COMPUTATION OF FAILURE/REPAIR FREQUENCY OF MULTI-STATE MONOTONE SYSTEMS

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Abstract

The paper deals with calculation methods for failure and repair frequencies of multi-state monotone systems, both for the instantaneous and steady state cases. Being based on the binary representation of multi-state structure, new general formula for the failure/repair frequency is derived. This formula is used to obtain simple rules for the calculation of failure/repair frequency. In particular, the use of the algebra of dual numbers is presented.

1. Introduction

The failure frequency, called also the rate of occurrence of failures (ROCOF), is defined as the mean number of failures per unit time. Let W(t) be the mean number of failures of an item (element or system) in time-interval (0, t]. When W(t) is absolutely continuous function in any finite time interval, then the failure frequency w(t) is defined as the density of W(t) with respect to the Lebesque measure on the real line, i.e.

$$W(t) = \int_{0}^{t} w(s) ds$$
, $w(t) = dW(t)/dt$ (a.e.).

The limiting (or steady-state) failure frequency $w(\infty)$ is defined as the limiting value of w(t) when t tends to infinity. The failure frequency is an important reliability measure of repairable items, since it may be used to compute the expected number of failures in given interval. Furthermore, $w(\infty)$ is equal to the reciprocal of the mean time between failures, and the following well known expressions hold true:

 $MUT = A(\infty)/w(\infty)$, $MDT = (1-A(\infty))/w(\infty)$,

where MUT = mean up-time, MDT = mean down time and $A(\infty)$ is the limiting (or steady state) availability of the item.

MUT and MDT are of practical importance, because they well enough characterise the reliability performance of repairable systems. Furthermore, these indices are often included into customer's requirements for reliability of newly designed systems, typically in industry and military areas. Therefore, calculations of MUT and MDT are needed during design and development phase in order to check if the requirements are met. According to the equation above, it is therefore important to calculate not only system availability, but also system failure frequency.

The repair (or restoration) frequency v(t) of an item is defined similarly as the failure frequency, by replacing failures with restorations (i.e. completion of repairs) of the item. That is, by integrating v(t) over given time-interval [a, b], we obtain the mean number of restorations of the item in that interval.

Considerable efforts have been devoted to the problem of finding the efficient calculation methods for the failure/repair frequency of binary monotone systems composed of independent binary components. See Amari (2000, 2002), Chang et al. (2004), Pavlov & Ushakov (1989),

Schneeweiss (1999) and the references given therein. The main objective of these researches was to obtain simple rules for transforming expressions of system availability/unavailability given in terms of element availability and unavailability into an expression for system failure frequency, and system repair frequency as well, both for time-dependent and steady-state cases.

In many real-life situations, however, the systems and their elements are capable of assuming a whole range of performance levels, varying from perfect functioning to the complete failure. A multi-state system (MSS) fails if its performance level is less than the desired performance level (demand). Beginning from the middle of 70s, the theory of binary systems is being replaced by the theory of MSS. The present state-of-art of the theory and practice of MSS may be found in recent monographs Kołowrocki (2004), Kuo and Zuo (2003), Levitin (2005), and Lisnianski and Levitin (2003).

In opposite to the binary case, rather little attention has been devoted to finding practical methods for computation of the frequency-type indices for MSS. Main results have been obtained by Murchland (1975), where very general relations for the computation of failure frequency and related indices were given. Similar relations were considered in Aven and Jensen (1999), Natvig and Streller (1984) and Franken et al. (1984) for the steady-state case of multi-state monotone systems (MMS). However the expressions obtained are stated in general form which is not very convenient for practical purpose due to its complexity. Another approach, based on the inclusion-exclusion principle applied to the set of prime implicants of an MSS was suggested by Bossche (1984, 1986). This approach has however big computational complexity. None of the results mentioned so far has the form of simple rules converting availability expression to failure frequency expression, as in binary case.

The main aim of this paper is to show how to calculate the failure/repair frequency of multistate systems using conversion rules being generalizations of the rules known from the binary systems. These multi-state conversion rules are obtained using a new general formula for the failure/repair frequency of MMS, which has very simple form. The presentation of these results is based on recent works of Korczak (2006, 2007), with some improvements. Moreover, is shown that the calculation can performed using the algebra of dual numbers.

2. Basic definitions and assumptions

2.1. Multi-state monotone systems and their binary representations

Let $\langle C, \mathbf{K}, \mathbf{K}_1, ..., \mathbf{K}_n, \phi \rangle$ be a multi-state system consisting of *n* multi-state elements with the index set $C = \{1, 2, ..., n\}$, where $\mathbf{K} = \{g(0), g(1), ..., g(M)\} \subseteq [0, +\infty)$ is the set of the system states, $\mathbf{K}_i = \{g_i(0), g_i(1), ..., g_i(M_i)\} \subseteq [0, +\infty)$ is the set of the states of element $i \in C$, and $\phi: \mathbf{V} \to \mathbf{K}$ is the system structure function, where $\mathbf{V} = \mathbf{K}_1 \times \mathbf{K}_2 \times ... \times \mathbf{K}_n$ is the space of element state vectors. We assume that the states of the system [element *i*] represent successive performance rates ranging from the perfect functioning level g(M) [$g_i(M_i)$] down to the complete failure level g(0) [$g_i(0)$], that is $0 \le g(0) < g(1) < ... < g(M)$ and $0 \le g_i(0) < g_i(1) < ... < g_i(M_i)$. The system is a *multi-state monotone system* (MMS) if its structure function ϕ is non-decreasing in each argument, $\phi(\mathbf{g}(\mathbf{0})) = g(0)$ and $\phi(\mathbf{g}(\mathbf{M})) =$ g(M), where $\mathbf{g}(\mathbf{0}) = (g_1(0), g_2(0), ..., g_n(0))$, $\mathbf{g}(\mathbf{M}) = (g_1(M_1), g_2(M_2), ..., g_n(M_n))$. We refer to Kuo and Zuo (2003), Levitin (2005) and Lisnianski and Levitin (2003) for detailed description and numerous examples of MMS. Throughput the paper, we will consider MMS only.

