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Dear members of the Gnedenko Forum:

I got a complain from a group of specialists of Maritime University, Gdynia, Poland (M.Sc. Krzysztof Dziedzicki, M.Sc. Marcin Tobiasz, and PhD. Mirosław Tomera) concerning the paper by Professor Roman Śmierzchalski in our journal "Reliabiity and Risk Analysis Theory and Applications", 2008, No.2, Vol.1.

The mentioned above group wrote that the paper "SAFE SHIP CONTROL SYSTEM" by R. Śmierzchalski (page 127) presents a word-in-the-word copy of a collective report.

After correspondence with the both sides involved in the conflict, we decided to exclude the mentioned paper from No. 2 of 2008 and delete it from the archive of our e-journal.

Igor Ushakov,
Editor-in-Chief

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The concept of "probabilistic logic" known in artificial intelligence needs a more thorough substantiation. A new approach to constructing probabilistic logic based on the N-tuple algebra developed by the author is proposed. A brief introduction is given to the N-tuple algebra and its properties that provide efficient paralleling of algorithms for solving problems of logical analysis of systems in computer implementation are generalized. Methods for solving direct and inverse problems of probabilistic simulation of logical systems are considered.

In this paper, the problem of finding sampling distributions for truncated laws is considered. This problem concerns the very important area of information processing in Industrial Engineering. It remains today perhaps the most difficult and important of all the problems of mathematical statistics that require considerable efforts and great skill for investigation. The technique discussed here is based on use of the unbiasedness equivalence principle, the idea of which belongs to the authors, and often provides a neat method for finding sampling distributions. It avoids explicit integration over the sample space and the attendant Jacobian but at the expense of verifying completeness of the recognized family of densities. Fortunately, general results on completeness obviate the need for this verification in many problems involving exponential families. The proposed technique allows one to obtain results for truncated laws via the results obtained for non-truncated laws. It is much simpler than the known techniques. The examples are given to illustrate that in many situations this technique allows one to find the results for truncated laws and to estimate system reliability in a simple way.

The bootstrap method is an extensive computational approach, based on Monte Carlo simulation, useful for understanding random samples and time series. It is a powerful tool, especially when only a small data set is used to predict the behaviour of systems or processes. This paper presents the results of an investigation on using bootstrap resampling (different types: uniform, importance based, block structured etc.) for time series appearing during software life cycle (mainly the software testing phase, and debugging), economics, and environment (air pollution generated by cement plants) in order to help the activity of staff working on risk management for software projects, risk management in finance, and those working on environment risk management.

In this paper financial management model of forward contracts insurance suggested in some works is considered by means of risk theory and heavy tailed technique. This model is based on a compensation principle. It attracted large interest and called active discussion among economists. So its mathematical analysis is initiated as economists so mathematicians.

This paper is devoted to algorithms of a calculation of ports reliabilities. A port is a no oriented graph with fixed initial and final nodes. As accuracy so asymptotic formulas are considered. Suggested algorithms have minimal numbers of arithmetical operations.

The main idea of Universal Generating Function is exposed in reliability applications. Some commonalities in this approach and the C++ language are discussed.
Igor Ushakov

Method of finding optimal spare stock for Mobile Repair Station is suggested. Numerical calculations are performed with use of real field data. It showed significant improvement: probability of first fix for suggested variant is 0.967 in comparison with 0.534 for existing variant.

The offered mathematical models and supporting them software tools complexes (M&STC) are purposed for a systems analysts from customers, designers, developers, users, experts of testing laboratories and certification bodies, as well as a staff of quality maintenance for any complex system etc. M&STC are focused on providing system standard requirements on the base of modeling random processes that exist for the life cycle of any complex system. Models implement original author's mathematical methodology based on probability theory, theory for regenerating processes and methods for system analysis. M&STC may be also used in training and education for specializations "System engineering", "Software engineering", "System safety and security", "Information systems".

BAYES-FIDUCIAL APPROUCH FOR AIRCRAFT SPECIFIED LIFE NOMINATION

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ABSTRACT

The problem of nomination of Retirement or Specified Life (SL) of aircraft on the base of full-scale fatigue test result processing is considered. SL can be defined (1) by requirement of fatigue failure probability limitation or (2) by economics reasons. For optimization problem the Bayes-fiducial (BF) approach is offered. BF decision is always a function of sufficient statistics and, by contrast with maximum likelihood method, it is based on the use of specific loss function. For the problem of failure probability limitation in case when sufficient statistics coincides with the sample itself (for example, for Weibull distribution) usually the Monte Carlo method is used but in this paper for the distributions with location and scale parameters an analytical solution is offered.

Some numerical examples for lognormal, Weibull distributions are given.

KEYWORDS

Bayes, p-bound, prediction limit, quantile, lognormal, Weibull, optimization

G. INTRODUCTION

In this paper we consider only the case when the operation reliability of aircraft is ensured by discarding the aircraft from service, if its service life exceeded the Retirement or Specified Life (SL). For discussion of inspection program (IP) development is planned another author paper but some short discussion of this problem already take place in [1,2,3,4].

There are at least two approaches to the SL choice on the base of experimental data: (1) it can be defined by requirement of fatigue failure probability limitation and (2) it can be defined by economics reasons. If the "weight" of loss induced by fatigue failure is estimated by some value b, which can be comparable with the "income" per service hour (it will be assumed, that the value of "income" per one service hour is equal to unit), then SL can be defined as operation time, corresponding to maximum of income expectation value. We'll consider both approaches. It should be mentioned also that SL can be chosen as (1) some number from $[0,\infty]$ and as (2) some number from set of two numbers $\{0, t^*_{SL}\}$. This corresponds to (1) nomination of Specified Life, t_{SL}, and (2) rejection or acceptance of predetermined (required) Specified Life, t^*_{SL}.

2. DEFINITION OF P-SET AND P-BOUND FOR RANDOM VARIABLES

To make possible the common approach for solution of the both problem SL nomination and IP development we need to remained the p-set function definition [4]. It is a special statistical decision function, which, in fact, is generalization of p-bound for random variable, definition of which was introduced by author some early [5,6,7].

P-set function is defined in following way.

Definition 1. Let Z and X are random vectors of m and n dimensions and we suppose that it is known the class $\{P_\theta\ ,\ \theta \in \Omega\}$ to which the probability distribution of the random vector *W=(Z,X)* is assumed to belong. Of the parameter θ, which labels the distribution, it is assumed known only that it lies in a certain set Ω, the parameter space. Let $S_Z(x) = \bigcup_{i=1}^{r} S_{Z,i}(x)$ denotes some set of disjoint sets of *z* values as function of *x*. If

$$\sup_{\theta} \sum_{i=1}^{r} P(Z \in S_{Z,i}(X)) = p \tag{1}$$

then statistical decision function $S_Z(x)$ is p-set function for r.v. Z on the base of a sample $x = (x_1, x_2, ..., x_n)$.

Remark. Later on the value *x*, observation of the vector *X*, would be interpreted as result of some test (for example, full-scale fatigue test of aircraft). For the problem SL nomination Z can be interpreted as some random variable equal to smallest fatigue life of N aircraft in service Z=min($Y_1, Y, ..., Y_N$). Then the problem is to find the function $\tau(x)$ for which

$$\sup_{\theta \in \Theta} P_\theta\{Z < \tau(X)\} = p.$$

For the problem of inspection planning Z would be interpreted as some random vector (T_d, T_c), where T_d, T_c are time moments when some fatigue crack become detectable or reaches critical size correspondingly. And in this case the problem is to find such sequence $t(x) = \{t_1(x), t_2(x), ...\}$ that

$$\sup_{\theta} \sum_{i=1}^{r} P(t_{i-1}(X) \le T_d < T_c < t_i(X)) = p\ ,$$

where $t_1(x) < t_2(x) < ... < t_r$ are time moments of inspections, t_0=0, t_r is time of aircraft odelling . The choice of odellin $t(x) = \{t_1(x), t_2(x), ...\}$ will be discussed in next paper but here we consider onle the problem of SL nomination.

For the most important case, when *m*=1 and *Z* is a random scalar, there are several useful definitions of special types of p-set functions $S_Z(x)$ which for this special case we denote by $\tau(x)$.

Definition 2. P-set function $\tau(x)$ is called a p-bound for r.v. Z if

$$\sup_{\theta \in \Theta} P_\theta\{Z < \tau(X)\} = p\ . \tag{2}$$

Definition 3. P-bound $\tau(x)$ is called a parameter-free (p.f.) p-bound for r.v. Z if

$$P_\theta\{Z < \tau(X)\} = p \text{ for all parameters } \theta \in \Omega. \tag{3}$$

Definition 4. P-bound for r.v. *Z* is called a right-hand binary (r.h.b. p-bound), if for each possible observation *x* of r.v. *X*, function $\tau(x)$ assigns only one of two decisions:

$$\tau(x) = -\infty \text{ if } x \in S;\ \tau(x) = \tau^*, \text{ if } x \in S^*, \tag{4}$$

where τ^* is some number, S^* and *S* are two complementary regions of the sample space .

So we see that the definition of p-bound can be considered as some generalization of definition of prediction limit. But it is some statistical decision function which cover both prediction limit and, in some may, testing statistical hypotheses.

We can say also that p.f. p-bound $\tau(x)$ is a p-quantile estimate of cdf $F_Z(x)$ and, as function of p, it is an estimate of inverse cumulative distribution function $F_Z^{-1}(p)$, but very specific estimate: expectation value $E(F_Z(\tau(X))) = p$.

If $Z=Y_{(k)}$ is kth order statistic of independent observations taken on Y, say $Y_1, Y_2, \dots, Y_m$, and strictly increasing c.d.f. of r.v. Y, $F_Y(x, \theta)$, has the same unknown parameter θ as the cdf of X_i, $i=1,2,\dots,n$, $k = [\beta m]$, where $0<\beta<1$, $[x]$-is a maximum integer less or equal to x, $m \to \infty$ then approximately

$$P\{Y_{(k)} < \tau(X)\} = P\{F_Y^{-1}(\beta) < \tau(X)\} = P\{F_Y(\tau(X)) > \beta\} = p$$

and $(-\infty, \tau(x))$ *is* β - content tolerance region at confidence level p.

The binary p-set function has, evidently, some close connection with testing statistical hypotheses: S^* and S are two complementary regions of the sample space just as S_0 and S_1 in the problem of hypotheses testing [6]. But there is some difference. Instead of problem to maximize the power of a test at a fixed level of significance (probability of first type of error) this time we need to get the maximum of probability of decision that reliability requirements are met at the fixed limitation of product of the probability of failure and probability of wrong decision (we think that reliability requirements are met but they are not met):

$$\sup_{\theta} P(Z \le \tau^*) P(X \in S^*) \le p \ .$$

3. P-BOUND FOR DISTRIBUTION WITH LOCATION AND SCALE PARAMETERS

It is easy to get $\tau(x)$ for distribution with location and scale parameters. As the main application of the problem under question we'll consider a problem of SL nomination for some fatigue-prone airframe structure. We suppose to have observations of fatigue lives of some identical units of this structure as a result of full-scale fatigue tests. Usually for fatigue life data processing both a lognormal and Weibull distributions are used. If we'll use logarithm scale (if we'll use $X = \ln(T)$ instead of T) then both these distributions become distributions with location and scale parameters. So we can say, that r.v. X has following structure: $X=\theta_0+\theta_1 \overset{0}{X}$, where θ_0, θ_1 are unknown parameters, r.v. $\overset{0}{X}$ has either standard normal c.d.f. $F_{\overset{0}{X}}(x) = \Phi(x)$ or standardized smallest extreme value (sev) c.d.f. $F_{\overset{0}{X}}(x) = 1 - \exp(-\exp(x))$ for lognormal or Weibull distributions of T correspondingly. For this case for the specified life nomination problem following theorem can be used (we give it without proof).

Theorem 1. Let

$$F_{X_i}(x,\theta) = F_{\overset{o}{X}}\left(\frac{x-\theta_0}{\theta_1}\right), \mathrm{i} = 1,\dots,\mathrm{n}, \qquad F_Z(x,\theta) = F_{\overset{o}{Z}}\left(\frac{x-\theta_0}{\theta_1}\right), \tag{5}$$

where $F_{\overset{0}{X}}(\cdot)$, $F_{\overset{0}{Z}}(\cdot)$ are known c.d.f. of $\overset{0}{X}$, $\overset{0}{Z}$, θ_0, θ_1 – are unknown location and scale parameters. And let the random variables, estimations of θ_0, θ_1, as function of $X = (X_1, X_2, \dots, X_n)$ can be described by the similar structural formulas:

$$\hat{\theta}_0 = \theta_0 + \theta_1 \overset{o}{\theta}_0, \quad \hat{\theta}_1 = \theta_1 \overset{o}{\theta}_1, \tag{6}$$

where $\overset{o}{\theta}_0, \overset{o}{\theta}_1$ - are random variables, corresponding to the estimates of θ_0, θ_1 using a sample of the same size n but when $\theta_0=0$, $\theta_1=1$. We refer to this type of estimates as "correct" estimates.

Then p.f. and r.h.b. p-bounds are described accordingly by formulae

$$\tau_1(x) = \hat{\tau}_1, \quad \tau_2(x) = \begin{cases} -\infty, & \hat{\tau}_2 \le \tau^*, \\ \tau^*, & \hat{\tau}_2 > \tau^*, \end{cases} \tag{7}$$

where $\hat{\tau}_i = \hat{\theta}_0 + t_i\hat{\theta}_1,\ i = 1,2,$

t_1 is p -quantile of r.v. $V_Z = (Z - \overset{0}{\theta}_0)/\overset{0}{\theta}_1$, t_2 is the root of equation : $\xi(t)=p$,

$$\xi(\mathrm{t}) = \sup_{\mathrm{c}} F_{\overset{o}{Z}}(c)(1 - F_{\overset{o}{\tau}(t)}(c)) = \sup_{\mathrm{c}} F_{\overset{o}{Z}}(c)F_{V_C}(t),$$

$$\overset{\mathrm{o}}{\tau}(t) = \overset{o}{\theta}_0 + \overset{o}{\theta}_1 t, \quad \mathrm{V_C} = (c - \overset{o}{\theta}_0)/\overset{o}{\theta}_1.$$

2. If one of the parameters θ_1 or θ_0 is known, then, as usually, we can transform the initial data ($x'_i = x_i/\theta_1$ *or* $x'_i = x_i - \theta_0,\ i \in 1,...,n$) in such a way that in previous formulae for τ we can put $\hat{\theta}_1 = 1$ *or* $\hat{\theta}_0 = 0$, and then

2.1 If it is known that the scale parameter $\theta_1 = 1$ then V_Z, V_C should be replaced by

$$U_Z = \overset{o}{Z} - \overset{o}{\theta}_0, \quad U_C = C - \overset{o}{\theta}_0;$$

function $\xi(t)$ should be replaced by the function $\xi_1(t) = \max_C F_{\overset{0}{Z}}(C)F_{U_C}(t)$,

but for $X = \min(X_1, X_2, ..., X_n)$ by function $\xi_1^1(t) = \max_C F_{\overset{0}{Z}}(C)(1 - F_{\overset{0}{X_1}}(t))^n$.

2.2. If it is known that the location parameter $\theta_0 = 0$ then V_Z, V_C should be replaced by

$$W_Z = \overset{o}{Z}/\overset{o}{\theta}_1, \quad W_C = C/\overset{o}{\theta}_1,$$

function $\xi(t)$ by the function $\xi_0(t) = \max_C F_{\overset{o}{Z}}(c)F_{W_C}(t)$, but if additionally

$X = \min(X_1, X_2, ..., X_n)$ by function $\xi_0^1(t) = \max_C F_{\overset{o}{Z}}(c)(1 - F_{\overset{0}{X}}(C/t))^n$.

◁ ◁

Let us remind that for the purpose of approximate calculation of c.d.f. for V_C, U_C, W_C the Monte Carlo method can be used or normal approximation of distributions of estimations $\overset{o}{\theta}_0, \overset{o}{\theta}_1$.

4. APPLICATION OF P-BOUND TO THE PROBLEM OF THE SPECIFIED LIFE NOMINATION

4.1. OPTIMALITY CRITERION FOR P.F. P-BOUND USED FOR AIRCRAFT SPECIFIED LIFE NOMINATION

Now we turn to a discussion of some preference orderings of decision procedures : choice of function $\tau(x)$. In framework of theorem 1 it is really the choice of estimates $\hat{\theta}_0$, $\hat{\theta}_1$ and risk function. Let $X = (X_1, X_2, ..., X_n)$, where X_i, $i = 1,...,n$, are fatigue lives of aircraft in (full-scale) laboratory test, $Z = \min(Y_1, Y_2, ..., Y_m)$,where Y_j, $j = 1,...,m$, are fatigue lives of aircraft in operation,

$F_{X_i}(t) = F_{Y_j}(t)$, $i = 1,...,n$, $j = 1,...,m$; p- allowed probability of failure in operation of at least one aircraft.

In application to the problem of required SL confirmation, when τ^* is required SL, we are interested in increasing of probability that $\tau(x) = \tau^*$. It is something similar to increasing of power of some test in testing some statistical hypothesis.

In application to the problem of some SL nomination we should get the maximum of expectation value of $\tau(X)$ provided that reliability requirements are met, it is if $\tau(X)$ is a p-bound for Z. To study the optimality of $\tau(x)$ we can use the Jensen's inequality. This inequality say that the function of complete sufficient statistic, which is unbiased estimation of its own mathematical expectation, provides the minimal risk if the correspondent loss-function is convex. Consider the simplest case, when θ_1 is known parameter. Let $\theta_t = \theta_0 + t\theta_1$ is some quantile. Random variable $\hat{\theta}_t = \tau(x) = \hat{\theta}_0 + t\theta_1$ is unbiased estimate of its own expectation (which in general case does not equal to θ_t). In problem under question the function $F_Z(\tau)$ can be considered as the loss-function. Then the expectation $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X))$ is the risk function. For normal and sev distributions of Y_j $j = 1,...,m$, $F_Z(\tau)$ is convex (and increasing one) if its value is small enough and we have minimum of $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X)) = p$ at the fixed expectation value of $\hat{\theta}_t = \tau(X)$, if $\tau(x)$ is a function of sufficient statistic. And, on the contrary, if $\tau(x)$ is a function of sufficient statistic and $P(Z < \tau(X)) = p$ then we have maximum of expectation value of $\tau(X)$ if p is small enough and probability $P(\tau(X) < c)$ is high enough for such c, that $F_Z(z)$ is convex if $z < c$. For example, for normal distribution $\Phi(z)$ is convex if $z<0$. The generalization of the Jensen's inequality for the case of multivariate sufficient statistic can be found in [9].

