BAYES-FIDUCIAL APPROUCH FOR AIRCRAFT SPECIFIED LIFE NOMINATION

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ABSTRACT

The problem of nomination of Retirement or Specified Life (SL) of aircraft on the base of full-scale fatigue test result processing is considered. SL can be defined (1) by requirement of fatigue failure probability limitation or (2) by economics reasons. For optimization problem the Bayes-fiducial (BF) approach is offered. BF decision is always a function of sufficient statistics and, by contrast with maximum likelihood method, it is based on the use of specific loss function. For the problem of failure probability limitation in case when sufficient statistics coincides with the sample itself (for example, for Weibull distribution) usually the Monte Carlo method is used but in this paper for the distributions with location and scale parameters an analytical solution is offered.

Some numerical examples for lognormal, Weibull distributions are given.

KEYWORDS

Bayes, p-bound, prediction limit, quantile, lognormal, Weibull, optimization

G. INTRODUCTION

In this paper we consider only the case when the operation reliability of aircraft is ensured by discarding the aircraft from service, if its service life exceeded the Retirement or Specified Life (SL). For discussion of inspection program (IP) development is planned another author paper but some short discussion of this problem already take place in [1,2,3,4].

There are at least two approaches to the SL choice on the base of experimental data: (1) it can be defined by requirement of fatigue failure probability limitation and (2) it can be defined by economics reasons. If the "weight" of loss induced by fatigue failure is estimated by some value b, which can be comparable with the "income" per service hour (it will be assumed, that the value of "income" per one service hour is equal to unit), then SL can be defined as operation time, corresponding to maximum of income expectation value. We'll consider both approaches. It should be mentioned also that SL can be chosen as (1) some number from $[0,\infty]$ and as (2) some number from set of two numbers $\{0, t^*_{SL}\}$. This corresponds to (1) nomination of Specified Life, t_{SL}, and (2) rejection or acceptance of predetermined (required) Specified Life, t^*_{SL}.

2. DEFINITION OF P-SET AND P-BOUND FOR RANDOM VARIABLES

To make possible the common approach for solution of the both problem SL nomination and IP development we need to remained the p-set function definition [4]. It is a special statistical decision function, which, in fact, is generalization of p-bound for random variable, definition of which was introduced by author some early [5,6,7].

P-set function is defined in following way.

Definition 1. Let Z and X are random vectors of m and n dimensions and we suppose that it is known the class $\{ P_\theta \ , \ \theta \in \Omega \}$ to which the probability distribution of the random vector *W=(Z,X)* is assumed to belong. Of the parameter θ, which labels the distribution, it is assumed known only that it lies in a certain set Ω, the parameter space. Let $S_Z(x) = \bigcup_{i=1}^{r} S_{Z,i}(x)$ denotes some set of disjoint sets of *z* values as function of *x*. If

$$\sup_{\theta} \sum_{i=1}^{r} P(Z \in S_{Z,i}(X)) = p \qquad (1)$$

then statistical decision function $S_Z(x)$ is p-set function for r.v. *Z* on the base of a sample $x = (x_1, x_2, ..., x_n)$.

Remark. Later on the value *x*, observation of the vector *X*, would be interpreted as result of some test (for example, full-scale fatigue test of aircraft). For the problem SL nomination Z can be interpreted as some random variable equal to smallest fatigue life of N aircraft in service Z=min($Y_1, Y, ..., Y_N$). Then the problem is to find the function $\tau(x)$ for which

$$\sup_{\theta \in \Theta} P_\theta \{Z < \tau(X)\} = p.$$

For the problem of inspection planning *Z* would be interpreted as some random vector (T_d, T_c), where T_d, T_c are time moments when some fatigue crack become detectable or reaches critical size correspondingly. And in this case the problem is to find such sequence $t(x) = \{t_1(x), t_2(x), ...\}$ that

$$\sup_{\theta} \sum_{i=1}^{r} P(t_{i-1}(X) \le T_d < T_c < t_i(X)) = p \ ,$$

where $t_1(x) < t_2(x) < ... < t_r$ are time moments of inspections, t_0=0, t_r is time of aircraft odelling . The choice of odellin $t(x) = \{t_1(x), t_2(x), ...\}$ will be discussed in next paper but here we consider onle the problem of SL nomination.