A vector $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbf{V}$ is said to be a *path* [*cut*] *vector to level* $c \in \mathbf{K}$ of an MMS if $\varphi(\mathbf{y}) \ge c$ [$\varphi(\mathbf{y}) < c$]. It is called a *minimal path* [*cut*] *vector to level* c if in addition $\mathbf{x} < \mathbf{y}$ [$\mathbf{x} > \mathbf{y}$] implies $\varphi(\mathbf{x}) < c$ [$\varphi(\mathbf{x}) \ge c$], where $\mathbf{x} < \mathbf{y}$ means $x_i \le y_i$ for i=1,...,n, and $x_i < y_i$ for some i. The set of all minimal path [cut] vectors to level c is denoted by \mathbf{U}_c [\mathbf{L}_c], where $\mathbf{U}_{g(0)} = \{\mathbf{g}(\mathbf{0})\}$ and $\mathbf{L}_{g(0)} = \emptyset$.

The state (performance level) of element *i* at time *t* is represented by a (random) variable $X_i(t)$, which takes values in \mathbf{K}_i . The state (performance level) X(t) of the system at time *t* is fully determined by the states of the elements through the multi-state structure function φ , i.e., $X(t) = \varphi(\mathbf{X}(t))$, where $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$.

Let us introduce level indicator processes $X_i(e,t) = \mathbf{1}(X_i(t) \ge e)$ and $X(d,t) = \mathbf{1}(X(t) \ge d)$, $e,d \ge 0$, where $\mathbf{1}(.)$ is the indicator function. Let $\varphi_d = \mathbf{1}(\varphi \ge d)$, $d \in \mathbf{K} - \{0\}$, be the system level indicators. They can be considered as functions of vector of binary variables $\underline{\mathbf{X}}(t) = [X_i(r,t): i \in \mathbf{C}, r \in \mathbf{K}_i - \{g_i(0)\}]$, so that $\varphi_d(\underline{\mathbf{X}}(t)) = X(d,t)$, resulting in the binary representation of MMS; see Block and Savits (1982), Korczak (2005) and Lisnianski and Levitin (2003) for more details.

From the definition of minimal path and minimal cut vectors, we obtain so-called, *min-path* and *min-cut forms*:

$$\varphi_d(\underline{\underline{\mathbf{X}}}(t)) = \max_{\mathbf{y}\in\mathbf{U}_d} \min_{i\in\mathbf{C}: \, y_i > g_i(0)} X_i(y_i, t), \ \varphi_d(\underline{\underline{\mathbf{X}}}(t)) = \min_{\mathbf{z}\in\mathbf{L}_d} \max_{i\in\mathbf{C}: \, z_i < g_i(M_i)} X_i(z_i \oplus_i 1, t)$$
(2.1)

where $r \oplus_i 1 = \min(\mathbf{K}_i \cap (r, \infty))$, for $r \in \mathbf{K}_i - \{g_i(M_i)\}$, is the next state in \mathbf{K}_i better than state *r*.

There are several algebraic forms of φ_d , which can be obtained from (2.1) using inclusionexclusion, SDP or other methods, see Korczak (2005, 2007). For example, the pseudo-polynomial form is given by:

$$\varphi_d(\underline{\underline{\mathbf{X}}}(t)) = \beta_0 + \sum_{k=1}^m \beta_k B_k(\underline{\underline{\mathbf{X}}}(t)), \qquad (2.2)$$

where

$$B_{k}(\underline{\underline{\mathbf{X}}}(t)) = \prod_{i \in C} \left(X_{i}(a(k,i),t) - X_{i}(b(k,i),t) \right),$$
(2.3)

 β_k are integer coefficients, $a(k,i), b(k,i) \in \mathbf{K}_i \cup \{g_i(M_i)+1\}$, a(k,i) < b(k,i) for all *i* and *k*, and the products B_k are non-trivial. The term $X_i(a(k,i),t) - X_i(b(k,i),t)$ reduces to $X_i(a(k,i),t)$ if $b(k,i) = g_i(M_i)+1$, to $1-X_i(b(k,i),t)$ if $a(k,i) = g_i(0)$, and to 1, if $b(k,i) = g_i(M_i)+1$ and $a(k,i) = g_i(0)$.

When reliability structure of a binary system is complex, the Shannon decomposition is frequently used to simplify the structure. The corresponding multi-state Shannon decomposition (or pivotal decomposition, or factoring) formulae are:

$$\varphi_{d}(\underline{\underline{\mathbf{X}}}(t)) = \sum_{r \in \mathbf{K}_{i}} (X_{i}(r,t) - X_{i}(r \oplus_{i} 1,t)) \varphi_{d}(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}(t)),$$

$$= \varphi_{d}(\underline{\mathbf{e}}_{i}(g_{i}(0)), \underline{\underline{\mathbf{X}}}(t)) + \sum_{r \in \mathbf{K}_{i}} X_{i}(r,t) [\varphi_{d}(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}(t)) - \varphi_{d}(\underline{\mathbf{e}}_{i}(r-_{i} 1), \underline{\underline{\mathbf{X}}}(t))], \qquad (2.4)$$

