ANALYSIS OF PORTS RELIABILITIES

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Introduction

This paper is devoted to algorithms of a calculation of ports reliabilities. A port is a no oriented graph with fixed initial and final nodes. As accuracy so asymptotic formulas are considered. Suggested algorithms have minimal numbers of arithmetical operations.

Using the article [1] results, in this paper algorithms of a calculation of asymptotic constants for ports reliabilities are constructed. These algorithms allow to estimate an influence of some arc reliability on a port reliability and to obtain an invariance condition when this influence is absent.

In solid state physics, surface physics and in nanotechnologies recursively defined ports are of large interest. An example of such a structure is in the monograph [2, fig. 7.7]: where each arc of a bridge scheme Γ is replaced by Γ. In this paper linear upper bounds of arithmetical operations numbers necessary to calculate as a reliability so its asymptotic constants are obtained. For a comparison it is worthy to say that a number of arithmetical operations necessary to calculate a reliability of a port increases as a geometrical progression of port arcs number.

1. Preliminaries

Consider a port Γ with a final number U of nodes, a set $W = \left\{ w = (u,v), u,v \in U \right\}$ of arcs and fixed initial u_0 and final u^0 nodes. Denote $\mathcal{R}$ a set of all ways R in the port Γ, which connect the nodes u_0 and u^0. Suppose that $\mathcal{R} \neq \varnothing$. Consider the sets

$$\mathcal{A} = \left\{ A \subset U, u_0 \in A, u^0 \notin A \right\}, \ L = L(A) = \left\{ (u,u') : u \in A, u' \notin A \right\}$$

and $\mathcal{L} = \left\{ L(A), A \in \mathcal{A} \right\}$ - the set of all sections in Γ. Correspond for each arc $w \in W$ a logic variable $\alpha(w) = I$ (the arc w works), where $I(B)$ is an indicator function of an event B. Denote a quantity which characterizes a connectivity between the nodes u_0, u^0 in Γ by

$$\beta = \bigvee_{R \in \mathcal{R}} \bigwedge_{w \in R} \alpha(w). \tag{1}$$

Suppose that $\alpha(w), w \in W$, are independent random variables,

$P(\alpha(w) = 1) = p_w(h), \ q_w(h) = 1 - p_w(h)$,

where h is some small parameter : $h \to 0$. In [1] the following statements are proved.

Theorem 1. *Suppose that* $p_w(h) \sim \exp\left(-h^{-d(w)}\right), h \to 0$, *where* $d(w) > 0, w \in W$. *Then* $-\ln P(\beta = 1) \sim h^{-D}$ *and* $D = D(\Gamma) = \min_{R \in \mathcal{R}} \max_{w \in \mathcal{R}} d(w)$.

Theorem 2. *Suppose that* $q_w(h) \sim \exp\left(-h^{-d_1(w)}\right), h \to 0$, *where* $d_1(w) > 0, w \in W$. *Then* $-\ln P(\beta = 0) \sim h^{-D_1}$ *and* $D_1 = D_1(\Gamma) = \max_{L \in \mathcal{L}} \min_{w \in \mathcal{L}} d_1(w)$.

Theorem 3. *Suppose that* $p_w(h) \sim h^{g(w)}, h \to 0$, *where* $g(w) > 0, w \in W$. *Then* $\ln P(\beta = 1) \sim G \ln h$ *and* $G = G(\Gamma) = \min_{R \in \mathcal{R}} \sum_{w \in \mathcal{R}} g(w)$.

Theorem 4. *Suppose that* $q_w(h) \sim h^{g_1(w)}, h \to 0$, *where* $g_1(w) > 0, w \in W$, *then* $\ln P(\beta = 0) \sim G_1 \ln h$ *and* $G_1 = G_1(\Gamma) = \min_{L \in \mathcal{L}} \sum_{w \in \mathcal{L}} g_1(w)$.

