CONSTRUCTION OF ACCEPTABILITY REGIONS FOR PARAMETRIC RELIABILITY OPTIMIZATION

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Abstract

The problem of representation and analysis of analog technical devices and systems acceptable regions is introduced. This problem occurs during designing and controlling in view of parametric dithering. The algorithms of constructing circumscribed parallelepiped, representation of acceptable region as a set of non-overlapping parallelepipeds are offered. The algorithm of acceptable region centre of mass computation is offered as the example of utilizing the acceptable region representation.

J.   Introduction

The engineering system parameters are subject to random variations and the variations may be considered as non-stationary stochastic processes. The conventional methods for choosing parameters (parametric synthesis) generally do not take account of parameters field deviations from their design values. As a result the engineering systems designed in such a manner are not optimal in the sense of their gradual failure reliability.

Construction of acceptable region is necessary for solving problems usually arising in reliability theory. The following topics are among them:

•   evaluation of the system's working capacity with account of parameters deviations from their design values,

•   evaluation of parameters sensitivity and emphasizing the key input parameters,

•   assigning tolerance values for parameters,

•   choosing the most fit (optimal) nominal values for parameters of system components.

•   reducing calculation time for parametric synthesis problem when the characteristics of random processes X(t) of system parameters variations are known [1].

The acceptable region construction in the space of input parameters is the one of steps in parametric synthesis solution. The task often can be cumbersome due to large dimension of the varying parameters space.

Most of known acceptable region construction methods can be applied on 3D space only. Some methods are efficient for a priori convex and simple connected regions. Other methods need information about form and orientation of the acceptable region.

2. Parametric reliability optimization problem

Suppose that we have a system which depends on a set of n input parameters $x=(x_1,\ldots,\ x_n)$. We will say that system is acceptable if $Y(x)$ satisfy the conditions (1):

$$a \leq Y \leq b \qquad (1)$$

where Y, a and b are m-vectors of system responses (output parameters) and their specifications, e.g. $Y_1(x)$ – average power, $Y_2(x)$ –delay, $Y_3(x)$ – gain.

The inequalities (1) define a region D_x in the space of input (system) parameters

$$D_x = \{\mathbf{x} \mid a \leq Y \leq b\} \tag{2}$$

D_x is called the region of acceptability for the system. Figure 1 illustrates such a region. The values given for a and b are the specifications for the system.
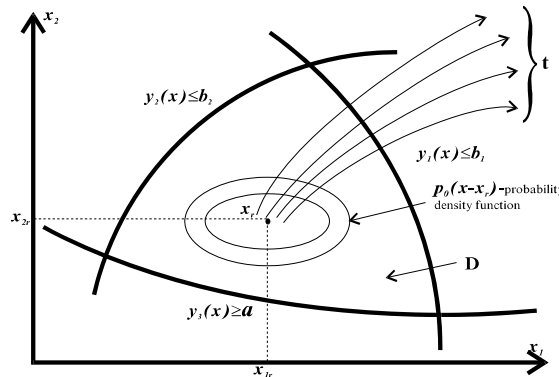
Fig. 1. Region of acceptability D_x defined by system response functions

The engineering system parameters are subject to random variations (aging, wear, temperature) and the variations may be considered as stochastic processes:

$$X(t) = \{X_1(t), ..., X_n(t)\}$$

In general parametric optimization (optimal parametric synthesis) problem can be stated as follows [1].

Given the characteristics of random processes $X(t)$ of system parameters variations, conditions of acceptability (1) and a service time T, find such a deterministic vector of parameter ratings (nominal values) $x_r = (x_{1r}, ..., x_{nr})$ that the reliability

$$P_r(\mathbf{x}_r, T) = P_r\{[X_1(t), X_2(t), ..., X_n(t)] \in D_x, \forall t \in [0, T]\} = \max \tag{3}$$

was maximal.

The practical algorithm of the stochastic criterion calculation is based on the conventional Monte Carlo method.