where $((r)_i, \mathbf{X}(t)) = (X_1(t), ..., X_{i-1}(t), r, X_{i+1}(t), ..., X_n(t)), \ \mathbf{\underline{e}}_i(r) = (\mathbf{1}(u \le r): u \in \mathbf{K}_i - \{g_i(0)\}), r \in \mathbf{K}_i, g_i(M_i) \oplus_i 1 = g_i(M_i) + 1, \text{ and } r_{-i} 1 = \max(\mathbf{K}_i \cap (-\infty, r)), \text{ for } r \in \mathbf{K}_i - \{g_i(0)\}, \text{ is the best state preceding state } r, \text{ and } g_i(0) -_i 1 = g_i(0), \text{ so that } \mathbf{\underline{e}}_i(g_i(0) -_i 1) = \mathbf{\underline{e}}_i(g_i(0)). \text{ Observe that for } r = g_i(k), \ \mathbf{\underline{e}}_i(r) = \mathbf{\underline{e}}_i(g_i(k)) = (\underbrace{1, ..., 1}_{k}, \underbrace{0, ..., 0}_{M_i - k}), \ \mathbf{\underline{e}}_i(g_i(0)) = (\underbrace{0, ..., 0}_{M_i}), \ \mathbf{\underline{e}}_i(g_i(M_i)) = (\underbrace{1, ..., 1}_{M_i}).$

Note that $\varphi((r)_i, \mathbf{x})$ is an extended structure function (i.e. it can be degenerated), taking its values in the set { $\varphi((r)_i, \mathbf{g}(\mathbf{0}))$, ..., $\varphi((r)_i, \mathbf{g}(\mathbf{M}))$ }. When $r > g_i(0)$, it may happen that $\varphi((r)_i, \mathbf{g}(\mathbf{0})) > g(0) = \varphi(\mathbf{g}(\mathbf{0}))$. However, if $\varphi((r)_i, \mathbf{g}(\mathbf{0})) < \varphi((r)_i, \mathbf{g}(\mathbf{M}))$, then $\varphi((r)_i, \mathbf{x})$ fits our definition of MMS with the lowest performance levels being not necessarily 0.

Unless otherwise stated, we make the following assumptions regarding stochastic properties of the elements of an MMS.

Assumption 2.1. The system's elements, that is, the stochastic processes $\{X_i(t)\}, i \in C$, are mutually s-independent.

Assumption 2.2. $\{X_i(t)\}, i \in C$, are regular jump processes, i.e.: have jump right-continuous sample paths with left-side limits, and have finite expected number of jumps in bounded intervals.

For any $r,s \in \mathbf{K}_i$, $r \neq s$, let $N_i^{r \to s}(t)$ be the number of transitions of element *i* from its state *r* to its state *s* in time interval (0, *t*]. Its expected value is denoted by $W_i^{r \to s}(t) = E[N_i^{r \to s}(t)]$. When $W_i^{r \to s}(t)$ is locally absolutely continuous on $[0, \infty)$, its density $w_i^{r \to s}(t)$ is called the instantaneous frequency of transitions from *r* to *s*:

$$W_i^{r \to s}(t) = \int_0^t w_i^{r \to s}(s) \, \mathrm{d}s \,, \quad w_i^{r \to s}(t) = \mathrm{d}W_i^{r \to s}(t) \,/\, \mathrm{d}t \,.$$
(2.5)

Assumption 2.3. (transient, or instantaneous case) For any $i \in C$, all the functions $W_i^{r \to s}(t)$, $r, s \in \mathbf{K}_i$, $r \neq s$, are locally absolutely continuous on $[0, \infty)$, i.e. the interstate frequencies $w_i^{r \to s}(t)$ exist.

Let $p_i(r; t) = \Pr\{X_i(t) = r\}$. The steady state frequency $w_i^{r \to s}(\infty)$ of transitions of element *i* from its state *r* to its state *s* and the steady state probability $p_i(r; \infty)$ that element *i* is in *r* are defined by:

$$w_i^{r \to s}(\infty) = \lim_{t \to \infty} w_i^{r \to s}(t), \quad p_i(r; \infty) = \lim_{t \to \infty} p_i(r; t).$$
(2.6)

Assumption 2.4. (steady state, or asymptotic case) For any $i \in C$ and $r,s \in \mathbf{K}_i$, $r \neq s$, the steady state frequencies $w_i^{r \to s}(\infty)$ and steady state probabilities $p_i(r; \infty)$ exist.

From assumptions 2.1 and 2.3, it follows that the processes $\{X_i(t)\}, i \in C$, have no common jump times with probability 1. And in consequence, any change of the system's state is caused, with probability one, by a jump of exactly one element.

2.2. Basic reliability measures of MMS

For an MMS, one can define various reliability and performance measures, see Aven and Jensen (1999), Levitin (2005), and Lisnianski and Levitin (2003). In this paper we will consider basic reliability indices only, which, however, may be used to calculate many other measures. For any fixed performance level d, $g(0) < d \le g(M)$, we define the system reliability measures like for binary systems, considering the sets $\mathbf{G}(d) = \mathbf{K} \cap [d,\infty)$ and $\mathbf{F}(d) = \mathbf{K} - \mathbf{G}(d)$ as up and down states respectively. The system availability to level d (or to demand d) is defined as $A(d,t) = \Pr{X(t) \ge d}$ = $E[X(d,t)] = E[\varphi_d(\underline{\mathbf{X}}(t))]$. The system unavailability to level d (or to demand d) is defined as $U(d,t) = \Pr{X(t) \le d} = 1 - A(d,t)$. A transition from $\mathbf{G}(d)$ to $\mathbf{F}(d)$ is called d-failure, and the reverse transition is called d-repair. The instantaneous failure [repair] frequency to level d (shortly: d-failure [repair] frequency) is denoted by w(d,t) [v(d,t)] and defined as the density of the function W(d,t) [V(d,t)], the expected number of d-failures [d-repairs] in (0, t], i.e.:

$$W(d,t) = \int_{0}^{t} w(d,s)ds, \quad V(d,t) = \int_{0}^{t} v(d,s)ds. \quad (2.7)$$

