RISK ANALYSIS ON THE BASIS OF PARTIAL INFORMATION ABOUT QUANTILES

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Abstract. Risk analysis under partial information about probability distributions of states of nature is studied. An efficient method is proposed for a case when initial information is elicited from experts in the form of interval quantiles of an unknown probability distribution. This method reduces a difficult to handle non-linear optimisation problem for computing the optimal action to a simple linear one. A numerical example illustrates the proposed approach.

Keywords: risk analysis, imprecise probability theory, optimisation problem, quantile, decision making, randomized strategy, expert opinions

INTRODUCTION

One of the main objectives of performing risk analyses is to support decision-making processes. Risk analysis provides a basis for comparing alternative concepts, actions or system configurations under uncertainty. A variety of methods has been developed for estimating losses and risks. When events occur frequently and when they are not very severe, it is relatively simple to estimate the risk exposure of an organization, as well as a reasonable premium when, for instance, an insurance transaction is made. Commonly used methods rely on variations of the principle of maximizing expected utility, tacitly assuming that all underlying uncertainty can adequately be described by a precise and completely known probability measure. However, when the uncertainty is complex and the quality of the estimates is poor, e.g., when evaluating low-probability, catastrophic events, the customary use of such rules together with overprecise data could be harmful as well as misleading. Therefore, it is necessary to extend the principle of maximizing expected utility to deal with complex uncertainty. This allows powerful evaluation under vague and numerically imprecise information. An efficient way for realizing such methods is the framework provided by imprecise probability theory [3,5,6].

Very often the initial data about unwanted events are elicited from experts, who are typically asked about quantiles of a random quantity (states of nature). Based on this information, and on the choice of a parameterized family of distribution functions, a fitted distribution function is chosen that represents the available information in some best way to some extent. However, as pointed out, for instance, in [2], experts better supply intervals of quantiles rather than point-values because their knowledge is not only of limited reliability, but also imprecise. Moreover, as discussed above, the choice of one particular distribution function fitted to the quantiles would lead to substantial errors in risk analysis. Therefore, new procedures for computing optimal actions under conditions of partial information about states of nature in the form of imprecise quantiles are proposed in the paper. Efficient methods for computing optimal unrandomized and randomized actions based on solving the linear optimisation problems are investigated. A numerical example illustrates the methods.

A GENERAL APPROACH TO RISK ANALYSIS AND IMPRECISE QUANTILES

Consider the basic model of decision theory: One has to choose an action from a non-empty, finite set $A=\{a_1,...,a_n\}$ of possible actions. The consequences of every action depend on the true, but unknown state of nature $t \in \Omega = \{t_1,...,t_m\}$. The corresponding outcome is evaluated by the utility function

$$u: \quad (A \times \Omega) \to \mathbf{R}$$
$$(a,t) \mapsto u(a,t)$$

and by the associated random variable u(a) on $(\Omega, Po(\Omega))$ taking the values $u(a,t)$. Alternatively a loss function $l(a,t)$ is assigned, which can be embedded into the framework proposed by setting $u(a,t) = -l(a,t)$. Often it makes sense to study randomized actions, which can be understood as a probability measure $\lambda = (\lambda_1,...,\lambda_n)$ on $(A, Po(A))$. Then $u(\cdot)$ and $\mathrm{u}(\cdot)$ are extended to randomized actions by defining $u(\lambda,t) := \sum_{s=1}^{n} u(a_s,t)\lambda_s$.

If the states of nature are produced by a perfect random mechanism (e.g. an ideal lottery), and the corresponding probability measure p on $(\Omega, Po(\Omega))$ is completely known, the Bernoulli principle is nearly unanimously favored. One chooses that action λ^* which maximizes the expected utility $\mathrm{E}_p\,\mathrm{u}(\lambda) := \sum_{j=1}^{m} u(\lambda,t_j)p(t_j)$ among all λ. Here E_p is the expectation operator with respect to the distribution p.

