TECHNIQUE FOR FINDING SAMPLING DISTRIBUTIONS FOR TRUNCATED LAWS WITH SOME APPLICATIONS TO RELIABILITY ESTIMATION

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ABSTRACT

In this paper, the problem of finding sampling distributions for truncated laws is considered. This problem concerns the very important area of information processing in Industrial Engineering. It remains today perhaps the most difficult and important of all the problems of mathematical statistics that require considerable efforts and great skill for investigation. The technique discussed here is based on use of the unbiasedness equivalence principle, the idea of which belongs to the authors, and often provides a neat method for finding sampling distributions. It avoids explicit integration over the sample space and the attendant Jacobian but at the expense of verifying completeness of the recognized family of densities. Fortunately, general results on completeness obviate the need for this verification in many problems involving exponential families. The proposed technique allows one to obtain results for truncated laws via the results obtained for non-truncated laws. It is much simpler than the known techniques. The examples are given to illustrate that in many situations this technique allows one to find the results for truncated laws and to estimate system reliability in a simple way.

KEYWORDS

Truncated law, Sampling distribution, Unbiasedness equivalence principle, Reliability estimation

1. INTRODUCTION

The truncated distributions have found many applications. Several examples have been given employing the truncated distributions in fitting rainfall data and animal population studies where observations usually begin after migration has commenced or concluded before it has stopped [1-2]. Similar situations arise with regard to aiming errors (range, deflection, etc.) in gunnery and other bombing accuracy studies. For example, in gun camera missions, the view angle of the camera defines a known truncation point for an exponentially distributed random variable, observable as some function of the radial error or the distance from the aiming point to the point of impact [3]. A situation for the truncated Poisson distribution would occur when one wishes to fit a distribution to Poisson-like data consisting of numbers of individuals in certain groups which posses a given attribute, but in which a group cannot be sampled unless at least a specified number of its members have the attribute. For example, the group may be a household of people, and the attribute measles; the specified number would then be one. Other examples arise in life testing and reliability problems, where if failure is caused by a wear-out mechanism or is a consequence of accumulated wear, then the length-of-life of a system can be expected to be of finite dimension. The object of the present paper is to obtain a sampling distribution for truncated law with a known truncation point and a minimum variance unbiased estimator of the reliability function for this model using the results obtained for non-truncated law. It is known that a sampling distribution for truncated law may be derived using, namely, the method based on characteristic functions [4], the method based on generating functions [5], or the combinatorial method [6]. In this paper, a much simpler technique than the above ones is proposed. It allows one to obtain the results for truncated laws more easily.

U. UNBIASEDNESS EQUIVALENCE PRINCIPLE

Suppose an experiment yields data sample $X^n = (X_1, \ldots, X_n)$ relevant to the value of a parameter θ (in general, vector). Let $L_X(x^n;\theta)$ denote the probability or probability density of X^n when the parameter assumes the value θ. Considered as a function of θ for given $X^n=x^n$, $L_X(x^n;\theta)$ is the likelihood function. If the data sample X^n can be summarized by a sufficient statistic **S**, one can write $L_{\mathbf{S}}(\mathbf{s};\theta) \propto L_X(x^n;\theta)$. Further, for any non-negative function $\omega(\mathbf{s})$, $\omega(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\theta)$ is also a likelihood function equivalent to $L_X(x^n;\theta)$. Suppose we recognize a function $\omega(\mathbf{s})$ such that $\omega(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\theta)$, regarded as a function of **s** for a given θ, is a density function. It can be shown that this is the sampling density of **S** if the family of recognized densities is complete.

The unbiasedness equivalence principle consists in the following. If

$$L_X(x^n;\theta,\vartheta)=[w(\theta,\vartheta)]^n L_X(x^n;\theta), \tag{1}$$

represents the likelihood function for the truncated law, where $w(\theta,\vartheta)$ is some function of a parameter (θ,ϑ) associated with truncation, ϑ is a known truncation point (in general, vector), then a sampling density for the truncated law is determined by

$$g_\vartheta(\mathbf{s};\theta) = \widehat{w}(\mathbf{s})\left[w(\theta,\vartheta)\right]^n g(\mathbf{s};\theta), \quad \mathbf{s} \in \mathsf{S}_\vartheta, \tag{2}$$

where

$$\widehat{w}(\mathbf{s})\left[w(\theta,\vartheta)\right]^n g(\mathbf{s};\theta) = \varphi(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta) \propto L_X(x^n;\theta,\vartheta), \tag{3}$$

$g(\mathbf{s};\theta)$ is a sampling density of a sufficient statistic $\mathbf{s}(X^n)$ (for a family of densities $\{f(x;\theta)\}$) determined on the basis of $L_X(X^n;\theta)$, $\widehat{w}(\mathbf{S})$ is *an unbiased estimator* of $1/[w(\theta,\vartheta)]^n$ with respect to $g(\mathbf{s};\theta)$, $\mathbf{s}\in\mathsf{S}$ (a sample space of a non-truncated sufficient statistic **S**), $\varphi(\mathbf{S})$ is a function of **S** for a given θ, which is equivalent to unbiased estimator $\widehat{w}(\mathbf{S})$ of $1/[w(\theta,\vartheta)]^n$, i.e.,

$$\varphi(\mathbf{S}) \propto \widehat{w}(\mathbf{S}) \tag{4}$$

or

$$\varphi(\mathbf{S}) = \widehat{w}(\mathbf{S})\left[w(\theta,\vartheta)\right]^n g(\mathbf{S};\theta)/L_{\mathbf{S}}(\mathbf{S};\theta,\vartheta), \tag{5}$$

$g_\vartheta(\mathbf{s};\theta)$ is the sampling density of a sufficient statistic **S** (for a family of densities $\{f_\vartheta(x;\theta)\}$) when the truncation parameter ϑ is known, S_ϑ is a sample space of a truncated sufficient statistic **S**.

