MEASURING RISK

Novosyolov A.
●
Institute of computational modeling SB RAS
Krasnoyarsk, Russia, anov@ksc.krasn.ru

Abstract. Problem of representation of human preferences among uncertain outcomes by functionals (risk measures) is being considered in the paper. Some known risk measures are presented: expected utility, distorted probability and value-at-risk. Properties of the measures are stated and interrelations between them are established. A number of methods for obtaining new risk measures from known ones are also proposed: calculating mixtures and extremal values over given families of risk measures.

Keywords: risk, risk measure, preference, expected utility, distorted probability, value-at-risk, mixture transform, extremal transforms.

INTRODUCTION

Quantifying risk is one of central problems in risk theory [1,2]. Risk measures are commonly used for the purpose. As of now there is a vast amount of different risk measures, including simple probability of an adverse event, second order measures (variance or standard deviation, beta) [3], quantile measures such as value-at-risk and its derivatives: conditional value-at-risk [4], expected shortfall [5] and some modifications.

A classic expected utility [6] is used more and more intensively in financial markets and other decision-making applications. Expected utility measures do form a wide class of risk measures, thus providing a flexible tool for decision-making under uncertainty. However any measure in this class is linear with respect to mixture of probability distributions, that may be undesirable in some cases.

Another wide class of risk measures was introdiced in [7]; these are the so called distorted probability measures. Fortunately they turned out to be nonlinear with respect to mixtures, so that they can represent human preferences in a more reliable fashion. One more attractive feature of risk measures in this class is that they are closely connected with other measures by some natural transforms. The latter provokes rigorous studying of the measures as possibly most appropriate tool for risk theory applications.

In the present paper we briefly describe properties of risk measures mentioned above and point out to interrelations among them. The following notation will be used throughout the paper. $(\Omega, \mathbf{A}, P)$ denotes a probability space with σ-algebra $\mathbf{A}$ and probability measure P; the latter may vary in some cases. *Risks* are represented by random variables, that is, measurable mappings from Ω to the measurable space $(\mathbf{R}, \mathbf{B})$, where $\mathbf{R}$ is the set of real numbers and $\mathbf{B}$ is a σ-algebra of its Borel subsets. Risks will be denoted by $X, Y, \ldots$ while their distribution functions by F_X, F_Y, etc. Denote also $\mathbf{X}$ the set of all risks and $\mathbf{F}$ the set of all distribution functions. *Risk measure* is any functional on $\mathbf{F}$. Introduce also partial orderings on $\mathbf{F}$ known as stochastic dominance of different orders. For a distribution function $F \in \mathbf{F}$ let $F^1 = F$ and $F^{(k)}$ for $k = 2,3,...$ are defined iteratively by

$$F^{(k)}(x) = \int_{-\infty}^{x} F^{(k-1)}(t)dt, \; x \in \mathbf{R}.$$

For $F, G \in \mathbf{F}$ we say that F preceeds G in the sense of stochastic dominance of order k ($F \leq_k G$) if $F^{(k)}(x) \geq G^{(k)}(x)$, $x \in \mathbf{R}$. For future reference denote W_a a degenerate (at a point $a \in \mathbf{R}$) distribution function and B_p a Bernoulli (with parameter $p \in (0,1)$) distribution function. Let also $\mathbf{W} = \{W_a, a \in \mathbf{R}\}$ be the class of all degenerate distribution functions, and $\mathbf{Be} = \{B_p, p \in (0,1)\}$ be the class of all Bernoulli distribution functions.

RISK MEASURES AND PREFERENCE

Let $\prec$ be a preference relation on $\mathbf{F}$, that is, a complete and transitive binary relation. We say that risk measure $\mu : \mathbf{F} \to \mathbf{R}$ represents the preference relation if for $F, G \in \mathbf{F}$

$$F \prec G \Leftrightarrow \mu(F) \le \mu(G) \qquad (1)$$

A perfect risk measure should represent preferences of specific individual or decision-maker. Since it is very unlikely that preference relation is known completely, representation theorems are usually based on reasonable assumptions that restrict the collection of available preferences to a rather narrow class, and provide an analytical form for risk measures representing preferences from that class.