Suppose that information about states of nature is represented as a set of r judgements $\underline{b}_i \le \mathrm{E}_p f_i \le \overline{b}_i$, i=1,...,r, on the expectations of some random quantities $f_1,...,f_r$. This set restricts all distributions p on $(\Omega, Po(\Omega))$ by a set M such that every distribution p from M satisfies all the inequalities. An action λ^* is optimal iff for all λ, $\underline{\mathrm{E}}_M\,\mathrm{u}(\lambda^*) \ge \underline{\mathrm{E}}_M(\mathrm{u}(\lambda))$. Here $\underline{\mathrm{E}}_M$ is the lower prevision (expectation) taken over all probability distributions p from M. Then the optimal action λ^* can be obtained by maximizing $\underline{\mathrm{E}}_M(\mathrm{u}(\lambda))$ subject to $\lambda_1 + ... + \lambda_n = 1$. This leads to the non-linear optimisation problem:

$$\min_{p \in M} \sum_{j=1}^{m} \sum_{s=1}^{n} u(a_s,t_j) \cdot \lambda_s \cdot p(t_j) \to \max_{\lambda_s \ge 0} \qquad (1)$$

subject to $\underline{b}_i \le \mathrm{E}_p f_i \le \overline{b}_i$, i=1,…,r, $\lambda_1 + ... + \lambda_n = 1$.

Similar expressions can be written in a case of the continuous set of states of nature $\Omega = [A,B]$. In this case, the expected utility is $\mathrm{E}_p\,\mathrm{u}(\lambda) := \int_A^B u(\lambda,t)p(t)dt$. Here $p(t)$ is a density function which is consistent with the set of initial judgements about states of nature. In the paper, we will consider the continuous set of states of nature.

In the probabilistic approach, experts are typically asked about quantiles of a random variable X defined on a continuous sample space Ω. The smallest number $t \in \Omega$, such that $\Pr\{X \le t\} = k/100$, is called the k% quantile and denoted qk%. In this approach, the experts are often asked to supply the 5%, 50% and 95% quantiles. In other words, an expert supplies t_1, t_2, t_3 such that $\Pr\{X \le t_1\} = 0.05$, $\Pr\{X \le t_2\} = 0.5$, $\Pr\{X \le t_3\} = 0.95$, respectively. Generally, if r experts provide their judgements about q_i quantiles, i=1,...,r, of an unknown cumulative probability distribution of the continuous random variable X, this information can be represented as $\Pr\{X \le t_i\} = q_i$, i=1,...,r. In terms of the imprecise probability theory, q_i can be viewed as identical lower and upper previsions (expectations) of the gamble $I_{[0,t_i]}(X)$, i.e., $\underline{\mathrm{E}}I_{[0,t_i]}(X) = \overline{\mathrm{E}}I_{[0,t_i]}(X)$. Here $I_{[0,t_i]}(X)$ is the indicator function taking the value 1 if $X \in [0,t_i]$ and 0 if $X \notin [0,t_i]$. However, judgements elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. In other words, experts provide some intervals of quantiles in the form $X_i = [\underline{t}_i, \overline{t}_i]$. This can be formally written as

$$\Pr\{X \leq [\underline{t}_i, \overline{t}_i]\} = q_i,\ i = 1,...,r \qquad (2)$$

Every interval X_i produces a set of probability distributions such that the lower distribution contains the point $q_i(\overline{t}_i)$ and the upper one contains the point $q_i(\underline{t}_i)$.

Decision making with imprecise quantiles

Let us define what $\underline{\mathrm{E}}_M\,\mathrm{u}(\lambda)$ means in the case when initial information about p is given in the form of quantile intervals. Suppose that we knew precise values of q_i quantiles t_i, i=1,...,r. Denote $T = (t_1,...,t_r)$ and the set of possible vectors T by $\{T\}$. Let $\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T)$ be the lower expectation of the function $u(\lambda)$ under condition of precise values T of quantiles. Since at least one of the points t_k belonging to the interval $X_i = [\underline{t}_i, \overline{t}_i]$ is a true value of the corresponding quantile, then there holds

$$\underline{\mathrm{E}}_M\,\mathrm{u}(\lambda) = \min_{\forall t \in X_i, i=1,...,r} \underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T).$$

By using the natural extension [3,4,5] for computing the lower prevision $\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T)$, we get the following linear programming problem:

$$\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T) = \max_{c, w_i}\left(c + \sum_{i=1}^{r} w_i q_i\right) \qquad (3)$$

subject to $w_i, c \in \mathbf{R}, i = 1,...,r,$ and $c + \sum_{i=1}^{r} w_i I_{[0,t_i]}(t) \leq u(\lambda, t), \forall t \in \Omega$.

UNRANDOMIZED STRATEGY

The unrandomized strategy supposes that $\lambda = (0,...,0,\lambda_s,0,...,0)$, $\lambda_s = 1$. Let us consider how to find the value s corresponding to the optimal action.

