BOTTLENECKS IN GENERAL TYPE LOGICAL SISTEMS WITH UNRELIABLE ELEMENTS

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In this paper a model of general type logical system with unreliable elements [1], [2] is considered. An asymptotic analysis of its work (failure) probability is made in appropriate conditions on work (failure) probabilities of the system elements. A concept of bottlenecks of this system is constructed on a suggestion that an increase (a decrease) of elements reliabilities lead to an increase (a decrease) of the system reliability.

A construction of general type logical system is founded on concepts of disjunctive and conjunctive normal forms (DNF and CNF) of a logical function. This approach allows obtaining main results in maximal general and convenient for engineering calculations form comparatively with recursive definitions of logical functions used in [3].

Denote Z the set which consists of $|Z|$ independent random logical variables z, $I \subseteq \left\{1, 2, ...2^{|Z|}\right\}$. Consider the logical function A represented in DNF

$$A = \bigvee_{i \in I} \left[\left(\bigwedge_{z \in Z_i} z \right) \bigwedge \left(\bigwedge_{z \in \overline{Z}_i} \overline{z} \right) \right]. \tag{1}$$

Here the family $\left\{ \left(Z_i, \overline{Z}_i \right), i \in I \right\}$ consists of the sets pairs $Z_i, \overline{Z_i} \subseteq Z$, $Z_i \bigcap \overline{Z_i} =$ Ø, and for $i \neq j$ $\left(Z_i, \overline{Z_i} \right) \neq \left(Z_j, \overline{Z_j} \right)$. Suppose that $p_z = P(z=1)$, $q_z = P(z=0)$, $p_z + q_z = 1$, and random variables $z \in Z$ are independent. The logical function A with random arguments $z \in Z$ is denoted by $\mathbf{A}$ and called the logical system.

Low reliable elements

Suppose that for $\forall z \in Z$

$$\exists \ c(z),\ c(z) > 0 : p_z = p_z(h) \sim \exp\left(-h^{-c(z)}\right),\ h \to 0. \tag{2}$$

Denote $C = \min_{i \in I} \max_{z \in Z_i} c(z)$,

$I' = \left\{ i \in I : \max_{z \in Z_i} c(z) = C \right\}$, $S_i = \left\{ z \in Z_i : c(z) = C \right\}$, $i \in I'$, $\mathcal{S} = \left\{ S_i,\ \ i \in I' \right\}$,

$N(\mathcal{S}) = \min\left(|S_i| : S_i \in \mathcal{S} \right)$, $\mathcal{T} = \left\{ \left\{ z_i \in S_i,\ \ i \in I' \right\} \right\}$, $N(\mathcal{T}) = \min\left(|T| : T \in \mathcal{T} \right)$

and let $\mathcal{S}', \mathcal{T}'$ are families of minimal by an inclusion sets from the families $\mathcal{S}$, $\mathcal{T}$,

$$\mathcal{S}''=\left\{S_i\in\mathcal{S}' :\ |S_i|=N(\mathcal{S})\right\},\ \mathcal{T}''=\left\{T\in\mathcal{T}' :\ |T|=N(\mathcal{T})\right\}.$$

Theorem 1. *If the formulas (1), (2) are true then*

$$-\ln P(\mathbf{A}=1)\sim N(\mathcal{S})h^{-C},\ h\to 0. \tag{3}$$

Proof. Rewrite the logical function A as follows

$$A=\bigvee_{i\in I}\left[\left(\bigwedge_{z\in Z_i} z\right)\bigwedge A_i\right],\ A_i=\bigvee_{k\in J_i}\left(\bigwedge_{z\in\overline{Z}_i}\overline{z}\right),\ J_i=\{k: Z_k=Z_i\}.$$

The formula (2) leads to $p_z=P(z=1)\to 0,\ \ h\to 0$, so

$$P(\mathbf{A_i}=0)=\prod_{z\in\overline{Z}_k:\ Z_k=Z_i} p_z\to 0,\ h\to 0.$$

If the obvious that

$$\begin{gathered}\sum_{i\in I}\prod_{z\in Z_i}p_zP(\mathbf{A_i}=1)-\sum_{i,j\in I,i\neq j}P\left(\left(\mathbf{A_i}\prod_{z\in Z_i}z=1\right)\bigcap\left(\mathbf{A_j}\prod_{z\in Z_j}z=1\right)\right)\le\\ \le P(\mathbf{A}=1)\le\sum_{i\in I}\prod_{z\in Z_i}p_zP(\mathbf{A_i}=1)\end{gathered} \tag{4}$$

As for $i\neq j$

$$P(\mathbf{A_i}\mathbf{A_j}=1)=P\left(\sum_{k\in J_i,\,n\in J_j}\left(\prod_{z\in\overline{Z}_k\bigcup\overline{Z}_n}\overline{z}\right)=1\right)\ge\prod_{z\in\overline{Z}_k\bigcup\overline{Z}_n}q_z\to 1,\ h\to 0\text{б},$$

and

$$\sum_{i,j\in I,i\neq j}P\left(\left(\mathbf{A}_i\prod_{z\in Z_i}z=1\right)\bigcap\left(\mathbf{A_j}\prod_{z\in Z_j}z=1\right)\right)=P(\mathbf{A_i}\mathbf{A_j}=1)\prod_{z\in Z_i\bigcup Z_j}p_z,$$