EXPECTED UTILITY MEASURE

Perhaps the first representation theorem of the sort is due to von Neumann and Morgenstern [6]. It states that under some assumptions (that actually mean linearity of preference with respect to mixture of distributions) there exists the unique (up to positive affine transforms) risk measure representing the relation. The resulting risk measure turns out to be the so called expected utility, that was well known for about 3 hundred years already. It has the form

$$\rho(F) = \int_{-\infty}^{\infty} U(x)dF(x), \; F \in \mathbf{F}, \qquad (2)$$

where U stands for utility function. Different preferences correspond to different utility functions. A disadvantage of risk measure (2) is that it is always linear with respect to mixture of distributions, that is, for any $F, G \in \mathbf{F}$ and any $\alpha \in [0,1]$ the following is always true:

$$\rho(\alpha F + (1-\alpha)G) = \alpha\rho(F) + (1-\alpha)\rho(G).$$

Experiments (eg. [8]) show that in many cases human preferences do not possess linearity, so risk measure (2) might be a very rough approximation to what is actually needed.

Let us state some properties of expected utility here. The following theorem may be found e.g. in [9].

Theorem 1. Expected utility ρ is monotone with respect to stochastic dominance of the 1st order if and only if the utility function U is nondecreasing. Expected utility ρ is monotone with respect to stochastic dominance of the 2nd order if and only if the utility function U is nondecreasing and concave.

A question of great practical importance is: how much additional information do one need to identify the utility function U that generate expected utility representing a specific linear preference relation? In other words, what is the characteristic class $\mathbf{G} \subseteq \mathbf{F}$ of distribution functions such that there exists the unique continuation of expected utility from $\mathbf{G}$ to $\mathbf{F}$? The final answer is contained in the following

Theorem 2. Let $\prec$ be a linear preference relation on $\mathbf{F}$. To specify the utility function representing $\prec$ it is necessary and sufficient to know values of ρ for all degenerate distributions $W_a, a \in \mathbf{R}$, except for any two of them that may be chosen arbitrarily. This means that $\mathbf{W}$ is essentially the characteristic class for expected utility measure.

It is sometimes more convenient to use the so called certainty equivalent instead of expected utility, The new functional c on $\mathbf{F}$ is defined as a real number posessing the same utility as a distribution, that is:

$$c(F) = U^{-1}(\rho(F)), \quad F \in \mathbf{F}.$$

The functional is well defined if utility function is strictly monotone, that is often the case for preferences monotone with respect to stochastic dominance.

DISTORTED PROBABILITY MEASURE

Let $g:[0,1]\to[0,1]$ be a nondecreasing real function with $g(0)=0, g(1)=1$. A distorted probability measure

$$\pi(F)=\int_{-\infty}^{0}[g(1-F(x))-1]dx+\int_{0}^{\infty}g(1-F(x))dx, \quad F\in\mathbf{F} \tag{3}$$

was introduced in [7], [10]. A function g is called a distortion function. In the special case $g(x)=x, x\in[0,1]$ this measure coincides with expectation: $\pi(F)=E_F$, and in all other cases it is essentially nonlinear in distribution. Note that $\pi(W_a)=a$, $a\in\mathbf{R}$ for any parameter function g, and state the representation family theorem for the risk measure.

Theorem 3. Let $\prec$ be a preference relation on $\mathbf{F}$ corresponding to a distorted probability measure. To specify the parameter function g representing $\prec$ it is necessary and sufficient to know values of π for all nondegenerate Bernoulli distributions $B_p, p\in(0,1)$. This means that $\mathbf{Be}$ is essentially the characteristic class for distorted probability measure.