V. EXAMPLES OF APPLICATIONS OF THE UNBIASEDNESS EQUIVALENCE PRINCIPLE TO FINDING SAMPLING DISTRIBUTIONS FOR TRUNCATED LAWS

Example 3.1 (*Sampling distribution for the left-truncated Poisson law*). Let the Poisson probability function be denoted by

$$f(x;\theta)=\frac{\theta^x}{x!}e^{-\theta},\quad x=0,1,2,\ldots\ . \tag{6}$$

The probability function of the restricted random variable, which is truncated away from some $\vartheta\geq 0$, is then

$$f_\vartheta(x;\theta)=w(\theta,\vartheta)f(x;\theta),\quad x=\vartheta+1,\vartheta+2,\ldots\ , \tag{7}$$

where

$$w(\theta,\vartheta)=\left(\sum_{j=\vartheta+1}^{\infty}\frac{\theta^j}{j\,!}e^{-\theta}\right)^{-1}=\left(1-\sum_{j=0}^{\vartheta}\frac{\theta^j}{j\,!}e^{-\theta}\right)^{-1}. \tag{8}$$

Consider a sample of n independent observations X_1, X_2, …, X_n, each with probability function $f_\vartheta(x;\theta)$, where the likelihood function is defined as

$$L_X(x^n;\theta,\vartheta)=\prod_{i=1}^{n}f_\vartheta(x_i;\theta)=\left[w(\theta,\vartheta)\right]^nL_X(x^n;\theta)=\left[w(\theta,\vartheta)\right]^n\prod_{i=1}^{n}f(x_i;\theta)$$

$$=\left[w(\theta,\vartheta)\right]^ne^{-n\theta}\frac{\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}, \tag{9}$$

and let

$$S=\sum_{i=1}^{n}X_i,\quad s=n(\vartheta+1),n(\vartheta+1)+1,\ldots\ . \tag{10}$$

It is well known that

$$S=\sum_{i=1}^{n}X_i,\quad s=0,1,\ldots\ . \tag{11}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. A result of [7] states that sufficiency is preserved under truncation away from any Borel set in the range of X. Hence, in the case at hand S is sufficient for $\{f_\vartheta(x;\theta)\}$. It can be verified that S is also complete.

For the sake of simplicity but without loss of generality, consider the case ϑ=0. This is at the same time the most important case for applications and the easiest with which to deal. It follows from (2) that

$$g_\vartheta(s;\theta) = \widehat{w}(s)\left[w(\theta,\vartheta)\right]^n g(s;\theta) = \frac{\theta^s n!}{(e^\theta - 1)^n s!} C_s^n, \quad s = n, n+1, \dots \,, \tag{12}$$

where

$$g(s;\theta) = \frac{(n\theta)^s}{s!} e^{-n\theta}, \quad s = 0, 1, \dots, \tag{13}$$

$$[\mathrm{w}(\theta,\vartheta)]^n = \frac{1}{(1-e^{-\theta})^n}, \tag{14}$$

$$\widehat{w}(s) = \frac{n!}{n^s} C_s^n, \tag{15}$$

C_s^n denotes the Stirling number of the second kind [8] defined by

$$C_s^n = \begin{cases} \dfrac{1}{n!}\displaystyle\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} j^s, & s = n, n+1, \dots, \\ 0, & s < n, \end{cases} \tag{16}$$

$$\begin{aligned} E\{\widehat{w}(s)\} &= \sum_{s=0}^{\infty} \widehat{w}(s) g(s;\theta) = \sum_{s=0}^{\infty} \frac{n!}{n^s} C_s^n \frac{(n\theta)^s}{s!} e^{-n\theta} \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} e^{-(n-j)\theta} \sum_{s=0}^{\infty} \frac{(j\theta)^s}{s!} e^{-j\theta} = (1-e^{-\theta})^n, \end{aligned} \tag{17}$$

This is the same result that of Tate and Goen [9]. Their proof was based on characteristic functions.