We set $A(g(0),t) \equiv U(u,t) \equiv 1$ and $w(g(0),t) \equiv w(u,t) \equiv A(u,t) \equiv U(g(0),t) \equiv 0$ for u > g(M). Binary-like reliability indices of the system elements are defined similarly, and are denoted as $A_i(u,t), U_i(u,t), w_i(u,t)$ and $v_i(u,t)$ for $i \in C$ and $g_i(0) < u \le g_i(M_i)$, with $A_i(g_i(0),t) \equiv U_i(s,t) \equiv 1, w_i(g_i(0),t) \equiv w_i(s,t) \equiv A_i(s,t) \equiv U_i(g_i(0),t) \equiv 0$ for $s > g_i(M_i)$. They can be calculated from known state probabilities $p_i(r; t)$ and interstate frequencies $w_i^{r \to s}(t)$:

$$A_{i}(u,t) = \sum_{\substack{r \in \mathbf{K}_{i} \\ r \ge u}} p_{i}(r;t), \quad w_{i}(u,t) = \sum_{\substack{r,s \in \mathbf{K}_{i} \\ s < u, r \ge u}} w_{i}^{r \to s}(t), \quad v_{i}(u,t) = \sum_{\substack{r,s \in \mathbf{K}_{i} \\ s < u, r \ge u}} w_{i}^{s \to r}(t).$$
(2.8)

The steady state (or limiting, asymptotic) reliability indices are defined as the limiting values of the corresponding instantaneous indices, by letting $t \rightarrow \infty$, if the limits exist. In steady state, the failure and repair frequencies of system, and each of its element as well, are equivalent:

$$w(d,\infty) = v(d,\infty), \quad w_i(r,\infty) = v_i(r,\infty). \tag{2.9}$$

The steady state system failure frequency is important for applications, since under rather mild assumptions, see Cocozza-Thivent (1997) and Cocozza-Thivent and Roussignol (2000) (for example when system's elements are modelled by irreducible time-continuous Markov chains, or by its functions), we have the following familiar relations:

$$MUT(d) = A(d,\infty)/w(d,\infty), \quad MDT(d) = U(d,\infty)/w(d,\infty), \tag{2.10}$$

where MUT(d) [MDT(d)] is the mean up-time [down-time] to level d of the system.

In the simplest case, when the stochastic evolution of element *i* is described by a

homogeneous time-continuous Markov chain with transition rate matrix $[\lambda_i(r,s): r,s \in \mathbf{K}_i]$, we have:

$$w_i^{r \to s}(t) = p_i(r;t)\lambda_i(r,s),$$
 (2.11)

where $t \ge 0$ or $t = \infty$ (for the limiting case).

2.2. System availability calculation

For any fixed $i \in C$, stochastic processes $\{X_i(e,t)\}, e \in \mathbf{K}_i - \{g_i(0)\}\)$, are dependent, as $1 \ge X_i(g_i(1),t) \ge X_i(g_i(2),t) \ge ... \ge X_i(g_i(M_i),t) \ge 0$. However, by stochastic independence of elements, the processes belonging to different elements are independent. Therefore, having $\varphi_d(\underline{\mathbf{X}}(t))$ written in a suitable form, and knowing availability/unavailability of independent elements, calculation of the system availability is very easy. For example, if $\varphi_d(\underline{\mathbf{X}}(t))$ is given in the form (2.2), then:

$$A(d,t) = \beta_0 + \sum_{k=1}^{m} \beta_k B_k(\underline{\underline{\mathbf{A}}}(t)), \qquad (2.12)$$

where $A_i(g_i(0),t) \equiv 1$, $A_i(u,t) \equiv 0$ for $u > g_i(M_i+1)$, and

$$B_k(\underline{\underline{\mathbf{A}}}(t)) = E[B_k(\underline{\underline{\mathbf{X}}}(t))] = \prod_{i \in C} \left(A_i(a(k,i),t) - A_i(b(k,i),t) \right).$$
(2.13)

with $\underline{\mathbf{A}}(t) = [A_i(r,t): i \in \mathbf{C}, r \in \mathbf{K}_i - \{g_i(0)\}].$

Applying the factoring formula (2.4), we get:

$$A(d,t) = \sum_{r \in \mathbf{K}_{i}} (A_{i}(r,t) - A_{i}(r \oplus_{i} 1,t)) A^{(i,r)}(d,t)$$

= $A^{(i,g_{i}(0))}(d,t) + \sum_{r \in \mathbf{K}_{i}} A_{i}(r,t) [A^{(i,r)}(d,t) - A^{(i,r-i)}(d,t)],$ (2.14)

where $A^{(i,r)}(d,t) = \Pr\{\varphi(\mathbf{X}(t)) \ge d \mid X_i(t) = r\} = \Pr\{\varphi((r)_i, \mathbf{X}(t)) \ge d\} = E[\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))]$ is the availability to level *d* of the system with indicator structure function $\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))$, or in other words, $A^{(i,r)}(d,t)$ is the availability of the system with structure function φ_d , given that element *i* is strapped in state *r*.