The constants G, G_1 [3] may be interpreted as a length of a shortest way or a minimal ability to handle of cross-sections in the port Γ correspondingly. In a definition of the constants D, D_1 a summation is replaced by a maximization. So the constants D, D_1 may be interpreted as a pseudo-length of the shortest way or a minimal pseudo-ability to handle in the port Γ.

Remark 1. Suppose that $\tau(w)$ are independent random variables which characterize life times of the arcs $w \in W$. Denote $P(\tau(w) > t) = p_w(h)$ and put the graph Γ life time equal to $\tau(\Gamma) = \min_{R \in \mathcal{R}} \max_{w \in R} \tau(w)$. If $h = h(t)$ is monotonically decreasing and continuous function and $h \to 0$, $t \to \infty$, then the theorems 1, 3 remain true if $P(\beta = 1)$ is replaced by $P(\tau(\Gamma) > t)$. If $h = h(t)$ is monotonically increasing and continuous function and $h \to 0$, $t \to 0$, then the theorems 2, 4 remain true if $P(\beta = 0)$ is replaced by $P(\tau(\Gamma) \leq t)$. So it is possible to consider widely used in the reliability theory the exponential and the Weibull distributions of arcs life times.

Denote $\overline{\Gamma}$ a port with the nodes set $U = \{u_0, u_1, u_2, u_3\}$ and with the arcs set (fig.1)

$$W = \{w_1 = (u_0, u_1), w_2 = (u_0, u_2), w_3 = (u_1, u_3), w_4 = (u_2, u_3), w_5 = (u_1, u_2)\},.$$

The node u_0 is initial and the node u_3 is final. The scheme $\overline{\Gamma}$ [4] is called the bridge scheme and the arc $w_5 -$ the bridge element in this scheme
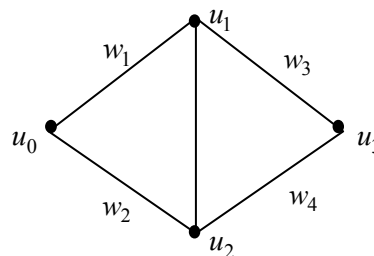
Fig. 1. Bridge scheme $\overline{\Gamma}$.

The scheme $\overline{\Gamma}$ reliability $P = P(p_1, ..., p_5)$ in a suggestion that the arcs $w_1, ..., w_5$ work independently with the probabilities $p_1, ..., p_5$ is calculated by the formula

$$P = p_5 \left[1 - (1 - p_1)(1 - p_2) \right] \left[1 - (1 - p_3)(1 - p_4) \right] + (1 - p_5) \left[1 - (1 - p_1 p_3)(1 - p_2 p_4) \right] \tag{2}$$

To make these calculations it is necessary $n(\overline{\Gamma}) = 14$ arithmetical operations.

2. Element wise analysis

Remark that in an accordance with the formula (1) the logical function $\beta = \beta(\alpha(w), w \in W)$ has all properties of the monotone structure [2, гл.7]:

a) $\beta(\alpha(w) = 1, w \in W) = 1$, b) $\beta(\alpha(w) = 0, w \in W) = 0$,

c) $\beta(\alpha_1(w), w \in W) \leq \beta(\alpha_2(w), w \in W)$, if $\alpha_1(w) \leq \alpha_2(w), w \in W$.

Fix an arc $v \in W$ and using the complete probability formula [2, §7.4] obtain the following formulas:

$$P(\beta = 1) = P(\alpha(v) = 1)F_v^1 + P(\alpha(v) = 0)F_v^0, \tag{3}$$

$$F_v^\delta = P\big(\beta(\alpha(w), w \in W, w \notin v; \alpha(v) = \delta) = 1\big), \delta = 0, 1,$$

and

$$F_v^0 \leq F_v^1. \tag{4}$$

Define the graph Γ_v^0 by an exclusion of the arc $v = (u, u')$ from the graph Γ and the graph Γ_v^1 by a gluing of the nodes u, u' in the graph Γ_v^0. Using the previous section results and the formulas (3), (4) obtain the following statements.