Example 3.2 (*Sampling distribution for the right-truncated exponential law*). Let the probability density function of the right-truncated exponential distribution be denoted by

$$f_\vartheta(x;\theta) = w(\theta,\vartheta) f(x;\theta), \quad 0 \le x \le \vartheta, \tag{18}$$

where

$$w(\theta,\vartheta) = \frac{1}{1-e^{-\vartheta/\theta}}, \tag{19}$$

$$f(x;\theta) = (1/\theta) e^{-x/\theta}, \quad x \in [0,\infty). \tag{20}$$

Consider a sample of n independent observations $X_1, X_2, \ldots, X_n$, each with density $f_\vartheta(x;\theta)$, where the likelihood function is determined as

$$L_X(x^n;\theta,\vartheta)=\prod_{i=1}^{n} f_\vartheta(x_i;\theta)=\left[w(\theta,\vartheta)\right]^n L_X(x^n;\theta)$$

$$=\left[w(\theta,\vartheta)\right]^n \prod_{i=1}^{n} f(x_i;\theta)=\left[w(\theta,\vartheta)\right]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{21}$$

It is well known that

$$S=\sum_{i=1}^{n} X_i, \quad s\in[0,\infty), \tag{22}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. It follows from (2) that

$$g_\vartheta(s;\theta)=\widehat{w}(s)\left[w(\theta,\vartheta)\right]^n g(s;\theta)=\frac{e^{-s/\theta}}{\Gamma(n)\theta^n(1-e^{-\vartheta/\theta})^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}[(s-(n-j)\vartheta)_+]^{n-1}$$

$$=\frac{e^{-s/\theta}}{\Gamma(n)\theta^n(1-e^{-\vartheta/\theta})^n}\sum_{j=1}^{n}\binom{n}{j}(-1)^{n-j}[(s-(n-j)\vartheta)_+]^{n-1}, \quad s\in[0,n\vartheta], \quad n\geq 1, \tag{23}$$

where $a_+=\max(0,a)$,

$$g(s;\theta)=\frac{s^{n-1}}{\Gamma(n)\theta^n}e^{-s/\theta}, \quad s\in[0,\infty), \tag{24}$$

$$[w(\theta,\vartheta)]^n=\frac{1}{(1-e^{-\vartheta/\theta})^n}, \tag{25}$$

$$\widehat{w}(\mathrm{s})=\frac{1}{s^{n-1}}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}[(s-(n-j)\vartheta)_+]^{n-1}, \tag{26}$$

$$E\{\widehat{w}(s)\}=\int_0^\infty \widehat{w}(s)g(s;\theta)ds$$

$$=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}e^{-(n-j)\vartheta/\theta}\int_0^\infty \frac{[(s-(n-j)\vartheta)_+]^{n-1}}{\Gamma(n)\theta^n}e^{-(s-(n-j)\vartheta)_+/\theta}d(s-(n-j)\vartheta)_+=(1-e^{-\vartheta/\theta})^n. \tag{27}$$

This is the same result that of Bain and Weeks [4]. Their proof was based on characteristic functions.

Example 3.3 (*Sampling distribution for the doubly truncated exponential law*). Consider an exponential distribution (20) that is doubly truncated at a lower truncation point (ϑ_1) and an upper truncation point (ϑ_2). The probability density function of the doubly truncated exponential distribution is defined as

$$f_{\vartheta}(x;\theta) = w(\theta,\vartheta) f(x;\theta), \quad \vartheta_1 \le x \le \vartheta_2, \tag{28}$$

where $\vartheta=(\vartheta_1,\vartheta_2)$,

$$w(\theta,\vartheta) = \frac{1}{e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta}}. \tag{29}$$

Consider a sample of n independent observations $X_1, X_2, \ldots, X_n$, each with density $f_{\vartheta}(x;\theta)$, where the likelihood function is determined as

$$L_X(x^n;\theta,\vartheta) = \prod_{i=1}^{n} f_{\vartheta}(x_i;\theta) = [w(\theta,\vartheta)]^n L_X(x^n;\theta)$$
$$= [w(\theta,\vartheta)]^n \prod_{i=1}^{n} f(x_i;\theta) = [w(\theta,\vartheta)]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{30}$$

It is well known that

$$S = \sum_{i=1}^{n} X_i, \quad s \in [0,\infty), \tag{31}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. It follows from (2) that

$$g_{\vartheta}(s;\theta) = \widehat{w}(s)[w(\theta,\vartheta)]^n g(s;\theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1)_+]^{n-1}$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n} \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1)_+]^{n-1}, \quad s \in [n\vartheta_1, n\vartheta_2],$$
$$n \ge 1, \tag{32}$$

where $a_+=\max(0,a)$, $g(s,\theta)$ is given by (24),

$$[w(\theta,\vartheta)]^n = \frac{1}{(e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n}, \tag{33}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-1}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1))_+]^{n-1}, \tag{34}$$

$$E\{\widehat{w}(s)\} = \int_0^{\infty} \widehat{w}(s) g(s;\theta) ds = (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n. \tag{35}$$

4. VALIDITY OF THE UNBIASEDNESS EQUIVALENCE PRINCIPLE

The theoretical results of this investigation into the validity of the proposed *unbiasedness equivalence principle* (UEP) for finding sampling distributions for truncated laws are largely contained in the theorem given below. We introduce the following notation and assumptions. Let X^n be a random variable taking on values x^n in a space X_ϑ, let A be a $\sigma-$field of subsets of X_ϑ, and let (θ,ϑ) be a parameter associated with truncation, where ϑ is a known truncation point. For all values of the parameter θ in some parameter space Θ, let P_ϑ be a probability measure on A; i.e., for any set A in A, $P_\vartheta(A;\theta)$ is the probability that X^n will belong to A when the parameter has the value θ. Let $\mathbf{S}=\mathbf{s}(X^n)$ be a statistic on the measurable space (X_ϑ,A) taking on values in a measurable space (S_ϑ, B). For each $\theta\in\Theta$, let G_ϑ be the probability distribution of $\mathbf{S}$ when X^n has the distribution P_ϑ; i.e., for any $B\in$B, $G_\vartheta(B;\theta) = P_\vartheta(\mathbf{s}^{-1}(B);\theta)$, where $\mathbf{s}^{-1}(B)$ is the set of points x^n in X_ϑ for which $\mathbf{s}(x^n)\in B$.