Proposition 1. Suppose that $q_1 \leq q_2 \leq ... \leq q_r$ and $\lambda = (0,...,0,\lambda_s,0,...,0)$, $\lambda_s = 1$. Denote q_0=0, q_{r+1}=1, t_0=A, t_{r+1}=B. Then the solution to problem (3) exists if (i) $t_1 \leq t_2 \leq ... \leq t_r$, (ii) $t_i < t_{i+1}$ for $q_i < q_{i+1}$, i=1,...,r, and this solution is

$$\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T) = \sum_{i=0}^{r} (q_{i+1} - q_i) \min_{t \in [t_i, t_{i+1}]} u(a_s, t).$$

Let us consider an approximate solution of the decision making problem in the case of interval quantiles. Let us divide the sample space Ω into N intervals by points $A = \tau_0, \tau_1, ..., \tau_{N-1}, \tau_N = B$. Then the set $\{T\}$ becomes finite and contains vectors of the form $(\tau_{l(0)}, \tau_{l(1)}, ..., \tau_{l(r)})$ such that $\tau_{l(i)} \in [\underline{t}_i, \overline{t}_i]$, i.e., $l(i)$ is an index of a point belonging to X_i.

Proposition 2. Suppose $t_i \in [\underline{t}_i, \overline{t}_i]$, i=1,...,r. If there exist such i and j that $\underline{t}_i > \overline{t}_j$ and $q_i < q_j$, then judgements are conflicting, otherwise the optimal action is

$$a_s = \arg\max_{s} \min_{s=1,...,n\ T \in \{T\}} \underline{\mathrm{E}}_M\,\mathrm{u} \cong \arg\max_{s} \min_{s=1,...,n\ (\tau_{l(0)}, \tau_{l(1)}, ..., \tau_{l(r)})} \sum_{i=0}^{r} (q_{i+1} - q_i) \min_{t \in [\tau_{l(i)}, \tau_{l(i+1)}]} u(a_s, t).$$

In particular, if all utility functions $u(a_s,t)$, s=1,…,n, are decreasing as t is increasing, then $a_{\text{opt}} = \arg\max_{s=1,\dots,n}\left(\sum_{i=0}^{r}(q_{i+1}-q_i)u(a_s,\overline{t}_{i+1})\right)$, if $u(a_s,t)$, s=1,…,n, are increasing, then $a_{\text{opt}} = \arg\max_{s=1,\dots,n}\left(\sum_{i=0}^{r}(q_{i+1}-q_i)u(a_s,\underline{t}_i)\right)$.

RANDOMIZED STRATEGY

The technique proposed in the previous sections leads to a series of non-linear optimisation problems in the case of the randomized strategy. Therefore, it is necessary to consider a different method for computing λ. Here the modification of an approach proposed by Augustin [1] based on using sets of extreme points is applied. The optimisation problem for computing the optimal randomised action is

$$\min_{p\in M}\int_A^B\left(\sum_{s=1}^{n}u(a_s,t)\lambda_s\right)p(t)dt \to \max_{\lambda_s\geq 0}, \qquad (4)$$

subject to $\lambda_1+\dots+\lambda_n=1$ and $\Pr\{X\leq[\underline{t}_i,\overline{t}_i]\}=q_i$, $i=1,\dots,r$.

Let us introduce the variable

$$G=\min_{p\in M}\int_A^B\left(\sum_{s=1}^{n}u(a_s,t)\lambda_s\right)p(t)dt$$

and consider the sense of (2). If to call the expectation $\underline{\mathrm{E}}_M(\mathrm{u}(\lambda)\,|\,T)$ and the set of constraints $\Pr\{X\leq t_i\}=q_i$, i=1,...,r, for every fixed T by an imprecise model, then (2) corresponds to the union of a set of imprecise models taken over all possible vectors T, i.e., the set M of distributions p restricted by constraints (1) is the union of sets M_T. According to [3], a set of extreme points of the united model is the union of extreme points of the imprecise models corresponding to vectors T, i.e., $extr(M)=\cup_{T\in\{T\}}extr(M_T)$. This implies that a set of problems (4) can be reduced to the problem:

$$\max_{\lambda_s\in\mathbf{R}_+,G\in\mathbf{R}} G$$

subject to

$$G\leq\int_A^B\left(\sum_{s=1}^{n}u(a_s,t)\lambda_s\right)p(t)dt,\ p\in\bigcup_{T\in\{T\}}extr(M_T),\ \sum_{s=1}^{n}\lambda_s=1. \qquad (5)$$