For the most important case, when *m*=1 and *Z* is a random scalar, there are several useful definitions of special types of p-set functions $S_Z(x)$ which for this special case we denote by $\tau(x)$.

Definition 2. P-set function $\tau(x)$ is called a p-bound for r.v. Z if

$$\sup_{\theta \in \Theta} P_\theta \{Z < \tau(X)\} = p \ . \qquad (2)$$

Definition 3. P-bound $\tau(x)$ is called a parameter-free (p.f.) p-bound for r.v. Z if

$$P_\theta \{Z < \tau(X)\} = p \ \text{ for all parameters } \theta \in \Omega. \qquad (3)$$

Definition 4. P-bound for r.v. *Z* is called a right-hand binary (r.h.b. p-bound), if for each possible observation *x* of r.v. *X*, function $\tau(x)$ assigns only one of two decisions:

$$\tau(x) = -\infty \ \text{ if } x \in S; \ \tau(x) = \tau^*, \ \text{ if } x \in S^*, \qquad (4)$$

where τ^* is some number, S^* and *S* are two complementary regions of the sample space .

So we see that the definition of p-bound can be considered as some generalization of definition of prediction limit. But it is some statistical decision function which cover both prediction limit and, in some may, testing statistical hypotheses.

We can say also that p.f. p-bound $\tau(x)$ is a p-quantile estimate of cdf $F_Z(x)$ and, as function of p, it is an estimate of inverse cumulative distribution function $F_Z^{-1}(p)$, but very specific estimate: expectation value $E(F_Z(\tau(X))) = p$.

If $Z=Y_{(k)}$ is kth order statistic of independent observations taken on Y, say Y_1, Y_2, ... , Y_m , and strictly increasing c.d.f. of r.v. Y , $F_Y(x, \theta)$, has the same unknown parameter θ as the cdf of X_i , $i=1,2,...,n$, k = [βm], where $0<\beta<1$, [x]-is a maximum integer less or equal to x, $m \to \infty$ then approximately

$$P\{Y_{(k)} < \tau(X)\} = P\{F_Y^{-1}(\beta) < \tau(X)\} = P\{F_Y(\tau(X)) > \beta\} = p$$

and $(-\infty, \tau(x))$ *is* β - content tolerance region at confidence level p.

The binary p-set function has, evidently, some close connection with testing statistical hypotheses: S^* and S are two complementary regions of the sample space just as S_0 and S_1 in the problem of hypotheses testing [6]. But there is some difference. Instead of problem to maximize the power of a test at a fixed level of significance (probability of first type of error) this time we need to get the maximum of probability of decision that reliability requirements are met at the fixed limitation of product of the probability of failure and probability of wrong decision (we think that reliability requirements are met but they are not met):

$$\sup_{\theta} P(Z \le \tau^*) P(X \in S^*) \le p \ .$$

3. P-BOUND FOR DISTRIBUTION WITH LOCATION AND SCALE PARAMETERS

It is easy to get $\tau(x)$ for distribution with location and scale parameters. As the main application of the problem under question we'll consider a problem of SL nomination for some fatigue-prone airframe structure. We suppose to have observations of fatigue lives of some identical units of this structure as a result of full-scale fatigue tests. Usually for fatigue life data processing both a lognormal and Weibull distributions are used. If we'll use logarithm scale (if we'll use $X = \ln(T)$ instead of T) then both these distributions become distributions with location and scale parameters. So we can say, that r.v. X has following structure: $X=\theta_0+\theta_1 \overset{0}{X}$, where θ_0, θ_1 are unknown parameters, r.v. $\overset{0}{X}$ has either standard normal c.d.f. $F_{\overset{0}{X}}(x) = \Phi(x)$ or standardized smallest extreme value (sev) c.d.f. $F_{\overset{0}{X}}(x) = 1 - \exp(-\exp(x))$ for lognormal or Weibull distributions of T correspondingly. For this case for the specified life nomination problem following theorem can be used (we give it without proof).