At the beginning, the random vector of parameters is generated (this vector means random manufacturing device realization), and then the internal parameters degradation is simulated using degradation model. For example, parameters variations can be approximated as follows

$$X(t) = \sum_{k=0}^{m} x_r u_k(t)$$

where x_k is a random variable; $\{u_k(t)\}^m_{k=0}$ are continuous deterministic functions of time.

The Monte Carlo method approximates $P_r(x_r, T)$ by the ratio of number of acceptable realizations (falling in region D_x)-N_a to the total number of trials – N.

$$P_r = {N_a}/{N},$$

Unfortunately, often the region D_x is unknown. It is given only implicitly through system's equations and the system response functions. If we do not know the region D_x a Monte Carlo evaluation of probability $P_r(x_r, T)$ at particular nominal value x_r requires N system analyses for each trial set of parameter x_r. Typically, hundreds of trials are required to obtain a reasonable estimation for $P_r(x_r, T)$.

Optimization requires evaluation of the probability $P_r(x_r, T)$ for many different values of parameters nominal values x_r.

The acceptable region constructing problem can be stated as follows.

The first step in constructing acceptable region is narrowing the search area in a space of internal parameters. A circumscribed parallelepiped is constructed for this purpose. The next step consists in approximation of D_x with a set of non-overlapping n-dimensional parallelepipeds within circumscribed parallelepiped.

3. Circumscribed box Construction

3.1. Definition of circumscribed box

Let information about variations of internal parameters is given as limits of their possible values, i.e.

$$x_{i\min} \leq x_i \leq x_{i\max}, i = \overline{1,n} \tag{5}$$

The region inside of space of internal parameters is a n-dimensional orthogonal parallelepiped called tolerance box B_d:

$$B_d = \{\mathbf{x} \in R^n \mid x_{i\min} \leq x_i \leq x_{i\max}, i = \overline{1,n}\} \tag{6}$$

with the volume
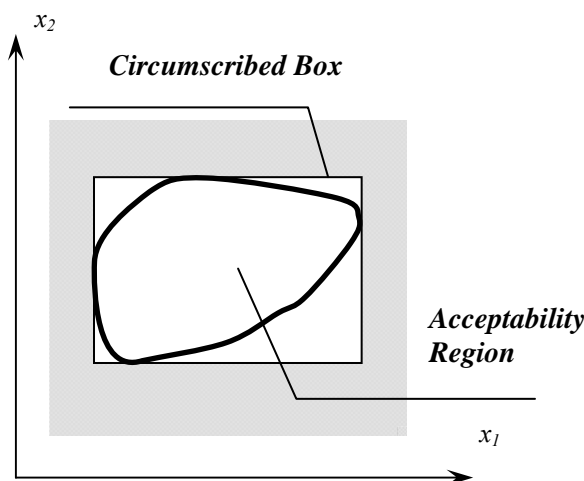
$$V_d = \prod_{i=1}^{n} (x_{i\max} - x_{i\min}) \tag{7}$$

The box B_0 circumscribed about acceptability region $D_{\mathbf{x}}$ is the n-dimensional orthogonal parallelepiped [2] within the tolerance box B_d.

$$B_o = \{\mathbf{x} \in R^n \mid a_i^0 \leq x_i \leq b_i^0 \ \forall i = \overline{1,n}\} \tag{8}$$

The circumscribed box's volume is represented as

$$V_0 = \prod_{i=1}^{n} (b_i^0 - a_i^0), \tag{9}$$

where $a_i^0 = \min_{\mathbf{x} \in D_x} x_i$, $b_i^0 = \max_{\mathbf{x} \in D_x} x_i$.

Fig. 2: The circumscribed box

The construction of the circumscribed box may narrow the area for construction of the acceptable region D_x and decreases computational cost of stochastic estimations. There's no need to

perform expensive scheme's simulations at the points outside the circumscribed box because the conditions (1) are not satisfied at those points.

The construction of circumscribed box and acceptability region's volume with the method of statistical testing is similar to computation of multiple integral with Monte-Carlo method.

The algorithm of circumscribed box construction

Let the tolerance box's boundaries (5) are known and the conditions of acceptability (1) are given.