3. Failure and repair frequency calculation

3.1. The main formula

According to general results obtained by Murchland (1975), we have:

$$w(d,t) = \sum_{i \in C} \sum_{\substack{r,s \in \mathbf{K}_i \\ r \neq s}} \Pr\{\varphi((r)_i, \mathbf{X}(t)) \ge d, \varphi((s)_i, \mathbf{X}(t)) < d\} w_i^{r \to s}(t)$$
(3.1)

$$v(d,t) = \sum_{i \in C} \sum_{\substack{r,s \in \mathbf{K}_i \\ r \neq s}} \Pr\{\varphi((r)_i, \mathbf{X}(t)) \ge d, \varphi((s)_i, \mathbf{X}(t)) < d\} w_i^{s \to r}(t)$$
(3.2)

for both monotone and non-monotone systems. For monotone systems considered in this paper we have $\Pr{\{\varphi((r)_i, \mathbf{X}(t)) \ge d, \varphi((s)_i, \mathbf{X}(t)) < d\}} = \mathbf{1}(r \ge s) \cdot (A^{(i,r)}(d,t) - A^{(i,s)}(d,t))$, hence these general expressions reduce to the following:

$$w(d,t) = \sum_{i \in C} \sum_{\substack{r,s \in \mathbf{K}_i \\ r > s}} (A^{(i,r)}(d,t) - A^{(i,s)}(d,t)) w_i^{r \to s}(t)$$
(3.3)

$$v(d,t) = \sum_{i \in C} \sum_{\substack{r,s \in K_i \\ r > s}} (A^{(i,r)}(d,t) - A^{(i,s)}(d,t)) w_i^{s \to r}(t)$$
(3.4)

The main disadvantages of these formulae are that they depend on a number of element's inter-state transition frequencies, and that the format of input data $\{w_i^{r\to s}(t)\}$ is different from the format of output data $\{w(d,t), v(d,t)\}$. As a result, recursive application of these formulae for complex systems with hierarchical structure is difficult, or even impossible. More convenient are formulae stated in terms of element's failure/repair frequencies, $w_i(r,t)$ and $v_i(r,t)$. Observe that for $r > s, r, s \in \mathbf{K}_i$:

$$A^{(i,r)}(d,t) - A^{(i,s)}(d,t) = \sum_{\substack{u \in \mathbf{K}_i:\\s < u \le r}} (A^{(i,u)}(d,t) - A^{(i,u-i)}(d,t)).$$
(3.5)

Substituting (3.5) into (3.3) and (3.5), interchanging the order of summation and using relations (2.8), we obtain:

$$w(d,t) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} [A^{(i,r)}(d,t) - A^{(i,r-i)}(d,t)] w_i(r,t), \qquad (3.6)$$

$$v(d,t) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} [A^{(i,r)}(d,t) - A^{(i,r-i)}(d,t)] v_i(r,t) .$$
(3.7)

According to (2.14),

$$A^{(i,r)}(d,t) - A^{(i,r_{-i}1)}(d,t) = \frac{\partial A(d,t)}{\partial A_{i}(r,t)},$$
(3.8)

where we consider any $U_i(r,t)$ appearing in the expression for A(d,t) as $1-A_i(r,t)$, so that $\partial U_i(r,t)/\partial A_i(r,t) = \partial (1-A_i(r,t))/\partial A_i(r,t) = -1$. Thus we have proved the following main result:

$$w(d,t) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} w_i(r,t) \frac{\partial A(d,t)}{\partial A_i(r,t)},$$
(3.9)

$$v(d,t) = \sum_{i \in \mathbf{C}} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} v_i(r,t) \frac{\partial A(d,t)}{\partial A_i(r,t)}.$$
(3.10)

Now the input and output data are of the same format as the failure/repair frequencies. Moreover, the expressions obtained are easy to remember, and are very similar to the formulae known from the binary system theory.

Since the expressions for w(d,t) and v(d,t) are similar, we will restrict our consideration to w(d,t). Repair frequency formulae can be obtained from failure frequency formulae by replacing w with v. Furthermore, for sake of brevity, we will time parameter t in what follows. Thus we will write:

 $A_i(r), U_i(r), w_i(r), A(d), U(d), w(d)$ instead of $A_i(r,t), U_i(r,t), w_i(r,t), A(d,t), U(d,t), w(d,t)$.

Let us consider some alterative forms of (3.9). When the system unavailability U(d) is given as a function of $U_i(r)$ and $A_i(r)$, then:

$$w(d,t) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} w_i(r) \frac{\partial U(d)}{\partial U_i(r)}, \qquad (3.11)$$

where we consider any $A_i(r)$ appearing in the expression for U(d) as $1-U_i(r)$, so that $\partial A_i(r)/\partial U_i(r) = \partial (1-U_i(r))/\partial U_i(r) = -1$.

When $U_i(r)$ and $A_i(r)$ appearing in the expression for A(d) or U(d) are considered as independent variables, so that $\partial U_i(r)/\partial A_i(r) = \partial A_i(r)/\partial U_i(r) = 0$, then according to the chain rule of differentiation, we may write (3.9) and (3.11) as:

$$w(d) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} w_i(r) \left(\frac{\partial A(d)}{\partial A_i(r)} - \frac{\partial A(d)}{\partial U_i(r)} \right),$$
(3.12)

$$w(d) = \sum_{i \in C} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} w_i(r) \left(\frac{\partial U(d)}{\partial U_i(r)} - \frac{\partial U(d)}{\partial A_i(r)} \right).$$
(3.13)