For the case when sufficient statistic coincides with the sample itself (for example,Weibull or smallest odelli value (sev) distribution) usually for prediction interval the Monte Carlo (MC) method is used [10]. Here we show that for the problem of p.f. p-bound, $\tau(x)$, calculation analytic solution can be found using Bayes-fiducial (BF) approach.

4.2. BAYES-FIDUCIAL APPROACH

This approach was offered in 1973 (see [5,6,7,8]). It was shown that using this approach we can get Pitmen's estimates of location and scale parameters and most powerful invariant test for testing statistical hypotheses ($H_0 : F(x) = F_0((x-\theta_0)/\theta_1)$; $H_1 : F(x) = F_1((x-\theta_0)/\theta_1)$). It can be used also for unbiased estimation. BF estimate, $\tau(x)$, of some function of parameter $\tau(\theta)$ is a function, which minimizes BF risk

$$\rho_{BF}(\tau_\theta, \tau_X) = \int L(\tau_\theta(\theta), \tau_X(x)) dF_{\tilde{\theta}}(\theta) \ ,$$

where $L(\tau_\theta(\theta), \tau_X(x))$ is loss function, $F_{\tilde{\theta}}(\theta)$ is fiducial distribution on parameter space [5,6].

There two advantages of BF approach:

1. As in a case of using a maximum likelihood (ML) estimates BF solution is always a function of sufficient statistics, but in contrast to ML the BF solution take into account the loss function.

2. We do not need to have a priori distribution of unknown parameters.

4.3. USING BF APPROACH FOR P.F. P-BOUND CALCULATION

Let the problem is to estimate p-quantile $\tau_p(\theta)$ for cdf $F_Z((x-\theta_0)/\theta_1)$ and loss function $L(\tau_\theta(\theta),\tau_X(x)) = (F_Z((\tau_p-\theta_0)/\theta_1) - F_Z((\tau_X(x)-\theta_0)/\theta_1))^2 = (p - F_Z((\tau_X(x)-\theta_0)/\theta_1))^2$ when we have sample $x=(x_1,x_2,...,x_n)$ from cdf $F_X((x-\theta_0)/\theta_1)$.

Let us denote by $\tau_X(x,p)$ the solution of BF equation , corresponding to the considered loss function

$$E_{\tilde{\theta}}\{F_Z((\tau_X(x,p)-\tilde{\theta}_0)/\tilde{\theta}_1\} = p\,, \tag{8}$$

where $\tilde{\theta}=(\tilde{\theta}_0,\tilde{\theta}_1)$, r.v. $\tilde{\theta}_0$, $\tilde{\theta}_1$ have fiducial distribution. Here $E_X(f(X))$ is expected value of $f(X)$ in accordance with cdf of *X*.

We can simplify solution of Eq.8. Instead of vector $x=(x_1,...,x_n)$ without loss of information we can consider vector $\varpi=(\hat{\theta}_0,\hat{\theta}_1,w_1,...,w_{n-2})$, where $\hat{\theta}_0,\hat{\theta}_1$ are correct parameter estimates (see (6)), $w_i=(x_i-\hat{\theta}_0)/\hat{\theta}_1$, $i=1,...,n$ -2 Then conditional fiducial distribution (at the fixed invariant $(w_1,...,w_{n-2})$) of random variables $\tilde{\theta}_0,\tilde{\theta}_1$ is defined by equation [5,6]

$$f_{\tilde{\theta}_0,\tilde{\theta}_1|w_1,...,w_n}(s_0,s_1) = h\frac{\hat{\theta}_1^{\,n-1}}{s_1^{n+1}}\prod_{i=1}^{n} f\left(\frac{\hat{\theta}_0+\hat{\theta}_1 w_i - s_0}{s_1}\right)ds_0 ds_1\,,$$

where *h* is just normalization factor. (Note: w_{n-1},w_n , $w_i=(x_i-\hat{\theta}_0)/\hat{\theta}_1$, are functions of vector ϖ).

If in (8) we use new notations:

$$U_0=(\hat{\theta}_0-s_0)/\hat{\theta}_1,\quad U_1=\hat{\theta}_1/s_1,\ \overset{0}{\tau}(x,p)=(\tau(x,p)-\hat{\theta}_0)/\hat{\theta}_1$$

then instead of (8) we get equation

$$E_{W_1,...,W_n}E_{U_0U_1|w_1,...,w_n}\left(F((\overset{0}{\tau}(x,p)-U_0)/U_1)\right)=p\,. \tag{9}$$

where random variables U_0,U_1 has conditional pdf

$$f_{U_0,U_1|w_1,...,w_n}(u_0,u_1) = h_w u_0^{n-2}\prod_{i=1}^{n} f(u_0+w_i u_1)\,, \tag{10}$$

where h_w is just normalization factor which depends only on invariant vector $w=(w_1,...,w_{n-2})$.

If $\overset{0}{\tau}(x,p)$ is solution of the equation

$$E_{U_0U_1|w_1,...,w_n}\left(F((\overset{0}{\tau}(x,p)-U_0)/U_1)\right)=p \tag{11}$$

then

$$\tau_X(x,p)=\hat{\theta}_0+\overset{0}{\tau}(x,p)\hat{\theta}_1 \tag{12}$$

is solution of Eq. (9) and Eq.8 because equation (11) takes place for every vector $w=(w_1,...,w_{n-2})$, cdf of which does not depend on $\theta=(\theta_0,\theta_1)$. So if (11) is true then (8) is true also.

It is very important that $\overset{0}{\tau}(x, p)$ in (11) does not depend on value of $\theta = (\theta_0, \theta_1)$ and for solution of this equation we can set $\theta_0 = 0$, $\theta_1 = 1$. If $\hat{\theta}_0, \hat{\theta}_1$ have the structures defined by (6) then probability $P(Z < \tau(X, p))$ does not depend on $\theta = (\theta_0, \theta_1)$ and we can find p_1 for which

$$P(Z < \tau(X, p_1)) = p.$$

So $\tau_X(x, p_1)$ is p-bound for random variable Z.

As is easy to see (see p.84 in [6]) the pdf (8.b) is conditional pdf of $\hat{\theta}_0, \hat{\theta}_1$ at the fixed $w = (w_1, ..., w_{n-2})$ for the case when $\theta_0 = 0$, $\theta_1 = 1$. This means that the values of p_1 and p coincide.

It is very important also that result does not depend on the choice of the type of correct statistics $\hat{\theta}_0, \hat{\theta}_1$ (see (14.a) and (14.b)), because vector $x = (x_1, ..., x_n)$ and vector $\varpi = (\hat{\theta}_0, \hat{\theta}_1, w_1, ..., w_{n-2})$ have one-one mapping at any choice of correct statistics.

H. Example 1. P-bound for lognormal distribution

Let r.v. T has a lognormal distribution and $t=(t_1, t_2, t_3)$=(45 952, 54 143, 65 440) is the sample from the same distribution. Then r.v. $X = \log(T)$ has a normal distribution $N(\theta_0, \theta_1^2)$ and $x=(x_1, x_2, x_3)$ = (10.735 10.899 11.089) is the sample from this distribution. The problem is to calculate the p.f. p-bound for independent r.v. $Z=\min(Y_1, ..., Y_m)$, where r.v. Y_i, $i = 1, ..., m$, has the normal distribution $N(\theta_0, \theta_1^2)$ also. We consider here only the case, when m=1, because for this case there is general analytical solution (see, for example p. 172 in [6])

$$\tau(x) = \hat{\theta}_0 + \hat{\theta}_1 t_{n-1,p}(1 + 1/n)^{1/2}, \tag{13}$$

where

$$\hat{\theta}_0 = \bar{x}\ ,\ \hat{\theta}_1 = (\sum (x_i - \bar{x})^2 /(n-1))^{1/2}$$

are estimates of expected value and standard deviation, $t_{k,q}$ is q-quantile from Student's distribution with k degree of freedom. So we can make comparison of this solution with the solution which we get, using new approach.

For considered data, using equation (13) for p=0.01 we calculate t_{St}=exp($\tau(x)$) = 13 162, which is the value of p-bound for r.v. T on the base of observations (t_1, t_2, t_3) .

Now let us consider the new approach. For normal distribution the conditional pdf has following form

$$f_{U_0, U_1 | w_1, ..., w_n}(u_0, u_1) = h_w u_0^{n-2} \prod_{i=1}^{n} \varphi(u_0 + w_i u_1),$$

where $\varphi(x) = \exp(-x^2/2)/(2\pi)^{1/2}$. After transformation the equation (11) has the following form

$$1 - a(\overset{0}{\tau}, \bar{z}, D_z)/\Gamma((n-1)/2) = p,$$

where

$$a(\overset{0}{\tau}, \bar{z}, D_z) = \int_0^{\infty} u^{(n-3)/2} \exp(-u) \Phi\left((2u/D_z(n+1))^{1/2} (\bar{z} - \overset{0}{\tau}) \right) du, \quad \bar{z} = \sum_1^n z_i / n,$$

$D_z = \sum_{i=1}^{n} (z_i - \bar{z})^2 / n$, $\Gamma(\cdot)$ is gamma function, , $\Phi(\cdot)$ is cdf of standard normal distribution.

Consider two types of statistics $\hat{\theta}_0, \hat{\theta}_1$, which for considered data has following values:

a) $\hat{\theta}_0 = \bar{x}$ =10.908, $\hat{\theta}_1 = (\sum (x_i - \bar{x})^2 /(n-1))^{1/2}$ =0.177 , (14. a)

b) $\hat{\theta}_0 = x_{1,n}$ =10.735, $\hat{\theta}_1 = x_{n,n} - x_{1,n}$ =0.354, (14. b)

where $x_{i,n}$ is ith order statistic of vector $x = (x_1,...,x_n)$.

In case a) we have $\overset{0}{\tau}$ = -7.889, in case b) we have $\overset{0}{\tau}$ = - 3.560.

Corresponding values of p-bound for r.v. *T* on the base of observations (t_1, t_2, t_3) are:

t_a = exp($\tau(x)$) = 13 523, t_b = exp($\tau(x)$) = 13 050.

It seems that the difference between t_a, t_b and t_{St}=13 162 is produced only by the problem to get required calculation accuracy.

I. Example 2. P-bound for Weibull distribution

Let we have the same sample $t=(t_1, t_2, t_3)$=(45 952, 54 143, 65 440) or $x=(x_1, x_2, x_3)$ = (10.735 10.899 11.089) but r.v. *T* has a Weibull distribution and, correspondingly $X = \log(T)$ has distribution of smallest extreme value with cdf $F_X(x) = 1 - \exp(-\exp((x-\theta_0)/\theta_1)$. In this case the equation (11) has following form

$$1 - a(\overset{0}{\tau}, \bar{z}, D_z) / b(\bar{z}, D_z) = p ,$$

where

$$a(\overset{0}{\tau}, \bar{z}, D_z) = \int_0^\infty u^{(n-2)} \left(\exp(-u\sum_{i=1}^n z_i) /(\sum_{i=1}^n \exp(uz_i) + m\exp(u\overset{0}{\tau}))^n \right) du ,$$

$$b(\bar{z}, D_z) = \int_0^\infty u^{(n-2)} \left(\exp(-u\sum_{i=1}^n z_i) /(\sum_{i=1}^n \exp(uz_i))^n \right) du ,$$

$$\bar{z} = \sum_1^n z_i / n , \quad D_z = \sum_{i=1}^n (z_i - \bar{z})^2 / n .$$

For *m*=1, *p*=0.01, using statistics (14. a) we get $\overset{0}{\tau}$ = -11.929, using statistics (14. b) we get $\overset{0}{\tau}$ = -5.424. Corresponding values of p-bound for r.v. *T* on the base of observations (t_1, t_2, t_3) are: t_a = exp($\tau(x)$) = 6 616, t_b = exp($\tau(x)$) = 6 752.

For m=500, *p*=0.2 using statistics (14.a) we get $\overset{0}{\tau}$ = -12.889, using statistics (14.b) we have $\overset{0}{\tau}$ = - 5.970. Corresponding values of p-bound for r.v. *T* on the base of observations (t_1, t_2, t_3) are: t_a = exp($\tau(x)$) = 5 584, t_b = exp($\tau(x)$) = 5 568.

Again, it seems that the difference between t_a and t_b is produced only by the problem to get required calculation accuracy.

Considered data really was considered in several papers and for m=500, p=0.2 Lowless (1973) obtaind prediction limit of 5623, Mee and Kushary (1994) – 5225. The Mann and Saunders (1969) result was only 766. For these calculation the Monte Carlo method was used [10].

5. USING BAYES-FIDUCIAL METHOD FOR SL NOMINATION WITH ECONOMICS OPTIMALITY CRITERION

Let the income of aircraft successful service during time t is equal to t but in case of failure the loss is equal to some negative value –b, where b is some large positive value. Then income of one aircraft service, r.v. U, is defined by formula

$$U = \begin{cases} t_{SL}, & if\ T > t_{SL}, \\ T - b, & if\ T \le t_{SL} \end{cases},$$

where T is random fatigue life, t_{SL} is some SL.

Expectation value of U

$$u(t_{SL}, \theta, b) = \int_0^{t_{SL}} (t\text{-}b)\, d\, F_T(t,\theta) + t_{SL}\,(1\text{-}\, F_T(t,\theta))$$

where $F_T(t,\theta)$ is c.d.f. of T.

In general case maximum of $u(t_{SL}, \theta, b)$ is reached at t^*_{SL}, which is the root of the equation

$$bf_T(t)/\,(1\text{-}\, F_T(t,\theta)) = 1\,.$$

For normal distribution of X=ln$T$ it can be written in following way

$$\theta_0 = t^*_{SL} - \theta_1\, \lambda^{-1}(\, t^*_{SL}\theta_1/b),$$

where $\lambda(z) = \varphi(z)/(1-\Phi(z))$ is failure rate function for standard normal distribution, $\lambda^{-1}(.)$ is inverse function. This equation allows very easy to get θ_0 as function of t^*_{SL} at the fixed θ_1 and then to find the inverse function:

$$t^*_{SL} = S^*(\theta_0, \theta_1, b)\,.$$

For b=346 000, θ_1=0.346 and θ_0=9.948 we have: t^*_{SL}=7936 (flights) . It is interesting to note that this value corresponds to the failure probability equal to 0.0026. This can be interpreted in following way. The failure of 2.6 aircraft (in flight) from 1000 aircraft can be considered as equivalent to the loss of 346000 hours of service time or loss of 346000/7936 = 43.6 aircraft (on the ground) of this types (the value t^*_{SL} = 7936 can be considered as the price of one aircraft of this type). Or in other words, failure of one aircraft (in flight) is equivalent to loss of 43.6/2.6 (approximately 16) aircraft of the same type (on the ground).

But we do not know parameters of c.d.f. of T and should estimate them using fatigue test data. Usually maximum likelihood estimate is considered as most appropriate. We show here that for considered problem the offered by outhor Bayes-fiducial approach is much more appropriate.

In accordance with Bayes approach the parameter θ_0 is r.v.. For the case of airframe it can be interpreted in following way. Design stress analysis of an airframe should meet some standard requirements (FAR, ...). These requirements in fact define only some mean value of θ_0 but of course, in every case there are some "occasional mistakes" and we have some specific (random) value of θ_0 for every aircraft type. And then there is a scatter of r.v. X (specific random fatigue life of some specific aircraft) at this random θ_0. The parameter θ_1 is function of technology level, and if one is not changed, then the parameter θ_1 is not changed also. So we suppose that θ_1 is known constant but θ_0 is random variable, $\tilde{\theta}_0$. Let $\pi(\theta_0)$ is a priory distribution density for $\tilde{\theta}_0$. Then c.d.f. of r.v. X will be

$$\tilde{F}_X(x) = \int_{-\infty}^{\infty} F_X((x\text{-}\theta_0)/\theta_1)\pi(\theta_0)\,d\theta_0\,.$$

It is well known, that if θ_1 is constant but r.v. $\tilde{\theta}_0$ has normal distribution with known both mean τ_0 and standard deviation τ_1 then distribution of X will be again normal with mean τ_0 and standard deviation $((\tau_1)^2+(\theta_1)^2)^{1/2}$. In this case t_{SL} again will be defined by equation (1), but θ_1 should be replaced by $\theta_{1\tau}=((\tau_1)^2+(\theta_1)^2)^{1/2}$.

In fact we do not know a priori distribution of $\tilde{\theta}_0$. For this case it is offered FB approach. Instead of posterior distribution of $\tilde{\theta}_0$ we offer to use already mentioned fiducial distribution [5]. In considered case fiducial distribution of $\tilde{\theta}_0$ again is normal with mean $\bar{x}$ and standard deviation $\theta_1/n^{1/2}$. Then for the purpose of calculation t_{SL} we again can use the same equation (1), but θ_0, θ_1 should be replaced by $\hat{\theta}_0=\bar{x}$ and $\theta_1(1+1/n)^{1/2}$ correspondingly. So using sample $x=(x_1,...,x_n)$, result of full-scale fatigue test, in case of ML approach the nominated SL is equal to $S^*(\bar{x},\theta_1,b)$, but for BF approach $t_{SL}(x)=S^*(\bar{x},\theta_1(1+1/n)^{1/2},b)$. By the use of Monte Carlo method for θ_0 = 9.948 , θ_1=0.346, b=346,000 we have got that the expectation value of r.v. U_X is equal to 2310, 4122, 5571, 6904 for BF approach but it is equal to -8624, 809, 4422, 6935 for ML approach for the same sample sizes n = 1, 2, 4, 100 . We see that for small n the expectation value of r.v. U_X is much more for BF than for ML approach.

