ANALYSIS OF ALTERNATING RENEWAL PROCESSES WITH DEPENDED COMPONENTS

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Abstract.

In the terms of operational calculus the probability characteristics of direct and reverse residual renewal time of alternating renewal process, where renewal time depends on life-time, are found.

1. Definitions, Motivations and Formulation of the Problem

1.1 According to [1] let's consider a system which fails after random life-time X1 and is fully renewed after the lapse of random time Y_1. The renewed system again fails after random life-time X_2 and is fully renewed after the lapse of random time Y_2 and so on (Fig.1). Time moments $T_1 = X_1$, $T2 = X_1 + Y_1 + X_2$,...., when the system fails is called moments of failure or moments of 0-renewal while time moments $S_1 = X_1 + Y_1$, $S_2 = X_1 + Y_1 + X_2 + Y_2$...., when renewals are ended are called the <u>moments of renewal</u> (or 1-renewal)

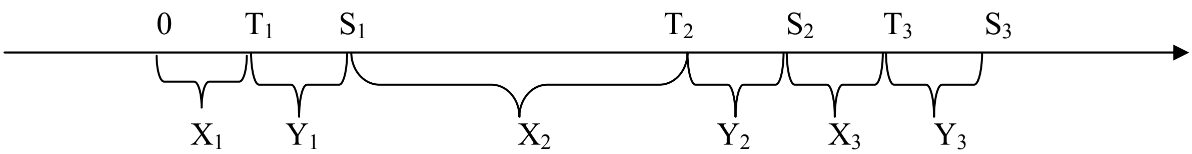

Fig.1. Realization of alternating renewal process

Definition. If $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are two sequences of equally distributed non-negative random variables when $n \geq 1$, then sequence $\{(T_n , S_n), n \geq 1\}$, as well as sequence $\{(X_n , Y_n), n \geq 1\}$ are called the alternating renewal processes.

Renewal process given in definition 1 can be equivalently described with process $\{Z(t)\ t \geq 0\}$ using relation

$$Z(t) = \begin{cases} 0, & \text{if } t \in [T_K , S_K] \\ 1, & \text{otherwise} \end{cases}$$

as realizations of process (X_n , Y_n) or (T_n , S_n) are one-to-one determined by realizations of process Z(t). According to definition, process Z(t) gives system state in moment t: Z(t) = 1, if the system is serviceable in moment t, and Z(t) = 0, if system is renewing (is non-serviceable) in moment t (Fig.2).

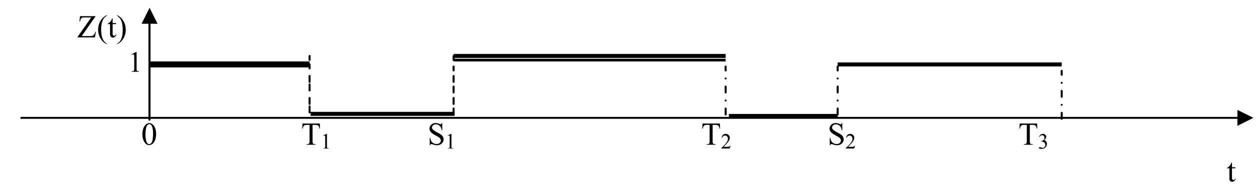

Fig.2. Realization of process Z(t)

For practical applications it is sometimes advisable to consider alternating processes $\{Y_1, (X_n + Y_n), n \geq 2\}$.

1.2 It is known that the existing renewal theory mainly studies alternating processes with independent components. This significantly limits its application, since in practice the problems modeled with alternating processes with depended (correlated) components arise very often and this dependence is so important that there is no possibility for its negligence.

In the well-known monograph of Cox [2], that has already become classic, such processes are not considered at all.

In two volumes by Feller on probability theory and its applications [3, 4] a rigorous mathematical investigation of the renewal processes is given. The alternating renewal processes (or renewal processes with two phases: active and passive) with independent components (phases) are also considered.

Alternating renewal processes are considered in the well-known monographs by Barlow and Proschan [5], Beichelt and Franken [1], etc. However, only in [1] the alternating renewal processes with dependent components are mentioned. The relations for different probability characteristics of alternating renewal processes with independent components are derived. It is shown that some of them are valid also if random variables X_n and Y_n (components of the process) are not independent and only the sums $(X_n + Y_n)$, n 1, 2 ... are independent.