W. Assume the family P = $\{P_\vartheta: \theta\in\Theta\}$ of probability distributions of X^n is dominated by a totally $\sigma-$finite measure μ over (X_ϑ,A), i.e., there exists, for all $\theta\in\Theta$, a non-negative A – measurable function $p_\vartheta(x^n;\theta)$ such that

$$P_\vartheta(A;\theta)=\int_A p_\vartheta(x^n;\theta)d\mu(x^n) \tag{36}$$

for all A$\in$A. (The integrand $p_\vartheta(x^n;\theta)$ is called the density of P_ϑ w.r.t. (with respect to) μ).

(ii) Assume that $\mathbf{s}(X^n)$ is sufficient for P. From the Halmos-Savage factorization theorem [10], $\mathbf{s}(X^n)$ is sufficient if and only if for each $\theta\in\Theta$ there exists a non-negative B-measurable function $L_\mathbf{S}(\mathbf{s}(x^n);\theta,\vartheta)$ on S_ϑ and a non-negative A – measurable function v on X_ϑ such that

$$p_\vartheta(x^n;\theta)=L_\mathbf{S}(\mathbf{s}(x^n);\theta,\vartheta)v(x^n) \quad (\mu). \tag{37}$$

(The symbol (μ) following a statement means that the statement holds except on a set of μ-measure zero). In (37), we will assume that $L_\mathbf{S}$ and v are finite (μ).

(iii) Assume we recognize some likelihood function $L_\mathbf{S}(\mathbf{s};\theta,\vartheta)$ equivalent to likelihood function $L_X(x^n;\theta,\vartheta)$. Define a $\sigma-$finite measure ρ over (X_ϑ,A) by

$$\rho(A)=\int_A v(x^n)d\mu(x^n), \quad \text{all } A\in\text{A}. \tag{38}$$

Then, from (36), (37), and (8),

$$P_\vartheta(A;\theta)=\int_A L_\mathbf{S}(\mathbf{s}(x^n);\theta,\vartheta)d\rho(x^n), \quad \text{all } A\in\text{A}. \tag{39}$$

(iv) Assume we recognize a totally σ-finite measure η over (S_ϑ, B) such that the measure $\rho\,\mathbf{s}^{-1}$ over (S_ϑ, B) is absolutely continuous w.r.t. η; i.e., $\eta(B)=0$ implis that $\rho\,\mathbf{s}^{-1}(B)=0$, where $\rho\,\mathbf{s}^{-1}(B)$ denotes the $\rho-$ measure of the inverse image of B.

(v) Assume we recognize a positive B-measurable function φ on S_ϑ such that

$$\int_{S_\vartheta} L_\mathbf{S}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s})\equiv 1 \tag{40}$$

for all $\theta\in\Theta$. Assume further that for any measurable set B of positive η – measure, there exists a $\theta\in\Theta$ and a measurable subset B_1 of B of positive η – measure over which $L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})$ is positive.

From (40), $\{L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}):\theta\in\Theta\}$ is a family of densities w.r.t. η. For $B\in$B, let

$$G_\vartheta(B;\theta)=\int_B L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s}). \tag{41}$$

Thus, (v) provides us with a family of densities, but at this stage we do not know if this recognized family is the family of sampling densities of $\mathbf{S}$.

(vi) Assume we recognize that the family $\{L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}):\theta\in\Theta\}$ is complete, i.e.,

$$\int_{S_\vartheta}\phi(\mathbf{s})L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s})\equiv 0 \quad \text{for all } \theta\in\Theta \tag{42}$$

implies

$$\phi(\mathbf{s})\equiv 0 \tag{43}$$

except on a set D with $G_\vartheta(D;\theta)=0$ for all $\theta\in\Theta$.

Theorem 1 (*Sampling distribution for truncated law*). Under assumptions (i) through (vi), G_ϑ has a density with respect to η and $L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})$ is a version of it, i.e.,

$$L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})=\widehat{w}(\mathbf{s})[w(\theta,\vartheta)]^n g(\mathbf{s};\theta) \tag{44}$$

is the sampling density, $g_\vartheta(\mathbf{s};\theta)$, of the sufficient statistic $\mathbf{s}(X^n)$.

