DISCRETE TIME MODELS OF FORWARD CONTRACTS INSURANCE

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Introduction

In this paper financial management model of forward contracts insurance suggested in [1,2] is considered by means of risk theory and heavy tailed technique. This model is based on a compensation principle. It attracted large interest and called active discussion among economists. So its mathematical analysis is initiated as economists so mathematicians.

Suppose that there are two insurance companies insuring both participants of some forward contract and working in discrete time with the net payouts $\xi + \eta$, $\xi - \eta$ during one step. Here ξ with $E\xi = a < 0$ is common random summand for net payouts of these two companies. Then η with $E\eta = 0$ is individual random claim of the first participant of the contract and $-\eta$ is individual claim of the second participant. The claim η (the claim $-\eta$) may be considered as a premium for $\eta < 0$ (for $\eta > 0$). Suppose that distribution functions (d.f.`s) $P(\xi \le x) = H(x)$, $P(\eta \le x) = S(x)$, $P(-\eta \le x) = L(x)$ and

$$\overline{H}(x) = o\left(\overline{S}(x)\right), \ \overline{H}(x) = o\left(\overline{L}(x)\right), \ x \to \infty \qquad (1)$$

with $\overline{F}(x) = 1 - F(x)$. Assume that ξ, η are independent random variables (r.v.`s) and $\xi$, $\eta$, $-\eta$ are subexponential r.v.`s.

Denote the one-step ruin probabilities of the companies with the initial capital x insuring the first and the second participants of the contract by

$$a_1(x) = P(\xi + \eta > x), \ a_2(x) = P(\xi - \eta > x)$$

and the one-step ruin probabilities of both companies and one of them by

$$a(x) = P\left[(\xi + \eta > x) \cap (\xi - \eta > x)\right].$$

Here $a_1(x), a_2(x)$ characterize individual risks of the insurance companies and $a(x)$ characterizes their group risk. Introduce $c(x) = P(\xi + \eta + \xi - \eta > x + x) = \overline{H}(x)$ one step ruin probability of these two companies aggregation. Here $c(x)$ characterizes as individual so group risks. The aggregation of these two companies allows to decrease individual risks $a_1(x), a_2(x)$ to small $\overline{H}(x)$ and to conserve the group risk $a(x)$ at small level $\overline{H}(x)$:

$$a_1(x) \sim \overline{S}(x), \ a_2(x) \sim \overline{L}(x), \ a(x) \sim c(x), \ x \to \infty. \qquad (2)$$

Main purpose of this paper is to obtain asymptotical comparisons analogous to (2) for individual and group risks in separate and aggregated insurance models. We speak about infinite horizon discrete time risk models without interest force with constant interest force and with stochastic interest force.

1. Preliminaries

Classes of distributions. Throughout, for a given r.v. X concentrated on $(-\infty, \infty)$ with a d.f. F then its right tail $\overline{F}(x) = P(X > x)$. For two d.f.'s F_1 and F_2 concentrated on $(-\infty, \infty)$ we write by $F_1 * F_2(x)$ the convolution of F_1 and F_2 and write by $F_1^{*2} = F_1 * F_1$

the convolution of F_1 with itself. All limiting relationships, unless otherwise stated, are for $x \to \infty$. Let $a(x) \geq 0$ and $b(x) > 0$ be two infinitesimals, satisfying

$$l^- \leq \liminf_{x \to \infty} \frac{a(x)}{b(x)} \leq \limsup_{x \to \infty} \frac{a(x)}{b(x)} \leq l^+ .$$

We write $a(x) = O(b(x))$ if $l^+ < \infty$, $a(x) = o(b(x))$ if $l^+ = 0$, and $a(x) \lesssim b(x)$ if $l^+ = 1$, $a(x) \gtrsim b(x)$ if $l^- = 1$, and $a(x) \sim b(x)$ if both.

Introduce the following classes of d.f.`s concentrated on $[0, \infty)$:

$$S = \left\{ F(x) : \lim_{x \to \infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2 \right\}, \quad \mathcal{L} = \left\{ F(x) : \forall t \; \lim_{x \to \infty} \frac{\overline{F}(x-t)}{\overline{F}(x)} = 1 \right\},$$

$$\mathcal{R}_{-\alpha} = \left\{ F(x) : \forall \theta > 0 \; \lim_{x \to \infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} = \theta^{-\alpha} \right\}, \quad 0 < \alpha < \infty, \quad \mathcal{R} = \bigcup_{0 < \alpha < \infty} \mathcal{R}_{-\alpha},$$

$$\mathcal{R}_{-\infty} = \left\{ F(x) : \forall \theta > 1 \; \lim_{x \to \infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} = 0 \right\},$$

$$S_* = \left\{ F(x) : \int_0^x \overline{F}(x-y)\overline{F}(y)dy \sim 2m_+ \overline{F}(x), \; x \to \infty, \; m_+ = \int_0^\infty \overline{F}(x)dx \right\}.$$

S is called the class of subexponential d.f.`s. $\mathcal{L}$ is called the class of long tailed d.f.`s. $\mathcal{R}$ (or $\mathcal{R}_{-\alpha}$) is called the class of regular varying d.f.`s (with index $\alpha$). $\mathcal{R}_{-\infty}$ is called the class of rapidly varying tailed d.f.`s.