In the given work, as distinct from [2-5], the independence of X_n and Y_n is not required.

1.3 The sequence of random variables $\{S_n = (X_n + Y_n), n \geq 1\}$ represents the ordinary renewal process, therefore all probability characteristics introduced and studied in the renewal theory are valid.

Sums $\{S_n = (X_n + Y_n), n \geq 1\}$ divide time axis on regeneration cycles, at the same time, time moments S_n, $n \geq 1$ are the points of regeneration.

For alternating renewal processes the probabilities $P(Z(t) = i, V_t^{(i)} > x)$, $P\{Z(t) = i, R_t^{(i)} > x\}$, $i = 0;1$ are of essential interest, where $V_t^{(1)}$ means residual life time (direct residual life time), while $V_t^{(0)}$ is residual (direct) renewal time, $R_t^{(1)}$ is reverse residual life time and $R_t^{(0)}$ is reverse residual renewal time [1].

For these probabilities, in the case of independent components of alternating renewal process, the expressions are derived with the help of renewal functions for processes $\{(X_n + Y_n), n \geq 1\}$ or $\{Y_1, (X_n + Y_n), n \geq 2\}$, respectively.

On the other hand, the mentioned renewal functions are the solutions of the corresponding renewal equations [1-5]. In [1-5] the Laplace-Stielties transforms of renewal functions are obtained. Therefore, it can be supposed that analogous transforms for the mentioned probabilities are known. In particular, all the enumerated calculations are done in [1] conformably to $P\{Z(t) = 1, V_t^{(1)} > x\}$.

Using the same methods, analogous results may be obtained conformably to P`$\{Z(t)=1, R_t^{(1)}>x\}$.

It is noted in [1] that expression for $P\{Z(t) = 1, V_t^{(1)} > x\}$ is also valid if random variables X_n and Y_n are not independent. Only the independence of sums $(X_n + Y_n)$, $n \geq 1$ is required.

On our part we will add, that after some transformations of calculations done in [1] we can receive analogous expressions for $P\{Z(t) = 1, R_t^{(1)} > x\}$ as well, in the case of dependent X_n and Y_n, $n \geq 1$.

At the same time when components of alternating renewal process components are dependent, the expressions of probabilities $P\{Z(t) = 0, V_t^{(0)} > x\}$ and $P\{Z(t) = 0, R_t^{(0)} > x\}$ are obtained neither directly with the help of the appropriate matching of renewal functions nor with the help of this probability reasoning that is used in [1] to derive the expressions of the mentioned renewal functions.

On the other hand, the latter probabilities have a considerable theoretical interest in the case of alternating renewal processes with depended (correlated) components. Besides, they have particular importance in practical applications, namely, within the framework of the structural and maintenance modeling of the technical systems.

Naturally, universally recognized scientist I. Ushakov distinguishes structural and maintenance models as one of the main directions of the modern Reliability Theory [5].

1.4 As a physical analog of alternating renewal process we can consider the above mentioned technical system if we assume that the wear and aging of the system's elements affect the system's reliability. Suppose such system fails when one of its elements loses serviceability. After such failure, as a rule, a detailed survey of the system is done and all elements that are in "critical" state by the signs of aging or wear are revealed. Repair (full renewal) of the system implies the substitution of all such elements, as well as of the failed element, with new ones. Besides, different operations are carried out directed to bring back the system to the initial state. In other words, as a result of such renewal, within the admissible errors, the renewed system can be considered as identical to the initial one.

At the same time, conformably to system life-time the quantity of worn-out and aged elements will be different. In particular, it is natural to suppose that this quantity increases with the increase of system life-time.

Thus, in such cases renewal time on average increases at the expense of the increase of the quantity of elements to be substituted.

The same can be said about the other renewal operations mentioned above.

As a result of the given reasoning we shall make a brief conclusion: the more the average life-time of the above described technical systems, the more the average renewal time or, in other words, renewal time depends on life-time.

Let's consider an example of technical system where the mentioned dependence shows up not as a result of physical and chemical processes of aging or wear but as a result of structural organization features of the standby system.