SUMMARY

BF approach for the specified life nomination using time test data is considered for both cases: probability of failure limitation and for the maximum of expected value of some specific function of preference (minimum of expected value of specific loss function).

BF approach has following advantages:

1. As in a case of using a maximum likelihood (ML) estimates BF solution is always a function of sufficient statistics, but in contrast to ML the BF solution take into account the loss function.

2. We do not need to have a priori distribution of unknown parameters.

It is given approximate analytical solution of the problem to get the maximum of expected value of SL. In case of economics optimality criterion it is shown also that for considered type of loss function the BF approach is more preferable than direct use of ML estimates. Numerical examples are provided

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CONSTRUCTION OF ACCEPTABILITY REGIONS FOR PARAMETRIC RELIABILITY OPTIMIZATION

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Abstract

The problem of representation and analysis of analog technical devices and systems acceptable regions is introduced. This problem occurs during designing and controlling in view of parametric dithering. The algorithms of constructing circumscribed parallelepiped, representation of acceptable region as a set of non-overlapping parallelepipeds are offered. The algorithm of acceptable region centre of mass computation is offered as the example of utilizing the acceptable region representation.

J. Introduction

The engineering system parameters are subject to random variations and the variations may be considered as non-stationary stochastic processes. The conventional methods for choosing parameters (parametric synthesis) generally do not take account of parameters field deviations from their design values. As a result the engineering systems designed in such a manner are not optimal in the sense of their gradual failure reliability.

Construction of acceptable region is necessary for solving problems usually arising in reliability theory. The following topics are among them:

- evaluation of the system's working capacity with account of parameters deviations from their design values,
- evaluation of parameters sensitivity and emphasizing the key input parameters,
- assigning tolerance values for parameters,
- choosing the most fit (optimal) nominal values for parameters of system components.
- reducing calculation time for parametric synthesis problem when the characteristics of random processes X(t) of system parameters variations are known [1].

The acceptable region construction in the space of input parameters is the one of steps in parametric synthesis solution. The task often can be cumbersome due to large dimension of the varying parameters space.

Most of known acceptable region construction methods can be applied on 3D space only. Some methods are efficient for a priori convex and simple connected regions. Other methods need information about form and orientation of the acceptable region.

2. Parametric reliability optimization problem

Suppose that we have a system which depends on a set of n input parameters $\boldsymbol{x}=(x_1,\ldots,\ x_n)$. We will say that system is acceptable if $\boldsymbol{Y(x)}$ satisfy the conditions (1):

$$\boldsymbol{a} \le \boldsymbol{Y} \le \boldsymbol{b} \qquad (1)$$

where $\boldsymbol{Y}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are m-vectors of system responses (output parameters) and their specifications, e.g. $Y_1(\boldsymbol{x})$ – average power, $Y_2(\boldsymbol{x})$ –delay, $Y_3(\boldsymbol{x})$ – gain.

The inequalities (1) define a region D_x in the space of input (system) parameters

$$D_x = \{\mathbf{x} \mid \boldsymbol{a} \le \boldsymbol{Y} \le \boldsymbol{b}\} \quad (2)$$

D_x is called the region of acceptability for the system. Figure 1 illustrates such a region. The values given for *a* and *b* are the specifications for the system.

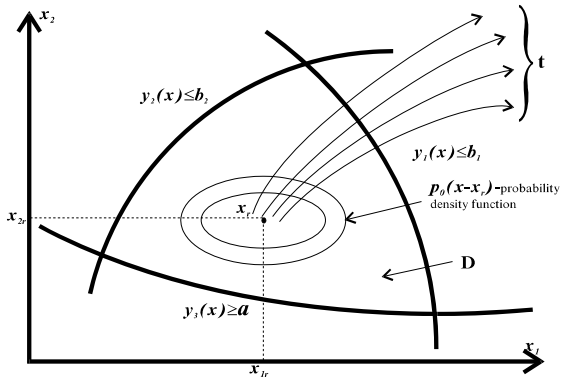

Fig. 1. Region of acceptability D_x defined by system response functions

The engineering system parameters are subject to random variations (aging, wear, temperature) and the variations may be considered as stochastic processes:

$$X(t) = \{X_1(t), ..., X_n(t)\}$$

In general parametric optimization (optimal parametric synthesis) problem can be stated as follows [1].

Given the characteristics of random processes ***X(t)*** of system parameters variations, conditions of acceptability (1) and a service time *T,* find such a deterministic vector of parameter ratings (nominal values) $\mathbf{x}_r=(x_{1r}, ..., x_{nr})$ that the reliability

$$P_r(\mathbf{x}_r, T) = P_r\{[X_1(t), X_2(t), ..., X_n(t)] \in D_x, \forall t \in [0, T]\} = \max \quad (3)$$

was maximal.

The practical algorithm of the stochastic criterion calculation is based on the conventional Monte Carlo method.

At the beginning, the random vector of parameters is generated (this vector means random manufacturing device realization), and then the internal parameters degradation is simulated using degradation model. For example, parameters variations can be approximated as follows

$$X(t) = \sum_{k=0}^{m} x_r u_k(t)$$

where x_k is a random variable; $\{u_k(t)\}^m{}_{k=0}$ are continuous deterministic functions of time.

The Monte Carlo method approximates $P_r(\mathbf{x}_r, T)$ by the ratio of number of acceptable realizations (falling in region D_x)-N_a to the total number of trials – *N*.

$$P_r = N_a / N,$$

Unfortunately, often the region D_x is unknown. It is given only implicitly through system's equations and the system response functions. If we do not know the region D_x a Monte Carlo evaluation of probability $P_r(\mathbf{x}_r, T)$ at particular nominal value $\mathbf{x}_r$ requires *N* system analyses for each trial set of parameter $\mathbf{x}_r$. Typically, hundreds of trials are required to obtain a reasonable estimation for $P_r(\mathbf{x}_r, T)$.

Optimization requires evaluation of the probability $P_r(\mathbf{x}_r, T)$ for many different values of parameters nominal values $\mathbf{x}_r$.

The acceptable region constructing problem can be stated as follows.

The first step in constructing acceptable region is narrowing the search area in a space of internal parameters. A circumscribed parallelepiped is constructed for this purpose. The next step consists in approximation of D_x with a set of non-overlapping n-dimensional parallelepipeds within circumscribed parallelepiped.

3. Circumscribed box Construction

3.1. Definition of circumscribed box

Let information about variations of internal parameters is given as limits of their possible values, i.e.

$$x_{i\min} \le x_i \le x_{i\max}, i = \overline{1,n} \tag{5}$$

The region inside of space of internal parameters is a n-dimensional orthogonal parallelepiped called tolerance box B_d:

$$B_d = \{\mathbf{x} \in R^n \mid x_{i\min} \le x_i \le x_{i\max}, i = \overline{1,n}\} \tag{6}$$

with the volume

$$V_d = \prod_{i=1}^{n} (x_{i\max} - x_{i\min}) \tag{7}$$

The box B_0 circumscribed about acceptability region $D_{\mathbf{x}}$ is the n-dimensional orthogonal parallelepiped [2] within the tolerance box B_d.

$$B_o = \{\mathbf{x} \in R^n \mid a_i^0 \le x_i \le b_i^0 \ \forall i = \overline{1,n}\} \tag{8}$$

The circumscribed box's volume is represented as

$$V_0 = \prod_{i=1}^{n} (b_i^0 - a_i^0), \tag{9}$$

where $a_i^0 = \min_{\mathbf{x} \in D_x} x_i$, $b_i^0 = \max_{\mathbf{x} \in D_x} x_i$.

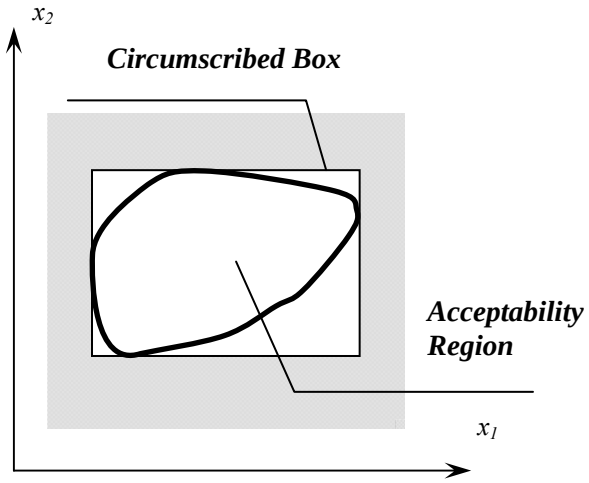

Fig. 2: The circumscribed box

The construction of the circumscribed box may narrow the area for construction of the acceptable region D_x and decreases computational cost of stochastic estimations. There's no need to

perform expensive scheme's simulations at the points outside the circumscribed box because the conditions (1) are not satisfied at those points.

The construction of circumscribed box and acceptability region's volume with the method of statistical testing is similar to computation of multiple integral with Monte-Carlo method.

The algorithm of circumscribed box construction

Let the tolerance box's boundaries (5) are known and the conditions of acceptability (1) are given.

The task consists in overlapping of the tolerance region with uniform points and testing satisfaction of the condition (1) at the each point.

The random point coordinates is a set of uniform numbers $\boldsymbol{\xi} = (\xi_1,...,\xi_n)$ distributed on

$$x_{i\min} \le \xi_i \le x_{i\max},\ i = \overline{1,n}\ .$$

To find components of the vectors $\mathbf{a}^O, \mathbf{b}^O$ we set the initial values $a_i^O = x_{i\max}$ and $b_i^O = x_{i\min}$, $i = \overline{1,n}$, $N_g = 1$. N_g is amount of the points where conditions (1) are satisfied.

An acceptability of the system (satisfaction of the conditions (1)) is checked at the every test. Depending on result of the test, the value of the indicator function $g(\boldsymbol{\xi})$ is set.

$$g(\boldsymbol{\xi}) = \begin{cases} 1, \textit{if conditions } (1) \textit{ are satisfied} \\ 0, \textit{if conditions } (1) \textit{ aren't satisfied} \end{cases}.$$

If acceptability conditions (1) are satisfied at the point ξ the following values are set: $a_i^O = \min(a_i^O, \xi_i)$, $b_i^O = \max(b_i^O, \xi_i)$, $i = \overline{1,n}$, $N_g = N_g + 1$.

Having performed N point simulations, we compute acceptability region's volume estimation:

$$\hat{V}_g / V_O = \frac{1}{N}\sum_{j=1}^{N} g(\boldsymbol{\xi}^j) \tag{10}$$

or

$$\hat{V}_g = V_O \frac{N_g}{N}, \tag{11}$$

where V_O - the circumscribed box's B_O volume, N_g - amount of points inside of the acceptability region.

This method enables us to solve several problems of practical importance:

a) Region D_x is multiply connected region.

b) Region D_x fall outside the limits of the tolerance box B_d.

c) Region D_x include the tolerance box ($B_0 \in D_x$, $N_g = N$).

d) Region D_x and tolerance box B_d are mutually disjoint ($D_x \cap B_0 = 0$, $N_g = 0$).

This method can be easily implemented on parallel computers with MPP architecture. The main condition for result accuracy is absence of interprocessor correlations [2, 3]. In the beginning of the process, there is an initialization of parallel random-number generator, then each process is initialized with the amount of tests to perform. Upon termination of calculations each of the processes passes computed ratings to the main processor which forms a final rating.

Following the given necessary tests amount N and processors amount k, necessary tests amount m is assigned to each processor (the sample volume on each processor), thereby $k \cdot m = N$.

The final estimation of the acceptable region is computed with the formula:

$$\hat{V}_g = V_0 \sum_{j=1}^{k} N_{gj} / N , \tag{12}$$

where N – necessary tests amount, N_G – the amount of points within the region of acceptability for each of the processors.

The boundaries of the circumscribed box B_O are computed with the formulas:

$$\begin{array}{l} \forall i = \overline{1,n} \quad a_i^O = \min(a_{ij}^O),\ j = \overline{1,k}, \\ \forall i = \overline{1,n} \quad b_i^O = \max(b_{ij}^O),\ j = \overline{1,k} \end{array} , \tag{13}$$

where $a_{ij}^O = \min\limits_{x \in D_{\mathbf{x}}} x_i$, $b_{ij}^O = \max\limits_{x \in D_{\mathbf{x}}} x_i$ for each of k processors.

The communication overheads are not considerable because they are reduced to primary initialization of the random numbers generator and obtaining of final estimations.

K. The multidimensional probing method

The region of acceptability within the tolerance box $D_{\mathbf{x}} \in B_d$ can be defined with the methods based on multidimensional probing of the tolerance box B_d.

Let's consider the algorithm of acceptability region construction. It is based on the concept of matrix tests on reliability.

The circumscribed box B_O is divided into a set of non-overlapping elementary boxes. Let's perform separation of circumscribed box B_O with the hyperplanes parallel to its bounds into $\mathbf{l} = (l_1,...,l_n)$ slices per each coordinate.

Then we have non-overlapping elementary boxes with bounds parallel to corresponding bounds of B_O. The length of elementary box's rib parallel to i-th coordinate axis is l_i times smaller than the corresponding rib of B_O.

So we have:

$$B_O = \bigcup_{k_1=1}^{l_1} \bigcup_{k_2=1}^{l_2} ... \bigcup_{k_n=1}^{l_n} B_{k_1,k_2,...k_n} . \tag{14}$$

The total amount of elementary boxes is

$$R = \prod_{i=1}^{n} l_i . \tag{15}$$

The step (i.e. the length of elementary box's rib) on the each parameter axis is

$$h_i = (b_i^0 - a_i^0) / l_i,\ i = \overline{1,n} . \tag{16}$$

Let's assign the central point of each elementary box as a point-"representative". The coordinates of point-representative for the elementary box $B_{k_1,k_2,...k_n}$ are computed as follows:

$$x_i = a_i^O + \frac{h_i(k_i - 1)}{2},\ i = \overline{1,n} . \tag{17}$$

If acceptability conditions (1) are satisfied at the point-representative of the elementary box $B_{k_1,k_2,...k_n}$ this elementary box belongs to a set of "good" elementary boxes, otherwise it belongs to a set of "bad" elementary boxes.

Note also that

$$\lim_{R \to \infty} \frac{M_g}{R} = V_{D_x} / V_{B_O} , \tag{18}$$

where M_g is the amount of "good" elementary boxes, R is the total amount of elementary boxes. V_{D_x} / V_{B_O} is the relation of acceptability region volume to the volume of circumscribed box B_O.

The figure consisting of "good" elementary boxes is defined as D_x^o. The figure D_x^o approximates the acceptable region D_x.

The n-dimensional array $C[l_1, l_2, ..., l_n]$ is corresponded to the circumscribed box B_O divided into elementary boxes $B_{k_1,k_2,...k_n}$. Each element of the array stores the state of corresponding elementary box (i.e. "good" or "bad", accordingly 1 or 0). The total amount of array elements can be found using (15). This array also can be called as n-dimensional matrix. In the text below it is called matrix.

Now consider the representation of n-dimensional array using 1-dimensional array by placing its elements sequentially in a row.

In common case the elements of that array *A[R]* are placed as follows:

$$\begin{aligned} A = \{&(1,1,1,...,1),\ (2,1,1,...,1),...,(l_1,1,1,...,1),\\ &(1,2,1,...,1),\ (2,2,1,...,1),...,(l_1,2,1,...,1),\\ &(1,3,1,...,1),\ (2,3,1,...,1),...,(l_1,3,1,...,1),...,\\ &(1,l_2,1,...,1),\ (2,l_2,1,...,1),...,(l_1,l_2,1,...,1),...,\\ &(1,l_2,1,...,l_{n-1},1),\ (2,l_2,...,2),...,(l_1,l_2,...,l_{n-1},l_n)\} \end{aligned} \quad . \tag{19}$$

The index p of the array element, corresponding to the elementary box $B_{k_1,k_2,...k_n}$ is computed as follows:

$$p = k_1 + l_1(k_2 - 1) + l_1 l_2(k_3 - 1) + ... + \prod_{i=1}^{n-1} l_i (k_n - 1) - 1. \tag{20}$$

In some kinds of tasks it is necessary to compute a set of indexes $(k_1, k_2, k_3, ..., k_n)$ having the array index p and the amount of quanta $(l_1, l_2, l_3, ..., l_n)$ per each coordinate axis. To compute a set of indexes $(k_1, k_2, k_3, ..., k_n)$ it is necessary to do the following sequentially:

$$\begin{aligned} k_n &= \left[\frac{p}{\prod_{i=1}^{n-1} l_i} \right] + 1, \\ k_{n-1} &= \left[\frac{p - k_n \cdot \prod_{i=1}^{n-1} l_i}{\prod_{i=1}^{n-2} l_i} \right] + 1, \\ &... \\ k_2 &= \left[\frac{p - k_n \cdot \prod_{i=1}^{n-1} l_i - k_{n-1} \cdot \prod_{i=1}^{n-2} l_i - ... - k_3 \cdot \prod_{i=1}^{2} l_i}{l_1} \right] + 1, \\ k_1 &= \left[p - k_n \cdot \prod_{i=1}^{n-1} l_i - k_{n-1} \cdot \prod_{i=1}^{n-2} l_i - ... - k_3 \cdot \prod_{i=1}^{2} l_i - k_2 \cdot l_1 \right] + 1 \end{aligned} \tag{21}$$

For example in the case of 3D parameter space the (21) looks as follows:

$$\begin{aligned} k_3 &= \left[\frac{p}{l_1 \cdot l_2} \right] + 1, \\ k_2 &= \left[\frac{p - k_3 \cdot l_1 \cdot l_2}{l_1} \right] + 1, \\ k_1 &= \left[p - k_3 \cdot l_1 \cdot l_2 - k_2 \cdot l_1 \right] + 1 \end{aligned}$$

The procedure of the array *A[R]* filling consists in follows.