The task consists in overlapping of the tolerance region with uniform points and testing satisfaction of the condition (1) at the each point.

The random point coordinates is a set of uniform numbers $\xi = (\xi_1,...,\xi_n)$ distributed on

$$x_{i\min} \le \xi_i \le x_{i\max}, i = \overline{1,n}.$$

To find components of the vectors $\mathbf{a}^O, \mathbf{b}^O$ we set the initial values $a_i^O = x_{i\max}$ and $b_i^O = x_{i\min}$, $i = \overline{1,n}, N_g = 1$. N_g is amount of the points where conditions (1) are satisfied.

An acceptability of the system (satisfaction of the conditions (1)) is checked at the every test. Depending on result of the test, the value of the indicator function $g(\xi)$ is set.

$$g(\xi) = \begin{cases} 1, \textit{if conditions (1) are satisfied} \\ 0, \textit{if conditions (1) aren't satisfied} \end{cases}.$$

If acceptability conditions (1) are satisfied at the point ξ the following values are set: $a_i^O = \min(a_i^O, \xi_i)$, $b_i^O = \max(b_i^O, \xi_i)$, $i = \overline{1,n}$, $N_g = N_g + 1$.

Having performed N point simulations, we compute acceptability region's volume estimation:

$$\hat{V}_g / V_O = \frac{1}{N} \sum_{j=1}^N g(\xi^j) \tag{10}$$

or

$$\hat{V}_g = V_O \frac{N_g}{N}, \tag{11}$$

where V_O - the circumscribed box's B_O volume, N_g - amount of points inside of the acceptability region.

This method enables us to solve several problems of practical importance:
a)   Region D_x is multiply connected region.
b)   Region D_x fall outside the limits of the tolerance box B_d.
c)   Region D_x include the tolerance box ($B_0 \in D_x$, $N_g = N$).
d)   Region D_x and tolerance box B_d are mutually disjoint ($D_x \cap B_0 = 0, N_g = 0$).

This method can be easily implemented on parallel computers with MPP architecture. The main condition for result accuracy is absence of interprocessor correlations [2, 3]. In the beginning of the process, there is an initialization of parallel random-number generator, then each process is initialized with the amount of tests to perform. Upon termination of calculations each of the processes passes computed ratings to the main processor which forms a final rating.

Following the given necessary tests amount N and processors amount k, necessary tests amount m is assigned to each processor (the sample volume on each processor), thereby $k \cdot m = N$.

The final estimation of the acceptable region is computed with the formula:

$$\hat{V}_g = V_0 \sum_{j=1}^{k} N_{gj} / N ,$$ (12)

where N – necessary tests amount, N_G – the amount of points within the region of acceptability for each of the processors.

The boundaries of the circumscribed box B_O are computed with the formulas:

$$\forall i = \overline{1,n} \quad a_i^O = \min(a_{ij}^O), \ j = \overline{1,k},$$
$$\forall i = \overline{1,n} \quad b_i^O = \max(b_{ij}^O), \ j = \overline{1,k}$$, (13)

where $a_{ij}^O = \min\limits_{x \in D_x} x_i$, $b_{ij}^O = \max\limits_{x \in D_x} x_i$ for each of k processors.

The communication overheads are not considerable because they are reduced to primary initialization of the random numbers generator and obtaining of final estimations.

к.   The multidimensional probing method

The region of acceptability within the tolerance box $D_X \in B_d$ can be defined with the methods based on multidimensional probing of the tolerance box B_d.

Let's consider the algorithm of acceptability region construction. It is based on the concept of matrix tests on reliability.

The circumscribed box B_O is divided into a set of non-overlapping elementary boxes. Let's perform separation of circumscribed box B_O with the hyperplanes parallel to its bounds into $\mathbf{l} = (l_1, ..., l_n)$ slices per each coordinate.

Then we have non-overlapping elementary boxes with bounds parallel to corresponding bounds of B_O. The length of elementary box's rib parallel to i-th coordinate axis is l_i times smaller than the corresponding rib of B_O.