Proposition 1. *The classes $\mathcal{R}$, S, $\mathcal{L}$ satisfy the formula [3] $\mathcal{R} \subset S \subset \mathcal{L}$.*

Proposition 2. *If $F \in S_*$ then [4] F, $1 - \frac{1}{m_+}\overline{F_I}(x) \in S$ with $\overline{F_I}(x) = \int_x^\infty \overline{F}(y)dy$.*

More generally, d.f. F concentrated on $(-\infty, \infty)$ is also said to belong to these classes if its right-hand distribution $\tilde{F}(x) = F(x)\mathbb{I}(x > 0)$ does.

Proposition 3. *Let F_1 and F_2 be two d.f.`s concentrated on $(-\infty, \infty)$. If $F_1 \in S$, $F_2 \in \mathcal{L}$ and $\overline{F_2}(x) = O(\overline{F_1}(x))$, then [5,Lemma 3.2] $F_1 * F_2 \in S$ and $\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.*

Proposition 4. *Suppose that X, Y are independent random variables with the d.f.`s F_1, F_2 concentrated on $(-\infty, \infty)$ and $F_1 \in \mathcal{L}$, $F_2(\infty = 1)$ then [6] $P(X - Y > t) \sim \overline{F_1}(t)$.*

Discrete time risk model under stochastic interest force. Consider a risk model with discrete time $n = 1, 2, \ldots$ and denote X_n the insurer`s net loss - the total claim amount minus the total incoming premium within period n and Y_n the discount factor from time n to time $n-1$. Here X_n

is called insurance risk and Y_n is called financial risk. These random variables are independent with d.f.`s $F(t)$, $G(t)$ relatively.

Let $\{X_n, n = 1, 2, ...\}$ be a sequence of independent and identically distributed (i.i.d.) r.v`s with generic random variable $X$, let $\{Y_n, n = 1, 2, ...\}$ be another sequence of i.i.d. positive r.v.`s with generic random variable Y, and let the two sequences be mutually independent. Denote

$$\Psi(x) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} X_k \prod_{j=1}^{k} Y_j > x \right).$$

Then $\Psi(x)$ is infinite-time ruin probability of the risk model under stochastic interest force with initial capital x.

Proposition 5. *Suppose that* $F \in S_*$ *and* $EX = m < 0$ *and* $P(Y = 1) = 1$ *then [7]*

$$\Psi(x) \sim \frac{1}{|m|} \overline{F_I}(x).$$

Proposition 6. *If* $F \in \mathcal{R}_{-\alpha}$, $0 < \alpha < \infty$, *and* $E \max\left\{ Y^{\alpha - \delta}, Y^{\alpha + \delta} \right\} < 1$ *for some* $0 < \delta < \alpha$ *then* *[8]*

$$\Psi(x) \sim \frac{EY^{\alpha}}{1 - EY^{\alpha}} \overline{F}(x).$$

Proposition 7. *Suppose that* $P(Y = 1 + r) = 1$ *for some* $0 < r < \infty$. *If* $F \in \mathcal{R}_{-\alpha}$ *for some* $0 < \alpha < \infty$ *then [9]*

$$\Psi(x) \sim \frac{\overline{F}(x)}{(1 + r)^{\alpha} - 1}.$$

If $F \in S \bigcap \mathcal{R}_{-\infty}$ *then* $\Psi(x) \sim \overline{F}((1 + r) x)$.

2. Asymptotic comparison of individual and group risks for forward contracts insurance

Suppose that at the step k the net payouts of both participants of the step k forward contract are $\xi_k + \eta_k$, $\xi_k - \eta_k$. Here random sequences $\{\xi = \xi_0, \xi_1, ...\}$ and $\{\eta = \eta_0, \eta_1, ...\}$ are independent. Each of these two random sequences consists of i.i.d.r.v`s with their own common d.f.`s $P(\xi \le t) = H(t)$, $P(\eta \le t) = S(t)$ correspondingly and $H(t)$, $S(t)$, $L(t) \in S$ with $L(t) = P(-\eta \le x)$.