3.2. Conversion rules for the failure frequency calculation

Fairly general conversion rule that converts an availability expression of an MMS into its failure frequency expression can be described as follows. Suppose that the availability A(d) is given in the following sum of products form:

$$A(d) = \beta_0 + \sum_{k=1}^{L} \beta_k \prod_{m \in \mathbf{E}_k} G_{k,m}(\underline{\mathbf{A}}), \qquad (3.14)$$

where \mathbf{E}_k is a non-empty index set, and $G_{k,m}(\underline{\mathbf{A}})$, $m \in \mathbf{E}_k$, are functions having no common relevant variable belonging to the same system's element, i.e. if $G_{k,m}(\underline{\mathbf{A}})$ depends on the variable $A_i(r)$ (belonging to element *i*), then other functions $G_{k,l}(\underline{\mathbf{A}})$), $l \neq m$, do not depend on variables $A_i(s)$, $s \in \mathbf{K}_i - \{g_i(0)\}$. This relevant variable disjointness property relates to each product separately. We assume that $G_{k,m}(\underline{\mathbf{A}})$ are differentiable with respect to each variable (the derivatives being 0 for of non-relevant variable). By applying (3.9) to A(d) given by (3.14), and using usual algebra and calculus, we obtain:

$$w(d) = \sum_{k=1}^{L} \beta_{k} \sum_{m \in \mathbf{E}_{k}} \left(\prod_{l \in \mathbf{E}_{k} - \{m\}} G_{k,l}(\underline{\underline{\mathbf{A}}}) \right) \cdot w_{k,m} = \sum_{k=1}^{L} \beta_{k} \left(\prod_{m \in \mathbf{E}_{k}} G_{k,m}(\underline{\underline{\mathbf{A}}}) \right) \sum_{m \in \mathbf{E}_{k}} \frac{w_{k,m}}{G_{k,m}(\underline{\underline{\mathbf{A}}})},$$
(3.15)

where, by convention, a/0 = 0 for any a, and

$$w_{k,m} = \sum_{i \in \mathbf{C}} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} \frac{\partial G_{k,m}(\underline{\mathbf{A}})}{\partial A_i(r)} \cdot w_i(r).$$
(3.16)

According to (3.9), when $G_{k,m}(\underline{\mathbf{A}})$ is the availability [unavailability] to a given level of a multi-state subsystem, then $w_{k,m}$ [$-w_{k,m}$] is its failure frequency to this level, and, in the steady state, $w_{k,m}/G_{k,m}(\underline{\mathbf{A}}) = 1/\text{MUT}_{k,m}$ [$-1/\text{MDT}_{k,m}$] of that subsystem to the given level.

By a suitable choice of functions $G_{k,m}(\underline{A})$ in the above general rule, we may obtain several special cases, being multi-state generalizations of conversion rules known for binary systems. By applying (3.15) with A(d) given by (2.12), we obtain:

$$w(d) = \sum_{k=1}^{m} \beta_k \sum_{z \in C} [w_z(a(k,z)) - w_z(b(k,z))] B_k^{(z)}(\underline{\underline{A}})$$

$$= \sum_{k=1}^{m} \beta_k B_k(\underline{\underline{A}}) \sum_{i \in C} \frac{w_i(a(k,i)) - w_i(b(k,i))}{A_i(a(k,i)) - A_i(b(k,i))}$$
(3.17)

where, a/0 = 0 for any a, and

$$B_k^{(z)}(\underline{\underline{\mathbf{A}}}) = \prod_{i \in C - \{z\}} (A_i(a(k,i)) - A_i(b(k,i))).$$
(3.18)

As a very simple example, let us consider an MMS with structure function $\varphi(\mathbf{X}) = \min(X_1, \max(X_2, X_3))$. Then $\varphi_d(\underline{\mathbf{X}}) = X_1(d)[1 - (1 - X_2(d))(1 - X_3(d))]$ and:

$$A(d) = A_1(d)[1 - U_2(d)U_3(d)] = A_1(d) - A_1(d)U_2(d)U_3(d).$$
(3.19)

Using (3.15)-(3.16) yields:

$$w(d) = w_1(d)[1 - U_2(d)U_3(d)] + A_1(d)[w_2(d)U_3(d) + U_2(d)w_3(d)],$$
(3.20)

or, in steady state:

$$w(d) = A_1(d)[1 - U_2(d)U_3(d)]/MUT_1(d) + A_1(d)U_2(d)U_3(d)[1/MDT_2(d) + 1/MDT_3(d)]$$

= $A_1(d)/MUT_1(d) - A_1(d)U_2(d)U_3(d)[1/MUT_1(d) - 1/MDT_2(d) - 1/MDT_3(d)].$
(3.21)

As a more general example, observe that Shannon's decomposition formulae (2.14) for the availability A(d) are of the form (3.14). Hence, by using (3.15) with A(d) given by (2.14), we obtain at once the following **Shannon's decomposition formulae** for the failure frequency w(d):

$$w(d) = \sum_{r \in \mathbf{K}_{i}} [(w_{i}(r) - w_{i}(r \oplus_{i} 1)) A^{(i,r)}(d) + (A_{i}(r) - A_{i}(r \oplus_{i} 1)) w^{(i,r)}(d)]$$

$$= \sum_{r \in \mathbf{K}_{i}} (A_{i}(r) - A_{i}(r \oplus_{i} 1)) A^{(i,r)}(d) \left(\frac{w_{i}(r) - w_{i}(r \oplus_{i} 1)}{A_{i}(r) - A_{i}(r \oplus_{i} 1)} + \frac{w^{(i,r)}(d)}{A^{(i,r)}(d)} \right)$$

$$= w^{(i,g_{i}(0))}(d) + \sum_{r \in \mathbf{K}_{i}} A_{i}(r) (A^{(i,r)}(d) - A^{(i,r-i)}(d)) \left(\frac{w_{i}(r)}{A_{i}(r)} + \frac{w^{(i,r)}(d) - w^{(i,r-i)}(d)}{A^{(i,r)}(d) - A^{(i,r-i)}(d)} \right),$$
(3.22)

where $w^{(i,r)}(d)$ is the failure frequency to level *d* of the system with indicator structure function $\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}})$, or in other words, $w^{(i,r)}(d)$ is the failure frequency to level *d* of the system with structure function φ , given that element *i* is strapped in state *r*.