So we have:

$$B_O = \bigcup_{k_1=1}^{l_1} \bigcup_{k_2=1}^{l_2} ... \bigcup_{k_n=1}^{l_n} B_{k_1, k_2, ... k_n} .$$ (14)

The total amount of elementary boxes is

$$R = \prod_{i=1}^{n} l_i .$$ (15)

The step (i.e. the length of elementary box's rib) on the each parameter axis is

$$h_i = (b_i^0 - a_i^0) / l_i, \ i = \overline{1,n}.$$ (16)

Let's assign the central point of each elementary box as a point-"representative". The coordinates of point-representative for the elementary box $B_{k_1, k_2, ... k_n}$ are computed as follows:

$$x_i = a_i^O + \frac{h_i(k_i - 1)}{2}, \ i = \overline{1,n}.$$ (17)

If acceptability conditions (1) are satisfied at the point-representative of the elementary box $B_{k_1, k_2, ... k_n}$ this elementary box belongs to a set of "good" elementary boxes, otherwise it belongs to a set of "bad" elementary boxes.

Note also that

$$\lim_{R \to \infty} \frac{M_g}{R} = V_{D_x} / V_{B_O} ,$$ (18)

where M_g is the amount of "good" elementary boxes, R is the total amount of elementary boxes. V_{D_x} / V_{B_O} is the relation of acceptability region volume to the volume of circumscribed box B_O.

The figure consisting of "good" elementary boxes is defined as D_x^o. The figure D_x^o approximates the acceptable region D_x.

The *n*-dimensional array $C[l_1, l_2, ..., l_n]$ is corresponded to the circumscribed box B_O divided into elementary boxes $B_{k_1, k_2, ... k_n}$. Each element of the array stores the state of corresponding elementary box (i.e. "good" or "bad", accordingly 1 or 0). The total amount of array elements can be found using (15). This array also can be called as n-dimensional matrix. In the text below it is called matrix.

Now consider the representation of *n*-dimensional array using 1-dimensional array by placing its elements sequentially in a row.

In common case the elements of that array *A[R]* are placed as follows:

$$A = \{(1,1,1,...,1), (2,1,1,...,1),...,(l_1,1,1,...,1),$$
$$(1,2,1,...,1), (2,2,1,...,1),...,(l_1,2,1,...,1),$$
$$(1,3,1,...,1), (2,3,1,...,1),...,(l_1,3,1,...,1),..., \qquad (19)$$
$$(1,l_2,1,...,1), (2,l_2,1,...,1),...,(l_1,l_2,1,...,1),...,$$
$$(1,l_2,1,...,l_{n-1},1), (2,l_2,...,2),...,(l_1,l_2,...,l_{n-1},l_n)\}$$

The index *p* of the array element, corresponding to the elementary box $B_{k_1, k_2, ... k_n}$ is computed as follows:

$$p = k_1 + l_1(k_2 - 1) + l_1 l_2(k_3 - 1) + ... + \prod_{i=1}^{n-1} l_i (k_n - 1) - 1. \qquad (20)$$

In some kinds of tasks it is necessary to compute a set of indexes $(k_1, k_2, k_3, ..., k_n)$ having the array index *p* and the amount of quanta $(l_1, l_2, l_3, ..., l_n)$ per each coordinate axis. To compute a set of indexes $(k_1, k_2, k_3, ..., k_n)$ it is necessary to do the following sequentially:

$$k_n = \left\lceil \frac{p}{\prod_{i=1}^{n-1} l_i} \right\rceil + 1,$$

$$k_{n-1} = \left\lceil \frac{p - k_n \cdot \prod_{i=1}^{n-1} l_i}{\prod_{i=1}^{n-2} l_i} \right\rceil + 1,$$

$$... \qquad (21)$$

$$k_2 = \left\lceil \frac{p - k_n \cdot \prod_{i=1}^{n-1} l_i - k_{n-1} \cdot \prod_{i=1}^{n-2} l_i - ... - k_3 \cdot \prod_{i=1}^{2} l_i}{l_1} \right\rceil + 1,$$

$$k_1 = \left\lceil p - k_n \cdot \prod_{i=1}^{n-1} l_i - k_{n-1} \cdot \prod_{i=1}^{n-2} l_i - ... - k_3 \cdot \prod_{i=1}^{2} l_i - k_2 \cdot l_1 \right\rceil + 1$$