Let the bounds $\mathbf{a}^O, \mathbf{b}^O$ of the circumscribed box and the amount of quanta per each coordinate is known. Initially the amount of "good" elementary boxes is $M_g = 0$.

For the each index $i_n = \overline{1, l_n}$; $i_{n-1} = \overline{1, l_{n-1}}$; …; $i_1 = \overline{1, l_1}$:

1. Compute the coordinates of the elementary box's $\mathbf{x}_{i_1 i_2 \ldots i_n}$ representative using the formula (17).
2. Compute the index *p* of the array *A* unit using the formula (20).
3. Compute the output system parameters $\mathbf{y}(\mathbf{x}_{i_1 i_2 \ldots i_n})$.
4. Check the acceptability conditions (1).
5. If conditions (1) are satisfied $A[p] = 1, \quad M_g = M_g + 1$.
6. Else $A[p] = 0$.

As the result we have an array *A[R]* that represents circumscribed box B_O divided into elementary boxes and the amount M_g of "good" elementary boxes. If $\frac{M_g}{R} \approx \frac{V_g}{V_O}$, the array *A[R]* represents the circumscribed box B_O and the acceptable region D_x**.**

Figure 3 illustrates rendered elementary boxes that represent approximation of the acceptable region.

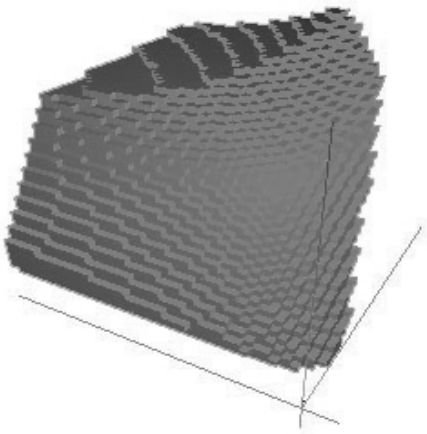

Fig. 3: Rendered "good" elementary boxes

The procedure of the array *A[R]* filling can be performed in parallel mode. This algorithm can be presented as a distributed process that requires l_1 processors for implementation. The algorithm of calculating the center of gravity for acceptable region constructed of elementary boxes.

In fact the distribution laws of system parameters variations and the characteristics of random parameters degradation processes X(t) are often unknown. In some cases the center of gravity for acceptable region may be the numerical solution of the parametric optimization problem.

Let's consider the acceptable region D_x^0 as a system of M unit-mass material points.

The center of gravity coordinates $\overline{C} = (x_{\overline{C}1}, ..., x_{\overline{C}n})$ of the body can be found as follows:

$$x_{\overline{C}i} = \frac{\sum_{j=1}^{M} I_i(j)}{M},$$

where M is normalized mass of D_x^0 (amount of elementary boxes D_x^0 consists of), $I_i(j)$ are static moments with respect to corresponding axis of each of the elementary boxes.

Static moment $I(B_{k_1,k_2,...,k_n})$ of the corresponding elementary box $B_{k_1,k_2,...,k_n}$ is calculated in the following way:

$$I_j(B_{k_1,k_2,...,k_n}) = \begin{cases} k_j, & if \quad B_{k_1,...k_n} \in D_x^0 \\ 0, & if \quad B_{k_1,...,k_n} \notin D_x^0 \end{cases}$$

for $j = \overline{1,n}$.

Since D_x^0 is represented with "good" elementary boxes inside the circumscribed box B_0 its center mass coordinates can be found on whole circumscribed box B_0 using the formula:

$$x_{\overline{C}i} = \frac{\sum_{k_1=1}^{l_1}\sum_{k_2=1}^{l_2}...\sum_{k_n=1}^{l_n} I_i(B_{k_1k_2,...k_n})}{M},$$

for $k_1 = \overline{1,l_1},...,k_n = \overline{1,l_n}$.

When center of gravity is being calculated simultaneously with construction of acceptable region using one-dimensional array the procedure of filling the array $C[l_1,l_2,...,l_n]$ (*A[R]*) and calculating the center of gravity $\overline{C} = (x_{\overline{C}1},...,x_{\overline{C}n})$ is in follows.

Let the boundaries $\mathbf{a}^O, \mathbf{b}^O$ of the circumscribed box, the amount of quanta $\mathbf{l} = (l_1,l_2,l_3,...,l_n)$ on each of the axes and the lengths $\mathbf{h} = (h_1,...,h_n)$ of quanta on each of the axes are known. Let also the amount of "good" matrix elements $M = 0$ and $\mathbf{SI} = (SI_1,...,SI_n)$ is the array of static moments sum with respect to *n* coordinate axes.

L. For $k_n = \overline{1,l_n}$; $k_{n-1} = \overline{1,l_{n-1}}$; …; $k_1 = \overline{1,l_1}$:

1.1. the coordinates $\mathbf{x}_{k1k2...kn}$ of point-representative for the current elementary box are calculated:

$$x_i = a_i^0 + h_i k_i - \frac{h_i}{2}, \quad i = \overline{1,n}.$$

1.2. The system response functions $y(x_{k1k2...kn})$ are calculated.

1.3. The acceptability conditions (1) are checked.

1.4. If conditions (1) are satisfied

$$C[k_1,k_2,...,k_n] = 1, M = M + 1,$$
$$SI_i = SI_i + k_i, \; i = \overline{1,n}$$

1.5. otherwise $C[k_1,k_2,...,k_n]$.

2. The gravity center coordinates for D_x^0 are calculated:

$$x_{\overline{C}i} = \frac{SI_i}{M}, \quad i = \overline{1,n}.$$

As the result the array $C[l_1,l_2,...,l_n]$ (*A[R]*) is filled, the amount *M* of "good" elementary boxes and the coordinates of the center of gravity $\overline{C}$ for D_x^0 are obtained.

In the case when the array $C[l_1,l_2,...,l_n]$ (*A[R]*) was filled previously the procedure of calculation of the gravity centre for D_x^0 consists in follows.

1. For $k_n = \overline{1,l_n}$; $k_{n-1} = \overline{1,l_{n-1}}$; …; $k_1 = \overline{1,l_1}$:

if $A(k_1, k_2, ..., k_n) = 1$, then $SI_i = SI_i + k_i, \quad i = \overline{1,n}$.

2. The gravity center coordinates for D_x^0 are calculated:

$$x_{\overline{C}i} = \frac{SI_i}{M}, \quad i = \overline{1,n}.$$

Hence it is necessary to perform exhaustive search of the array $C[l_1, l_2, ..., l_n]$ (*A[R]*) elements for calculating of the gravity center of the body.

Having found the center of gravity for D_x^0 the inverse mapping on acceptable region is performed the following way:

$$x_{Ci} = \frac{b_i^0 - a_j^0}{l_i} x_{\overline{C}i}, \quad i = \overline{1,n}.$$

Conclusion

The circumscribed box may narrow the area for construction of the acceptable region and decreases computational cost of stochastic estimations. Computation of internal parameters ratings with lack of distribution density functions for both input and output parameters requires obtaining of acceptable region characteristics. On the basis of the proposed methods and the algorithms a computer-aided reliability-oriented design system called CARD has been developed. The CARD system states mathematical models and calculates ratings of component parameters with the highest precision, acceptability (manufacturing yield) or reliability of engineering systems under design.

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MATCHING OF CRITERIA THE DISCERNMENT OF THE FUNCTIONAL CHARACTERISTICS OF INDEXES OF RELIABILITY OF PLANTS EES

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References on variation of reliability on the curves received at analysis of statistical data can appear erratic if not to consider a random in character of assessments of indexes of reliability. The comparison method of criteria of a discernment of the functional characteristics indexes of reliability reduced at ordinal and nominal dials of variation of argument.

While in service the equipment and systems of plants EES there is a necessity for a reliability analysis of their activity. The reliability analysis implies an assessment and matching of some indexes of reliability (IR), describing those or other properties. As a result, of analysis the certain references on build-down of working costs formed. The greatest propagation was received with data on «weak links » plant, about conditions and character of originating of failures, a type of failures and so forth. These data in many respects determine volume of plan repair work, measures on perfecting system of maintenance, perfecting of methods verification availability index.

The solution so important for build-down of working costs of problems, in an essential degree is at a loss a small amount of information about availability index of the equipment and systems of plants EES. The averaged IR and their empirical characteristics (EC) often do not mirror a singularity of particular plant, and individual IR and matching them EC, application of special methods and the approaches considering a random in character of assessments of IR require and the statistical odelling es orientated on check. Under EC IR we shall agree to fathom empirical regularity of IR in function of some varieties of indications (VI). Instances EC are regularity variation of IR on calendar years, duration of exploitation, a season and day, depending on the class-room of a voltage, the dispatcher numbers of the equipment, systems and electric sets, configuration items, etc. Real regularity of variation of IR in function VI we shall agree to name the functional characteristics (FC).

The urgency of a problem of the account of a random in character assessments of IR causes steadfast notice of technicians. Are developed series of criteria for matching assessments of the same type IR and their characteristics for continuous random quantities [1]. At a reliability analysis, not less characteristics of IR which scale of measurement of argument concerns to the classroom ordinal or nominal [2] often are used. For these scales of measurement, the criteria considering a random in character of watched regularities require the perfection since insufficiently full mirror as modes of an assessment of the fundamental and additional IR, and the discrete character of variation of argument. Therefore, there are reasons to believe, that the number will increase them in due course. Thus, there is a problem of matching of criteria for the purpose characteristics of their reliability (probability of a correct solution).

It is known, that in theory checks of statistical hypothesizes the preference is returned criterion, for which at the fixed value of an error of first kind, an error of second kind the least. As it noted in [1], comparison of statistical criterions constitutes rather a challenge of modern mathematical statistics.

The most simple and illustrative mode is graphical map of characteristics of intercoupling of errors first [α(x)] and second [β(x)] stems in the form of function $\beta(x) = f[\alpha(x)]$, or in the form of intercoupling of power of criterion $W(x) = 1 - \beta(x)$ and α(x). However, a seeming ease of this

mode is deceptive. For a case history of originating difficulties, we shall survey sequence of a presence of dependence $\beta(x) = f[\alpha(x)]$. She provides following determinations and evaluations:

1. Shaping of suppositions (hypotheses) concerning character of variation of examined dependence. As agency of a random in character of assessments IR is considered, normally surveyed two hypotheses. Considered, that actually a development all VI equiprobable (hypothesis H_1). For example, assemblies of the cutout have equal reliability, and the watched divergence of assessments VI coupled only to a small amount of information, random.

The second (alternative) hypothesis (H_2) also is natural – the watched regularity of variation of IR mirrors a real quantitative ratio of significance VI.

2. Account of distribution functions $F(x_i / H_1)_s$ and $F(x_i / H_2)_s$ where x_i – statistician of i-th criterion i=1,s; s – number of compared criteria. If the distribution function of the discrete random quantity is known, formulas of account, as a rule, are known $F(x_i / H_1)$ and $F(x_i / H_2)$. For example, if the model of experiment matches to a binomial low distribution, formulas of account $F(x_i / H_1)$ and $F(x_i / H_2)$ will differ only, accordingly, with usage for account $F(x_i / H_1)$ hypothetical probability, and for account $F(x_i / H_2)$ - empiric probability.

If the distribution function of a random quantity is unknown, that occurs for a greater unit of IR, allocations $F(x_i / H_1)$ and $F(x_i / H_2)$ evaluated by a method of simulation modeling. An instance of such characteristics are regularity of variation of an average of failures of cutouts of various class-rooms of a voltage, variation of an emergency shut-down coefficient depending on duration of exploitation and others;

3. Account of allocations $\alpha(x_i)$ and $\beta(x_i)$. The solution of this problem simple enough would seem

$$\alpha(x_i / H_1) = 1 - F(x_i / H_1) \quad (1)$$

$$\beta(x_i / H_2) = F(x_i / H_2) \quad (2)$$

However, such inference is fair, if assessments of expectation of a statistician x_i for H_1 and H_2 satisfy to a following condition:

$$M^*(x_i / H_1) < M^*(x_i / H_2) \quad (3)$$

Otherwise, i.e. when

$$M^*(x_i / H_1) > M^*(x_i / H_2) \quad (4)$$

following equalities are fair

$$\alpha(x_i / H_2) = 1 - F(x_i / H_2) \quad (5)$$

$$\beta(x_i / H_1) = F(x_i / H_1) \quad (6)$$

If neglect a parity of means statistician x_i serious errors in an adoption of a decision are possible. The account of a parity $M(x_i / H_1)$ и $M(x_i / H_2)$ it is especially important at automatic application of criteria in program models. The graphical case history of a short of an erratic solution reduced on fig.1.

As follows from fig.1, not account parities $M^*(x / H_1)$ and $M^*(x / H_2)$ leads to sharp variation of critical value of quintiles of allocations $\alpha(x / H_2)$ and $\beta(x / H_1)$. If at physically correct comprehension of errors the first and second stem, critical value of quintiles at $\alpha_k = \beta_k = 0{,}1$ are accordingly peer $\overline{X}^{(1)}$ and $\overline{X}^{(2)}$, that at their erratic comprehension ($M^*(x / H_1) > M^*(x / H_2)$), these quintiles are accordingly peer $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$, that $\overline{X}^{(1)} << \underline{X}^{(1)}$ and $\overline{X}^{(2)} >> \underline{X}^{(2)}$. If $\overline{X}^{(1)} < \overline{X}^{(2)}$, that $\underline{X}^{(1)} > \underline{X}^{(2)}$;

4. Construction of dependence $\beta(x_i) = f[\alpha(x_i)]$. To build this dependence it is necessary to consider following singularities:

4.1. Levels of discrete samplings of allocations $F(x_i / H_1)$ and $F(x_i / H_2)$ can completely and partially differ. Here there is in view of not the partial overlapping of spacing of possible value of argument of allocations and not a complete divergence of these spicing. Difference of discrete samplings watched on the interval overlapping of possible value. It is established, that a necessary condition of existence of generic points of a discrete sampling is proportionality x_m to value $\varepsilon = 1/n_\Sigma$, where n_Σ - total number of failures

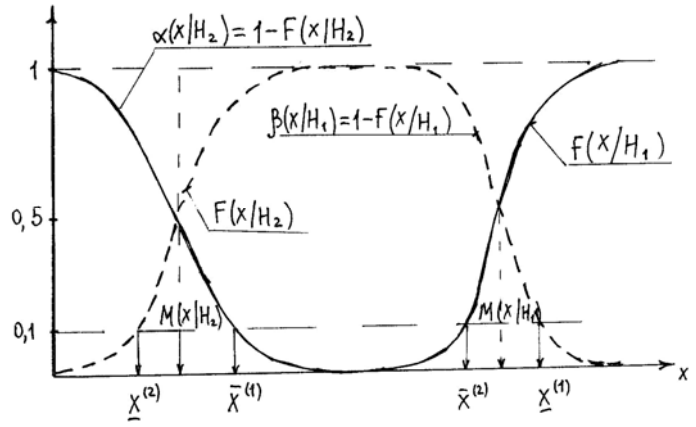

Fig. 1. A graphical case history of aftereffects of disregard a parity $M^*(x / H_1)$ and $M^*(x / H_2)$

4.2. Between dependences $\beta(x) = f[\alpha(x)]$, builted for conditions (3) and (4), there is a divergence. In the first event, we have dependence of probability erratic disallowance hypothesis H_1 in function of probability erratic disallowance hypothesis H_2, i.e. $\beta(x / H_2) = f[\alpha(x / H_1)]$, and in the second event $\beta(x / H_1) = f[\alpha(x / H_2)]$. In discover the reflecting noted in item.3 serious errors in an adoption of a decision.

Therefore, it is necessary to compare not with value of error of second kinds at fixed error figures of the first stem, and an error at adoption of hypothesis H_2 for the fixed error figure at disallowance hypothesis H_1.

Graphical case history of difference of curves $\beta(x / H_2) = f[\alpha(x / H_1)]$ and $\beta(x / H_1) = f[\alpha(x / H_2)]$ it reduced on fig.2. These curves are builted for criterion of matching of an assessment of chances of failure Q^* with hypothetical probability Q_0, where $Q^* = n_i / n_\Sigma = 3/60 = 0.05$, and $n_\Sigma = \sum_{i=1}^{m_r} n_i$

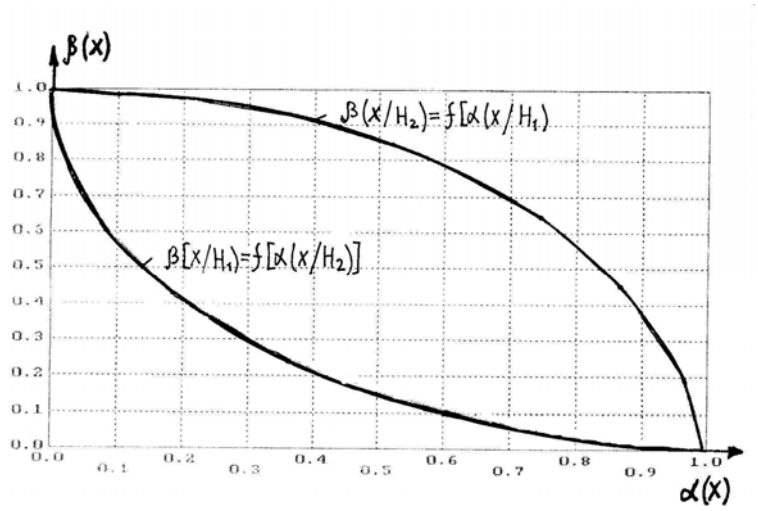

Fig. 2. A graphical case history of difference curves of intercoupling of errors of the first and second stem.

To simplify the subsequent account, to consider (3) and (4), we shall agree probability erratic disallowance hypotheses H_2 to designate through $Sh(x/H_2)$ and probability erratic disallowance hypotheses H_1 to designate through $Sh(x/H_1)$.

