STOCHASTIC APPROACH TO SAFETY AT SEA ASSESSMENT

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Abstract. A general approach for analysing spatial survival in the plane is suggested. Two types of harmful random events are considered: points with fixed coordinates and moving points. A small normally or tangentially oriented interval is moving along a fixed route in the plane, crossing points of initial Poisson random processes. Each crossing leads to termination of the process with a given probability. The probability of passing the route without termination is derived. A safety at sea application is discussed.

Keywords: Spatial point process, Survival probability, Random field, Rate of occurrence.

1. INTRODUCTION: ONE-DIMENSIONAL CASE

A model of survival in the plane is presented in, based on the following simple reasoning used in the one-dimensional case. Consider a system subject to stochastic point influences (shocks). Each shock can lead with a given probability to a fatal failure of a system, resulting in termination of the process, and this will be called an "accident". The probability of performance without accidents in the time interval $(0,t]$ is of interest. It is natural to describe the situation in terms of stochastic point processes.

Denote by $h(t)$ the rate of occurrence or just the rate function of the corresponding point process of shocks $\{N(t);t>0\}$. It is well known ([2, p.31]) that for orderly processes, assuming the limits exist,

$$h(t)=\lim_{\Delta t\to 0}\frac{\Pr\{N(t,t+\Delta t)=1\}}{\Delta t}=\lim_{\Delta t\to 0}\frac{E[N(t,t+\Delta t)]}{\Delta t}. \tag{1}$$

Assume now that a shock occurring in $(t,t+dt]$ independently of the previous shocks leads to an accident with probability $\theta(t)$, and does not cause any changes in the system with probability $1-\theta(t)$. Denote by T_a a random time to an accident and by $F_a(t)=\Pr\{T_a\le t\}$ the corresponding distribution function (DF). If $F_a(t)$ is absolutely continuous, then

$$P(t)=1-F_a(t)=\exp\left\{-\int_0^t \lambda_a(x)dx\right\}, \tag{2}$$

where $\lambda_a(t)$ is a hazard rate, corresponding to $F_a(t)$ and $P(t)$ is the survival function: probability of performance without accidents in $(0,t]$. Assuming that $\{N(t);t>0\}$ is the nonhmogeneous Poisson processes:

$$\lambda_a(t)=\theta(t)h(t). \tag{3}$$

For the time-dependent case this result was proved in Block et al. [1]. Considering the Poisson point processes in the plane, we shall construct the corresponding hazard rate "along the fixed curve". An obvious application of this model is assessing the probability of a safe performance of a ship moving along a fixed route [4].

2. OBSTACLES WITH FIXED COORDINATES

Denote by $\{N(B)\}$ the nomhomogeneous Poisson point process in the plane (the random number of points in $B \subset \Re^2$, where B belongs to the Borel σ-algebra in $\Re^2$). We shall consider points as prospective point influences on our system (shallows for the ship, for instance). Similar to the one-dimensional definition (1) the rate $h_f(\xi)$ can be formally defined as

$$h_f(\xi) = \lim_{S(\delta(\xi)) \to 0} \frac{E[N(\delta(\xi))]}{S(\delta(\xi))} , \tag{4}$$

where $B = \delta(\xi)$ is the neighborhood of ξ with area $S(\delta(\xi))$ and diameter tending to zero.

Assume for simplicity that $h_f(\xi)$ is a continuous function of ξ in an arbitrary closed circle in $\Re^2$. Let R_{ξ_1,ξ_2} be a fixed continuous curve connecting ξ_1 and ξ_2 - two distinct points in the plane. We shall call R_{ξ_1,ξ_2} a route. A point (a ship in our application) is moving in one direction along the route. Every time it "crosses the point" of process $\{N(B)\}$ an accident can happen with a given probability. We are interested in assessing probability of moving along R_{ξ_1,ξ_2} without accidents. Let r be the distance from ξ_1 to the current point of the route (coordinate) and $h_f(r)$ denote the rate in $(r, r+dr]$(a one-dimensional parameterization).