Proof. We show first that (43) and the second part of (v) imply that $\phi(\mathbf{s})\equiv 0$ (η). For suppose there exists a measurable B with $\eta(B)>0$ and $\phi(\mathbf{s})\neq 0$ over B. Then $B\subset D$, so $G_\vartheta(B;\theta)=0$ for all $\theta\in\Theta$. But, from (v), there exists a $B_1\subset B$ for which $G_\vartheta(B_1;\theta)>0$ for some θ, contradicting $G_\vartheta(B;\theta)=0$ for all $\theta\in\Theta$. Now, by a theorem in [10], there exists a non-negative measurable function ψ on S_ϑ such that

$$\int Q_\vartheta(\mathbf{s}(x^n);\theta)d\rho(x^n)=\int Q_\vartheta(\mathbf{s};\theta)\psi(\mathbf{s})d\eta(\mathbf{s}) \tag{45}$$

for every measurable function Q_ϑ, in the sense that if either integral exists, then so does the other and the two are equal. In (45), let $Q_\vartheta(\mathbf{s},\theta)=\chi_B L_S(\mathbf{s};\theta,\vartheta)$, where χ_B is the characteristic function of B ($B\in$B). Then there exists a $\psi(\mathbf{s})$ such that

$$\int_{\mathbf{s}^{-1}(B)} L_S(\mathbf{s}(x^n);\theta,\vartheta)d\rho(x^n)=\int_B L_S(\mathbf{s};\theta,\vartheta)\psi(\mathbf{s})d\eta(\mathbf{s}) \tag{46}$$

for all $B\in$B. Note that the left side of (45) is $G_\vartheta(B;\theta)$.

In (42), let $\phi(\mathbf{s})=1-[\psi(\mathbf{s})/\varphi(\mathbf{s})]$. From (40) and (46),

$$\int_{S_\vartheta}\left[1-\frac{\psi(\mathbf{s})}{\varphi(\mathbf{s})}\right]L_S(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s})=0 \tag{47}$$

for all $\theta\in\Theta$. Thus, from (43), $\psi(\mathbf{s})=\varphi(\mathbf{s})$ almost everywhere (η), and, from (47),

$$L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}) = \widehat{w}(\mathbf{s})\left[w(\theta,\vartheta)\right]^n g(\mathbf{s};\theta) \tag{48}$$

is a version of the density of G_ϑ with respect to η. □

X. FINDING RELIABILITY ESTIMATORS FOR TRUNCATED LAWS VIA THE UNBIASEDNESS EQUIVALENCE PRINCIPLE

Consider a system that is required to operate for a given 'mission time', t. The reliability of this system for the right-truncated distribution of time-to-failure with the probability density function $f_\vartheta(x;\theta)$ may be defined as

$$R(t) = \Pr(x \geq t) = \int_t^{\vartheta} f_\vartheta(x;\theta)dx. \tag{49}$$

Due to the Rao-Blackwell and Lehmann-Scheffé theorem [11] a minimum variance unbiased (MVU) estimator for R may be obtained as

$$\widehat{R}(t) = \int_t^{\vartheta} f_\vartheta(x;\mathbf{s})dx, \tag{50}$$

where X may be any one of the observations (X_1, …, X_n) from $f_\vartheta(x;\theta)$, $\mathbf{S}$ is a complete sufficient statistic for $\{f_\vartheta(x;\theta)\}$, and $f_\vartheta(x;\mathbf{s})$ is the conditional distribution of X given $\mathbf{S}$=$\mathbf{s}$; $f_\vartheta(x;\mathbf{s})$ is obtained
as

$$f_\vartheta(x;\mathbf{s}) = \frac{f_\vartheta(x,\mathbf{s};\theta)}{g_\vartheta(\mathbf{s};\theta)} = \frac{\widehat{w}_f(x,\theta,\vartheta)}{\widehat{w}(\mathbf{s})}, \tag{51}$$

where

$$f_\vartheta(x,\mathbf{s};\theta) = \widehat{w}_f(x,\theta,\vartheta)[w(\theta,\vartheta)]^n g(\mathbf{s};\theta) \tag{52}$$

is the joint probability density of X and $\mathbf{S}$, $\widehat{w}_f(x,\theta,\vartheta)$ is an unbiased estimator of

$$w_f(x,\theta,\vartheta) = \frac{f_\vartheta(x;\theta)}{[w(\theta,\vartheta)]^n}. \tag{53}$$

with respect to $g(\mathbf{s};\theta)$.
It should be noted that (50) can be obtained by different method as

$$\widehat{R}(t) = \frac{\widehat{w}_R(t,\theta,\vartheta)}{\widehat{w}(\mathbf{s})}, \tag{54}$$

where $\widehat{w}_R(t,\theta,\vartheta)$ is an unbiased estimator of

$$w_R(t,\theta,\vartheta) = \frac{R(t)}{[w(\theta,\vartheta)]^n} \tag{55}$$

with respect to $g(\mathbf{s};\theta)$.

Example 5.1 (*MVU estimator of reliability for the right truncated exponential distribution*). Let $X^n=(X_1, \ldots, X_n)$ be a random sample of size n from a population with density (18). Then it follows from (50) (or (54)) that the MVU estimator of $R(t)$ is obtained as

$$\widehat{R}(t) = \frac{\sum_{j=0}^{n-1}(-1)^j \binom{n-1}{j}\left([(S-(j\vartheta+t))_+]^{n-1} - [(S-(j+1)\vartheta)_+]^{n-1}\right)}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}[(S-j\vartheta)_+]^{n-1}}. \tag{56}$$

As a particular case, if $\vartheta \to \infty$ that is the variable X is assumed unrestricted, the corresponding MVU estimator of reliability reduces to

$$\widehat{R}(t) = [(1-t/S)_+]^{n-1}. \tag{57}$$

For instance, suppose that the following failure times, in hours, are available from a given system: 4.2, 9.8, 16, 20 and that the truncation point ϑ=25 hours and the mission time t=5 hours. Clearly s=50 hours. Substituting these values in (56), the estimate of reliability is obtained as $\widehat{R}(t) = 0.824$. Had we assumed, however, that the observations are coming from the complete population, the estimate of reliability would have been, from (57), $\widehat{R}(t) = 0.729$.