The subsequent treating of singularities of matching of criteria of discernment distribution functions of variation of IR at nominal and ordinal dials of argument we shall continue on a particular instance.

5. To have a possibility to evaluate reliability of a solution, EC has been received by a method of statistical modeling, by:

а) software prototyping n_Σ random numbers ξ with an even distribution in the interval [0,1];

б) compliance test of these (n_Σ) random numbers to the uniform law Kolmogorov's criterion;

в) arrangement n_Σ random numbers in m_r peer spacing by comparison ξ_v with the upper boundary values m_r spacing by formula

$$i/m_r < \xi_v \le (i+1)/m_r \qquad \text{c i=1,}(m_r+1)$$

г) assessments of probability of a development set VI by formula $Q_i^* = n_i / n_\Sigma$.

The first criterion is based on the supposition of correspondence of probability of a development of each of i=1, m_r VI to binomial low. Critical value of errors of the first and second stem for each spacing were sampled in view of theorem Touke according to which $\alpha_{K,i} = \alpha_K / m_r$ and $\beta_{K,i} = \beta_K / m_r$ c i=1, m_r. Let's designate it conditionally through K_B.

The second criterion based on an assessment of allocation of the greatest divergences of simulated implementation of allocations $F(i)$ и $F^*(i)$, where $F(i) = i/m_r$; $F^*(i) = \sum_{i=1}^{m_r} n_i / n_\Sigma$; $n_\Sigma = \sum_{i=1}^{m_r} n_i$ .. We shall designate it conditionally through K_δ

In table 1 value of argument X and conforming discontinuous distributions are reduced $\alpha(x_1/H_1)$, $\beta(x_1/H_2)$, $\alpha(x_2/H_1)$ and $\beta(x_2/H_2)$, where $x_2 = x_m \cdot n_\Sigma$. As follows from this table, to the same x there match various value $\alpha(x_1/H_1)$ and $\alpha(x_2/H_1)$, that brings ambiguity of

comparison of criteria and comparison bears that $\beta(x_1 / H_2)$ and $\beta(x_2 / H_2)$ at fixed $\alpha(x / H_1)$ it is impossible, and consequently, and it is erratic.

M. To reduce $\alpha(x_1 / H_1)$ and $\alpha(x_2 / H_1)$ to the same argument, we shall compare argument X with quotients of a significance of power of the criteria computed by formula:

$$A(x) = \left[1 - \beta(x / H_2)\right] / \alpha(x / H_1) = W(x / H_2) / \alpha(x / H_1) \quad (7)$$

at $M^*(x / H_1) < M^*(x / H_2)$

$$B(x) = \beta(x / H_1) / \left[1 - \alpha(x / H_2)\right] = W(x / H_1) / \alpha(x / H_2) \quad (8)$$

at $M^*(x / H_1) > M^*(x / H_2)$

Numerical values of allocations $\alpha(x / H_1)$ and $\beta(x_{2,i} / H_2)$

Table 1

x_i	$\alpha(x_{1,i} / H_1)$	$\beta(x_{2,i} / H_2)$	$\alpha(x_{2,i} / H_1)$	$\beta(x_{2,i} / H_2)$
2	0,8861	0,0015	0,9810	0,0030
3	0,7471	0,0063	8322	0,0659
4	0,5672	0,0202	0,5275	0,3336
5	0,3899	0,0512	0,2178	0,6613
6	0,2312	0,1081	0,0579	0,8721
7	0,1241	0,1958	0,0090	0,9590
8	0,0596	0,3120	-	0,9860
9	0,0258	0,4464	-	0,9940
10	0,0100	0,5834	-	0,9980
11	0,0036	0,7079	-	0,9990
12	0,0011	0,8097	-	-
13	0,0003	0,8848	-	-

Outcomes of accounts $A(x_1)$ and $A(x_2)$ are reduced in table 2. As follows from table 2, at the fixed value of argument X quotient of a significance $A(x_1)$ for criterion K_B it is more, than value $A(x_2)$ for criterion K_δ.

Outcomes of accounts of empirical value of quotients $A(x_1)$ and $B(x_2)$

Table 2

x_i	$A(x_{1,i})$	$A(x_{2,i})$
2	1,13	1,02
3	1,13	1,12
4	1,73	1,26
5	2,47	1,56
6	3,86	2,21
7	6,48	4,56
8	11,54	-
9	21,46	-
10	41,66	-

It is necessary to mark, that comparison of criteria should conducted not for the same arguments x, and for critical value $x_K = X\left[\alpha(x / H_1) < \alpha_K\right]$, i.e. to argument with the greatest value

$\alpha(x/H_1)$, satisfying to a condition $\alpha(x/H_1) < \alpha_K$. For example, according to table 1 for criterion K_B at $\alpha_K = 0.05$ value $x_{1,K} = 9$, and for criterion K_δ - pearly 8.

7. Comparison of criteria can carried out and a little differently. In a fig 3 curves are reduced $Sh(x/H_2) = f[Sh(x/H_1)]$. For conforming critical value of probability $Sh(x/H_1)$ value are determined $Sh(x/H_2)$. In the received statement of conditions of comparison of criteria, than $Sh(x/H_2)$ for matching $Sh(x/H_1)$ it is less, that the criterion is more preferable. According to a fig 3 it is criterion K_B.

8. Despite of a seeming finality the tasks in view separate, multiply the checked out facts were not matched with noted in item 6 and 7 outcomes of comparison of criteria. To them concerned:

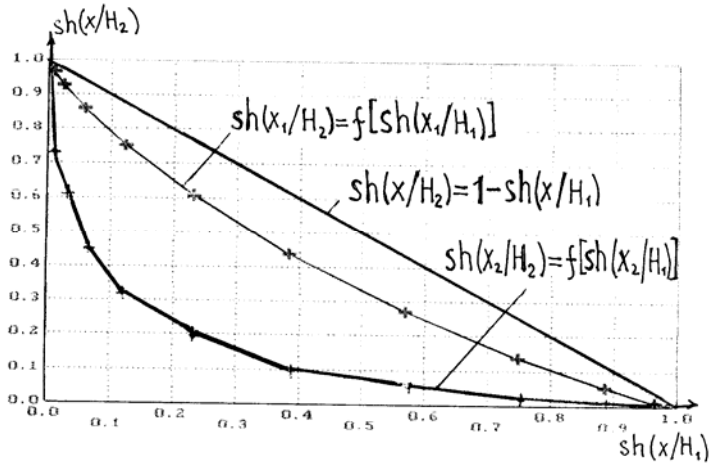

Fig. 3. The Graphical case history of matching of criteria

8.1. If $F(x/H_1) = F(x/H_2)$, i.e. H_1=H_2, that irrespective of type FC function $Sh(x/H_2) = f[Sh(x/H_1)]$ looks like $Sh(x/H_2) = 1 - Sh(x/H_1)$. If $H_1 \neq H_2$, that $Sh(x/H_2) \neq 1 - Sh(x/H_1)$;

8.2. The empirical value of the greatest divergence between EC and FC is less, the incurvation (bulge) of curves is less $Sh(x/H_2) = f[Sh(x/H_1)]$ and the more error figure $Sh(x/H_2)$;

8.3. At small difference EC and EC, including practically insignificant, a solution of matching EC and FC will be: «the information has not enough for an adoption of a decision »;

8.4. With magnifying of number of experiences n_Σ and correspondences $F^*(i)$ to even distribution, value $Sh(x/H_2)$ increases.

9. These data allow to conclude, that if EC is received by statistical modeling on some allocation $F(i)$ with i=1,m_r, that value $Sh(x/H_2)$ characterizes probability of an erratic deflection of hypothesis H_2 owing to a random in character of assessments of the IR computed for each of m_r VI. In this case it is easy to explain, why with decrease of a divergence between EC and FC value $Sh(x/H_2)$ at the fixed number of "experiences" increases (the less divergence, the more than data it is necessary for a discernment of this divergence) and why with magnifying of a divergence between EC and FC $Sh(x/H_2)$ diminished

10. If to receive, that at equiprobable development VI value $Sh(x/H_2) = \overline{Sh}(x/H_2)$, that value

$$\Delta Sh(x/H_2) = \left| Sh(x/H_{2S}) - \overline{Sh}(x/H_2) \right| \qquad (9)$$

it will be proportional to an error in a discernment of difference EC from the uniform law. At the fixed value $Sh(x_K/H_1)$ the preference is returned criterion with greater value $Sh(x/H_2)$, and with allowance for item.6 the preference is returned criterion, for which greatest divergence of quotients A(x) or B(x) and units of the fixed value $sh(x_K/H_1)$ the least on matching with other criteria.

Systematizing the above-stated, the method and algorithm of matching S of criteria is represented the following amalgamated sequence of evaluations:

1. Allocations pay off $F(x/H_1)$ and $F(x/H_2)$;
2. Subject to the conditions (3) and (4) allocations are formed $Sh(x/H_1)$ and $Sh(x/H_2)$;
3. Real critical value of probability erratic disallowance hypotheses H_1 by formula are determined

$$Sh_d(x/H_1) = \max\{Sh(x/H_1) < \alpha_K\} \qquad (10)$$

4. It is determined $Sh(x/H_2)$, matching $Sh_d(x/H_2)$;
5. The probability erratic disallowance hypotheses H_2, caused by algorithm of criterion by formula is evaluated

$$\Delta Sh(x/H_2) = \left| 1 - Sh_d(x/H_1) - Sh(x/H_2) \right|$$

6. The preference returned criterion, for which $\Delta Sh(x/H_2)$ the least.

Inference

1. The account of an error of second kind in conditions when aftereffects from erratic solutions are indiscernible, so important, as well as an error of first kind. The disregard to physical nature of both errors leads in practical accounts to incorrect solutions;

2. Known references with reference to criteria of a discernment of the functional characteristics of indexes of reliability at ordinal and nominal dials of argument are unacceptable for matching criteria;

3. Erratic solutions at usage of these references originate owing to:

Insufficient sharpness of the gear of the account of physical nature of errors of the first and second stem. Such "gear" the parity of means of argument of allocations can minister $F(x/H_1)$ and $F(x/H_1)$;

Difference of levels of a discrete sampling of arguments allocations $\alpha(x/H_1)$ and $\beta(x/H_2)$. Characteristics $\beta(x) = f(\alpha(x))$ should be under construction for the same value x;

Comparisons of error of second kinds of criteria at the fixed value of an error of first kind. It is necessary to compare with an error in disallowance hypotheses H_2 at the fixed value erratic disallowance hypotheses H_1;

Differences erratic disallowance hypothesis H_1 for equal levels of a discrete sampling of statistician of compared criteria. Overcoming of this nonconformity is reached by comparison of criteria on regularities of variation of a relative significance of power of criteria under formulas (7) and (8)

Irregular interpreting matching $Sh(x/H_1)$ probabilities $Sh(x/H_2)$, only as probabilities erratic to deny hypothesis H_2. Actually $Sh(x/H_2)$ characterizes probability erratic disallowance hypotheses H_2 owing to roundedness of statistical data for a discernment of the functional junctions;

4. The method, algorithm and programs model of matching of criteria of a discernment of the functional characteristics of indexes of reliability of plants EES is developed. Probability of the supervision of reliability of a solution were ensured with a solution technique of "inverse problem" when empirical the characteristics of indexes of reliability were simulated on the sampled regularity of variation VI.

5. Matching of the criterion based on binomial model of probability of development VI and criterion, the greatest deflection of empirical and hypothetical characteristics based on value bears to doubtless advantage of the second criterion.

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ON ENGINEERING RELIABILITY CONCEPTS AND BIOLOGICAL AGING

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ABSTRACT

In this study, various stochastic approaches to biological aging modeling are discussed. We assume that an organism acquires a random resource at birth. Death occurs when the accumulated damage (wear) exceeds this initial value. Another source of death of an organism is also taken into account, when it occurs as a consequence of a shock or of a demand for energy, which is a generalization of the Strehler-Mildvan's model. Biological age, based on the observed degradation, is also defined. Finally, aging properties of imperfectly repaired systems are discussed. We show that aging slows down with age in this case. This presents another possible explanation for human mortality rate plateaus.

Keywords: damage accumulation, redundant systems, degradation of organisms, aging distributions, mortality rate, biological age.

1. INTRODUCTION

There is extensive published literature on numerous biological theories of aging. Various stochastic mortality models are reviewed, for instance, in Yashin *et al.* (2000). Most authors agree that the nature of aging is associated with "biological wearing" or "wear and tear". Reliability theory possesses the well-developed tools for odelling wear in technical systems; therefore it is natural to apply this technique to biological aging (Finkelstein, 2005). Because even the simplest organisms are much more complex than the technical systems that are usually considered in reliability analysis, these analogies should not be interpreted too literally. Therefore, the implications of the corresponding stochastic odelling should be considered carefully.

Populations of biological organisms – unlike populations of technical devices – evolve in accordance with evolutionary theory. Various maintenance and repair problems (including those with limited resources) have been intensively studied by reliability theory. However, the notion of reproduction, which is crucial for bio-demography, has not been considered – although stochastic birth and death processes can certainly be useful for the corresponding odelling. On the other hand, popular evolutionary theories (e.g., Kirkwood's "disposable soma" concept (Kirkwood, 1977, 1997)) try to link mortality, fertility, maintenance and repair, but do not yet possess the sufficient biological knowledge and mathematical tools for considering appropriate stochastic models of repair and maintenance in a proper evolutionary context. This means that existing and future reliability models could enrich biological aging theory and *vise versa*: for example, a disposable soma concept can be helpful for the optimal allocation of spare parts in some "structurally homogeneous" engineering systems.

It is worth noting that evolutionary theories tend towards a rather controversial view in that all damage, in principle, is repairable and that natural selection can shape the lifetime trajectory of damage and repair, constrained only by physical limitations of available resources (Steinsaltz and Goldwasser, 2006). However, not all damage in organisms can be reversed: for example, damage to the central nervous system and heart tissue is usually irreversible. In any case, the importance of different repair mechanisms for the survival of organisms is evident, which brings into play stochastic odelling of all types of repairable systems: perfectly, minimally and imperfectly repairable ones. This topic has been partially studied in reliability theory, but there are still many problems.

The future general theory of aging will probably be built on the basis of future unified biological theories that will use stochastic reliability approaches as an important analytical tool. Some interesting discussion on general "quality management" of organisms and the pros and cons of exploiting the existing reliability approaches for biological aging are presented in Steinsaltz and Goldwasser (2006).

Vaupel's (2003) conjecture that "after reproduction ceases, the remaining trajectory of life is determined by forces of wear, tear, and repair acting on the momentum produced by the Darwinian forces operating earlier in life" resulted in the reliability odelling of Finkelstein and Vaupel (2006). These authors state: "As the force of natural selection diminishes with age, structural reliability concepts can be profitably used in mortality analysis. It means that the design of the structure is more or less fixed at this stage and reliability laws govern its evolution in time. However, it does not mean that these concepts cannot be used for mortality odelling at earlier ages, but in this case they should be combined with the laws of natural selection".

In accordance with a conventional definition, reliability of a technical object is the probability of performing a designed function under given conditions and in a given interval of time (Hoyland and Rausand, 1993). This definition can be applied for a probabilistic description of a lifespan of organisms T, where its designed function is just to be alive. For example, the main demographic model for the lifetime of humans is the Gompertz (1825) law of mortality, defined by the exponentially increasing mortality rate $\mu(t)$:

$$F(t) = \Pr(T \le t) = 1 - \exp\left\{-\frac{\alpha}{\beta}[\exp\{\beta t\} - 1]\right\}, \qquad (1)$$

$$\mu(t) = \alpha \exp\{\beta t\}, \quad \alpha > 0, \beta > 0.$$

This is a direct descriptive way to model the lifespan random variable T. It is well known that human mortality data, at least for adults, perfectly comply with this model.

In accordance with reliability terminology, the Gompertz law belongs to a family of increasing failure rate (IFR) distributions. This is the simplest and most commonly used in reliability theory aging family of distributions. It is widely used for description of various degradation processes in engineering systems (Barlow and Proschan, 1975). There were a number of attempts in the past to justify the exponential form of the human mortality rate by some mechanism or model, but most of these exploited additional assumptions, either explicitly or implicitly equivalent to the desired exponentiality. (Strehler and Mildvan, 1960; Witten, 1985; Koltover,1997; Gavrilov and Gavrilova, 2001).

In what follows, we consider several important applications of reliability-based stochastic reasoning, unified by concepts of aging and degradation. This is partially a review paper of relevant approaches that are mostly developed or generalized by the author in the field of engineering reliability, but modified and adjusted to the description of biological aging.

In Section 2, the resource-based models are considered – an organism at birth acquires some random resource (vitality) and the death occurs when this resource is 'consumed'. For the first time, we use here the unified approach to deal both with the cases of continuous and discrete

resources. Specifically, we show that the reliability theory of aging (Gavrilov and Gavrilova, 2001) is a simple particular case of our more general model and, moreover, does not necessarily result in the Gompertz law.

Section 3 is devoted to the definition of the virtual age of a degrading object. We consider this topic as being one of the most important for the future research, as it creates the possibility of comparing life spans of organisms in different environments.

In Section 4, we suggest a new generalization of the Strehler and Mildvan (1960) vitality model with shocks, and show that the necessary condition that is omitted in original and subsequent publications is the assumption of the Poisson property of the shock process.

Finally, Section 5 deals with aging in repairable systems (imperfect repair). The results of this section are based on our recent mathematical findings (Finkelstein, 2007). An important interpretation for biological aging is that under certain assumptions, aging can slow down for individuals of advanced ages, which is already observed for human populations (mortality plateau).

We are convinced that mathematical, reliability-based modeling of aging is an important part of biological aging research. We show how straightforward stochastic approaches can work, in principle, for some settings. These approaches are probably oversimplified and should be developed in the future to more closely match the real biological situation.