For example in the case of 3D parameter space the (21) looks as follows:

$$k_3 = \left\lceil \frac{p}{l_1 \cdot l_2} \right\rceil + 1,$$

$$k_2 = \left\lceil \frac{p - k_3 \cdot l_1 \cdot l_2}{l_1} \right\rceil + 1,$$

$$k_1 = \left\lceil p - k_3 \cdot l_1 \cdot l_2 - k_2 \cdot l_1 \right\rceil + 1$$

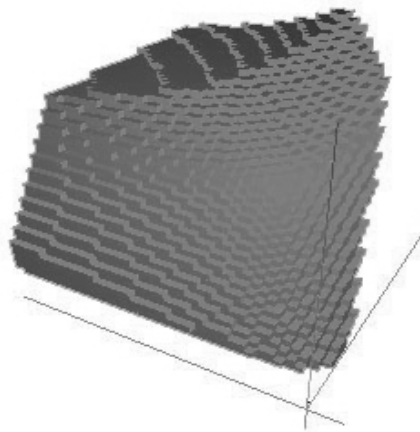The procedure of the array *A[R]* filling consists in follows.

Let the bounds $\mathbf{a}^O, \mathbf{b}^O$ of the circumscribed box and the amount of quanta per each coordinate is known. Initially the amount of "good" elementary boxes is $M_g = 0$.

For the each index $i_n = \overline{1, l_n}$; $i_{n-1} = \overline{1, l_{n-1}}$; ...; $i_1 = \overline{1, l_1}$:

1. Compute the coordinates of the elementary box's $\mathbf{x}_{i_1 i_2 \dots i_n}$ representative using the formula (17).

2. Compute the index *p* of the array *A* unit using the formula (20).

3. Compute the output system parameters $\mathbf{y}(\mathbf{x}_{i_1 i_2 \dots i_n})$.

4. Check the acceptability conditions (1).

5. If conditions (1) are satisfied $A[p] = 1$, $M_g = M_g + 1$.

6. Else $A[p] = 0$.

As the result we have an array *A[R]* that represents circumscribed box B_O divided into elementary boxes and the amount M_g of "good" elementary boxes. If $\dfrac{M_g}{R} \approx \dfrac{V_g}{V_O}$, the array *A[R]* represents the circumscribed box B_O and the acceptable region D_x.

Figure 3 illustrates rendered elementary boxes that represent approximation of the acceptable region.

Fig. 3: Rendered "good" elementary boxes

The procedure of the array *A[R]* filling can be performed in parallel mode. This algorithm can be presented as a distributed process that requires l_1 processors for implementation. The algorithm of calculating the center of gravity for acceptable region constructed of elementary boxes.

In fact the distribution laws of system parameters variations and the characteristics of random parameters degradation processes X(t) are often unknown. In some cases the center of gravity for acceptable region may be the numerical solution of the parametric optimization problem.

Let's consider the acceptable region D_x^0 as a system of *M* unit-mass material points.

The center of gravity coordinates $\overline{C} = (x_{\overline{C}1}, \dots, x_{\overline{C}n})$ of the body can be found as follows:

$$x_{\overline{C}i} = \frac{\sum\limits_{j=1}^{M} I_i(j)}{M},$$

where M is normalized mass of D_x^0 (amount of elementary boxes D_x^0 consists of), $I_i(j)$ are static moments with respect to corresponding axis of each of the elementary boxes.

Static moment $I(B_{k_1,k_2,...,k_n})$ of the corresponding elementary box $B_{k_1,k_2,...,k_n}$ is calculated in the following way:

$$I_j(B_{k_1,k_2,...,k_n}) = \begin{cases} k_j, & if \quad B_{k_1,...k_n} \in D_x^0 \\ 0, & if \quad B_{k_1,...,k_n} \notin D_x^0 \end{cases}$$

for $j = \overline{1,n}$.