Define individual risks of the companies with initial capitals x insuring net payouts $\{\xi_1 + \eta_1, \xi_2 + \eta_2, ...\}$, $\{\xi_1 - \eta_1, \xi_2 - \eta_2, ...\}$ separately by

$$A_1(x) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k + \eta_k) \prod_{j=1}^{k} Y_j > x \right), \quad A_2(x) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k - \eta_k) \prod_{j=1}^{k} Y_j > x \right).$$

Analogously define group risk of separately working insurance companies by

$$A(x) = P\left[\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k + \eta_k) \prod_{j=1}^{k} Y_j > x \right) \bigcap \left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k - \eta_k) \prod_{j=1}^{k} Y_j > x \right) \right]$$

and common individual and group risk of aggregated company by

$$C(x) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k + \eta_k + \xi_k - \eta_k) \prod_{j=1}^{k} Y_j > 2x\right) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} \xi_k \prod_{j=1}^{k} Y_j > x\right).$$

Lemma 1. *Suppose that the condition (1) is true then*

$$P(\xi + \eta > x) \sim \overline{S}(x), \ P(\xi - \eta > x) \sim \overline{L}(x).$$

Proof. This statement arises from the proposition 3.

Lemma 2. *The formula* $P(\xi - |\eta| > x) \sim \overline{H}(x)$ *is true.*

Proof. This statement arises from the propositions 1, 4.

Lemma 3. *The following inequality takes place*

$$A(x) \ge R(x), \ R(x) = P\left(\sup_{1 \le i < \infty} \sum_{k=1}^{i} (\xi_k - |\eta_k|) \prod_{j=1}^{k} Y_j > x\right).$$

Proof. This statement arises from the inequalities $a \ge -|a|, \ -a \ge -|a|$, which are true for all real a.

Theorem 1. *Suppose that the condition (1) is true and* $H(t), \ S(t), \ L(t) \in S_*, \ P(Y = 1) = 1$ *then*

$$A_1(x) \sim \frac{\overline{S_I}(x)}{|a|}, \ A_2(x) \sim \frac{\overline{L_I}(x)}{|a|}, \ C(x) \sim \frac{\overline{H_I}(x)}{|a|}, \ R(x) \sim \frac{\overline{H_I}(x)}{|a - E|\eta\|}.$$

Proof. This statement arises from the propositions 5 and from the lemmas 1, 2.

Theorem 2. *Suppose that the condition (1) is true and for some* $\alpha_1, \alpha_2, \alpha_3, \ 0 < \alpha_1, \alpha_2 < \alpha_3$, *d.f.'s* $S(x) \in \mathcal{R}_{-\alpha_1}, \ L(x) \in \mathcal{R}_{-\alpha_2}, \ H(x) \in \mathcal{R}_{-\alpha_3}, \ E \max\left\{Y^{\alpha_i - \delta(\alpha_i)}, Y^{\alpha_i + \delta(\alpha_i)}\right\} < 1$ *for some* $0 < \delta(\alpha_i) < \alpha_i, \ 1 \le i \le 3$, *then*

$$A_1(x) \sim \frac{EY^{\alpha_1}}{1 - EY^{\alpha_1}} \overline{S}(x), \ A_2(x) \sim \frac{EY^{\alpha_2}}{1 - EY^{\alpha_2}} \overline{L}(x), \ C(x) \sim R(x) \sim \frac{EY^{\alpha_3}}{1 - EY^{\alpha_3}} \overline{H}(x).$$

Proof. This statement arises from the propositions 6 and from the lemmas 1, 2.

Theorem 3. *Suppose that the condition (1) is true and* $P(Y = 1 + r) = 1$ *for some* $0 < r < \infty$ *If for some* $\alpha_1, \alpha_2, \alpha_3, \ 0 < \alpha_1, \alpha_2 < \alpha_3$, *d.f.'s* $S(x) \in \mathcal{R}_{-\alpha_1}, \ L(x) \in \mathcal{R}_{-\alpha_2}, H(x) \in \mathcal{R}_{-\alpha_3}$ *then*

$$A_1(x) \sim \frac{\overline{S}(x)}{(1+r)^{\alpha_1} - 1}, \ A_2(x) \sim \frac{\overline{L}(x)}{(1+r)^{\alpha_2} - 1}, \ C(x) \sim R(x) \sim \frac{\overline{H}(x)}{(1+r)^{\alpha_3} - 1}.$$

If d.f.'s $S(x), \ L(x), H(x) \in S \cap \mathcal{R}_{-\infty}$ *then*

$$A_1(x) \sim \overline{S}((1+r)x), \ A_2(x) \sim \overline{L}((1+r)x), \ C(x) \sim R(x) \sim \overline{H}((1+r)x).$$

Proof. This statement arises from the propositions 7 and from the lemmas 1, 2.

Theorem 4. *If the conditions of the theorem 1 (the theorem 2 or the theorem 3) are true then for some* $0 < k < \infty$

$$A(x) \gtrsim kC(x), \ C(x) = o(A_1(x)), \ C(x) = o(A_2(x)).$$

Proof. This statement arises from the lemma 3 and from the theorems 1-3.

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