Now we have to find the extreme points for each $T\in\{T\}$. Let us rewrite the available information about quantiles corresponding to T in the following form:

$$\int_A^{t_1}p(t)dt=q_1,\ \int_{t_1}^{t_2}p(t)dt=q_2-q_1,\dots,\int_{t_r}^{B}p(t)dt=1-q_r \ .$$

All equalities can be considered independently in the sense that they do not have common variables. If we approximately represent the integrals as sums, then the i-th hyperplane produced by the i-th equality has the following extreme points:

$$(q_i-q_{i-1},0,\dots,0),\ (0,q_i-q_{i-1},\dots,0),\dots,(0,0,\dots,q_i-q_{i-1}).$$

Hence the set M_T has the extreme points of the form:

$$p(t)=\sum_{i=0}^{r}(q_{i+1}-q_i)\delta(t-\tau_i),\ \tau_i\in[t_i,t_{i+1}],$$

where $\delta(t-\tau_i)$ is the Dirac function which has unit area concentrated in the immediate vicinity of the point τ_i; $t_0=A$, $t_{r+1}=B$, $q_0=0$, $q_{r+1}=1$.

After substituting these extreme points into constraints (5), we get

$$G \leq \sum_{s=1}^{n} \lambda_s \sum_{i=0}^{r} \int_{t_i}^{t_{i+1}} u(a_s, t) p(t) dt, \forall p \in extr(M_T).$$

If we take one set of extreme points by fixed T, then there holds

$$G \leq \sum_{s=1}^{n} \lambda_s \sum_{i=0}^{r} u(a_s, \tau_i)(q_{i+1} - q_i), \forall \tau_i \in [t_i, t_{i+1}]. \tag{6}$$

Let us consider an approximate solution of the decision making problem in the case of interval quantiles. By dividing the sample space Ω into N intervals by points $A = \tau_0, \tau_1, ..., \tau_{N-1}, \tau_N = B$ (see the section "Unrandomized strategy"), we get a finite set of constraints

$$G \leq \sum_{s=1}^{n} \lambda_s \sum_{i=0}^{r} u(a_s, \tau_i)(q_{i+1} - q_i), \forall \tau_i \in [\tau_{l(i)}, \tau_{l(i+1)}], \ \forall l(i) \tag{7}$$

Proposition 3. Suppose $t_i \in [\underline{t}_i, \overline{t}_i]$, i=1,...,r. If there exist such i and j that $\underline{t}_i > \overline{t}_j$ and $q_i < q_j$, then judgements are conflicting, otherwise the optimal randomized action is approximately defined by solving the following linear programming problem:

$$\max_{\lambda_s \in \mathbf{R}_+, G \in \mathbf{R}} G,$$

subject to (7) and $\sum_{s=1}^{n} \lambda_s = 1$.

Since the right side of (7) has to be as small as possible and $q_{i+1} - q_i \geq 0$, then the set of constraints is reduced to one constraint in the case of increasing or decreasing utility functions. By considering the set $\{T\}$, we can say that constraints (7) have to be satisfied for arbitrary values t_i and t_{i+1} such that $t_i \in X_i$ and $t_{i+1} \in X_{i+1}$, i=0,...,r. It is obvious that $\min_{\tau_i \in [t_i, t_{i+1}]} u(a_s, \tau_i)$ by $t_i = \underline{t}_i$ for increasing utility functions (by $t_i = \overline{t}_{i+1}$ for decreasing utility functions) is less than by any $t_i \geq \underline{t}_i$ ($t_i \leq \overline{t}_i$). This implies that we remain one constraint

$$G \leq \sum_{s=1}^{n} \lambda_s \sum_{i=0}^{r} (q_{i+1} - q_i) u(a_s, \underline{t}_i) \tag{8}$$

in the case of increasing utility functions or one constraint

$$G \leq \sum_{s=1}^{n} \lambda_s \sum_{i=0}^{r} (q_{i+1} - q_i) u(a_s, \overline{t}_{i+1}) \tag{9}$$

in the case of decreasing utility functions.