Theorem 1. Let

$$F_{X_i}(x,\theta) = F_{\overset{o}{X}}\left(\frac{x-\theta_0}{\theta_1}\right), \mathrm{i} = 1,...,\mathrm{n}, \qquad F_Z(x,\theta) = F_{\overset{o}{Z}}\left(\frac{x-\theta_0}{\theta_1}\right), \tag{5}$$

where $F_{\overset{0}{X}}(\cdot)$, $F_{\overset{0}{Z}}(\cdot)$ are known c.d.f. of $\overset{0}{X}$, $\overset{0}{Z}$, θ_0, θ_1 – are unknown location and scale parameters. And let the random variables, estimations of θ_0, θ_1, as function of $X = (X_1, X_2, ..., X_n)$ can be described by the similar structural formulas:

$$\hat{\theta}_0 = \theta_0 + \theta_1 \overset{o}{\theta}_0, \quad \hat{\theta}_1 = \theta_1 \overset{o}{\theta}_1, \tag{6}$$

where $\overset{o}{\theta}_0, \overset{o}{\theta}_1$ - are random variables, corresponding to the estimates of θ_0, θ_1 using a sample of the same size n but when $\theta_0=0$, $\theta_1=1$. We refer to this type of estimates as "correct" estimates.

Then p.f. and r.h.b. p-bounds are described accordingly by formulae

$$\tau_1(x) = \hat{\tau}_1, \quad \tau_2(x) = \begin{cases} -\infty, \ \hat{\tau}_2 \le \tau^*, \\ \tau^*, \ \hat{\tau}_2 > \tau^*, \end{cases} \tag{7}$$

where $\hat{\tau}_i = \hat{\theta}_0 + t_i \hat{\theta}_1, \ i = 1,2,$

t_1 is p-quantile of r.v. $V_Z = (\overset{0}{Z} - \overset{0}{\theta}_0)/\overset{0}{\theta}_1$, t_2 is the root of equation : $\xi(t)=p$,

$$\xi(\mathrm{t}) = \sup_{\mathrm{c}} F_{\overset{o}{Z}}(c)(1 - F_{\overset{o}{\tau}(t)}(c)) = \sup_{\mathrm{c}} F_{\overset{o}{Z}}(c) F_{V_C}(t),$$

$$\overset{\mathrm{o}}{\tau}(t) = \overset{o}{\theta}_0 + \overset{o}{\theta}_1 t, \quad \mathrm{V_C} = (c - \overset{o}{\theta}_0)/\overset{o}{\theta}_1.$$

2. If one of the parameters θ_1 or θ_0 is known, then, as usually, we can transform the initial data ($x'_i = x_i/\theta_1$ *or* $x'_i = x_i - \theta_0$, $i \in 1,...,n$) in such a way that in previous formulae for τ we can put $\hat{\theta}_1 = 1$ *or* $\hat{\theta}_0 = 0$, and then

2.1 If it is known that the scale parameter $\theta_1 = 1$ then V_Z, V_C should be replaced by

$$U_Z = \overset{o}{Z} - \overset{o}{\theta}_0, \ \ U_C = C - \overset{o}{\theta}_0;$$

function $\xi(t)$ should be replaced by the function $\xi_1(t) = \max_C F_{\overset{0}{Z}}(C) F_{U_C}(t)$,

but for $X = \min(X_1, X_2,..., X_n)$ by function $\xi_1^1(t) = \max_C F_{\overset{0}{Z}}(C)(1 - F_{\overset{0}{X}_1}(t))^n$.

2.2. If it is known that the location parameter $\theta_0 = 0$ then V_Z, V_C should be replaced by

$$W_Z = \overset{o}{Z} / \overset{o}{\theta}_1, \ \ W_C = C / \overset{o}{\theta}_1,$$

function $\xi(t)$ by the function $\xi_0(t) = \max_C F_{\overset{o}{Z}}(c) F_{W_C}(t)$, but if additionally

$X = \min(X_1, X_2,..., X_n)$ by function $\xi_0^1(t) = \max_C F_{\overset{o}{Z}}(c)(1 - F_{\overset{0}{X}}(C/t))^n$.

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Let us remind that for the purpose of approximate calculation of c.d.f. for V_C, U_C, W_C the Monte Carlo method can be used or normal approximation of distributions of estimations $\overset{o}{\theta}_0, \overset{o}{\theta}_1$.