Theorem 5. *Suppose that $p_w(h) \sim \exp\big(-h^{-d(w)}\big), h \to 0$, where $d(w) > 0, w \in W$. Then*

$$-\ln P(\beta = 1) \sim h^{-D} \text{ where } D = \min\Big[\max\big(d(v), D(\Gamma_v^1)\big), D(\Gamma_v^0)\Big], \ D(\Gamma_v^1) \leq D(\Gamma_v^0).$$

Theorem 6. *Suppose that $q_w(h) \sim \exp\big(-h^{-d_1(w)}\big), h \to 0$, where $d_1(w) > 0, w \in W$. Then*

$$-\ln P(\beta = 0) \sim h^{-D_1} \text{ where } D = \min\Big[\max\big(d_1(v), D_1(\Gamma_v^1)\big), D_1(\Gamma_v^0)\Big], \ D_1(\Gamma_v^1) \leq D_1(\Gamma_v^0).$$

Theorem 7. *If $p_w(h) \sim h^{g(w)}, h \to 0$, where $g(w) > 0, w \in W$. Then $\ln P(\beta = 1) \sim G \ln h$ and*

$$G = \min\Big[g(v) + G(\Gamma_v^1), G(\Gamma_v^0)\Big], \ G(\Gamma_v^1) \leq G(\Gamma_v^0).$$

Theorem 8. *If $q_w(h) \sim h^{g_1(w)}, h \to 0$, where $g_1(w) > 0, w \in W$. Then $\ln P(\beta = 0) \sim G_1 \ln h$ and*

$$G_1 = \min\Big[g_1(v) + G_1(\Gamma_v^1), G_1(\Gamma_v^0)\Big], \ G_1(\Gamma_v^1) \leq G_1(\Gamma_v^0).$$

Remark 2. *The constants D, D_1, G, G_1 do not depend on $d(v), d_1(v), g(v), g_1(v)$ correspondingly if and only if $D(\Gamma_v^1) = D(\Gamma_v^0), D_1(\Gamma_v^1) = D_1(\Gamma_v^0), G(\Gamma_v^1) = G(\Gamma_v^0), G_1(\Gamma_v^1) = G_1(\Gamma_v^0)$.* The fig. 2, 3 show how the parameters $d(v), g(v)$ influence on the constants $D(\Gamma), G(\Gamma)$.
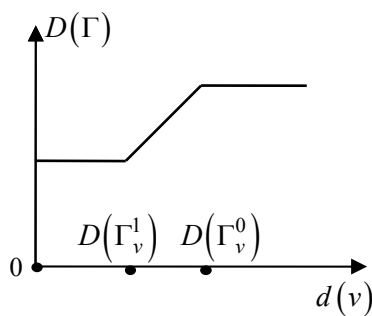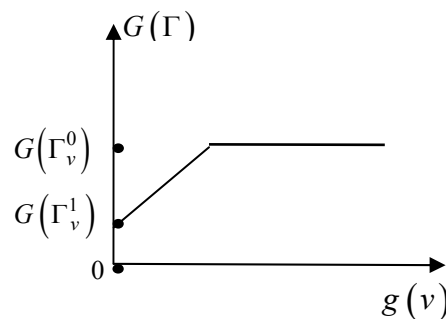

Fig. 2.

Fig. 3.