Proposition 4. The linear optimisation problems with constraints (8) or (9) have the following solution:

$$G = \max_{s=1,...,n} \sum_{i=0}^{r} (q_{i+1} - q_i) u(a_s, \tau_i),$$

$$\lambda_k = \begin{cases} 1, & k = \arg_s \max_{s=1,...,n} \sum_{i=0}^{r} (q_{i+1} - q_i) u(a_s, \tau_i) \\ 0, & otherwise \end{cases},$$

where $\tau_i = \underline{t}_i$ for increasing utility functions, $\tau_i = \overline{t}_{i+1}$ for decreasing utility functions.

Proposition 4 implies that the randomised optimal action for the considered decision problem is equivalent to the unrandomized one (see Proposition 2).

NUMERICAL EXAMPLE

Suppose experts provide 5%, 50%, 95% quantiles of the probability distribution of a random variable defined on the sample space $\Omega = [0, 120]$. This implies r=3 and q_1=0.05, q_2=0.5, q_3=0.95. Expert judgements are given in Table 1. Suppose we have to choose one of two actions $\{a_1, a_2\}$ in accordance with utility functions

$$u(a_1,t) = \exp(-0.1t),\ u(a_2,t) = 0.5 - 0.012x .$$

Table 1. Interval quantiles provided by experts

5%		50%		95%	
Lower	Upper	Lower	Upper	Lower	Upper
2	4	12	15	19	19

Since the utility functions are decreasing, it follows from Proposition 4 or Proposition 2 that

$$k = \arg\max_{s} \max_{s=1,\dots,n} \sum_{i=0}^{r} (q_{i+1} - q_i) u(a_s, \overline{t}_{i+1}) .$$

If s=1, then we get the lower expected utility

$$(q_1 - q_0)u(a_1,\overline{t}_1) + (q_2 - q_1)u(a_1,\overline{t}_2) + (q_3 - q_2)u(a_1,\overline{t}_3) + (q_4 - q_3)u(a_1,\overline{t}_4)$$
$$= (0.05-0)\exp(-0.1\times 4) + (0.5-0.05)\exp(-0.1\times 15)$$
$$+(0.95-0.5)\exp(-0.1\times 19) + (1-0.95)\exp(-0.1\times 20) = 0.208.$$

If s=2, then the lower expected utility is

$$(q_1 - q_0)u(a_2,\overline{t}_1) + (q_2 - q_1)u(a_2,\overline{t}_2) + (q_3 - q_2)u(a_2,\overline{t}_3) + (q_4 - q_3)u(a_2,\overline{t}_4)$$
$$= (0.05-0)(0.5-0.012\times 4) + (0.5-0.05)(0.5-0.012\times 15)$$
$$+(0.95-0.5)(0.5-0.012\times 19) + (1-0.95)(0.5-0.012\times 20) = 0.302.$$

The above numerical results imply that the optimal action is a_2.

SOME REMARKS ABOUT DISCRETE STATES OF NATURE

If the set of states of nature is discrete, $\Omega = \{t_1,\dots,t_m\}$, then information about interval quantiles can be represented as

$$q_i \le \Pr\{X \le [\underline{t}_i, \overline{t}_i]\} \le q_{i+1},\ 1 - q_i \le \Pr\{X \ge [\underline{t}_i, \overline{t}_i]\} \le 1 - q_{i-1}, i = 1,\dots,r .$$

In this case, the linear programming problem for computing $\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T)$ is of the form:

$$\underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T) = \max_{c,c_i,d_i w_i,v_i} \left(c + \sum_{i=1}^{r} (c_i q_i - d_i q_{i+1} + w_i(1-q_i) - v_i(1-q_{i-1})) \right)$$

subject to $c_i, d_i w_i, v_i \in \mathbf{R}_+, c \in \mathbf{R},\ i = 1,\dots,r,$ and

$$c + \sum_{i=1}^{r} \left((c_i - d_i) I_{[0,t_i]}(t) + (w_i - v_i) I_{[t_i,m]}(t) \right) \le u(\lambda,t), \forall t \in \Omega .$$

Generally, it is difficult to find any solution to the above problem in the explicit form. However, this problem can be numerically solved for every $T \in \{T\}$, and the optimal action is computed by maximizing $\min_{T\in\{T\}} \underline{\mathrm{E}}_M(\mathrm{u}(\lambda) \mid T)$ over all possible actions.

CONCLUSION

Computationally simple algorithms have been obtained for calculating optimal actions under partial information about quantiles. It is worth noticing that we have focused in this paper on the basic decision problem. However, the ideas of this paper should be also applicable to more complex decision problems, for example, multi-criteria decision making, or the case where additional sample information is available.

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