Example 5.2 *(MVU estimator of reliability for the right-truncated gamma distribution). Let $X^n=(X_1, \ldots, X_n)$* be a random sample of size n from a population with density

$$f_\vartheta(x;\theta) = w(\theta,\vartheta)\frac{1}{\Gamma(\delta)}\sigma^{-\delta}x^{\delta-1}e^{-x/\sigma}, \quad 0 < x \le \vartheta, \quad \sigma > 0, \quad \delta > 0, \tag{58}$$

where ϑ is point of truncation, $\theta=(\sigma,\delta)$, and $w(\theta,\vartheta)$ is such that

$$w(\theta,\vartheta)\int_0^\vartheta \frac{1}{\Gamma(\delta)}\sigma^{-\delta}x^{\delta-1}e^{-x/\sigma}dx = 1. \tag{59}$$

This distribution has found applications in a number of diverse fields, for instance, in fitting of length-of-life data under fatigue. Note that for δ=1, the right-truncated gamma distribution reduces to the right-truncated exponential distribution with parameter σ. Although, this distribution is a special case of gamma distribution and gives a good fit to length-of-life data in many situations, it is not suitable since its use carries the implication that at any time future life-length is independent of past history.

To find MVU estimator of $R(t)$ we apply the above technique. If the shape parameter δ in (58) is assumed to be known, then it is well known that

$$S = \sum_{i=1}^{n} X_i \tag{60}$$

is a complete sufficient statistic for σ. The probability density function of the sampling distribution of S is given by

$$g_{\vartheta}(s;\theta)=\widehat{w}(s)\left[w(\theta,\vartheta)\right]^{n}g(s;\theta)=\frac{[w(\theta,\vartheta)]^{n}\Gamma^{n}(\delta)}{\sigma^{n\delta}\Gamma(n\delta)}e^{-s/\sigma}$$

$$\times\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}[(s-r\vartheta)_{+}]^{n\delta-1}\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right),\quad s\in(0,n\vartheta),\tag{61}$$

where

$$\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right)=\sum_{\substack{\{r_0,r_1,\ldots,r_{\delta-1}\}:\\ \sum_{j=0}^{\delta-1}r_j=r}}\frac{r!\varpi!}{\prod_{j=0}^{\delta-1}(r_j!)\sum_{j=0}^{\delta-1}(j!)^{r_j}}\binom{n\delta-1}{\varpi}\left(\frac{\vartheta}{(s-r\vartheta)_{+}}\right)^{\varpi},\tag{62}$$

$$\varpi=\sum_{j=0}^{\delta-1}jr_j.\tag{63}$$

The joint distribution of X and S is given by

$$f_{\vartheta}(x,s;\theta)=\widehat{w}_f(x,\theta,\vartheta)[w(\theta,\vartheta)]^{n}g(s;\theta)=\frac{[w(\theta,\vartheta)]^{n}\Gamma^{(n-1)}(\delta)}{\sigma^{n\delta}\Gamma((n-1)\delta)}x^{\delta-1}e^{-s/\sigma}$$

$$\times\sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}(s-r\vartheta-x)^{(n-1)\delta-1}\,\Delta\left((n-1)\delta-1,\frac{\vartheta}{(s-r\vartheta-x)_{+}}\right).\tag{64}$$

Thus the conditional distribution of X given S is

$$f_{\vartheta}(x;s)=\frac{f_{\vartheta}(x,s;\theta)}{g_{\vartheta}(s;\theta)}=\frac{\widehat{w}_f(x,\theta,\vartheta)}{\widehat{w}(s)}=\frac{\Gamma(n\delta)}{\Gamma(\delta)\Gamma((n-1)\delta)}$$

$$\times\sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}x^{\delta-1}[(s-r\vartheta-x)_{+}]^{(n-1)\delta-1}\,\Delta\left((n-1)\delta-1,\frac{\vartheta}{(s-r\vartheta-x)_{+}}\right)$$

$$\times\left(\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}(s-r\vartheta)^{n\delta-1}\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right)\right)^{-1}.\tag{65}$$

Hence the MVU estimator of $R(t)$ at time t is given by

$$\widehat{R}(t)=\int_{t}^{\vartheta}f_{\vartheta}(x;s)dx=\frac{\widehat{w}_R(t,\theta,\vartheta)}{\widehat{w}(s)}=\frac{\Gamma(n\delta)}{\Gamma(\delta)\Gamma((n-1)\delta)}\sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}$$

$$\times \int_t^{\vartheta} x^{\delta-1}[(s-r\vartheta-x)_+]^{(n-1)\delta-1}\Delta\left((n-1)\delta-1, \frac{\vartheta}{(s-r\vartheta-x)_+}\right)dx$$

$$\times\left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}(s-r\vartheta)^{n\delta-1}\Delta\left(n\delta-1, \frac{\vartheta}{(s-r\vartheta)_+}\right)\right)^{-1}. \quad (66)$$

It may be remarked that the result (66) though at the first look appears quite unwieldy is not so in practical applications, particularly when the sample size is small.