So from the formula (4) obtain

$$P(\mathbf{A}=1)\sim\sum_{i\in I}\prod_{z\in Z_i}p_z\sim\sum_{i\in I}\exp\left(-\sum_{z\in Z_i}h^{c(z)}\right),\ h\to 0. \tag{5}$$

Denote $C_i=\max_{z\in Z_i}c(z), K_i=\{z\in Z_i : c(z)=C_i\}$. The formulas

$$\sum_{z\in Z_i}h^{-c(z)}\sim h^{-C_i}|K_i|,\ h\to 0,$$

and (5) give

$$P(\mathbf{A}=1)\sim\sum_{i\in I}\exp\left(-h^{C_i}(1+o(1))|K_i|\right),\ h\to 0.$$

Consequently,

$$\begin{aligned}P(\mathbf{A}=1)&\sim\sum_{i\in I'}\exp\left(-h^{-C}(1+o(1))|S_i|\right)\sim\sum_{i\in I':|S_i|=N(\mathcal{S})}\exp\left(-h^{-C}(1+o(1))|S_i|\right)=\\&=\exp\left(-h^{-C}(1+o(1))N(\mathcal{S})\right)\left|\{i\in I' : |S_i|=N(\mathcal{S})\}\right|.\end{aligned}$$

As

$$\ln\left[\exp\left(-h^{-C}\left(1+o(1)\right)N(\mathcal{S})\right)\left|\left\{i\in I':\left|S_i\right|=N(\mathcal{S})\right\}\right|\right]\sim\ln\exp\left(-h^{-C}\left(1+o(1)\right)N(\mathcal{S})\right)=$$
$$=-h^{-C}\left(1+o(1)\right)N(\mathcal{S})\sim-h^{-C}N(\mathcal{S}),\ h\to 0.$$

So formula (3) is true.

Remark 1. *Suppose that* $\tau(z)$ *are independent random variables equal to life times of logical elements* z *, and* $h=h(t)$ *- is monotonically decreasing and continuous function,* $h\to 0,\ \ t\to\infty$ *. Then the asymptotic*

$$P\left(\tau(z)>t\right)=p_z(h)\sim\exp\left(-h^{-c(z)}\right),\ t\to\infty.$$

Is character for the Weibull distribution which is widely used in life time models of complex systems with old and so low reliable elements [4], [5].

Highly reliable elements

Suppose that for $\forall z\in Z$

$$\exists c(z),\ c(z)>0: q_z=q_z(h)\sim\exp\left(-h^{-c(z)}\right),\ h\to 0 \tag{6}$$

Consider the logical function A represented in CNF

$$A=\bigwedge_{i\in I}\left[\left(\bigvee_{z\in Z_i}z\right)\bigvee\left(\bigvee_{z\in\overline{Z}_i}\overline{z}\right)\right]. \tag{7}$$

Theorem 2. *If the formulas (6), (7) are true then*

$$-\ln P(\mathbf{A}=0)\sim N(\mathcal{S})h^{-C},\ h\to 0. \tag{8}$$

Remark 2. *Suppose that* $\tau(z)$ *are independent random variables equal to life times of logical elements* z *, u* $h=h(t)$ *- is monotonically increasing and continuous function,* $h\to 0,\ t\to 0$ *. Then the asymptotic*

$$P\left(\tau(z)\le t\right)=q_z(h)\sim\exp\left(-h(t)^{-c(z)}\right),\ t\to 0$$

Is character for the Weibull distribution which is widely used in life time models of complex systems with young and so high reliable elements.

Mixing case

Suppose that the sets $X_i, V_i, \overline{X}_i, \overline{V}_i \subseteq Z$ are nonintersecting. For $\forall z \in X_i \bigcup \overline{X}_i$ the formula (2) is true and for $\forall z \in V_i \bigcup \overline{V}_i$ the formula (6) taking place, $i \in I$. So low reliable and high reliable elements in the system **A** are present simultaneously

Theorem 3. *Suppose that*

$$A = \bigvee_{i \in I} \left[\left(\bigwedge_{z \in X_i \bigcup V_i} z \right) \bigwedge \left(\bigwedge_{z \in \overline{X}_i \bigcup \overline{V}_i} \overline{z} \right) \right] \quad (9)$$

Then for $Z_i = X_i \bigcup \overline{V}_i \neq \varnothing$, $\overline{Z}_i = V_i \bigcup \overline{X}_i$, $i \in I$, *the formula (3) is true .*

Suppose that

$$A = \bigwedge_{i \in I} \left[\left(\bigvee_{z \in X_i \bigcup V_i} z \right) \bigvee \left(\bigvee_{z \in \overline{X}_i \bigcup \overline{V}_i} \overline{z} \right) \right].$$

Then for $Z_i = V_i \bigcup \overline{X}_i \neq \varnothing$, $\overline{Z}_i = X_i \bigcup \overline{V}_i$, $i \in I$, *the formula (8) is true.*

Concept of bottlenecks

Define bottlenecks in logical system **A**

Theorem 4. *Suppose that* $\varepsilon_0 = \min\left(\left|C - c(z)\right| > 0 : z \in Z\right)$.