N. UNOBSERVED OVERALL RESOURCE

Following Finkelstein (2003), we assume that an organism at birth ($t = 0$) acquires an overall unobserved random resource R with a distribution function $F_0(r): F_0(r) = P(R \le r)$. We also assume that the process of an organism's aging is described by an increasing, deterministic for simplicity cumulative damage function $W(t)$ ($W(0) = 0$) (to be called "wear"). The wear increment in $[t, t+dt)$ is defined as $w(t) + o(dt)$. Additionally, let $W(t) \to \infty$ as $t \to \infty$. Under these assumptions, we arrive at the well-known in reliability theory the accelerated life model (ALM):

$$P(T \le t) \equiv F(t) = F_0(W(t)) \equiv P(R \le W(t)), \tag{2}$$

$$W(t) = \int_0^t w(u)du; \quad w(t) > 0; \quad t \in [0, \infty).$$

Death occurs when the wear $W(t)$ reaches R.

Substituting the deterministic wear $W(t)$ in (2) by the increasing stochastic process $W_t, t \ge 0$ leads to the following relationship (Finkelstein, 2003):

$$F(t) = P(T \le t) = P(R \le W_t) = E[F_0(W_t)], \tag{3}$$

where the expectation is defined with respect to $W_t, t \ge 0$. As the mortality rate is a conditional characteristic, it cannot be obtained from (3) as a simple expectation: $\mu(t) = E[w_t \mu_0(W_t)]$ and the proper conditioning should be performed (Yashin and Manton (1997)):

$$\mu(t) = E[w_t \mu_0(W_t) \mid T > t], \tag{4}$$

where w_t denotes the stochastic rate of diffusion: $dW_t \equiv w_t dt$, and the baseline mortality rate $\mu_0(t)$ is defined by the distribution $F_0(t)$.

A good candidate for $W_t, t \ge 0$ is the gamma process, which, according to definition, has stationary independent increments and $W_t - W_s$ ($t > s$) has the gamma density with scale 1 and shape $(t - s)$. The Wiener process can also sometimes be used for odelling wear, but it does not possess the monotonicity property, which is natural for the processes of wear.

Example 1. As a specific case of the unobserved reserve model, consider now a discrete resource $R = N$ with a distribution: $F_0(n) \equiv P(N \le n)$. The following simple reliability interpretation is meaningful: Let N be a random number of initially (at $t = 0$) operable independent and identically distributed (i.i.d.) components with constant failure rates λ. Assume that these components form a parallel system, which, according to Gavrilov and Gavrilova (2001), can model the lifetime of an organism (the generalization to the series-parallel structure is straightforward). In each realization, $N = n, n \ge 1$, our degradation process $W_t, t \ge 0$ for this setting is just a counting process for the corresponding process of pure death: when the number of events (failures of components) reaches n, the death of an organism occurs. The transitions rates of the corresponding Markov chain are: $n\lambda, (n-1)\lambda, (n-2)\lambda, ...$ Denote by $\mu_n(t)$ the mortality rate, which describes T_n - the time to death random variable for the fixed $N = n, n = 1,2,...$. ($n = 0$ is excluded, as there should be operable components at $t = 0$). Similar to (4), the mortality rate is given as the following conditional expectation with respect to N:

$$\mu(t) = E[\mu_N(t) \mid T > t]. \tag{5}$$

Note that for small t:

$$\mu(t) \approx E[\mu_N(t)] = \sum_{n=1}^{\infty} P_n \mu_n(t), \tag{6}$$

where $P_n \equiv P(N = n)$, but the limiting transition, as $t \to 0$, should be performed carefully in this case. It is clear that as $t \to \infty$:

$$\mu(t) \to \lambda. \tag{7}$$

This is because the conditional probability (on condition that the system is operable) that only one component is operable, tends to 1.

Assume that N is Poisson-distributed with parameter η. Taking into account that the system should be operable at $t = 0$:

$$P_n = \frac{\exp\{-\eta\}\eta^n}{n!(1-\exp\{-\eta\})}; \; n = 1,2,....$$

It can be shown that (Steinsaltz and Evans, 2004):

$$F(t) = P(T \le t) = \frac{1-\exp\{-\eta \exp\{-\lambda t\}\}}{1-\exp\{-\eta\}}. \tag{8}$$

The corresponding mortality rate is:

$$\mu(t) = \frac{F'(t)}{1-F(t)} = \frac{\eta\lambda \exp\{-\lambda t\}}{\exp\{\eta \exp\{-\lambda t\}\} - 1}. \tag{9}$$

It can be seen that the mortality plateau (7) exists for the mortality rate (9) as well. This function is far from the exponentially increasing Gompertz law. In fact, the Gompertz law can erroneously result, if approximation (6) is used formally, as in Gavrilov and Gavrilova (2001).

O. DEGRADATION AND VIRTUAL (BIOLOGICAL) AGE

The previous section is helpful for discussing an important and challenging notion of virtual (biological) age. Assume for simplicity, as previously, that deterioration of an organism can be modeled by a single, predictable, increasing stochastic process with independent increments $W_t, t \ge 0$. Observing its state at time t can give, under certain assumptions, an indication of a 'true' age, which is defined by the level of the observed deterioration. We shall call this characteristic an *information-based virtual (biological) age* of a system or of an organism. If, for example, someone of 50 years old looks like and has vital characteristics (blood pressure, level of cholesterol, etc) that

are of an 'ordinary' 35- year-old individual, we could say that this observation indicates that his virtual (biological) age can be estimated as 35 years. This is, of course, a rather vague statement, which could be made more precise for some simple, specific model settings and under certain assumptions.

Example 2. Consider a system of $n+1$ components (one initial component and n cold standby identical ones) with constant failure (mortality) rates λ, which starts operating at $t=0$. Note that in the previous example we had described a system with a hot (loaded) redundancy. The failure occurs when the last component fails. Thus $W_t, t \geq 0$ in this case is just a counting process (number of failed components) for the stopped Poisson process with rate λ. A possible biological interpretation: the limited number of repairs (Vaupel and Yashin, 1987) or cell replications. The mortality rate of the described system is an increasing function of the form (Hoyland and Rausand, 1993):

$$\mu(t) = \frac{\lambda \exp\{-\lambda t\}(\lambda t)^n / n!}{\exp\{-\lambda t\}\sum_0^n \frac{(\lambda t)^i}{i!}}. \tag{10}$$

Consider the following conditional expectation:

$$D(t) \equiv E[N(t) \mid N(t) \leq n] = \frac{\exp\{-\lambda t\}\sum_0^n i\frac{(\lambda t)^i}{i!}}{\exp\{-\lambda t\}\sum_0^n \frac{(\lambda t)^i}{i!}} \tag{11}$$

where $N(t)$ is the number of events in the interval $[0,t]$ for the Poisson process with the rate λ. As we observe an operable system, relationship (11) defines the expected value of the number of its failures (measure of degradation) on condition of survival in $[0,t]$. The function $D(t)$ is monotonically increasing, $D(0)=0$ and $\lim_{t\to\infty} = n$. This function defines an average degradation curve for the defined system. Assume that at time t we observe k failed components. This is the measure of observed degradation in our system. Denote the corresponding (information-based) virtual age by $V(t)$. Our definition of $V(t)$ for this specific model is:

$$V(t) = D^{-1}(k), \tag{12}$$

where $D^{-1}(t)$ is an inverse function to $D(t)$, which is obviously also increasing. If $k = D(t)$, then: $V(t) = D^{-1}(D(t)) = t$.

When the observed degradation k at time t is less than the expected $D(t)$, then the corresponding virtual age is less than a calendar age t and vise versa.

If n is sufficiently large, then $D(t) = \lambda t$ and in accordance with (12):

$$V(t) = \frac{k}{\lambda}.$$

Equivalently, as the function $D(t)$ is linear in this specific case, the virtual age $V(t)$ is equal to the expected age at which the number of observed failures is k.

A general case of degrading objects can be considered in the same way. Let D_t be an increasing, smoothly varying (predictable) stochastic process of degradation with a mean $D(t)$. Assume for simplicity that this is a *process with independent increments*, and therefore it possesses the Markov property. Similar to (12), observation, d_t at time t defines the virtual age. Formally:

Definition. *Let D_t be an increasing, predictable, with independent increments stochastic process of degradation with a mean $D(t)$, and let d_t be an observation at time t.*

Then the virtual age is defined as:

$$V(t) = D^{-1}(d_t).$$

Alternatively, the virtual age can be defined as the mean age for the process to reach the level d_t. Usually, obtaining $D(t)$ is easier then obtaining the mean time to reach the threshold d_t and, therefore, the foregoing definition is more convenient.

Thus, considering degradation in a simple reliability structure resulted in a general definition and in a helpful for studying of aging notion of virtual age. Note that, the approach of this section is heuristic and further mathematical justification will be published elsewhere.

P. SHOCK MODELS AND DEGRADATION

Technical systems and organisms are usually subject to shocks, which are harmful events that occur randomly in time and magnitude, and that can cause a failure or death, respectively. We assume for simplicity that durations of shocks are negligible. In mechanical and electronic systems, for example, shocks occur when the applied load exceeds the strength. Diseases, viruses, heart attacks or, more generally, demands for energy (as in the Strehler-Mildvan model to be discussed in this section), can be interpreted as shocks for organisms. The stochastic theory of shocks was extensively studied in reliability literature, although there are still a lot of open questions from theoretical and practical points of view. Traditionally, two basic cases – the cumulative shock model and the extreme shock model – were considered. The former means that the system fails when the cumulative shock magnitude enters some critical region (Sumita and Shantikumar, 1985). The latter means that the system breaks down as soon as the magnitude of an individual shock goes into some given critical region (Shantikumar and Sumita, 1983). In what follows in this section, we will revisit the Strehler-Mildvan model in more general assumptions and justify this approach from the probabilistic point of view, proving that it is valid only under the additional assumption that the shocks (demands for energy) occur in accordance with the Poisson process.

Consider a univariate first-passage-type model with shocks. Let, as previously, $W_t, t \geq 0$ denote an increasing stochastic process of damage accumulation (e.g. the gamma process) and $R(t)$ be a function that defines a corresponding boundary. In Section 2, it was a random constant: $R(t) \equiv R$. Assume for simplicity that $R(t)$ is deterministic.

Let $P_t, t \geq 0$ be a point process of shocks with rate $\lambda(t)$ and independent from $W_t, t \geq 0$. Assume that each shock, independently from the previous ones, results in death with probability $\theta(t)$ and is "survived" with the complementary probability $1-\theta(t)$. This can be interpreted in the following way: each shock has a random magnitude $Y_i = Y, i = 1,2,...$ with a distribution function $\Psi(y)$. The death at age t occurs when this magnitude exceeds the margin: $R(t) - w(t)$, where $w(t)$ denotes the increasing sample path of the process of degradation. Therefore:

$$\theta(t) = \Pr(Y > R(t) - w(t)) = 1 - \Psi(R(t) - w(t)).$$

In the original Strehler-Mildvan model (Strehler and Mildvan, 1960), which was widely applied to human mortality data, our $R(t) - w(t)$ has a meaning of vitality of organisms. It was also supposed that this function linearly decreases with age and that the distribution function $\Psi(y)$ is exponential (Yashin *et al*, 2000). We do not need these stringent assumptions for the forthcoming considerations.

It is worth noting that the rate $\lambda(t)$ does not define an arbitrary point process. However, it can be defined via its complete intensity function (Cox and Isham, 1980):

$$\lambda(t;H_t)=\lim_{\Delta t\to 0}\frac{\Pr\{N(t,t+\Delta t)=1\mid H_t\}}{\Delta t},$$

where H_t specifies the point process up to time t (history). Thus $\lambda(t;H_t)dt$ can be interpreted as the probability of a shock occurrence in $[t,t+dt)$, given the process history up to t. Therefore, the conditional mortality rate in our model is:

$$\mu_c(t,H_t)dt=\Pr\{T\in[t,t+dt)\mid H_t,T(H_t)\geq t\}=\theta(t)\lambda(t,H_t)dt, \qquad (13)$$

where condition $T(H_t)\geq t$ means that all shocks in $[0,t)$ were survived. It is clear from the definition of the Poisson process that only for this specific case equation (13) did reduce to the usual, non-history-dependent mortality rate $\mu(t)$ (unfortunately, the Strehler-Mildvan model did not consider this crucial assumption):

$$\mu_c(t,H_t)=\theta(t)\lambda(t)=\mu(t). \qquad (14)$$

Therefore, the conventional exponential representation for the corresponding survival function ($\overline{F}(t)\equiv 1-F(t)$) is

$$\overline{F}(t)=\exp\left\{-\int_0^t\theta(u)\lambda(u)du\right\} \qquad (15)$$

and this completes the proof for the specific case of the Poisson process of shocks for the case when shocks are the only source of death. The technical proof of this fact can be found, for example, in Brown and Proschan (1983). Another meaningful interpretation of this result is via the thinning of the initial Poisson process with the rate $\lambda(t)$, which results in the Poisson process with the rate $\theta(t)\lambda(t)$. Therefore, the survival function up to the first event in this process (death) is given by equation (15).

We have derived equations (14) and (15) for the sample paths $w(t)$ and deterministic $R(t)$. A general case of the processes $W_t, t\geq 0$ and $R_t, t\geq 0$ can be also considered under reasonable assumptions. The probability $\theta(t)$ turns to a stochastic process $\theta_t, t\geq 0$, whereas the mortality rate $\mu(t)$ also starts to be stochastic, and conditioning similar to those in equations (4) and (6) should be used.

Equation (14) states that the resulting mortality rate is just a simple product of the rate of the Poisson process and of the probability $\theta(t)$. Therefore, its shape can be easily analyzed. When $R(t)-w(t)$ is decreasing, the probability of death $\theta(t)$ is increasing with age, which is consistent with the conventional accumulation of degradation reasoning. If, additionally, the rate of shocks $\lambda(t)$ is not decreasing, or decreasing not faster than $\theta(t)$ is increasing, the resulting mortality rate $\mu(t)$ is also increasing. In conventional settings, $R(t)$ is usually assumed to be a constant: therefore, $R-w(t)$ is decreasing automatically. On the other hand, it can be easily seen that, in principle, certain reasonable combinations of shapes of functions $\theta(t)$ and $\lambda(t)$ can result in decreasing or ultimately decreasing mortality rates (negative senescence). For example, $R(t)$ can increase faster than $w(t)$ - an organism is 'earning or obtaining' additional vitality in the course of life. This approach, in fact, deals with two dependent sources of death: degradation and shocks.

Example 3. Following our previous examples, assume that the degradation process is given by the counting measure of the Poisson process with rate λ and that there are no deaths due to

direct degradation. On the other hand, let the traumatic mortality rate be constant for the degradation level n (number of events in the Poisson process): $\mu_n, n = 0,1,2,....$ It is reasonable to assume that mortality rates are increasing with degradation: $\mu_0 < \mu_1 < \mu_2 < ....$ The stochastic mortality rate (the mortality rate process) can be compactly written via the corresponding indicator function as:

$$\mu_t = \sum_{n=0}^{\infty} \mu_n I(S_n \le t < S_{n+1}); t \ge 0,$$

where S_n is the time of the nth event arrival, $S_0 = 0$. The observed (marginal) mortality rate $\mu(t)$ can be, in principle, obtained from this equation by direct integration on condition that there were no deaths in $[0,t)$, but the resulting formula is cumbersome.

Q. AGING OF REPAIRABLE SYSTEMS

Although it is widely admitted by the evolutionary and non-evolutionary theories of aging that repair and repair mechanisms on all levels play a crucial role in senescence, little had been done in terms of stochastic repair odelling in organisms. On the one hand, it is clear that different theories require different "machinery", on the other, there are certain general principles and approaches developed (or to be developed) by reliability theory and the theory of stochastic processes that can be applied to various biological setting.

Consider some hypothetical repairable object – to be called for convenience a component – which starts functioning at $t = 0$. Assume, as usual, that repair is perfect (after the repair a component is as good as new. The sequence of independent, identically distributed inter-arrival times $\{T_i\}_{i\ge1}$ with a common distribution function $F(t)$ forms a standard renewal process. The repair times in this case are given by the sequence $T_1, T_1 + T_2, T_1 + T_2 + T_3, ...$ Assume that the generic $F(t) \in$ IFR, which means that the corresponding failure rate $\lambda(t)$ is not decreasing. Therefore $F(t)$ is an aging distribution. What can be said about the aging properties of the renewal process? It is reasonable to conclude that as the repair is perfect, there is no aging in this process, as after each perfect repair the age of a component is 0. Thus, the perfect repair clearly does not lead to accumulation of damage in the described sense. But this is not so when the repair is not perfect, which is definitely the case in nature and in most technical systems. Note that even the complete overhaul of a system, which is usually considered as a perfect repair, is not such, as even switched off standby items also age.

Let us call a period between two successive repairs a 'cycle'. We have two major possibilities. The first is when the imperfect repair reduces wear of the last cycle only. It is clear that, in this case, the overall wear increases and under some reasonable assumptions this operation only decreases the rate of accumulation of wear for the process. This ant-aging mechanism is described in Finkelstein (2003). The situation starts to be much more interesting, at least from the modeling point of view, when the current repair reduces the overall accumulated wear. We shall model this setting in the following way: Assume now that the repair at $t = t_1$ (realization of T_1) decreases the age of a system not to 0 as in the case of a perfect repair, but to $v_1 = qt_1, 0 < q < 1$, and the system starts the second cycle with this initial age in accordance with the distribution of the remaining lifetime $1 - \overline{F}(v_1 + t)/\overline{F}(v_1)$. The constant q defines the quality of repair. The forthcoming results can be generalized to the cases of random quality of repair (the time-dependent $q(t)$ can be also considered).