Let $(\gamma_n^+(r), \gamma_n^-(r))$ be a small interval of length $\gamma_n(r) = \gamma_n^+(r) + \gamma_n^-(r)$ in a normal to R_{ξ_1,ξ_2} in the point with coordinate r, where upper indexes denote the corresponding direction Let $\overline{R}$ be the length of $R_{\xi_1,\xi_2}: \overline{R} \equiv | R_{\xi_1\xi_2} |$ and assume that: $\overline{R} >> \gamma_n(r), \forall r \in [0, R]$. The interval $(\gamma_n^+(r), \gamma_n^-(r))$ is moving along R_{ξ_1,ξ_2}, crossing points of a random field. (For our application it is reasonable to assume the following model for the symmetrical $(\gamma_n^+(r) = \gamma_n^-(r))$ equivalent interval: $\gamma_n(r) = 2\delta_s + 2\delta_o(r)$, where $2\delta_s, 2\delta_o(r)$ are the diameters of a ship and of an obstacle, respectively, and for simplicity it is assumed that all obstacles have the same diameter. There can be other models as well). Using definition (4), the *equivalent rate* of occurrence of points, $h_{e,f}(r)$ along the route can be defined as

$$h_{ef}(r) = \lim_{\Delta r \to 0} \frac{E[N(B(r, \Delta r, \gamma_n(r))]}{\Delta r} , \tag{5}$$

where $N(B(r, \Delta r, \gamma_n(r))$ is the random number of points crossed by the interval $\gamma_n(r)$, moving from r to $r + \Delta r$.

It can be easily seen, as in Finkelstein [4], that for $\Delta r \to 0$ and $\gamma_n(r)$ sufficiently small:

$$E[N(B(r, \Delta r, \gamma_n(r))] = \int_{B(r, \Delta r, \gamma_n(r))} h_f(\xi) dS(\delta(\xi)) \overset{\Delta r \to 0}{=} \gamma_n(r) h_f(r) dr [1 + o(1)],$$

which leads to the expected relation for the equivalent rate of the corresponding one-dimensional point process (which is obviously also nonhomogeneous Poisson):

$$h_{ef}(r) = \gamma_n(r) h_f(r)[1 + o(1)]. \tag{6}$$

Hence, r-parameterization along the fixed route reduces the problem to the one-dimensional setting of Section 1.

As in the one-dimensional case, assume that crossing of a point with a coordinate r leads to an accident with probability $\theta_f(r)$ (and is survived with probability $\overline{\theta}_f(r) = 1 - \theta_f(r)$). Denote by R a random distance from the initial point of the route ξ_1 till a point on the route where an accident had occurred. Similar to (2)-(3), probability of passing the route R_{ξ_1,ξ_2} without accidents can be derived in the following way:

$$\Pr\{R > \overline{R}\} \equiv P(\overline{R}) = 1 - F_{af}(\overline{R}) = \exp\left\{ -\int_0^{\overline{R}} \lambda_{af}(r) dr \right\} \tag{7}$$

$$\lambda_{af}(r) \equiv \theta_f(r) h_{ef}(r) \tag{8}$$

Assume that the hazard rate $\lambda_{af}(r)$ is now a stochastic process defined, for instance, as in Yashin and Manton [8] by an unobserved covariate stochastic process $Y = Y_r, r \geq 0$. Denote the corresponding hazard rate process by $\lambda_{af}(Y,r)$. It is well known (see, e.g. Kebir [7]) that under certain assumptions in this the following equation holds:

$$P(\overline{R}) = E\left[\exp\left\{-\int_0^{\overline{R}} \lambda_{af}(Y,r)dr\right\}\right], \tag{9}$$

which can be written via the conditional hazard rate process [8] as

$$P(\overline{R}) = \exp\left\{-\int_0^{\overline{R}} E\left[\lambda_{af}(Y,r) \mid R > r\right]dr\right\} = \exp\left\{-\int_0^{\overline{R}} \overline{\lambda}_{af}(r)dr\right\}, \tag{10}$$

where $\overline{\lambda}_{af}(r)$ is the corresponding equivalent or observed hazard rate:

$$\overline{\lambda}_{af}(r) = E\left[\lambda_{af}(Y,r) \mid R > r\right]. \tag{11}$$

