose that* $q_w(h)\sim\exp\left(-h^{-c_1(w)}\right)$, $h\to 0$, *where* $c_1(w)>0, w\in W$. *Then*

$$-\ln P\big(\beta(u)=0\big)\sim h^{-D_1(u)},\quad D_1(u)=\max_{R\in\mathcal{R}(u)}\min_{w\in R}c_1(w). \tag{2}$$

Theorem 3. *Suppose that* $p_w(h)\sim h^{g(w)}$, $h\to 0$, *where* $g(w)>0, w\in W$. *Then*

$$-\ln P\big(\beta(u)=1\big)\sim T(u)\ln h,\quad T(u)=\min_{R\in\mathcal{R}(u)}\sum_{w\in R}g(w). \tag{3}$$

Theorem 4. *Suppose that* $q_w(h)\sim h^{g(w)}$, $h\to 0$, *where* $g(w)>0, w\in W$. *Then*

$$-\ln P\big(\beta(u)=0\big)\sim T_1(u)\ln h,\quad T_1(u)=\min_{L\in\mathcal{L}(u)}\sum_{w\in L}g(w). \tag{4}$$

Statement 1. *Suppose that all* $c(w)$ *(all* $c_1(w)$*),* $w\in W$*, are different. Then there is the single edge* $w(u)$ *(there is the single edge* $w_1(u)$*), so that* $c(w(u))=D(u)$ *(*$c_1(w_1(u))=D_1(u)$*). It is called the critical edge.*

2. Graphs generated by dynamic systems

Suppose that the set U consists of non-intersected subsets $U_0,U_1,...,U_m$, and the set U_0 contains the single vertex u_0, which is called initial. All edges of the oriented graph Γ are represented as (u_i,u_j), $1\le i<j\le m$, $u_i\in U_i$, $u_j\in U_j$, and each vertexis accessible from the initial vertex u_0. Described graphs are generated by dynamic systems with a delay. In this section we calculate $D(u),D_1(u),T(u)$ and find critical edges $w(u),w_1(u)$ for a fixed u_0.

A main idea of this section is an application of the Floyd algorithm [3], when a solution is calculated for all $u\in U$. To construct fast algorithms it is natural to constrict a class of considered graphs. An idea of such a constriction is illustrated in [4] but for a fixed u.

Suppose that $D(u_0)=D_1(u_0)=T(u_0)=0$, for all $u\in U_1$ put

$$D(u)=D_1(u)=T(u)=c(u),\ w(u)=w_1(u)=(u_0,u).$$

For $u\in U$ define $S(u)=\{v:(v,u)\in W\}$, $|S(u)|$ a number of elements in the finite set $S(u)$. Assume that for all $u\in U_1,...,U_k$ the meanings $D(u)$, $D_1(u)$, $T(u),w(u)$, $w_1(u)$ are defined. Take $u\in U_{k+1}$ and in an accordance with the formulas (1), (2) put

$$D(u)=\min_{v\in S(u)}\max\big(c(v,u),D(v)\big),\ D_1(u)=\max_{v\in S(u)}\min\big(c(v,u),D_1(v)\big), \tag{5}$$

$$T(u) = \min_{v \in S(u)} \left(c(v,u) + T(v) \right), \ k \geq 1. \tag{6}$$

To calculate each element from the set $D(u),\ D_1(u),\ T(u),\ u \in U$ it is necessary $2|S(u)|-1$ arithmetic operations and this number can not be decreased. So the algorithm (5), (6) is optimal. And if for fixed $u \in U$ $D(u), D_1(u), T(u)$ are calculated by the algorithm (5), then we find $D(v), D_1(v), T(v)$ for all nodes v from which the vertex u is accessible.