Thus, the reduction of wear is modeled by the corresponding reduction in age after the repair. Note that, as the failure rate of a component $\lambda(t)$ is increasing, the described operation also

decreases its value and the failure rate at the beginning of the new cycle is smaller, than it was at the end of the previous one. The forthcoming cycles are defined in a similar way to form a process of general repair (Kijima, 1989; Finkelstein, 2007). The sequence of ages after the i th repair $\{V_i\}_{i\geq 0}$ in this model is defined as:

$$V_0 = 0; V_1 = qT_1; V_2 = q(V_1 + T_2),....,V_i = q(V_{i-1} + T_i),.... \quad (16)$$

and distributions of the corresponding inter-arrival times for realizations v_i are given by:

$$\overline{F}_i(t) = \frac{\overline{F}(v_{i-1} + t)}{\overline{F}(v_{i-1})}, i \geq 1.$$

Denote the distribution of age at the start of the $(i+1)$th cycle by $A^S_{i+1}(v)$, $i = 1,2,....$ ($v = 0$ at the start of the first cycle) and by $A^E_i(v), i = 1,2,...$ the corresponding age distribution at the end of the previous i th cycle. It is clear that, in accordance with our model:

$$A^S_{i+1}(v) = A^E_i(v/q)), i = 1,2,...$$

This can be easily seen, as

$$A^S_{i+1}(v) = \Pr(V^S_{i+1} \leq v) = \Pr(qV^E_i \leq v) = \Pr(V^E_i \leq v/q),$$

where V^S_{i+1} is a random age at the start of $(i+1)$ th cycle, whereas V^E_i -is a random age at the end of the previous one. The following results (Finkelstein, 2007) state that the age processes under consideration are stochastically increasing and are tending to a limiting distribution.

a. *Random ages at the end (start) of each cycle in the general repair model (16) form the stochastically increasing sequences:*

$$\overline{A}^E_{i+1}(v) > \overline{A}^E_i(v), (\overline{A}^S_{i+2}(v) > \overline{A}^S_{i+1}(v)), t > 0; v > 0, i = 1,2,.... \quad (17)$$

b. *There exist limiting distributions for ages at the start and at the end of cycles:*

$$\lim_{i\to\infty} A^E_i(v) = A^E_L(v), (\lim_{i\to\infty} A^S_i(v) = A^S_L(v)). \quad (18)$$

The corresponding interpretation is simple and meaningful. Indeed, as the ages at the start (end) of the cycles are random, they should be compared stochastically. The simplest and the most natural ordering is the ordering of the corresponding distribution functions at every point of support. This is usually called stochastic ordering or stochastic dominance. It follows from (17) that the sequences of the corresponding mean ages at the start (end) of each cycle are also increasing. Thus, the process as a whole is aging, because the ages at the start (end) of the cycles are stochastically increasing with i and the failure rates of inter-arrival times are also increasing functions. The process can be described as "stochastic sliding' to the right along the generic failure rate $\lambda(t)$, which can definitely be qualified as aging. On the other hand, it follows from (18) that the sequences of ages have a finite limit, which means that aging of the process slows down and asymptotically vanishes!

If the repair process in parts of organisms decreases the accumulated wear and not only the wear of the last cycle, then the mortality rate (as a function of degradation) of these parts and of an organism as a whole, slows down at advanced ages and can even tend to a constant t (mortality plateau). Therefore, our model can explain the deceleration of human mortality at advanced ages (see, for example, Thatcher (1999)) and even approaching the mortality plateau. It is worth noting that another possible explanation of the mortality deceleration phenomenon is via the concept of population heterogeneity (see Vaupel *et al* (1979) for basic facts and Finkelstein and Esaulova (2006) for mathematical details in a general frailty model).

It can be shown under reasonable assumptions that in the case of a minimal repair, which does not reduce wear, or when repair reduces the damage only of the last cycle, the corresponding point process can be described by inter-arrival times of a non-homogeneous Poisson process with

increasing rate. The ages at the start (end) of the cycles in this process tend to infinity as $i \to \infty$. Thus, this model shows a different asymptotic behavior than the one considered previously.

Example 4. The reduction of accumulated damage was modeled via the reduction of age (or the decrease in the failure rate). This is a reasonable approach, as under some assumptions, the process of damage accumulation can be "translated" into the corresponding IFR model. In order to illustrate the limiting behavior of our model in a time-free, direct damage-based reasoning, consider the following simplified setting. Assume that each event from the orderly (without multiple occurrences) point process results in a unit damage, which is immediately reduced by the repair mechanism to $q, 0 < q < 1$. Therefore, accumulation of damage in this model is given by the following series: after the first repair it is q; after the second repair it is $q(q+1) = q + q^2$; after the third repair it is $q(q(q+1)+1) = q + q^2 + q^3$,….Therefore, the accumulated damage increases with each cycle and tends to the limiting, stable value:

$$D_l = \frac{q}{1-q},$$

which defines the accumulated damage plateau.

R. CONCLUSIONS

Under a conventional assumption that the process of biological aging is a process of "wear and tear" we consider several approaches that are useful for odelling and odelling the lifetimes of organisms. All these approaches are united by the accumulation of damage concept, which allows the incorporation and generalization of engineering-reliability thinking to a wider class of objects. Aging is an extremely complex biological process, but it does not mean that it cannot be odellin by some relatively simple stochastic tools.

Repairable and non-repairable systems are considered. We prove that, even in the case of imperfect repair, the resulting process of aging under reasonable assumptions slows down with time and asymptotically fades out. This gives another possible explanation of the human mortality rate plateau.

Using the obtained results, we plan to combine them in future work with optimization under constraints tools, developed in reliability theory, in a suitable evolutionary-theory-based manner.

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"SAFETY" AS A CHARACTERISTIC OF ONE OF THE SINGLE RELIABILITY PROPERTIES

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Abstract

The paper deals with the problems of interrelation between two most important properties of technical (production) systems: safety and reliability that were considered historically separately. However, recently both properties have proven to be increasingly more interrelated, which makes their joint study topical. And the safety may be treated as the most important reliability property, determining to a great extent all the remaining single reliability properties. The relation between the notions of "technical safety" and "energy safety" is also studied.

Keywords. Object, single properties, safety, reliability, technical perfection, interrelation, costs, failures, effects.

In 1980 the terminology in [1], possibly for the first time in the world practice, included such properties as stabilability, survivability, controllability and also **safety** as the object's principal properties forming a complex reliability property of energy systems in addition to those indicated in the then State Standard 13337-75 [2] (or in the current State Standard 27.002-89).

We will consider expediency of applying the notion of safety in reliability on the example of power plants, though in our opinion all the information given below is true for other technical (production) objects.

As concerns safety that is defined as an object's property to prevent people and environment from dangerous situations, the terminology in [1] explains: "the necessity to consider "safety" property in the context of "reliability" can be shown on the following example. Reliability …of a system can be low … because of high probability for people and environment to be injured at any failure of operation … This is a low safety and hence a low reliability".

However, so far some doubts are cast upon the need for the notion "safety" within the framework of "reliability". In our opinion it is caused by the following factors.

Over the course of the whole history of electric power industry development the safety problem has been given great attention thanks to physicotechnical features of electromagnetism. Electromagnetic fields, high currents, discharges and charges, high voltages as well as high temperatures, pressures, etc. in the electric power industry are known to be hazardous for people and environment. For safety reasons concentration in relatively small volumes of considerable amounts of substances dangerous for man and also substances with considerable own energy reserve is also important.

A great number of problems in safe control of power plants stipulated, from the outset, independent consideration of safety as an energy object's property without its relations with other properties and particularly with the reliability property. And the safety problems were solved so successfully that the safety level of an object seemingly had no effect on its reliability.

Hence, it was believed that safety and reliability are two different object's properties that are not connected to one another. Therefore, even nowadays some specialists suppose that safety and reliability are properties to be studied in parallel, independently of each other.

On the other hand, however, danger is always realized as an industrial accident, as a result of a certain type of random events (failure flows):

- failures of production equipment elements;
- natural phenomena;

- erroneous actions of personnel;
- random external effects of man-induced or social nature, i.e. failures that can be simultaneously failures in a sense of safety and other single reliability properties.

The events of the recent two-three decades have confirmed validity of the statement about existing interrelation and urgency of considering this interrelation between safety and reliability. Accidents at nuclear power plants, for example, involve not only dangerous consequences, but long-term failures of their operation and serviceability, long restoration works. Attempts to decrease safe distances between conductors at different potentials; utilization of toxic SF_6 gas insulation and as a result complete change of geometrical forms, sizes, designs and methods of safe maintenance of power units and also their reliability indices; increased environmental pollution; introduction of advanced technologies ("warm" and cryogenic superconductivities) and many other achievements of the technological progress – these combined factors revealed in practice the safety impact on the complex property of object reliability. In [1], therefore, it was proposed that safety should be considered as a single reliability property along with its other properties.

Safety is surely a specific single property of reliability, since it reflects the level of performing only the functions induced solely by the fact of object creation rather than its purpose, as distinct from other properties specified by both the purpose and the fact of object creation. The safety, however, should be considered in terms of reliability as the rest of single properties because the hazard for people and environment arises, as was mentioned, as a result of failures in the process of object operation.

Besides, the link of safety with other reliability properties is presented in [1] because of the fact that some notions applied to determine such properties as longevity, survivability also take into account safety problems. So, in reliability "the limiting state" is defined as an object state, at which it must not operate further due to *unavoidable violation of safety requirements* or unavoidable decrease in operability level or inadmissible decrease in operation efficiency". "The contingency armor is a minimum power supply level, at which industrial or some other consumers stop operating without damage of facilities, product, half-finished material, raw material and at preservation of minimum required sanitation, fireproof conditions and *safety* support".

Nonetheless some specialists continue expressing doubt about validity to refer safety to a single reliability property. Their main arguments are the following:

- a small number of studies treating safety as a reliability property and numerous studies continuing to consider safety separately. In other words, there is no need for the safety property in the reliability aspect;
- the emerging notion "energy safety (security)" makes it expedient to concentrate on it.

As was already mentioned, the first argument seems to be explained by the specific feature of safety as an object's property.

The second argument concerns to a great extent the fact that no matter how "mighty" this or that language is, there are always situations, when we have not enough words for defining different objective factors. The way out, as usual, is to apply adjectives to the basic notion. The same we have in this case: different types of safety are denoted as purely technical (including firefighting, chemical, radiation, nuclear, etc.) safety, industrial (production) safety, technological safety, environmental safety, (electric) energy safety and so on. Reliability is related to technical and environmental safety, however in no way to energy safety (more "energy security" named) that is a component of the economic security and correspondingly the national safety (national security). The technical safety differs from the energy security in the depth of damaging factors. The area of study on the energy security applies to global (and possibly catastrophic) consequences for a country on vast territories and for a long time. Besides, the energy security is based on the reasonable reliability of energy systems.

As a single reliability property the paper discusses exactly the technical safety including environmental safety. In this case all the arguments concerning safety apply primarily to energy objects (equipment, installations, devices) and to a lesser extent to electric power systems.

As distinct from other single reliability properties the technical safety is considerably regulated and characterized in such documents (in Russia) as "The rules of power plant installation", "The rules of safe operation", "The safety regulations", Constructions norms and regulations, etc.

A latest document regulating safety is the Federal Law "On technical regulation" [3]. The Law is entirely devoted to all types of safety only, without reliability. Item 2 of clause 6 of the Law contains the statement: "Approval of technical regulations for other purposes is inadmissible". In our opinion, as a result the role of the Law considerably decreases, and the sphere of applicability is unreasonably reduced. It is obvious that not only safety, but reliability as a complex property of objects, and quality of all types of products, etc. should be certified at all levels. In this aspect the Law "On technical regulation" needs adjustments.

Both safety and other single reliability properties, primarily failure-free operation, maintainability, depend on the accepted sizes of switchgears, height of wire suspension, width of service corridors, height of fencing, applied protection devices, etc. Surely, the accepted sizes of installations can be substantially decreased, making thus the corresponding objects cheaper. However, in this case the safety margins and correspondingly the object safety and reliability will decrease, since the number of overlaps between phases and to the ground increases because of different reasons.

Reliable operation of power units is ensured only, provided the strength of insulation used exceeds possible maximum voltage levels of insulation during continuous operating conditions and at short-term overvoltages. The insulation strength increase, however, leads to the pronounced increase of power unit cost and proves to be economically inexpedient. An excessive decrease of electric strength of insulation can cause heavy emergencies (equipment failures).

Grounding devices to protect from overvoltages, provide normal operating conditions and also create safe conditions for operation of power units are also an important element of power units along with insulation strength.

Recently the number of different aspects of safety to be considered has increased. One of the problem aspects concerns a direct impact of power system objects on the safety of people and environment. This aspect in turn is divided into safety at normal operation of power system objects and the safety of objects during their emergencies (faults). This statement can be illustrated by an example of the known views that a coal-fired power plant at its normal operation is more dangerous for people and environment due to large-scale emissions of harmful substances than a nuclear power plant. However, in the case of a failure of nuclear reactors at NPP its danger becomes much higher than the danger of failure at TPP.

The second aspect applies to the safety hazard because of interruption in power supply to consumers. Violation in power supply of consumers can lead to runaway of their dangerous production processes which can become uncontrollable and entail negative consequences for people and environment not only within the corresponding production, but in neighboring and remote areas.

Finally, another aspect of safety problem to be considered and which is related to reliability is the problem of protecting both a human being from technical (industrial) facilities and facilities from dangerous actions of the human being that are performed either unconsciously or unskillfully, by mistake, because of poor discipline, as a result of subversive act or terrorist act, etc. A typical example in this respect is the Chernobyl disaster. Nobody disputes that the personnel of the plant was to blame (they treated carelessly the nuclear reactor).

The above said confirms once again the role of safety within the complex property of reliability. Such events have started to occur increasingly often lately and lead not only to the

dangerous consequences but to long-term failures of operation and operability of the facilities. Obviously there are common reasons for decrease in safety and reliability.

Speaking of the single reliability properties an emphasis should be placed on the fact that they have different degrees of importance. Therefore while studying and ensuring them it is important to follow a certain sequence.

The sequence in consideration of the single properties is not so important for the facilities to be studied in terms of reliability in the course of operation. Each property can be estimated when necessary separately from one another depending on the situation. However, for designing the sequence of ensuring the single properties is very important. For example it is senseless to ensure high survivability prior to ensuring a necessary level of safety, longevity, failure-free operation, maintainability and stabilability (that has to be ensured previously by the appropriate means).

It is known that:

- single properties are interdependent and interrelated;
- the same means are applicable to ensure different single properties;
- a relative cost of ensuring the required level of different single properties varies;
- in a correctly ranged sequence it is cheaper to ensure each subsequent single property if all the previous properties are ensured.

Taking into account the above facts the rational sequence of single properties of reliability should be as follows:

1. Safety
2. Longevity
3. Failure-free operation
4. Maintainability
5. Stabilability
6. Survivability
7. Controllability
8. Resource supply
9. Storability

In this list "safety" takes the first place though it should be noted that in [1] "safety", without any ground, closed the list of single properties. The priority of safety as a single property of reliability can be substantiated by the following reasons.

Ensuring safety of an object is a very costly measure. In fact it is a considerable share of the whole object cost, which is particularly well seen in the electric power plant whose cost is determined, as was already mentioned, by the distances between the plant's elements, which are approved in terms of safety. This also affects the other properties of the object. It has been noted in the technical literature lately that the cost of ensuring safety grows because of the need to take into account all aspects of safety that were considered above.

The next factor characterizing safety as a single property of reliability is a high degree of its standardization level. In fact ensuring the appropriate level of safety implies meeting the required safety standards. The requirements for meeting the safety standards are "strict", up to the point: if safety standards of a constructed object are not met the construction of the object is considered to be inadmissible and it is naturally not necessary to consider the other reliability properties of the object.

It should be noted that the safety level to be ensured depends on the level of "wealth" of the country where the objects are constructed. More developed countries can afford a higher level of safety for their technical and industrial objects. These countries have higher living standards, longer life interval and higher cost of life. At the same time the reliability standards are also high. As an example we can mention such a normative index of reliability as a probability of shortage-free power supply. Whereas in the USSR it was taken equal to 0.996 in the Western countries this index was 0.9996.

The second property in the list is longevity. Longevity is also so important because of considerable costs and a large impact it has on the subsequent single properties. Considerable costs are related to the fact that to increase longevity (technological life span, service life) it is necessary to apply expensive advanced materials and technologies that possess the properties of thermal stability, chemical stability, higher durability etc.

The analysis of other properties from the above list can be continued, however we do not do it because the paper is devoted to the property of safety. It is important to pay attention to another point: the properties indicated in the list relative to the resulting effect of the object operation (element, system) can be subdivided into internal and external in terms of their manifestation. The external properties include only one property of failure-free operation which characterizes the resulting effect of the object operation, i.e. power supply to consumers. All the remaining properties characterize behavior of the object itself in terms of reliability and manifest themselves externally by influencing failure-free operation (continuity and duration of interruptions in power supply). Schematically these interrelations between the properties are presented in Fig.1. The figure shows role and place of safety in the complex property of reliability.

In conclusion it is sensible to emphasize the following.

Technical (production, industrial) safety is a very important component of the complex property of the object reliability. The absence of the need to consider it as a property of reliability in the past for energy systems is explained by the great attention paid exclusively to the safety problem itself which, in turn, predetermined a high level of safety within reliability. Currently the situation is changing essentially due to complication of the safety problem and a great impact on the remaining single properties of reliability.

As to the need and absence of the need to study one property or another in terms of reliability we can show two characteristic examples. Nowadays there is a single property which is very much in demand. This is supply of different resources – financial, labor, material and particularly primary energy resources (different fuels). In the USSR with centralized vertically-integrated management of the economy this property (supply of resources) was less urgent since there was the guideline of first-priority fuel supply to power plants. Therefore there were practically no power supply limitations due to interrupted fuel supply to power plants in those times, which can hardly be said about the current situation.

The second example is related to the energy security as it is currently considered in Russia. During soviet times we did not pay so much attention to the energy security problem for the reason of no need for it because the country was considered to have surplus energy resources and the Unified Energy System was the most efficient in the world. Replacement of vertical integration by horizontal, privatization of energy sectors and quasi market relations in the economy made the problem of energy security in Russia extremely urgent and not only at the national level but at the regional level as well despite the fact that the country still has surplus energy resources. It may sound ironic but it is a fact that the national energy security was first completely ruined and now the heroic efforts are made to restore it under new conditions.