As follows from (10), equation (11) can constitute a reasonable tool for obtaining $P(\overline{R})$, but the corresponding explicit derivations can be performed only in some simplest specific cases. On the other hand, it can help to analyze some important properties. Assume, for instance, that probability $\theta_f(r)$ is indexed by a parameter $Y : \theta_f(Y,r)$. Let Y be interpreted as a non-negative random variable with support in $[0,\infty)$ and the probability density function $\pi(y)$. In the sea safety application this randomization can be due to the unknown characteristics of the navigation (or (and) collision avoidance) onboard system, for instance (we are pooling from the population of ships). There can be other interpretations as well. Thus, the specific case, when Y in relations (9) and (10) is a random variable, is considered. The observed failure rate $\overline{\lambda}_{af}(r)$ then is the corresponding mixture failure rate:

$$\overline{\lambda}_{af}(r) = \int_0^{\infty} \lambda_{af}(y,r)\pi(y \mid r)dy, \tag{12}$$

where $\pi(y \mid r)$ is the conditional probability density function of Y given that $R > r$ [5]

$$\pi(y \mid r) = \frac{\pi(y)P(y,r)}{\int_0^{\infty} P(y,r)\pi(y)dy}, \tag{13}$$

and $P(y,r)$ is defined similar to (7), where $\lambda_{af}(r)$ is substituted by $\theta_f(y,r)h_{ef}(r)$.

Relations (12) and (13) constitute a tool for analyzing the shape of the observed failure rate $\overline{\lambda}_{af}(r)$. As shown in [5,6], the shape of $\overline{\lambda}_{af}(r)$ can differ dramatically from the shape of the conditional failure rate $\lambda_{af}(y,r)$ and this fact should be taken into consideration in applications. Assume, for example, a specific multiplicative form of parameterization:

$$\theta_f(Y,r)h_{ef}(r) = Y\theta_f(r)h_{ef}(r).$$

It is well known that, if $\theta_f(r)h_f(r)$ is constant in this case, than the observed failure rate is decreasing. But it turns out that even, if $\theta_f(r)h_{ef}(r)$ is sharply increasing, $\overline{\lambda}_{af}(r)$ can still decrease at least for sufficiently large r! [6]. Thus, the random parameter changes the aging properties of the corresponding distribution functions.

For the "highly reliable systems" when, for instance, $h_f(r) \to 0$ uniformly in $r \in [0, \overline{R}]$, one can easily obtain obvious approximations. On the other hand, applying Jensen's inequality to the right hand side of (9), a simple lower bound for $P(\overline{R})$ can be also derived:

$$P(\overline{R}) \geq \exp\left\{E\left[-\int_0^{\overline{R}} \lambda_{af}(Y,r)dr\right]\right\} = \exp\left\{-\int_0^{\overline{R}} E[\lambda_{af}(Y,r)]dr\right\}. \tag{14}$$

3. CROSSING THE LINE PROCESS

Consider a random process of continuous curves in the plane to be called paths. We shall keep in mind an application when ships' routes on a chart represent paths, while the rate of the stochastic processes to be defined represents the intensity of navigation in the given sea area. The specific case of stationary random lines in the plane is called a *stationary line process.*

It is convenient to characterize a line in the plane by its (ρ,ψ) coordinates, where ρ is the perpendicular distance from the line to a fixed origin, and ψ is the angle between this perpendicular line and a fixed reference direction. A random process of undirected lines can be defined as a point process on the cylinder $\Re_+ \times S$, where $\Re_+ = (0,\infty)$ and S denote both the circle group and its representations as $(0,2\pi]$. Thus each point on the cylinder is equivalent to the line in $\Re^2$ and for the finite case the point process (and associated stationary line process) can be described. The following result is stated in Daley and Vere-Jones [5, p.389]. Let V be a fixed line in $\Re^2$ with coordinates (ρ_v,α) and let N_V be the point process on V *generated by its intersections with the stationary line process.* Then N_V is a stationary point process on V with rate h_V given by