Note that distorted probability measure may be represented in the form

$$\pi(F)=-\int_{0}^{1}F^{-1}(v)dg(1-v), \quad F\in\mathbf{F} \tag{4}$$

Simple consequences of (4) are monotonicity of distorted probability measure with respect to first order stochastic dominance, and the fact that Value-at-risk measure

$$\tau_\lambda(F)=F^{-1}(\lambda),\ F\in\mathbf{F} \tag{5}$$

is a special case of (3) with

$$g_\lambda(v)=\begin{cases}0, & v<1-\lambda\\ 1, & v\geq 1-\lambda\end{cases}$$

where $\lambda\in(0,1)$ is a parameter.

FAMILY-GENERATED RISK MEASURES

Since risk measures are used to represent individual preference among probability distributions, they should catch attitude of an individual to risk. Constructing new risk measures may provide flexible tool for the purpose. In the present section several ways of obtaining new risk measures from given families are presented and studied to some extent.

Let Λ be a parameter set endowed with a structure of probability space $(\Lambda,\mathbf{C},Q)$. Next, let $\mathbf{\Lambda}=\{\mu_\lambda,\lambda\in\Lambda\}$ be a family of risk measures, id est, functionals $\mu_\lambda:\mathbf{F}\to\mathbf{R}$. Consider the following functionals generated using this family.

Mixture risk measure $\mathbf{M}_\Lambda:\mathbf{F}\to\mathbf{R}$.

$$\mathbf{M}_\Lambda(F)=\int_\Lambda\mu_\lambda(F)dQ(\lambda), \quad F\in\mathbf{F}. \tag{6}$$

Maximal risk measure $M^\Lambda:\mathbf{F}\to\mathbf{R}$.

$$M^{\Lambda}(F) = \sup_{\lambda \in \Lambda} \mu_{\lambda}(F), \quad F \in \mathbf{F}. \tag{7}$$

Minimal risk measure $M_{\Lambda} : \mathbf{F} \to \mathbf{R}$.

$$M_{\Lambda}(F) = \inf_{\lambda \in \Lambda} \mu_{\lambda}(F), \quad F \in \mathbf{F}. \tag{8}$$

Now let us state some results for these derivative measures.

Theorem 4. Let $\mathbf{\Lambda}$ be a family of risk measures such that each $\mu_{\lambda}, \lambda \in \Lambda$ is expected utility measure with utility function U_{λ}. Then mixture risk measure (6) is also an expected utility measure with utility function $U(x) = \int_{\Lambda} U_{\lambda}(x) dQ(\lambda), x \in \mathbf{R}$.

However extremal measures (7) and (8) for a family of expected utilities do not in general constitute an expected utility. Informally the class of expected utilities is closed with respect to mixtures and is not closed with respect to taking exprema.

Theorem 5. Let $\mathbf{\Lambda}$ be a family of risk measures such that each $\mu_{\lambda}, \lambda \in \Lambda$ is distorted probability measure with distortion function g_{λ}. Then mixture risk measure (6) is also a distorted probability measure with distortion function $g(v) = \int_{\Lambda} g_{\lambda}(v) dQ(\lambda), v \in [0,1]$.

So the class of distorted probability measures of risk is also closed with respect to mixtures. The following fact is somewhat surprising: any distorted probability measure may be represented by a mixture of Value-at-risk measures, a very special case of distorted probability measures.

Theorem 6. Let $\Lambda = (0,1)$ be endowed with a probability space structure by σ-algebra of Borel subsets and a distribution function G. Let further $\mathbf{\Lambda}$ be a family of Value-at-risk measures (5). Then the mixture risk measure (6) is a distorted probability measure with distortion function $g(v) = 1 - G(1 - v), v \in (0,1)$.

Clearly any distortion function g may be obtained by appropriate choice of mixing distribution function $G(v) = 1 - g(1 - v), v \in (0,1)$. Note that a similar spectral representation of distorted probability measures via expected shortfall family was presented in [5].

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