To define critical edges it is necessary to complement the formulas (5) by

$$w(u) = w_1(u) = (u_0, u), \text{ if } u_0 \in S(u),$$

$$w(u) = \begin{cases} w(v), \ \text{if} \ D(u) = \max\left(D(v), c(v,u)\right) > c(v,u), \\ (v,u), \ \text{if} \ D(u) = \max\left(D(v), c(v,u)\right) > D(v), \end{cases} \tag{7}$$

$$w_1(u) = \begin{cases} w_1(v), \ \text{if} \ D_1(u) = \max\left(D_1(v), c(v,u)\right) < c(v,u), \\ (v,u), \ \text{if} \ D_1(u) = \max\left(D_1(v), c(v,u)\right) < D(v). \end{cases} \tag{8}$$

3. Graphs with integer-valued lengths of edges

In this section we consider a calculation of $T(u)$ for all $u \in U$ in graphs with integer-valued lengths of edges. Suppose that $g(w)$, $w \in W$, are natural numbers, $g(w) \leq \overline{g} < \infty$ and define

$$G_\Gamma = \sum_{w \in W} g(w). \tag{9}$$

Divide each edge of the graph Γ into edges with unit lengths by an introduction of intermediary nodes. As a result obtain the graph Γ^1 with the nodes set U^1, $U \subseteq U^1$ and with the edges set W^1. Denote $N(u^1)$ the minimal number of the graph edges in ways, which connect the nodes u_0, u^1. It is easy to obtain that

$$N(u) = G(u), \ u \in U. \tag{10}$$

Consider now an algorithm of $N(u^1),\ u^1 \in U^1$ calculation.

Suppose that all nodes of the graph. Γ^1 are not marked. Mark the vertex u_0, and put $U_0^1 = \{u_0\}$. Then construct a recurrent procedure of non-intersected sets $U_k^1,\ k \geq 0,$ definition. Suppose that the sets U_k^1, $V_k^1 = \bigcup_{0 \leq i \leq k} U_i^1$ are known and all nodes of the set V_k^1 are marked and all other nodes are not marked. Define the set. U_{k+1}^1 as a set of all unmarked nodes from U^1, which are connected directly with some vertexfrom the set U_k^1. By a definition the set U_{k+1}^1 satisfies the formula

$$U_{k+1}^1 = \left\{u^1 : N\left(u^1\right) = k+1\right\}.$$

Mark all nodes of the set U_{k+1}^1 and define the set $V_{k+1}^1 = V_k^1 \bigcup U_{k+1}^1$.

Estimate a number of operations which are necessary to calculate U_{k+1}^1 if each vertexof the graph Γ is connected directly with no more l nodes. Then a number of operations to define U_{k+1}^1 does not exceed $l\left|U_k^1\right|$. Define M by the formula

$$V_0^1 \subset V_1^1 \subset \ldots \subset V_M^1 = V_{M+1}^1 = \ldots,$$

then to construct the sequence $U_1^1,\ldots,U_M^1$ it is necessary no more lG_Γ operations where $lG_\Gamma \le l^2 \overline{g}\left|U\right|$. Compare these results with the results of Deikstra [4], in a case when $c(w)$ is not integer-valued. To calculate $D(u)$, $u \in U$ in a general case it is necessary no more $K_1\left|U\right|^2$ operations and for a dendriform graph - no more $K_2\left|U\right|\ln\left|U\right|$ operations, where $K_1, K_2 < \infty$.

4. Superposition of graphs

Fix in the graph Γ some vertex v_0. Assume that Γ' is non-oriented graph with the nodes set $U' = \{1',\ldots,m'\}$, $U \bigcap U' = \varnothing$ and with the edges set W' $(i',j'),(i',i) \notin W'$. Distinguish in the graph Γ' initial and final nodes u_0', v_0' and in the set U - two nodes $\overline{u},\overline{v}$ so that $\overline{w} = \left(\overline{u},\overline{v}\right) \in W$. Denote by $\mathcal{R}'$ the set of all ways R' of the graph Γ' from u_0' to v_0'.