4. APPLICATION OF P-BOUND TO THE PROBLEM OF THE SPECIFIED LIFE NOMINATION

4.1. OPTIMALITY CRITERION FOR P.F. P-BOUND USED FOR AIRCRAFT SPECIFIED LIFE NOMINATION

Now we turn to a discussion of some preference orderings of decision procedures : choice of function $\tau(x)$. In framework of theorem 1 it is really the choice of estimates $\hat{\theta}_0$, $\hat{\theta}_1$ and risk function. Let $X = (X_1, X_2,..., X_n)$, where X_i, $i = 1,...,n$, are fatigue lives of aircraft in (full-scale) laboratory test, $Z = \min(Y_1, Y_2,..., Y_m)$,where Y_j, $j = 1,...,m$, are fatigue lives of aircraft in operation,

$F_{X_i}(t) = F_{Y_j}(t)$, $i = 1,...,n$, $j = 1,...,m$; *p*- allowed probability of failure in operation of at least one aircraft.

In application to the problem of required SL confirmation, when τ^* is required SL, we are interested in increasing of probability that $\tau(x) = \tau^*$. It is something similar to increasing of power of some test in testing some statistical hypothesis.

In application to the problem of some SL nomination we should get the maximum of expectation value of $\tau(X)$ provided that reliability requirements are met, it is if $\tau(X)$ is a *p*-bound for *Z*. To study the optimality of $\tau(x)$ we can use the Jensen's inequality. This inequality say that the function of complete sufficient statistic, which is unbiased estimation of its own mathematical expectation, provides the minimal risk if the correspondent loss-function is convex. Consider the simplest case, when θ_1 is known parameter. Let $\theta_t = \theta_0 + t\theta_1$ is some quantile. Random variable $\hat{\theta}_t = \tau(x) = \hat{\theta}_0 + t\theta_1$ is unbiased estimate of its own expectation (which in general case does not equal to θ_t). In problem under question the function $F_Z(\tau)$ can be considered as the loss-function. Then the expectation $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X))$ is the risk function. For normal and sev distributions of Y_j $j = 1,...,m$, $F_Z(\tau)$ is convex (and increasing one) if its value is small enough and we have minimum of $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X)) = p$ at the fixed expectation value of $\hat{\theta}_t = \tau(X)$, if $\tau(x)$ is a function of sufficient statistic. And, on the contrary, if $\tau(x)$ is a function of sufficient statistic and $P(Z < \tau(X)) = p$ then we have maximum of expectation value of $\tau(X)$ if p is small enough and probability $P(\tau(X) < c)$ is high enough for such *c*, that $F_Z(z)$ is convex if $z < c$. For example, for normal distribution $\Phi(z)$ is convex if $z<0$. The generalization of the Jensen's inequality for the case of multivariate sufficient statistic can be found in [9].

For the case when sufficient statistic coincides with the sample itself (for example,Weibull or smallest odelli value (sev) distribution) usually for prediction interval the Monte Carlo (MC) method is used [10]. Here we show that for the problem of p.f. p-bound, $\tau(x)$, calculation analytic solution can be found using Bayes-fiducial (BF) approach.

4.2. BAYES-FIDUCIAL APPROACH

This approach was offered in 1973 (see [5,6,7,8]). It was shown that using this approach we can get Pitmen's estimates of location and scale parameters and most powerful invariant test for testing statistical hypotheses ($H_0 : F(x) = F_0((x-\theta_0)/\theta_1)$; $H_1 : F(x) = F_1((x-\theta_0)/\theta_1)$). It can be used also for unbiased estimation. BF estimate, $\tau(x)$, of some function of parameter $\tau(\theta)$ is a function, which minimizes BF risk

$$\rho_{BF}(\tau_\theta, \tau_X) = \int L(\tau_\theta(\theta), \tau_X(x)) dF_{\tilde{\theta}}(\theta) ,$$

where $L(\tau_\theta(\theta), \tau_X(x))$ is loss function, $F_{\tilde{\theta}}(\theta)$ is fiducial distribution on parameter space [5,6].

There two advantages of BF approach:

1. As in a case of using a maximum likelihood (ML) estimates BF solution is always a function of sufficient statistics, but in contrast to ML the BF solution take into account the loss function.