3.3. Application of dual numbers

A dual number is one of the form $a + \varepsilon b$, where a and b are real numbers and ε is an algebraic (imaginary) unit having the formal property that $\varepsilon^2 = 0$. The set \mathbb{D} of all dual numbers is a commutative ring with basic algebraic operations defined by:

$$(a + \varepsilon b) + (c + \varepsilon d) = a + c + \varepsilon (b + d), \quad (a + \varepsilon b) \cdot (c + \varepsilon d) = ab + \varepsilon (ad + bc).$$

Since $\varepsilon^2 = 0$, a pure dual number εd has no inversion, so the ring \mathbb{D} is not a field (has zero divisors). However, if $c \neq 0$, then $1/(c + \varepsilon d) = 1/c - \varepsilon d/c^2$.

The concept of the dual number was introduced by Clifford (1873) and the name was given by Study (1903). Application areas of dual numbers include kinematics, geometry, mechanics, robotics, etc. We refer to (Angeles 1998), (Dimentberg 1978), (Fischer 1999), (Veretennikov and Sinitsyn 2006) and (Yaglom 1968, 1979) for more detailed historical account, discussion of properties and applications of dual numbers.

Observe that

$$\prod_{k=1}^{n} (x_k + \varepsilon y_k) = \prod_{k=1}^{n} x_k + \varepsilon \sum_{k=1}^{n} y_k \prod_{\substack{m=1 \\ m \neq k}}^{n} x_m .$$
(3.23)

Let $f(x_1, x_2, ..., x_n) = x_1 \cdot x_2 \cdot ... \cdot x_n$, where $x_1, x_2, ..., x_n$ are real variables. By replacing each x_i by dual number $x_i + \varepsilon y_i$, we obtain dual function *F* of *n* dual numbers. According to (3.23), we have the following representation for the function *F*:

$$F(\mathbf{x} + \varepsilon \mathbf{y}) = \prod_{k=1}^{n} (x_k + \varepsilon y_k) = f(\mathbf{x}) + \varepsilon \sum_{k=1}^{n} y_k \frac{\partial f(\mathbf{x})}{\partial x_k}, \qquad (3.24)$$

where $\mathbf{x} + \varepsilon \mathbf{y} = (x_1 + \varepsilon y_1, ..., x_n + \varepsilon y_n), \mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n).$

It is easy to see that this representation holds true for dual functions F defined in the above way using a real analytic functions $f(\mathbf{x})$. In particular, it holds for polynomial, multi-linear and more general functions $f(\mathbf{x})$ defined by elementary algebraic expressions.

Let $A(d) = A(d; \underline{\mathbf{A}})$ be given in appropriate algebraic form. e.g. in the form (2.12) or (3.14). Replacing each variable $A_i(r)$ by dual variable $A_i(r) + \varepsilon w_i(r)$ in $A(d; \underline{\mathbf{A}})$ we obtain dual function $A^{\circ}(d; \underline{\mathbf{A}} + \varepsilon \underline{\mathbf{w}})$, where $\underline{\mathbf{w}} = (w_i(r): i \in \mathbf{C}, r \in \mathbf{K}_i - \{g_i(0)\})$. According to representation (3.24) and formula (3.9), we have:

$$A^{\circ}(d; \underline{\mathbf{A}} + \varepsilon \underline{\mathbf{w}}) = A(d; \underline{\mathbf{A}}) + \varepsilon w(d; \underline{\mathbf{A}}, \underline{\mathbf{w}}) = A(d) + \varepsilon w(d).$$
(3.25)

This formula leads to simple calculation method of failure frequency using dual number algebra:

- (1) write A(d) in appropriate algebraic form,
- (2) replace all $A_i(r)$ [$U_i(r)$] by dual variables $A_i(r) + \varepsilon w_i(r)$ [$1 (A_i(r) + \varepsilon w_i(r)) = U_i(r) \varepsilon w_i(r)$],
- (3) perform calculation using dual number algebra to obtain dual number $a + \varepsilon b$,
- (4) w(d) = b.

For example, if $A(d) = A_1(d)(1 - U_2(d)U_3(d))$, then

 $A^{\circ}(d; \underline{\mathbf{A}} + \varepsilon \underline{\mathbf{w}}) = \{A_1(d) + \varepsilon w_1(d)\}(1 - \{U_2(d) - \varepsilon w_2(d)\}\{U_3(d) - \varepsilon w_3(d)\})$ = $\{A_1(d) + \varepsilon w_1(d)\}\{[1 - U_2(d)U_3(d)] + \varepsilon[w_2(d)U_3(d) + U_2(d)w_3(d)]\}$ = $A_1(d)(1 - U_2(d)U_3(d)) + \varepsilon\{w_1(d)[1 - U_2(d)U_3(d)] + A_1(d)[w_2(d)U_3(d) + U_2(d)w_3(d)]\}$ = $a + \varepsilon b = A(d) + \varepsilon w(d)$; (compare with (3.20)).

4. Extension to random demand rate

Many real technical systems operate under demand randomly changing in time. Examples of such systems are power generating systems, transportation systems, distributed computer networks and production systems. We refer to Levitin (2005) and Lisnianski and Levitin (2003) for more examples and further discussion.