Define the superposition $\overline{\Gamma} = \Gamma \overset{\overline{w}}{\otimes} \Gamma'$ of the graphs Γ,Γ' with a replacement of the edge $\left(\overline{u},\overline{v}\right)$ from the graph Γ by the graph Γ' and with an aliasing of the nodes $\overline{u}$ with u_0' and of the nodes $\overline{v}$ with v_0' correspondingly. Denote by $\overline{U}$ the nodes set, by $\overline{W}$ - the edges set and by $\overline{\mathcal{R}}$ - the set of ways from the vertex u_0 to the vertex v_0 in the graph $\overline{\Gamma}$. Put $\mathcal{R}$ the set of ways from u_0 to v_0 in the graph Γ, $\mathcal{R}'$ - the set of ways from u_0' to v_0' in the graph Γ'. Analogously define $\overline{\mathcal{L}},\mathcal{L},\mathcal{L}'$ the sets of sections in the graphs $\overline{\Gamma},\Gamma,\Gamma'$ with pairs of initial and final nodes $\left(\overline{u_0},\overline{v_0}\right)$, $\left(u_0,v_0\right)$, $\left(u_0',v_0'\right)$ correspondingly. Define

$$\beta = \underset{R\in\mathcal{R}}{\vee}\ \underset{w\in R}{\wedge}\alpha(w),\ \overline{\beta} = \underset{\overline{R}\in\overline{\mathcal{R}}}{\vee}\ \underset{w\in\overline{R}}{\wedge}\alpha(w)$$

characteristics of a connectivity between the nodes u_0, v_0 in the graphs Γ, $\overline{\Gamma}$ correspondingly.

Then from the theorems 1-4 it is possible to obtain asymptotic formulas for the superposition $\overline{\Gamma}$.

Theorem 5. *Suppose that* $p_w(h) \sim \exp\left(-h^{-c(w)}\right)$, $h \to 0$, *where* $c(w) > 0, w \in \overline{W}$. *Then*

$$-\ln P\left(\overline{\beta} = 1\right) \sim h^{-\overline{D}},\ \ \overline{D} = \min_{R\in\mathcal{R}}\max_{w\in R}\overline{c}(w),$$

$$\overline{c}(w) = c(w),\ w \neq \overline{w},\ \overline{c}(\overline{w}) = \min_{R' \in \mathcal{R}'} \max_{w \in R'} c(w).$$

Theorem 6. *Suppose that* $q_w(h) \sim \exp\left(-h^{-c_1(w)}\right),\ h \to 0$*, where* $c_1(w) > 0,\ w \in \overline{W}$*. Then*

$$-\ln P\left(\overline{\beta} = 0\right) \sim h^{-\overline{D}_1},\ \overline{D}_1 = \min_{L \in \mathcal{L}} \max_{w \in R} \overline{c}_1(w),$$

$$\overline{c}_1(w) = c_1(w),\ w \neq \overline{w},\ \overline{c}_1(\overline{w}) = \min_{L' \in \mathcal{L}'} \max_{w \in L'} c_1(w).$$

Theorem 7. *Suppose that* $p_w(h) \sim h^{g(w)},\ h \to 0$*, where* $g(w) > 0,\ w \in \overline{W}$*. Then*

$$\ln P\left(\overline{\beta} = 1\right) \sim \overline{T} \ln h,\ \overline{T} = \min_{R \in \mathcal{R}} \sum_{w \in R} \overline{g}(w),$$

$$\overline{g}(w) = g(w),\ w \neq \overline{w},\ \overline{g}(\overline{w}) = \min_{R' \in \mathcal{R}'} \sum_{w \in R'} g(w).$$

Theorem 8. *Suppose that* $q_w(h) \sim h^{g(w)},\ h \to 0$*, where* $g(w) > 0,\ w \in \overline{W}$*. Then*

$$\ln P\left(\overline{\beta} = 0\right) \sim \overline{T}_1 \ln h,\ \overline{T}_1 = \min_{L \in \mathcal{L}} \sum_{w \in L} \overline{g}_1(w),$$

$$\overline{g}_1(w) = g_1(w),\ w \neq \overline{w},\ \overline{g}_1(\overline{w}) = \min_{L' \in \mathcal{L}'} \sum_{w \in L'} g_1(w).$$

It is obvious that the formulas from these theorems allow calculating asymptotic of a reliability for superposition of networks with unreliable elements rationally. These formulas may be used to calculate a reliability in recursively defined networks which are widely used in the solid state physics and in the nanotechnology.