2. We do not need to have a priori distribution of unknown parameters.

4.3. USING BF APPROACH FOR P.F. P-BOUND CALCULATION

Let the problem is to estimate p-quantile $\tau_p(\theta)$ for cdf $F_Z((x-\theta_0)/\theta_1)$ and loss function $L(\tau_\theta(\theta),\tau_X(x)) = (F_Z((\tau_p-\theta_0)/\theta_1) - F_Z((\tau_X(x)-\theta_0)/\theta_1))^2 = (p - F_Z((\tau_X(x)-\theta_0)/\theta_1))^2$ when we have sample $x=(x_1,x_2,...,x_n)$ from cdf $F_X((x-\theta_0)/\theta_1)$.

Let us denote by $\tau_X(x,p)$ the solution of BF equation , corresponding to the considered loss function

$$E_{\tilde\theta}\{F_Z((\tau_X(x,p)-\tilde\theta_0)/\tilde\theta_1\} = p, \tag{8}$$

where $\tilde\theta = (\tilde\theta_0,\tilde\theta_1)$, r.v. $\tilde\theta_0$, $\tilde\theta_1$ have fiducial distribution. Here $E_X(f(X))$ is expected value of $f(X)$ in accordance with cdf of *X*.

We can simplify solution of Eq.8. Instead of vector $x=(x_1,...,x_n)$ without loss of information we can consider vector $\varpi = (\hat\theta_0,\hat\theta_1,w_1,...,w_{n-2})$, where $\hat\theta_0,\hat\theta_1$ are correct parameter estimates (see (6)), $w_i = (x_i-\hat\theta_0)/\hat\theta_1$, $i=1,...,n$ -2 Then conditional fiducial distribution (at the fixed invariant $(w_1,...,w_{n-2})$) of random variables $\tilde\theta_0,\tilde\theta_1$ is defined by equation [5,6]

$$f_{\tilde\theta_0,\tilde\theta_1|w_1,...,w_n}(s_0,s_1) = h\frac{\hat\theta_1^{\,n-1}}{s_1^{n+1}}\prod_{i=1}^n f\left(\frac{\hat\theta_0+\hat\theta_1 w_i - s_0}{s_1}\right)ds_0 ds_1,$$

where h is just normalization factor. (Note: w_{n-1},w_n , $w_i=(x_i-\hat\theta_0)/\hat\theta_1$, are functions of vector ϖ).

If in (8) we use new notations:

$$U_0 = (\hat\theta_0 - s_0)/\hat\theta_1,\quad U_1 = \hat\theta_1/s_1,\ \overset{0}{\tau}(x,p) = (\tau(x,p)-\hat\theta_0)/\hat\theta_1$$

then instead of (8) we get equation

$$E_{W_1,...,W_n}E_{U_0U_1|w_1,...,w_n}\left(F((\overset{0}{\tau}(x,p)-U_0)/U_1)\right) = p. \tag{9}$$

where random variables U_0,U_1 has conditional pdf

$$f_{U_0,U_1|w_1,...,w_n}(u_0,u_1) = h_w u_0^{n-2}\prod_{i=1}^n f(u_0+w_i u_1), \tag{10}$$

where h_w is just normalization factor which depends only on invariant vector $w=(w_1,...,w_{n-2})$.

If $\overset{0}{\tau}(x,p)$ is solution of the equation

$$E_{U_0U_1|w_1,...,w_n}\left(F((\overset{0}{\tau}(x,p)-U_0)/U_1)\right) = p \tag{11}$$

then

$$\tau_X(x,p) = \hat\theta_0 + \overset{0}{\tau}(x,p)\hat\theta_1 \tag{12}$$

is solution of Eq. (9) and Eq.8 because equation (11) takes place for every vector $w=(w_1,...,w_{n-2})$, cdf of which does not depend on $\theta=(\theta_0,\theta_1)$. So if (11) is true then (8) is true also.

It is very important that $\overset{0}{\tau}(x,p)$ in (11) does not depend on value of $\theta=(\theta_0,\theta_1)$ and for solution of this equation we can set $\theta_0=0$, $\theta_1=1$. If $\hat{\theta}_0,\hat{\theta}_1$ have the structures defined by (6) then probability $P(Z<\tau(X,p))$ does not depend on $\theta=(\theta_0,\theta_1)$ and we can find p_1 for which

$$P(Z<\tau(X,p_1))=p.$$

So $\tau_X(x,p_1)$ is p-bound for random variable Z.