Example. Consider the port $\overline{\Gamma}$ (fig. 1) with independently working arcs $w_1, \ldots, w_5$ and show how the element w_5 reliability influences on the port reliability on an example of the constants $D(\overline{\Gamma}), G(\overline{\Gamma})$ from the theorems 5, 7. Define the port $\overline{\Gamma}_{w_5}^0$ by an exclusion of the arc w_5 from the graph $\overline{\Gamma}$ and the port $\overline{\Gamma}_{w_5}^1$ by a gluing of the nodes u_1, u_2 in the graph $\overline{\Gamma}_{w_5}^0$.
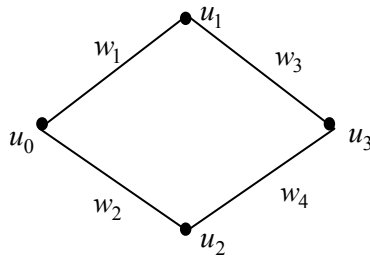
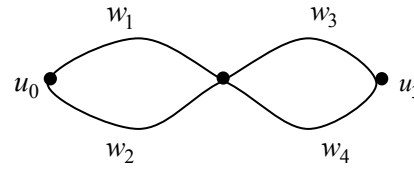**Fig. 4.** Port $\overline{\Gamma}^0_{w_5}$ **Fig.5.** Port $\overline{\Gamma}^1_{w_5}$

If $p_{w_i}(h) \sim \exp\left(-h^{-d_i}\right), h \to 0$, with $d_i = d(w_i) > 0$, then it is easy to obtain the formulas

$$D\left(\overline{\Gamma}^0_{w_5}\right) = \min\left(\max(d_1, d_3), \max(d_2, d_4)\right), \quad D\left(\overline{\Gamma}^1_{w_5}\right) = \max\left(\min(d_1, d_2), \min(d_3, d_4)\right),$$

$$D(\overline{\Gamma}) = \min\left[\max\left(d_5, D\left(\Gamma^1_{w_5}\right)\right), D\left(\Gamma^0_{w_5}\right)\right].$$

Here the equality $D\left(\overline{\Gamma}^0_{w_5}\right) = D\left(\overline{\Gamma}^1_{w_5}\right)$ is true in one of the following eight conditions:

1) $d_3 \geq d_1 > d_2$, 2) $d_3 \geq d_1 = d_2$, 3) $d_4 \geq d_1 = d_2$, 4) $d_4 \geq d_2 > d_1$,

5) $d_1 \geq d_3 > d_4$, 6) $d_1 \geq d_3 = d_4$, 7) $d_2 \geq d_3 = d_4$, 8) $d_2 \geq d_4 > d_3$.

If $p_{w_i}(h) \sim h^{g_i}, h \to 0$, with $g_i = g(w_i) > 0, i = 1, ..., 5$, then it is easy to obtain the formulas

$$G\left(\Gamma^0_{w_5}\right) = \min\left((g_1 + g_3), (g_2 + g_4)\right), \quad G\left(\Gamma^1_{w_5}\right) = \min(g_1, g_2) + \min(g_3, g_4),$$

$$G(\overline{\Gamma}) = \min\left[g_5 + G\left(\Gamma^1_{w_5}\right), G\left(\Gamma^0_{w_5}\right)\right].$$

Here the equality $G\left(\Gamma^0_{w_5}\right) = G\left(\Gamma^1_{w_5}\right)$ is true in one of the following two conditions:

1) $g_4 \geq g_3, g_2 \geq g_1$, 2) $g_3 \geq g_4, g_1 \geq g_2$.

Remark 3. If the graph $\overline{\Gamma}'$ is constructed by an addition of the arc $w_6 = (u_0, u_3)$ to the port $\overline{\Gamma}$ then $D(\overline{\Gamma}') = \min(d_6, D(\overline{\Gamma})), G(\overline{\Gamma}') = \min(g_6, G(\overline{\Gamma}))$ where d_6, g_6 are appropriate parameters of the arc w_6. As the graph $\overline{\Gamma}'$ is complete (each two its nodes is connected by some arc) so these formulas may be spread to a case when we take interest to a connectivity of each two nodes of the graph $\overline{\Gamma}'$ (this scheme is an analog of a transformer electrical scheme). For this purpose it is necessary to renumber the graph $\overline{\Gamma}'$ nodes.

3. Ports superposition

Define recursively a class of bridge schemes $\mathcal{B}$:

1) the arcs $w_1, w_2, ...,$ working independently with the probabilities $p_1, p_2, ...,$ belong to $\mathcal{B}$,

2) if the ports $\Gamma_1, ..., \Gamma_5 \in \mathcal{B}$ consist of nonintersecting sets of arcs then their superposition $\Gamma' = \overline{\Gamma}(\Gamma_1, ..., \Gamma_5)$ belongs to $\mathcal{B}$.