1. *For any* $S_i \in \mathcal{S}$ *and each* ε, $0 < \varepsilon < \varepsilon_0$, *the property* ***(B)*** *is true: the replacement* $c(z)$ *by* $c(z) - \varepsilon$ *for all* $z \in S_i$ *leads to the replacement* $C \to C - \varepsilon$.
2. *If a set* $S \subseteq Z$ *and satisfies the condition* ***(B)***, *then* $S_* \in \mathcal{S} : S_* \subseteq S$.
3. *For any* $T \in \mathcal{T}$ *and each* ε, $0 < \varepsilon < \varepsilon_0$, *the property* ***(C)*** *is true: the replacement* $c(z)$ *by* $c(z) + \varepsilon$ *for all* $z \in T$ *leads to the replacement* $C \to C + \varepsilon$.
4. *If a set* $T \subseteq Z$ *and satisfies the condition* ***(C)***, *then* $\exists T_* \in \mathcal{T} : T_* \subseteq T$.

Proof. Proof the statements 1, 3, as the statements 2, 4 are trivial.

1. If $c(z)$ is replace by $c(z) - \varepsilon$, $z \in S_i$, then

$\max_{z \in Z_i} c(z) = C - \varepsilon$, $\max_{z \in Z_i} c(z) \geq C - \varepsilon$, $j \neq i \Rightarrow \min_{i \in I} \max_{z \in Z_i} c(z) = C - \varepsilon$.

2. If $c(z)$ is replace by $c(z) + \varepsilon$, $z \in T$, then

$\max_{z \in Z_i} c(z) = C + \varepsilon$, $i \in I'$, $\max_{z \in Z_i} c(z) \geq C + \varepsilon$, $j \neq I' \Rightarrow \min_{i \in I} \max_{z \in Z_i} c(z) = C + \varepsilon$.

Corollary 1. *The statements 2, 4 of the theorem 4 establish that the families* $\mathcal{S}'$, $\mathcal{S}''$, $\mathcal{T}'$, $\mathcal{T}''$ *and the numbers* C, $N(\mathcal{S})$, $N(\mathcal{T})$ *do not depend on a view of DNF (of KNF) of the logical function* A.

Proof. Suppose that the theorem 1 condition is true, all other case is considered analogically. Denote by A_1, A_2 - DNF, which define the logical function A, $\mathcal{S}_1$, $\mathcal{S}_2$ are families of subsets Z, created by A_1, A_2, and $\mathcal{S}'_1$, $\mathcal{S}'_2$ are families of minimal sets from the families $\mathcal{S}_1$, $\mathcal{S}_2$, correspondingly. If the set $S_1 \in \mathcal{S}'_1$ then it satisfies the property (**B**) and so $\exists\, S_2 \in \mathcal{S}'_2 : S_2 \subseteq S_1$. Analogously if $S_2 \in \mathcal{S}'_2$ then there is $S_1^* \in \mathcal{S}'_1 : S_1^* \subseteq S_2$. Consequently $S_1^* \subseteq S_2 \subseteq S_1$ and the families $\mathcal{S}'_1$, $\mathcal{S}'_2$ definition leads to the equality $S_1^* = S_2 = S_1$ and so $\mathcal{S}'_1 = \mathcal{S}'_2$. Thus, the family $\mathcal{S}'$ does not depend on a view of logical function A DNF. Similar statements may be proved for the families $\mathcal{T}', \mathcal{T}'', \mathcal{S}''$. For the numbers $N(\mathcal{T})$, C, $N(\mathcal{S})$ the statements of the corollary 1 may be obtain from the formula (3).

Remark 3. *The statements 1 (the statements 3) of the theorem 4 establishes that an increase of elements $z \in S$ reliabilities for any set $S \in \mathcal{S}$ (a decrease of elements $z \in T$ reliabilities for any set $T \in \mathcal{T}$) leads to an increase (to a decrease) of system* **A** *reliability. The corollary1 allows to call sets from the families $\mathcal{S}'$, $\mathcal{S}''$, $\mathcal{T}'$, $\mathcal{T}''$ by bottlenecks in logical system* **A**.

Remark 4. *Suppose that $\forall z \in Z$ the condition (2) or the condition (6) are replaced by*

$$\exists\ c(z),\ d(z),\ c(z) > 0,\ d(z) > 0 : p_z = p_z(h) \sim \exp\left(-d(z)h^{-c(z)}\right),\ h \to 0,$$

or by

$$\exists\ c(z),\ d(z),\ c(z) > 0,\ d(z) > 0 : q_z = q_z(h) \sim \exp\left(-d(z)h^{-c(z)}\right),\ h \to 0,$$

correspondingly. Then to obtain the formula (3) or the formula (8) correspondingly it is enough to redefine $|S|$, $S \subseteq Z$, and put (besides of number of elements in a set S):

$|S| = \sum_{z \in S} d(z)$.

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