As is easy to see (see p.84 in [6]) the pdf (8.b) is conditional pdf of $\hat{\theta}_0,\hat{\theta}_1$ at the fixed $w=(w_1,...,w_{n-2})$ for the case when $\theta_0=0$, $\theta_1=1$. This means that the values of p_1 and p coincide.

It is very important also that result does not depend on the choice of the type of correct statistics $\hat{\theta}_0,\hat{\theta}_1$ (see (14.a) and (14.b)), because vector $x=(x_1,...,x_n)$ and vector $\varpi=(\hat{\theta}_0,\hat{\theta}_1,w_1,...,w_{n-2})$ have one-one mapping at any choice of correct statistics.

H. Example 1. P-bound for lognormal distribution

Let r.v. T has a lognormal distribution and $t=(t_1,t_2,t_3)$=(45 952, 54 143, 65 440) is the sample from the same distribution. Then r.v. $X=\log(T)$ has a normal distribution $N(\theta_0,\theta_1^2)$ and $x=(x_1,x_2,x_3)=$ (10.735 10.899 11.089) is the sample from this distribution. The problem is to calculate the p.f. p-bound for independent r.v. $Z=\min(Y_1,...,Y_m)$, where r.v. Y_i, $i=1,...,m$, has the normal distribution $N(\theta_0,\theta_1^2)$ also. We consider here only the case, when m=1, because for this case there is general analytical solution (see, for example p. 172 in [6])

$$\tau(x)=\hat{\theta}_0+\hat{\theta}_1 t_{n-1,p}(1+1/n)^{1/2}, \tag{13}$$

where

$$\hat{\theta}_0=\bar{x}\ ,\ \hat{\theta}_1=(\sum(x_i-\bar{x})^2/(n-1))^{1/2}$$

are estimates of expected value and standard deviation, $t_{k,q}$ is q-quantile from Student's distribution with k degree of freedom. So we can make comparison of this solution with the solution which we get, using new approach.

For considered data, using equation (13) for p=0.01 we calculate t_{St}=exp($\tau(x)$) = 13 162, which is the value of p-bound for r.v. T on the base of observations (t_1,t_2,t_3) .

Now let us consider the new approach. For normal distribution the conditional pdf has following form

$$f_{U_0,U_1|w_1,...,w_n}(u_0,u_1)=h_w u_0^{n-2}\prod\nolimits_{i=1}^n \varphi(u_0+w_i u_1),$$

where $\varphi(x)=\exp(-x^2/2)/(2\pi)^{1/2}$. After transformation the equation (11) has the following form

$$1-a(\overset{0}{\tau},\bar{z},D_z)/\Gamma((n-1)/2)=p,$$

where

$$a(\overset{0}{\tau},\bar{z},D_z)=\int_0^\infty u^{(n-3)/2}\exp(-u)\Phi\left((2u/D_z(n+1))^{1/2}(\bar{z}-\overset{0}{\tau})\right)du,\quad \bar{z}=\sum_1^n z_i/n,$$

$$D_z=\sum_{i=1}^n(z_i-\bar{z})^2/n,$$ $\Gamma(\cdot)$ is gamma function, , $\Phi(\cdot)$ is cdf of standard normal distribution.

Consider two types of statistics $\hat{\theta}_0,\hat{\theta}_1$, which for considered data has following values:

a) $\hat{\theta}_0 = \bar{x}$ =10.908, $\hat{\theta}_1 = (\sum (x_i - \bar{x})^2 /(n-1))^{1/2}$ =0.177 , (14. a)

b) $\hat{\theta}_0 = x_{1,n}$ =10.735, $\hat{\theta}_1 = x_{n,n} - x_{1,n}$ =0.354, (14. b)

where $x_{i,n}$ is ith order statistic of vector $x = (x_1,..., x_n)$.

In case a) we have $\overset{0}{\tau} =$ -7.889, in case b) we have $\overset{0}{\tau} =$ - 3.560.

Corresponding values of p-bound for r.v. T on the base of observations (t_1, t_2, t_3) are:

t_a = exp($\tau(x)$) = 13 523, t_b = exp($\tau(x)$) = 13 050.