Let technical system contain main and standby elements undergoing failures.

In the system there functions continuous, reliable control of serviceability that instantly detects the failure of the main, as well as standby element. There also is one unit for replacement (switching over) and repair that immediately starts the necessary maintenance operation. After the failure of the main element it is replaced with a standby one if in this moment it is serviceable (I type failure).

If in the moment of main element failure the standby one is unserviceable and therefore is repaired, it replaces the failed main element after repair (II type failure). Thus, the functioning process of the standby system represents the sequence of regeneration cycles. Each cycle consists of life-time of the main element and downtime (idle time) equal to replacement time in case of I type failure and to the sum of durations of the remained renewal and following replacement in case of II type failure. If life-time of the main element in each cycle of regeneration denote through X_1, X_2 ...., and idle time - through Y_1, Y_2,...., then the sequence $\{(X_n , Y_n), n \geq 1\}$ forms alternating renewal process. Thus, standby system is reduced to single-unit one and its subsequent functioning can be realized from this point of view.

As the duration of n-th idle time of such system Y_n, $n \geq 1$ depends on in what state is the standby element in the moment of the failure of the main one, it is easy to guess that it depends on X_n, $n \geq 1$. As we see, there exists an alternating renewal process with depended components and this dependence is "created" not with physical-chemical but with structural peculiarities of the considered system.

The reduction of two-unit technical system to single-unit one at different suppositions respective to statistic characteristics of life-time, replacement time and renewal time is given in [6]. Thus, alternating renewal process with depended components is "created".

Other kinds of technical systems with depended (correlated) life-time (failure time) and repair time are considered in [7, 8].

2. Probabilistic Analysis

Below we propose the method of determination of probabilities $P\{Z(t) = 0, V_t^{(0)} > x\}$ and $P\{Z(t) = 0, R_t^{(0)} > x\}$ in the terms of operational calculus, in the presumption, that X_n and Y_n are depended. Namely Y_n depends on X_n, $n \geq 1$ and this dependence is given with the conditional distribution function:

$$G(t, v) = P\{Y_n < t \mid X_n = v\}, n \geq 1 \tag{1}$$

It is clear that distribution function (unconditional) of Y_n is expressed through the following integral:

$$G(t) = \int_0^t G(t, v) dF(v) \tag{2}$$

where $F(v) = P\{X_n < v\}$, $n \geq 1$

As within the given article such notations will not be used in a different sense, in order to simplify the calculations these probabilities are denoted respectively with:

$$V(t,x)=P\{Z(t)=0,V_t^{(0)}>x\} \tag{3}$$

$$R(t,x)=P\{Z(t)=0,R_t^{(0)}>x\} \tag{4}$$

Denote with $Q(u)$ the distribution function of sum $\{X_n + Y_n\}$, $n \geq 1$:

$$Q(u) = P\{X_n + Y_n < u\}, \quad n \geq 1, \tag{5}$$

Evidently:

$$Q(u) = \int_0^u G(u - v, v) dF(v) \tag{6}$$

Theorem 1. Function $V(t,x)$ is the solution of the following convolution type second order Volterra integral equation:

$$V(t, x) = \int_0^t [1 - G(t - u + x, u)] dF(u) + \int_0^t V(t - u, x) dQ(u) \tag{7}$$

Proof. Event B(t,x) = {Z(t) = 0, $V_t^{(0)}$ >x} can be realized simultaneously with one of the two incompatible events: A(t) and $\overline{A}(t)$.

1. A(t) = {moment t is covered with the first regeneration cycle of renewal process $S_n = (X_n + Y_n), n \geq 1$};

2. $\overline{A}(t)$ = {moment t is not covered with the first regeneration cycle of renewal process $S_n = (X_n + Y_n), n \geq 1$}.

Simultaneous execution of events A(t) and B(t,x) can be presented as: in interval (0,t) in some moment of time u system fails; renewal time is more than t-u+x; the probability of this event with consideration of all possible values of variable u is:

$$\int_0^t [1 - G(t - u + x)]dF(u) \tag{8}$$

Simultaneous execution of events $\overline{A}(t)$ and B(t,x) can be presented as: in interval (0,t) in some moment of time u the first cycle of regeneration ends and event B(t-u,x) is executed; the probability of this event with consideration of all possible values of variable u is:

$$\int_0^t V(t-u,x)dQ(u) \tag{9}$$

Sum of (8) and (9) gives the right part of (7) that proves the theorem.