Since D_x^0 is represented with "good" elementary boxes inside the circumscribed box B_0 its center mass coordinates can be found on whole circumscribed box B_0 using the formula:

$$x_{\overline{C}i} = \frac{\sum_{k_1=1}^{l_1}\sum_{k_2=1}^{l_2}...\sum_{k_n=1}^{l_n}I_i(B_{k_1k_2,...k_n})}{M},$$

for $k_1 = \overline{1,l_1},...,k_n = \overline{1,l_n}$.

When center of gravity is being calculated simultaneously with construction of acceptable region using one-dimensional array the procedure of filling the array $C[l_1,l_2,...,l_n]$ (*A[R]*) and calculating the center of gravity $\overline{C} = (x_{\overline{C}1},...,x_{\overline{C}n})$ is in follows.

Let the boundaries $\mathbf{a}^O, \mathbf{b}^O$ of the circumscribed box, the amount of quanta $\mathbf{l} = (l_1,l_2,l_3,...,l_n)$ on each of the axes and the lengths $\mathbf{h} = (h_1,...,h_n)$ of quanta on each of the axes are known. Let also the amount of "good" matrix elements $M = 0$ and $\mathbf{SI} = (SI_1,...,SI_n)$ is the array of static moments sum with respect to n coordinate axes.

   L.   For $k_n = \overline{1,l_n}$; $k_{n-1} = \overline{1,l_{n-1}}$; ...; $k_1 = \overline{1,l_1}$:

    1.1. the coordinates $\mathbf{x}_{k1k2...kn}$ of point-representative for the current elementary box are calculated:

$$x_i = a_i^0 + h_i k_i - \frac{h_i}{2}, \quad i = \overline{1,n}.$$

    1.2. The system response functions $y(x_{k1k2...kn})$ are calculated.

    1.3. The acceptability conditions (1) are checked.

    1.4. If conditions (1) are satisfied

$$C[k_1,k_2,...,k_n] = 1, M = M + 1,$$
$$SI_i = SI_i + k_i, i = \overline{1,n}$$

    1.5. otherwise $C[k_1,k_2,...,k_n]$.

2. The gravity center coordinates for D_x^0 are calculated:

$$x_{\overline{C}i} = \frac{SI_i}{M}, \quad i = \overline{1,n}.$$

As the result the array $C[l_1,l_2,...,l_n]$ (*A[R]*) is filled, the amount M of "good" elementary boxes and the coordinates of the center of gravity $\overline{C}$ for D_x^0 are obtained.

In the case when the array $C[l_1,l_2,...,l_n]$ (*A[R]*) was filled previously the procedure of calculation of the gravity centre for D_x^0 consists in follows.

    1. For $k_n = \overline{1,l_n}$; $k_{n-1} = \overline{1,l_{n-1}}$; ...; $k_1 = \overline{1,l_1}$:

$$\text{if } A(k_1, k_2, \ldots, k_n) = 1, \text{ then } SI_i = SI_i + k_i, \quad i = \overline{1, n}.$$

2. The gravity center coordinates for D_x^0 are calculated:

$$x_{\overline{C}i} = \frac{SI_i}{M}, \quad i = \overline{1, n}.$$

Hence it is necessary to perform exhaustive search of the array $C[l_1, l_2, \ldots, l_n]$ (*A[R]*) elements for calculating of the gravity center of the body.

Having found the center of gravity for D_x^0 the inverse mapping on acceptable region is performed the following way:

$$x_{Ci} = \frac{b_i^0 - a_j^0}{l_i} x_{\overline{C}i}, \quad i = \overline{1, n}.$$

Conclusion

The circumscribed box may narrow the area for construction of the acceptable region and decreases computational cost of stochastic estimations. Computation of internal parameters ratings with lack of distribution density functions for both input and output parameters requires obtaining of acceptable region characteristics. On the basis of the proposed methods and the algorithms a computer-aided reliability-oriented design system called CARD has been developed. The CARD system states mathematical models and calculates ratings of component parameters with the highest precision, acceptability (manufacturing yield) or reliability of engineering systems under design.

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