It seems that the difference between t_a, t_b and t_{St} =13 162 is produced only by the problem to get required calculation accuracy.

I. Example 2. P-bound for Weibull distribution

Let we have the same sample $t=(t_1, t_2, t_3)$=(45 952, 54 143, 65 440) or $x=(x_1, x_2, x_3)$= (10.735 10.899 11.089) but r.v. T has a Weibull distribution and, correspondingly $X = \log(T)$ has distribution of smallest extreme value with cdf $F_X(x) = 1 - \exp(-\exp((x-\theta_0)/\theta_1)$. In this case the equation (11) has following form

$$1 - a(\overset{0}{\tau}, \bar{z}, D_z)/b(\bar{z}, D_z) = p \,,$$

where

$$a(\overset{0}{\tau}, \bar{z}, D_z) = \int_0^\infty u^{(n-2)} \Big(\exp(-u \sum_{i=1}^n z_i)/(\sum_{i=1}^n \exp(uz_i) + m \exp(u \overset{0}{\tau}))^n \Big) du \,,$$

$$b(\bar{z}, D_z) = \int_0^\infty u^{(n-2)} \Big(\exp(-u \sum_{i=1}^n z_i)/(\sum_{i=1}^n \exp(uz_i))^n \Big) du \,,$$

$$\bar{z} = \sum_1^n z_i / n \,, \quad D_z = \sum_{i=1}^n (z_i - \bar{z})^2 / n \,.$$

For m=1, p=0.01, using statistics (14. a) we get $\overset{0}{\tau}$ = -11.929, using statistics (14. b) we get $\overset{0}{\tau}$ = -5.424. Corresponding values of p-bound for r.v. T on the base of observations (t_1, t_2, t_3) are: t_a = exp($\tau(x)$) = 6 616, t_b = exp($\tau(x)$) = 6 752.

For m=500, p=0.2 using statistics (14.a) we get $\overset{0}{\tau} =$ -12.889, using statistics (14.b) we have $\overset{0}{\tau} =$ - 5.970. Corresponding values of p-bound for r.v. T on the base of observations (t_1, t_2, t_3) are: t_a = exp($\tau(x)$) = 5 584, t_b = exp($\tau(x)$) = 5 568.

Again, it seems that the difference between t_a and t_b is produced only by the problem to get required calculation accuracy.

Considered data really was considered in several papers and for m=500, p=0.2 Lowless (1973) obtaind prediction limit of 5623, Mee and Kushary (1994) – 5225. The Mann and Saunders (1969) result was only 766. For these calculation the Monte Carlo method was used [10].

5. USING BAYES-FIDUCIAL METHOD FOR SL NOMINATION WITH ECONOMICS OPTIMALITY CRITERION

Let the income of aircraft successful service during time t is equal to t but in case of failure the loss is equal to some negative value –b, where b is some large positive value. Then income of one aircraft service, r.v. U, is defined by formula

$$U = \begin{cases} t_{SL}, & if\ T > t_{SL}, \\ T - b, & if\ T \le t_{SL} \end{cases},$$

where T is random fatigue life, t_{SL} is some SL.

Expectation value of U

$$u(t_{SL}, \theta, b) = \int_0^{t_{SL}} (t-b)\, d F_T(t,\theta) + t_{SL}(1 - F_T(t,\theta))$$

where $F_T(t,\theta)$ is c.d.f. of T.

In general case maximum of $u(t_{SL}, \theta, b)$ is reached at t^*_{SL}, which is the root of the equation

$$b f_T(t)/(1 - F_T(t,\theta)) = 1 .$$

For normal distribution of X=ln$T$ it can be written in following way

$$\theta_0 = t^*_{SL} - \theta_1 \lambda^{-1}(t^*_{SL}\theta_1/b),$$

where $\lambda(z) = \varphi(z)/(1-\Phi(z))$ is failure rate function for standard normal distribution, $\lambda^{-1}(.)$ is inverse function. This equation allows very easy to get θ_0 as function of t^*_{SL} at the fixed θ_1 and then to find the inverse function:

$$t^*_{SL} = S^*(\theta_0, \theta_1, b).$$

For b=346 000, θ_1=0.346 and θ_0=9.948 we have: t^*_{SL}=7936 (flights) . It is interesting to note that this value corresponds to the failure probability equal to 0.0026. This can be interpreted in following way. The failure of 2.6 aircraft (in flight) from 1000 aircraft can be considered as equivalent to the loss of 346000 hours of service time or loss of 346000/7936 = 43.6 aircraft (on the ground) of this types (the value t^*_{SL} = 7936 can be considered as the price of one aircraft of this type). Or in other words, failure of one aircraft (in flight) is equivalent to loss of 43.6/2.6 (approximately 16) aircraft of the same type (on the ground).