Theorem 2. Function R(t,x) is the solution of convolution type second order Volterra integral equation:

$$R(t,x) = \begin{cases} \int_0^{t-x} [1 - G(t-u,x)]dF(u) + \int_0^t R(t-u,x)dQ(u) & \text{if} \quad x < t \\ 0 & \text{if} \quad x \geq t \end{cases} \tag{10}$$

Proof. As in the initial moment of time the life time of the system begins, it is easy to guess that the probability of event {Z(t) = 0, $R_t^{(0)}$ >x} for all x ≥ t is equal to zero.

Denote C(t, x) = {Z(t) = 0, $R_t^{(0)}$ >x}, x < t. Event C(t,x) can be realized simultaneously with one of events A(t) and $\overline{A}(t)$ (see proof of theorem 1).

Simultaneous execution of events A(t) and C(t,x) can be presented as: in interval (0, t-x), in some moment of time u the system fails; its renewal time is more than t-u; the probability of this event with consideration of all possible values of variable u is:

$$\int_0^{t-x} [1 - G(t-u,x)]dF(u) \tag{11}$$

Simultaneous execution of events $\overline{A}(t)$ and C(t,x) can be presented as: in interval (0,t) in some moment of time u the first cycle of regeneration is ended; event {Z(t-u) = 0, $R_{t-u}^{(0)}$ >x} is executed; the probability of this event with consideration of all possible values of variable u equals:

$$\int_0^t R(t-u,x)dQ(u) \tag{12}$$

Sum of (11) and (12) gives the right part of (10) when x < t, that proves the theorem.

Applying Laplace transformation in respect to t to (7) and (10) we get:

$$\overline{V}(s,x) = A(s,x) + \overline{Q}(s)\overline{V}(s,x) \tag{13}$$

Here

$$\overline{V}(s,x) = \int_0^\infty e^{-st} \; V(t,x)dt$$

$$A(s,x) = \int_0^\infty e^{-st} \left(\int_0^t [1 - G(t-u+x,u)dF(u)] \right) dt$$

$$\overline{Q}(s) = \int_0^\infty e^{-st} dQ(u)$$

From (7) we easily get:

$$\overline{\mathrm{V}}(\mathrm{s},\mathrm{x}) = \mathrm{A}(\mathrm{s},\mathrm{x})/(1 - \overline{\mathrm{Q}}(\mathrm{s})) \tag{14}$$

Similarly, from (10) we get:

$$\overline{R}(s,x) = B(s,x)/(1 - \overline{Q}(s)) \tag{15}$$

Here

$$\overline{R}(s,x) = \int_0^\infty e^{-st} R(t,x)dt$$

$$B(s,x) = \int_0^\infty e^{-st} \left(\int_0^{t-x} [1 - G(t-x,x)]dF(x) \right) dt$$

As a rule, the reverse transformation of transforms (14) and (15) is rarely successful. However, all numerical characteristics of the considered events, random variables and processes can be obtained from them. Investigations in these directions will be the continuation of the given work.

3. Remarks and Conclusions

The main motivation for conducting this research is the statement, that residual life-time and residual renewal (repair) time are the important characteristics of renewable technical systems. Analysis of such characteristics is carried out within the framework of the renewal theory [1]. However, when life-time and renewal time of technical system are interdependent (correlated), probabilistic analysis of residual life-time and residual renewal time by the methods of classical renewal theory is difficult, if not impossible (subsection **1.2**). At the same time the technical systems, in which there is no possibility for negligence of such correlation are widespread (subsections **1.2, 1.3, 1.4**).In subsection 1.4 two kinds of` such systems are described.

In section **2** the alternating renewal process in which the renewal time depends on the life-time is examined. Probabilistic characteristics of direct and reverse residual renewal time are studied. The method is worked out for obtaining the Laplace transforms of these characteristics (expressions (14) and (15)). Undoubtedly these results have a wide practical application.

Currently the author is working for obtaining analogous results for alternating renewal process in which the life-time depends on the previous renewal time.

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