$$h_V = h\int_S |\cos(\psi-\alpha)| P(d\psi), \tag{15}$$

where h is the constant rate of the stationary line process and $P(d\psi)$ is the probability that an arbitrary line has orientation ψ (first order directional rose on S). If the line process is isotropic, then $h_V = 2h/\pi$. The rate h is induced by the random measure defined by the total length of lines inside any closed bounded convex set in $\Re^2$. Assume that the line process is (homogeneous) Poisson in the sense that the point process N_V generated by its intersections with an arbitrary V is a Poisson point process.

Consider now a stationary temporal Poisson line process in the plane. Similar to N_V, the Poisson point process $\{N_V(t); t>0\}$ of its intersections with V in time can be defined. The constant rate of this process, $h_V(1)$, as usual, defines the probability of intersection (by a line from a temporal line process) of an interval of a unit length in V and in a unit interval of time given these units are substantially small.

Let V_{ξ_1,ξ_2} be a finite line route, connecting ξ_1 and ξ_2 in $\Re^2$ and r, as in the previous section, is the distance from ξ_1 to the current point of V_{ξ_1,ξ_2}. Then $h_V(1)drdt$ is the probability of intersecting V_{ξ_1,ξ_2} by the temporal line process in $(r,r+dr)\times(t,t+dt); \forall r \in (0,\overline{R}),\, t>0$.

A point (a ship) starts moving along V_{ξ_1,ξ_2} at $\xi_1, t=0$ with a given speed $v(t)$. We assume that an accident happens with a given probability when "it intersects" the line from the (initial) temporal line process. A regularization procedure, involving dimensions (of a ship, in particular) can be performed in the following way: an attraction interval

$$(r-\gamma_{ta}^-, r+\gamma_{ta}^+) \subset V_{\xi_1,\xi_2},\;\; \gamma_{ta}^+,\gamma_{ta}^- \geq 0,\;\; \gamma_{ta}(r) = \gamma_{ta}^+(r) + \gamma_{ta}^-(r) << \overline{R},$$

where the subscript "ta" stands for tangential, is introduced. The attraction interval (which can be defined by the ship's dimensions) is moving along the route, attached to the point itself with changing in time coordinate:

$$r(t) = \int_0^t v(s)ds, \quad t \leq t_{\overline{R}}, \tag{16}$$

where $t_{\overline{R}}$ is the total time on the route. Similar to (5), we can construct the *equivalent rate of intersections*, $h_{e,m}(r)$, assuming for simplicity constant speed $v(t)=v_0$ and γ_{ta}:

$$E\left[N_V\left((r,r+\Delta r),\Delta t\right)\right]=h_V(1)\Delta r\Delta t\ . \tag{17}$$

Thus the equivalent rate is also constant

$$h_{em}=\Delta t\, h_V(1)=\frac{\gamma_{ta}}{v_0}h_V(1), \tag{18}$$

where $\Delta t=\dfrac{\gamma_{ta}}{v_0}$ is the time needed for the moving attraction interval to pass the interval $(r,r+\Delta r)$ as $\Delta r\to 0$. As assumed earlier, the intersection can lead to an accident. Let the corresponding probability of an accident θ_m be also constant. Then, using results of sections 1 and 2, the probability of moving along the route V_{ξ_1,ξ_2} without accidents is:

$$P(\overline{R})=\exp\{-\theta_m h_{e,m}\overline{R}\}\ , \tag{19}$$

The non-linear generalization is rather straightforward. The line route V_{ξ_1,ξ_2} turns into the continuous curve R_{ξ_1,ξ_2} and lines of the stochastic line process turn also into continuous curves. Eventually

$$P(\overline{R})=\exp\left\{-\int_0^{\overline{R}}\theta_m(r)h_{em}(r)dr\right\} \tag{20}$$

and assuming independence of fixed and moving obstacles, relations (7) and (20) can be combined in an obvious way.

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