A number of arcs in the superposition Γ' is $m(\Gamma') = m(\Gamma_1) + ... + m(\Gamma_5)$ where $m(\Gamma_i)$ is a number of arcs in the port Γ_i. The reliability of the superposition Γ' equals to $P(P_1, ..., P_5)$ and is calculated by the formula (2) and needs

$$n_p(\Gamma') = n_p(\overline{\Gamma}) + n_p(\Gamma_1) + ... + n_p(\Gamma_5)$$

arithmetical operations where $n(\Gamma_i)$ is a number of arithmetical operations necessary to calculate the

reliability P_i.

If $n_p(\Gamma_i) \le n_p(\overline{\Gamma})(m(\Gamma_i)-1), 1 \le i \le 5$, then

$$n_p(\Gamma') \le n_p(\overline{\Gamma})(m(\Gamma')-1). \tag{5}$$

So a number of arithmetical operations necessary to calculate the reliability of the port $\Gamma' \in \mathcal{B}$ has a bound which is linear increasing by a number of the port Γ' arcs.

For the superposition $\Gamma' = \Gamma(\Gamma_1,...\Gamma_5)$ of the ports $\Gamma_1,...\Gamma_5 \in \mathcal{B}$ it is easy to obtain the recurrent formulas

$$D(\Gamma') = \min_{R \in \mathcal{R}} \max_{i:w_i \in R} D(\Gamma_i), D_1(\Gamma') = \max_{L \in \mathcal{L}} \min_{i:w_i \in L} D_1(\Gamma_i) \tag{6}$$

$$G(\Gamma') = \min_{R \in \mathcal{R}} \sum_{i:w_i \in R} G(\Gamma_i), G_1(\Gamma') = \max_{L \in \mathcal{L}} \sum_{i:w_i \in L} G_1(\Gamma_i) \tag{7}$$

Here $\mathcal{R}, \mathcal{L}$ are the sets of ways and cross sections in the graph $\overline{\Gamma}$. The constants

$$D(\Gamma_i), D_1(\Gamma_i), G(\Gamma_i), G_1(\Gamma_i), i=1,...,5,$$

are calculated by the theorems 1-4 formulas. The formulas (6), (7) allow analogously to (5) to construct linear by $m(\Gamma')$ upper bounds for numbers of arithmetical operations $n_D(\Gamma'), n_{D_1}(\Gamma'), n_G(\Gamma'), n_{G_1}(\Gamma')$ which are necessary to calculate the constants $D(\Gamma'), D_1(\Gamma'), G(\Gamma'), G_1(\Gamma')$:

$$n_D(\Gamma') \le n_D(\overline{\Gamma})(m(\Gamma')-1), \ n_{D_1}(\Gamma') \le n_{D_1}(\overline{\Gamma})(m(\Gamma')-1),$$

$$n_G(\Gamma') \le n_G(\overline{\Gamma})(m(\Gamma')-1), \ n_{G_1}(\Gamma') \le n_{G_1}(\overline{\Gamma})(m(\Gamma')-1).$$

For a comparison remark that a number of arithmetical operations necessary to define the shortest way length or the minimal cross sections ability to handle in general type graphs [5] is significantly larger.

Remark 4. The constructed algorithm of a recursive definition of a port reliability for the class $\mathcal{B}$ with the generating scheme $\overline{\Gamma}$ and the upper bound (5) may be spread to a case of a finite set $\mathcal{G} = \{\Gamma\}$ of generating schemes with a replacement $n(\overline{\Gamma})$ in the formula (5) by $\max_{\Gamma \in \mathcal{G}} n(\Gamma)$. For example it is possible to construct $\mathcal{G}$ by the graphs with two arcs which are connected parallel and sequentially.

References

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