As a particular case, if $\vartheta \to \infty$ that is the random variable X is assumed unrestricted, the distribution of the sufficient statistics from equation (61) reduces to

$$g(s;\boldsymbol{\theta}) = \frac{1}{\Gamma(n\delta)\sigma^{n\delta}} s^{n\delta-1}e^{-s/\sigma}, \quad s\in(0,\infty) \quad (67)$$

and the corresponding MVU estimator of reliability at time t is given by

$$\widehat{R}(t) = \frac{\Gamma(n\delta)}{s^{\delta-1}}\sum_{j=0}^{\delta-1}\frac{s^j t^{\delta-1-j}(1-t/s)^{(n-1)\delta+j}}{(\delta-1-j)![(n-1)\delta+j]!}, \quad (68)$$

which corresponds to Basu's [12] equation (9).

Y. CONCLUSIONS

The authors hope that this work will stimulate further investigation using the approach on specific applications to see whether obtained results with it are feasible for realistic applications.

It will be noted that the similar approach also can be used to find the sampling distribution for truncated law when some or all of its truncation parameters are left unspecified.

For instance, consider Example 3.3, where it is assumed that the truncation parameter $\boldsymbol{\vartheta}=(\vartheta_1,\vartheta_2)$ is unknown. It is known that the statistic $(X_{(1)}, X_{(n)}, S)$, where

$$X_{(1)} = \min_{1\le i\le n} X_i, \quad (69)$$

$$X_{(n)} = \max_{1\le i\le n} X_i, \quad (70)$$

and

$$S=\sum_{i=2}^{n-1} X_i, \quad (71)$$

is a complete sufficient statistic for a set of parameters ($\vartheta_1, \vartheta_2, \theta$). In this case, the likelihood function of a sample is determined as

$$L_X(x_{(1)}, x_{(n)}, x_2, ..., x_{n-1}; \theta, \vartheta) = n(n-1) f_\vartheta(x_{(1)}; \theta) f_\vartheta(x_{(n)}; \theta) \prod_{i=2}^{n-1} f_\vartheta(x_i; \theta)$$

$$= h_\vartheta(x_{(1)}, x_{(n)}; \theta) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} \prod_{i=2}^{n-1} f(x_i; \theta) = h_\vartheta(x_{(1)} x_{(n)}; \theta) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} \frac{1}{\theta^{n-2}} e^{-\sum_{i=2}^{n-1} x_i / \theta}, \tag{72}$$

where

$$h_\vartheta(x_{(1)}, x_{(n)}; \theta) = n(n-1)[F_\vartheta(x_{(n)}; \theta) - F_\vartheta(x_{(n)}; \theta)]^{n-2} f_\vartheta(x_{(1)}; \theta) f_\vartheta(x_{(n)}; \theta),$$

$$x_{(1)} < x_{(\mathrm{n})}, \quad x_{(1)}, x_{(n)} \in [\vartheta_1, \vartheta_2], \tag{73}$$

is the joint probability density function of the order statistics $x_{(1)}$ and $x_{(n)}$), $F_\vartheta(\cdot)$ is the probability distribution function. It is well known that

$$S = \sum_{i=2}^{n-1} X_i, \quad s \in [0, \infty), \tag{74}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. It follows from (2) and (72) that

$$g_\vartheta(s; \theta) = \widehat{w}(s) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} g(s; \theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n-2)\theta^{n-2}[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s - (n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_+]^{n-3}$$

$$= \frac{e^{-s/\theta}}{\Gamma(n-2)\theta^{n-2}[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}} \sum_{j=1}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s - (n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_+]^{n-3},$$

$$s \in [(n\text{-}2)x_{(1)}, (n\text{-}2)x_{(n)}], \quad n \geq 3, \tag{75}$$

where

$$g(s; \theta) = \frac{s^{n-3}}{\Gamma(n-2)\theta^{\mathrm{n-2}}} e^{-s/\theta}, \quad s \in [0, \infty), \tag{76}$$

$$\left[w(\theta, x_{(1)}, x_{(n)})\right]^{n-2} = \frac{1}{[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}}, \tag{77}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-3}} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s-(n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_{+}]^{n-3}, \tag{78}$$

$$E\{\widehat{w}(s)\} = \int_0^\infty \widehat{w}(s) g(s;\theta) ds = [e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}. \tag{79}$$

Thus, the sampling distribution of the sufficient statistic ($X_{(1)}$, $X_{(n)}$, S) for ($\vartheta_1, \vartheta_2, \theta$) is given by

$$g_{\vartheta}(x_{(1)}, x_{(n)}, s;\theta) = h_{\vartheta}(x_{(1)}, x_{(n)};\theta) g_{\vartheta}(s;\theta). \tag{80}$$

In other words, we have the following results.