But we do not know parameters of c.d.f. of T and should estimate them using fatigue test data. Usually maximum likelihood estimate is considered as most appropriate. We show here that for considered problem the offered by outhor Bayes-fiducial approach is much more appropriate.

In accordance with Bayes approach the parameter θ_0 is r.v.. For the case of airframe it can be interpreted in following way. Design stress analysis of an airframe should meet some standard requirements (FAR, ...). These requirements in fact define only some mean value of θ_0 but of course, in every case there are some “occasional mistakes” and we have some specific (random) value of θ_0 for every aircraft type. And then there is a scatter of r.v. X (specific random fatigue life of some specific aircraft) at this random θ_0. The parameter θ_1 is function of technology level, and if one is not changed, then the parameter θ_1 is not changed also. So we suppose that θ_1 is known constant but θ_0 is random variable, $\tilde{\theta}_0$. Let $\pi(\theta_0)$ is a priory distribution density for $\tilde{\theta}_0$. Then c.d.f. of r.v. X will be

$$\tilde{F}_X(x) = \int_{-\infty}^{\infty} F_X((x-\theta_0)/\theta_1)\pi(\theta_0)d\theta_0 .$$

It is well known, that if θ_1 is constant but r.v. $\tilde{\theta}_0$ has normal distribution with known both mean τ_0 and standard deviation τ_1 then distribution of X will be again normal with mean τ_0 and standard deviation $((\tau_1)^2+(\theta_1)^2)^{1/2}$. In this case t_{SL} again will be defined by equation (1), but θ_1 should be replaced by $\theta_{1\tau}=((\tau_1)^2+(\theta_1)^2)^{1/2}$.

In fact we do not know a priori distribution of $\tilde{\theta}_0$. For this case it is offered FB approach. Instead of posterior distribution of $\tilde{\theta}_0$ we offer to use already mentioned fiducial distribution [5]. In considered case fiducial distribution of $\tilde{\theta}_0$ again is normal with mean $\bar{x}$ and standard deviation $\theta_1/n^{1/2}$. Then for the purpose of calculation t_{SL} we again can use the same equation (1), but θ_0, θ_1 should be replaced by $\hat{\theta}_0=\bar{x}$ and $\theta_1(1+1/n)^{1/2}$ correspondingly. So using sample $x=(x_1,...,x_n)$, result of full-scale fatigue test, in case of ML approach the nominated SL is equal to $S^*(\bar{x},\theta_1,b)$, but for BF approach $t_{SL}(x)=S^*(\bar{x},\theta_1(1+1/n)^{1/2},b)$. By the use of Monte Carlo method for θ_0 = 9.948 , θ_1=0.346, b=346,000 we have got that the expectation value of r.v. U_X is equal to 2310, 4122, 5571, 6904 for BF approach but it is equal to -8624, 809, 4422, 6935 for ML approach for the same sample sizes n = 1, 2, 4, 100 . We see that for small n the expectation value of r.v. U_X is much more for BF than for ML approach.

SUMMARY

BF approach for the specified life nomination using time test data is considered for both cases: probability of failure limitation and for the maximum of expected value of some specific function of preference (minimum of expected value of specific loss function).

BF approach has following advantages:

1. As in a case of using a maximum likelihood (ML) estimates BF solution is always a function of sufficient statistics, but in contrast to ML the BF solution take into account the loss function.
2. We do not need to have a priori distribution of unknown parameters.

It is given approximate analytical solution of the problem to get the maximum of expected value of SL. In case of economics optimality criterion it is shown also that for considered type of loss function the BF approach is more preferable than direct use of ML estimates. Numerical examples are provided

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