Let D(t) be the demand rate at time *t*. The fixed (time-independent) demand rate $D(t) \equiv d$ was considered in section 3. Now we consider randomly changing in time demand rate $\{D(t)\}$. We assume that the process $\{D(t)\}$ takes its state in finite set $\mathbf{D} \subseteq [0, \infty)$ and that it satisfies Assumptions 2.2 and 2.3. The system with performance process $X(t) = \varphi(\mathbf{X}(t))$ is operating at time *t*, if $X(t) \ge D(t)$. Otherwise the system is failed. It is assumed that the processes $\{X(t)\}$ and $\{D(t)\}$ are independent. We show how to apply the results of section 4 to the case of randomly changing demand.

Let the demand states in **D** be indexed in decreasing order:

 $0 \le d(m) \le d(m-1) \le \dots \le d(1), m \ge 1.$

Let $\mathbf{H} = \{1, 2, ..., m\}$ be the index set of demand levels and let L(t) be the index of demand level at time *t*, so that D(t) = d(L(t)). Define a function $\psi: \mathbf{H} \times \mathbf{K} \to \{0,1\}$ by:

$$\Psi(k, x) = \mathbf{1}(x \ge d(k)). \tag{4.1}$$

Since d(k) is decreasing in k, $\psi(k, x)$ is a monotone increasing binary structure, which can be considered as the structure function of a binary system consisting of two multi-state elements. The first element corresponds to the demand level index and its stochastic behaviour is described by $\{L(t)\}$. The second element corresponds to the original system with stochastic behaviour described by $\{X(t)\}$. We have:

$$\psi(L(t), X(t)) = \mathbf{1}(X(t) \ge d(L(t))) = \sum_{k=1}^{m} (L(k, t) - L(k+1, t))X(d(k), t)), \quad (4.2)$$

where $L(k, t) = \mathbf{1}(L(t) \ge k)$, L(m+1, t) = 0 and $X(c, t) = \mathbf{1}(X(t) \ge c)$. Therefore:

$$A(t) = \sum_{k=1}^{m} \Pr\{L(t) = k\} A(d(k), t) = \sum_{k=1}^{m} (A_L(k; t) - A_L(k+1; t)) A(d(k), t),$$
(4.3)

where $A_L(k; t) = \Pr\{L(t) \ge k\} = \Pr\{L(k, t) = 1\}.$

Let $w_L^{j \to l}(t)$ be the frequency of transitions of the process $\{L(t)\}$ from state *j* to state *l* at time *t* (assumed to exist). Then we can apply the factoring formula (3.22) to obtain the failure frequency w(t) and the repair frequency v(t) of the system operating under random demand $\{D(t)\}$:

$$w(t) = \sum_{k=1}^{m} (w_L(k,t) - w_L(k+1,t)) A(d(k),t) + \sum_{k=1}^{m} (L(k,t) - L(k+1,t)) w(d(k),t), \qquad (4.4)$$

$$v(t) = \sum_{k=1}^{m} (v_L(k,t) - v_L(k+1,t)) A(d(k),t) + \sum_{k=1}^{m} (L(k,t) - L(k+1,t)) v(d(k),t), \quad (4.5)$$

where

$$w_L(k,t) = \sum_{j=k}^{m} \sum_{l=1}^{k-1} w_L^{j \to l}(t), \quad v_L(k,t) = \sum_{j=1}^{k-1} \sum_{l=k}^{m} w_L^{j \to l}(t), \quad (4.6)$$

with $w_L(1,t) \equiv w_L(m+1,t) \equiv 0$ and $v_L(1,t) \equiv v_L(m+1,t) \equiv 0$.

The failure [repair] frequency of system with variable demand rate has two contributors, designated by $w^{(L)}(t)$ and $w^{(X)}(t) [v^{(L)}(t)$ and $v^{(X)}(t)]$:

1) $w^{(L)}(t)$ and $v^{(L)}(t)$ are related to failures caused by changes of demand rate, and correspond to the first sum in equations (4.4) and (4.5) respectively, and

2) $w^{(X)}(t)$ and $v^{(X)}(t)$ are related to failures caused by changes of the state of the system of elements, and correspond to the second sum in these equations.

Of course, for the steady state $(t = \infty)$, the failure and repair frequencies, and their two separate contributors as well, coincide: $w(\infty) = v(\infty)$, $w^{(L)}(\infty) = v^{(L)}(\infty)$, $w^{(X)}(\infty) = v^{(X)}(\infty)$.

Notice that the results presented in this section also include, as a special case, the random demand, which does not change in time: $D(t) \equiv D$ and consequently, $L(t) \equiv L$ (*D* and *L* are just random variables). Then all $w_L^{j\to l}(t)$ are equal to 0, and thus the demand related contributors $w^{(L)}(t) \equiv v^{(L)}(t) \equiv 0$, i.e. the first sum in each right-hand side of each equation (4.4) and (4.5) disappear.

5. Conclusions

New general formula for the failure/repair frequency of a multi-state monotone system was derived in the paper. Using this formula simple conversion rules from an availability or unavailability expression into an expression for failure/repair frequency, were obtained. Application of dual number algebra was also discussed.

As further investigations in the area, we may mention:

- 1) developing other efficient algorithms, considering, for example, application of the Universal Generating Function (UGF) technique, Levitin (2005);
- 2) considering some statistical dependencies among system's elements (e.g. common-cause failures), and between demand rate and elements performance processes as well;
- 3) obtaining approximations useful for analysing very complex and large systems;
- 4) generalisation to multi-state systems which are not necessarily monotone (thought in this case not so simple conversion rules are expected).

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