5. Asymptotic analysis of bridge scheme

The simplest superposition of graphs is parallel-sequential graphs. But there are graphs widely used in the reliability theory, which are not parallel - sequential. One of them is a bridge scheme.

Consider the non-oriented graph Γ with the nodes set $U = \{u_i, i = 0,...,3\}$ and with the edges set $W = \{w_j, j = 1,..,5\}$, where

$$w_1 = (u_0, u_1),\ w_2 = (u_0, u_2),\ w_3 = (u_1, u_3),\ w_4 = (u_2, u_3),\ w_5 = (u_1, u_2).$$

The vertex u_0 is initial and the vertex u_3 is final. The edge w_5 is a bridge element in the graph Γ. The graph Γ is called the bridge scheme in the reliability theory. Define the Γ_1 by a deleting of the edge w_5 from the graph Γ. Introduce the graph Γ_2 by an aliasing of the nodes u_1, u_2 in the graph Γ_1.

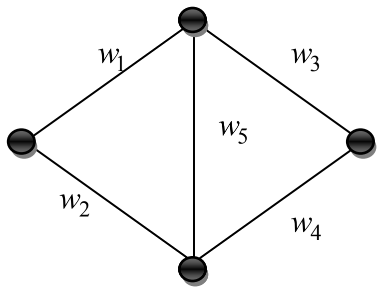

Fig. 1. Graph Γ.

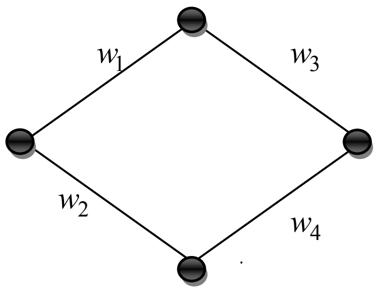

Fig. 2. Graph Γ_1.

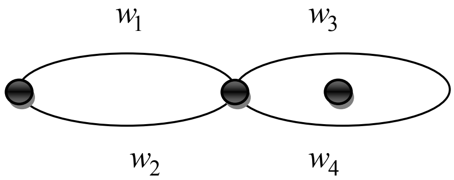

Fig. 3. Graph Γ_2.

Suppose that the edges $w_1,...,w_5$ work independently and define positive numbers $c\left(w_i\right)=c_i, 1\le i\le 5$,

$$C_1=\min\Big(\max\left(c_1,c_3\right),\max\left(c_2,c_4\right)\Big),C_2=\max\Big(\min\left(c_1,c_2\right),\min\left(c_3,c_4\right)\Big),C_2\le C_1.$$

If random logical variables β,β_1,β_2 characterize the nodes u_0,u_3 connectivity in the graphs Γ,Γ_1,Γ_2, correspondingly, then from the complete probability formula we have:

$$P\left(\beta=1\right)=p_{w_5}\left(h\right)P\left(\beta_2=1\right)+\Big(1-p_{w_5}\left(h\right)\Big)P\left(\beta_1=1\right),\ P\left(\beta_1=1\right)\le P\left(\beta_2=1\right). \tag{11}$$

From the theorem 1 and the equalities (11) obtain) the statement which characterizes a role of the bridge element.

Theorem 9. *If* $p_w\left(h\right)\sim\exp\Big(-h^{-c(w)}\Big)$, $h\to 0$*, where* $c\left(w\right)>0, w\in W$*, then*

$$-\ln P\left(\beta=1\right)\sim h^{-D},\ \ D=\min\Big(C_1,\max\left(C_2,c_5\right)\Big). \tag{12}$$

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