In a singly truncated case, when a truncation point on the left, ϑ_1, is unknown, a sampling distribution of the sufficient statistic ($X_{(1)}$, S) for (ϑ_1, θ) is given by

$$g_{\vartheta_1}(x_{(1)}, s;\theta) = h_{\vartheta_1}(x_{(1)};\theta) g_{\vartheta_1}(s;\theta), \tag{81}$$

where

$$X_i \sim f_{\vartheta_1}(x_i;\theta) = w(\theta, \vartheta_1) f(x_i;\theta) = \frac{1}{1 - F(\vartheta_1;\theta)} f(x_i;\theta), \quad x_i \geq \vartheta_1, \quad i = 1, \ldots, n, \tag{82}$$

$$h_{\vartheta_1}(x_{(1)};\theta) = n[1 - F_{\vartheta_1}(x_{(1)};\theta)]^{n-1} f_{\vartheta_1}(x_{(1)};\theta) \tag{83}$$

is the probability density function of the order statistic $X_{(1)}$,

$$g_{\vartheta_1}(s;\theta) = \widehat{w}(s)\left[w(\theta, x_{(1)})\right]^{n-1} g(s;\theta), \tag{84}$$

$s \equiv s(X_2, \ldots, X_n)$.

In a singly truncated case, when a truncation point on the right, ϑ_2, is unknown, a sampling distribution of the sufficient statistic ($X_{(n)}$, S) for (ϑ_2, θ) is given by

$$g_{\vartheta_2}(x_{(n)}, s;\theta) = h_{\vartheta_2}(x_{(n)};\theta) g_{\vartheta_2}(s;\theta), \tag{85}$$

where

$$X_i \sim f_{\vartheta_2}(x_i;\theta) = w(\theta, \vartheta_2) f(x_i;\theta) = \frac{1}{F(\vartheta_2;\theta)} f(x_i;\theta), \quad x_i \leq \vartheta_2, \quad i = 1, \ldots, n, \tag{86}$$

$$h_{\vartheta_2}(x_{(n)};\theta) = n[F_{\vartheta_2}(x_{(n)})]^{n-1} f_{\vartheta_2}(x_{(n)};\theta) \tag{87}$$

is the probability density function of the order statistic $X_{(n)}$,

$$g_{\vartheta_2}(s;\theta) = \widehat{w}(s)\left[w(\theta, x_{(n)})\right]^{n-1} g(s;\theta), \tag{88}$$

$s \equiv s(X_1, \ldots, X_{n-1})$.

In a doubly truncated case, when a lower truncation point, ϑ_1, and an upper truncation point, ϑ_2, are unknown, a sampling distribution of the sufficient statistic $(X_{(1)}, X_{(n)}, S)$ for $(\vartheta_1, \vartheta_2, \theta)$ is given by

$$g_{\vartheta}(x_{(1)}, x_{(n)}, s; \theta) = h_{\vartheta}(x_{(1)}, x_{(n)}; \theta) g_{\vartheta}(s; \theta). \tag{89}$$

where

$$X_i \sim f_{\vartheta}(x_i;\theta) = w(\theta, \vartheta_1, \vartheta_2) f(x_i;\theta) = \frac{1}{F(\vartheta_2;\theta) - F(\vartheta_1;\theta)} f(x_i;\theta), \quad \vartheta_1 \le x_i \le \vartheta_2,\ i = 1, \ldots, n, \tag{90}$$

$$h_{\vartheta}(x_{(1)}, x_{(n)}; \theta) = n[F_{\vartheta}(x_{(n)};\theta) - F_{\vartheta}(x_{(1)};\theta)]^{n-2} f_{\vartheta}(x_{(n)};\theta) \tag{91}$$

is the joint probability density function of the order statistic $X_{(1)}$ and $X_{(n)}$,

$$g_{\vartheta}(s;\theta) = \widehat{w}(s)\left[w(\theta, x_{(1)}, x_{(n)})\right]^{n-2} g(s;\theta), \tag{92}$$

$s \equiv s(X_2, \ldots, X_{n-1})$.

If, say, we deal with a left-truncated exponential distribution,

$$f_{\vartheta_1}(x;\theta) = w(\theta, \vartheta_1) f(x;\theta), \quad \vartheta_1 \le x < \infty, \tag{93}$$

where

$$w(\theta, \vartheta_1) = \frac{1}{e^{-\vartheta_1/\theta}} \tag{94}$$

and a truncation point on the left, ϑ_1, is unknown, then it follows immediately from (81) that the sampling distribution of the sufficient statistic $(X_{(1)}, S)$, $S = \sum_{i=2}^{n} X_i$, for (ϑ_1, θ) is given by

$$g_{\vartheta_1}(x_{(1)}, s;\theta) = h_{\vartheta_1}(x_{(1)};\theta) g_{\vartheta_1}(s;\theta) = \left[n\left(\frac{e^{-x_{(1)}/\theta}}{e^{-\vartheta_1/\theta}}\right)^{n-1} \frac{1}{e^{-\vartheta_1/\theta}} \frac{1}{\theta} e^{-x_{(1)}/\theta} \right]$$

$$\times \left[\frac{[s-(n-1)x_{(1)}]^{n-2}}{s^{n-2}} \frac{1}{[e^{-x_{(1)}/\theta}]^{n-1}} \frac{s^{n-2}}{\Gamma(n-1)\theta^{n-1}} e^{-s/\theta} \right] = \frac{n}{\theta} e^{-n(x_{(1)}-\vartheta_1)/\theta} \frac{[s-(n-1)x_{(1)}]^{n-2}}{\Gamma(n-1)\theta^{n-1}} e^{-[s-(n-1)x_{(1)}]/\theta}, \tag{95}$$

which corresponds to the well-known result [11].

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