For the same reasons and for the reasons considered in this paper it becomes urgent to consider safety within the complex property of reliability. Generalizing we can say that the demand for safety is determined by the urgency and extent to which the property is ensured.

The question of what method to choose for studying safety remains open. In our opinion there can be two options. Safety can be studied within reliability as one of its single properties and it can be studied separately since the danger for people and the environment, as was mentioned above, may arise under normal operation of object, i.e. in the absence of failures. This can be related to the low technical perfection of an object. Technical perfection is the property of higher rank. It includes the properties of safety, reliability and some others. On the other hand it is obvious that people in danger do not care about the reasons for low safety: unreliability or technical

imperfection. In any case safety should be ensured. Besides it is also obvious that safety in terms of perfection and safety in terms of reliability are interrelated.

A suggestion, though disputable, has been put forward lately to consider technical safety within the framework of energy security. However, in any case, it is indisputable that solutions to the problems of safety and practices of ensuring safety affect reliability of an object and safety should be taken into account in the study of reliability.

Conclusion

In our opinion it is more justified and correct to consider technical safety in terms of all aspects of its manifestation as one of the most important single properties of reliability of a technical object.

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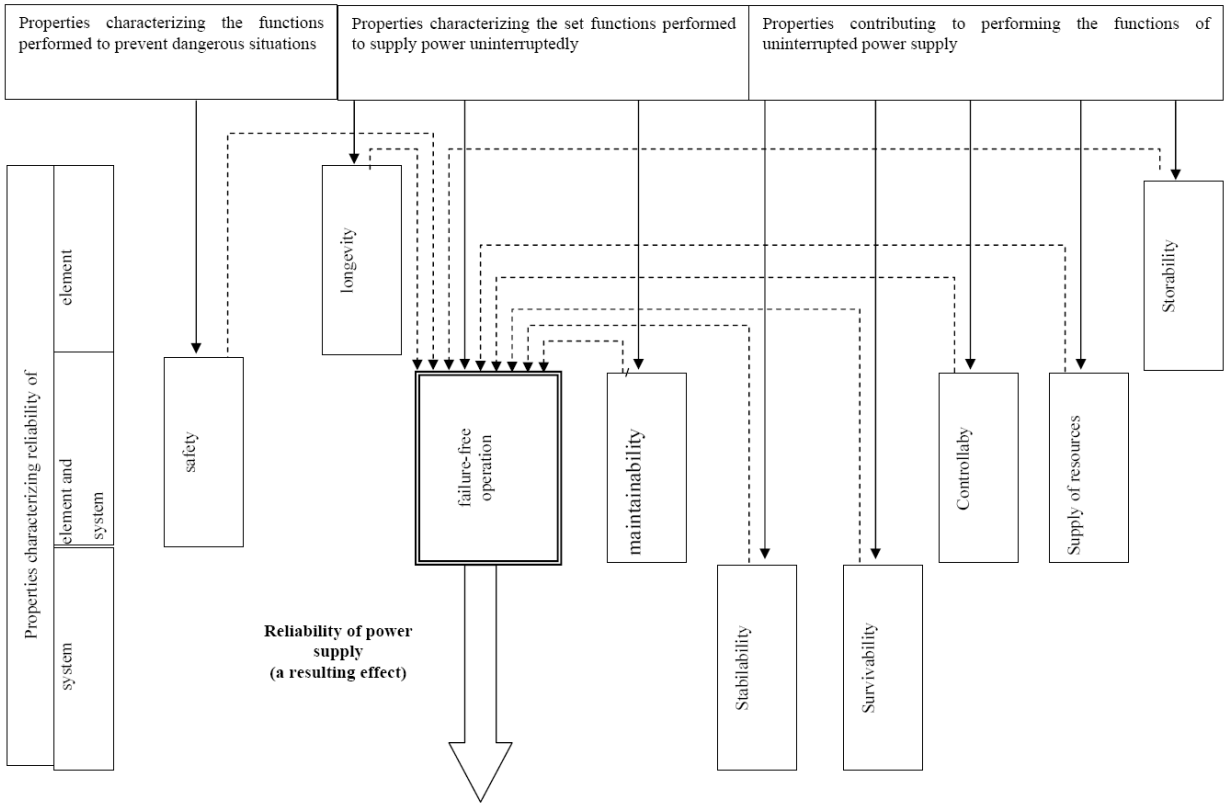

Fig. 1. Interrelations of single reliability propertie

N-TUPLE ALGEBRA-BASED PROBABILISTIC LOGIC

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Abstract

The concept of "probabilistic logic" known in artificial intelligence needs a more thorough substantiation. A new approach to constructing probabilistic logic based on the N-tuple algebra developed by the author is proposed. A brief introduction is given to the N-tuple algebra and its properties that provide efficient paralleling of algorithms for solving problems of logical analysis of systems in computer implementation are generalized. Methods for solving direct and inverse problems of probabilistic simulation of logical systems are considered.

INTRODUCTION

The *N*-tuple algebra based on the known properties of Cartesian products [1] was developed for solving certain problems in artificial intelligence, in particular, for simulating logical systems and in order to reduce the complexity of algorithms of logical inference [2, 3]. The foundations of the N-tuple algebra and potentialities of its application in probabilistic simulation were presented in [4–6]. Further investigations of this system have shown that the class of problems solved on its ground can be extended substantially. In addition, the structures of the N-tuple algebra can be programmed relatively easily and have a natural parallelism; therefore, their application in software–hardware support of logical and logical–probabilistic analysis of systems allows one to reduce the cost of development of programs and the required computational resources.

In this paper, we give a brief introduction to N-tuple algebra taking into account the correction of certain terminology introduced earlier and its capabilities in solving the inverse problem of logical–probabilistic analysis are also considered, i.e., the restoration of probability distributions of simple events based on data on the probabilities of complex events. Such problems were posed within the framework of probabilistic logic [7–9].

1. BASIC CONCEPTS AND STRUCTURES OF THE N-TUPLE ALGEBRA

The *N*-tuple algebra contains a number of definitions and more than 30 theorems, which are used in order to obtain the following results:

(1) It is substantiated that it is isomorphic to the system such as the theories of multiplace relations, of propositional calculus, and of predicate calculus.

(2) This system is embedded in the probability space.

(3) The algorithmic foundation for solving various problem of logical analysis of systems (logical inference, search for correct hypotheses and "hidden axioms", probabilistic analysis, etc.) is developed.

In order to avoid the consideration of original works [2–6], we present here in short the basic concepts and relationships of N-tuple algebra necessary for understanding probabilistic relations. Moreover, in the author's opinion, this section is useful because of the appropriate correction in the previous notation.

The *N*-tuple algebra is based on the concept of a flexible universe of discourse. Let a totality of different sets called *sorts* be given. We assign a certain set of attributes to every sort (in the previous publications, the term "coordinate", which does not seem to be completely adequate, was used). The *domain* of each attribute is the set that is equal to the corresponding sort. In mathematical logic, the domains of definition of variables correspond to attribute domains. A *flexible universe* consists of a totality of *partial universes*, Cartesian products of domains for a given sequence of attributes. The sequence of attributes that determines a given partial universe is called a *relation diagram*.

The N-tuple algebra contains five structures (N-tuple algebra objects) such as an elementary *n*-tuple, *C*-*n*-tuple, *C*-system, *D*-*n*-tuple, and *D*-system. Objects of the N-tuple algebra formed in the same partial universe are called similar.

Suppose that a partial universe in the form of a Cartesian product of arbitrary sets is given $S = X_1 \times X_2 \times \ldots \times X_n$. Clearly, *S* can be represented as a space of features with *attributes X*. The domains of these attributes correspond to feature *scales*. Then, we can form in the space *S* the following substructures:

(1) Projections, which are subspaces in which only certain attributes from the set of attributes generating *S* are used.

(2) The Cartesian products in the given relation diagram; certain subsets of the sets *X* represented in the given relation diagram are components of these Cartesian products.

Consider examples of these substructures. Let $S = X \times Y \times Z$, where $X = \{a, b, c, d\}$, $Y = \{f, g, h\}$, and $Z = \{a, b, c\}$. The Cartesian products $X \times Y$, $X \times Z$, etc. or particular sets, e.g. *X*, may be projections of this space. For simplicity, we assume that in this system a unique attribute corresponds to each sort.

Within the limits of the space *S* or some its projection, we can give the corresponding substructures in the form of Cartesian products. For example, the Cartesian product

$R[XYZ] = \{b, d\} \times \{f, h\} \times \{a, b\}$

is an example of such a substructure. Here, the expression [*XYZ*] is a relation diagram. It can be easily tested that $R \subseteq S$ (a property of Cartesian products). Similarly, a certain subset of elementary *n*-tuples of the projection $Y \times Z$ can be represented as the Cartesian product $Q[YZ] = \{f, g\} \times \{a, c\}$. Cartesian products represent the sets of elementary *n*-tuples. If necessary, these sets can be listed, although it is not necessary in performing operations with the structures of N-tuple algebra.

An *elementary n-tuple* is an element of a Cartesian product or its projection; i.e., a sequence of elements, each of which belongs to the domain of the corresponding attribute. For example, the Cartesian product $Q[YZ] = \{f, g\} \times \{a, c\}$ contains the following set of elementary *n*-tuples: $\{(f, a), (f, c), (g, a), (g, c)\}$.

A *C*-*n*-*tuple* is an *n*-tuple given in the complete space or in some its projection with components generated by subsets of the corresponding domains of attributes. A *C*-*n*-tuple is interpreted as the Cartesian product of these components; i.e., as a certain subset of elementary *n*-tuples. Square brackets are employed for denoting *C*-*n*-tuples. For example, the relations *R* and *Q* presented above can be represented as *C*-*n*-tuples

$R[XYZ] = [\{b, d\}\ \{f, h\}\ \{a, b\}]$; $Q[YZ] = [\{f, g\}\ \{a, c\}]$.

A C-n-tuple that has at least one empty component is empty. In the N-tuple algebra, if we deal with models of propositional or predicate calculus, this proposition is taken as an axiom, which has an interpretation based on the properties of Cartesian products.

To generalize operations that are applied frequently to structures with various relation diagrams, we introduce *dummy components*. These components have two types. One of these components is used in *C*-*n*-tuples and is designated "∗". Another dummy component (∅) involved in *D*-*n*-tuples is considered in what follows.

Dummy components "∗" designate the sets equal to the domains of the corresponding attributes; they can be inserted in the corresponding *C*-*n*-tuple instead of missing attributes and thus introduce new attributes in it. For example, the *C*-*n*-tuple $Q[YZ] = [\{f, g\}\ \{a, c\}]$ can be represented in the relation diagram [*XYZ*] in the form of the *C*-*n*-tuple $[*\ \{f, g\}\ \{a, c\}]$ using a dummy component. Since the dummy component in *Q* corresponds to the attribute *X*, we have the equality

$[*\ \{f, g\}\ \{a, c\}] = [\{a, b, c, d\}\ \{f, g\}\ \{a, c\}]$.

The *intersection* of similar *C*-*n*-tuples is performed componentwise. The result of intersection is the *C*-*n*-tuple that contains the intersection of the components of the source *C*-*n*-tuples related to the same attribute, e.g.,

$[\{b, d\}\ \{f, h\}\ \{a, b\}] \cap [*\ \{f, g\}\ \{a, c\}] = [\{b, d\}\ \{f\}\ \{a\}]$.

The result of intersection of *C*-*n*-tuples may be an empty set (empty *C*-*n*-tuple)

$[\{b, d\}\ \{f, h\}\ \{a, b\}] \cap [*\ \{g\}\ \{a, c\}] = \varnothing$,

since the intersection of the second components of these *C*-*n*-tuples is an empty set.

Many relations given as subsets of a Cartesian product cannot always be represented by a single *C*-*n*-tuple. Therefore, it is reasonable to introduce a universal structure that is the union of similar *C*-*n*-tuples.

A *C*-*system* is a structure that is the union of an arbitrary number of similar *C*-*n*-tuples. As *C*-*n*-tuples, *C*-systems are confined by square brackets. For example, for the space *S* given above, we can define a certain relation *P* as a *C*-system

$$R[XYZ] = \begin{bmatrix} \{a,d\} & * & \{b,c\} \\ \{b,d\} & \{f,h\} & \{a,c\} \\ \{b,c\} & \{g\} & \{b\} \end{bmatrix}.$$

The fact that a *C*-*n*-tuple (C_m) is included in another *C*-*n*-tuple (C_n) is tested componentwise, $C_m \subseteq C_n$ if and only if all components C_m are included in the corresponding components of C_n. Based on the properties of Cartesian products, we are able to find conditions for which the union of two *C*-*n*-tuples C_m and C_n can be transformed into a single *C*-*n*-tuple. There are two such conditions

(1) if $C_m \subseteq C_n$, then $C_m \cup C_n = C_n$;

(2) if C_m and C_n differ only in the *i*th component, then $C_m \cup C_n$ can be represented as a single *C*-*n*-tuple that have all component the same except for the *i*th component, which becomes equal to the union of the corresponding components from C_m and C_n.

If we know how to obtain the intersection of *C*-*n*-tuples, we can formulate the algorithm for finding the intersection of a *C*-*n*-tuple with a *C*-system and a *C*-system with a *C*-system. For this purpose, we should represent *C*-*n*-tuples as conventional sets whose elements are similar elementary *n*-tuples. Then, the *C*-system that contains the *C*-*n*-tuples *A*, *B*, …, *L* is the union of these sets. On this ground, using the law of algebra of sets, in particular, the distributive law, we can easily obtain the corresponding algorithms for calculating the intersection of the corresponding structures.

Algorithm 1. The calculation of the intersection of a *C*-*n*-tuple *P* with a *C*-system *Q*:

(1) calculate the intersection of the *C*-*n*-tuple *P* with each *C*-*n*-tuple from *Q*;

(2) eliminate empty *C*-*n*-tuples from the obtained results;

(3) form a *C*-system from the remaining *n*-tuples;

(4) terminate the algorithm.

Algorithm 2. Calculation of the intersection of a *C*-system *P* with a *C*-system *Q*:

(1) calculate the intersection of the *C*-*n*-tuple from *P* with each *C*-*n*-tuple from *Q*;

(2) eliminate empty *C*-*n*-tuples from the obtained results;

(3) form a *C*-system from the remaining *n*-tuples;

(4) terminate the algorithm.

As an example, we calculate the intersection of two *C*-systems given on the space *S* defined above (this means that the symbol "∗" in the second position of *C*-*n*-tuples corresponds to the set $X_2 = \{f, g, h\}$)

$$P[XYZ] = \begin{bmatrix} \{a,b,d\} & \{f,h\} & \{b\} \\ \{b,c\} & * & \{a,c\} \end{bmatrix},\ Q[XYZ] = \begin{bmatrix} \{a,d\} & * & \{b,c\} \\ \{b,d\} & \{f,h\} & \{a,c\} \\ \{b,c\} & \{g\} & \{b\} \end{bmatrix}.$$

(1) We calculate the intersection of all pairs of *C*-*n*-tuples belonging to different *C*-systems

$[\{a, b, d\}\ \{f, h\}\ \{b\}] \cap [\{a, d\} * \{b, c\}] = [\{a, d\}\ \{f, h\}\ \{b\}]$;
$[\{a, b, d\}\ \{f, h\}\ \{b\}] \cap [\{b, d\}\ \{f, h\}\ \{a, c\}] = \varnothing$;
$[\{a, b, d\}\ \{f, h\}\ \{b\}] \cap [\{b, c\}\ \{g\}\ \{b\}] = \varnothing$;
$[\{b, c\} * \{a, c\}] \cap [\{a, d\} * \{b, c\}] = \varnothing$;
$[\{b, c\} * \{a, c\}] \cap [\{b, d\}\ \{f, h\}\ \{a, c\}] = [\{b\}\ \{f, h\}\ \{a, c\}]$;
$[\{b, c\} * \{a, c\}] \cap [\{b, c\}\ \{g\}\ \{b\}] = \varnothing$.

(2) From the remaining nonempty *C*-*n*-tuples, we form the *C*-system

$$P \cap Q = \begin{bmatrix} \{a,d\} & \{f,h\} & \{b\} \\ \{b\} & \{f,h\} & \{a,c\} \end{bmatrix}.$$

Even this relatively simple example shows certain opportunities to reduce the complexity of the algorithms using the *N*-tuple algebra. The same result can be obtained if we convert preliminarily the source *C*-systems to the set of elementary *n*-tuples. However, this increases the complexity of computations since the *C*-system *P*, *C*-system *Q*, and the *C*-system $P \cap Q$ contain 24, 20, and 8 elementary *n*-tuples, respectively.

The union of *C*-*n*-tuples and *C*-systems is computed much simpler. For this purpose, we need to form a new *C*-system from the united structures that contains all *C*-*n*-tuples of these structures. Then, we may unite certain *C*-*n*-tuples in particular cases. It is necessary to remember that the implemented algorithms of union and intersection, as well as testing the inclusion of the structures of the N-tuple algebra, make sense only when these structures are similar or are transformed into similar structures with the help of addition of dummy attributes.

If it is required to compute the *complement* of a *C*-*n*-tuple, then, using conventional methods from the theory of multiplace functions, we should perform the following operations:

(1) to split into elementary n-tuples the *C*-*n*-tuple *R* and the partial universe *S* corresponding to it;

(2) eliminate elementwise all elementary *n*-tuples belonging to *R*.

It is clear that this operation is laborious in general. However, it is simplified essentially if we employ the following relations. Let us first define the notion of complement to the component of a *C*-*n*-tuple. If a multiplace relation is defined in the space such that each its attribute is represented by a certain set, then it is obvious that the universe for the component of the *C*-*n*-tuple is the domain of the attribute that corresponds to it (partial universe), and the set that contains all elements of this partial universe that do not belong to this component is the complement of the component. For example, assume that, in the space $S = X \times Y \times Z$, a *C*-*n*-tuple $R = [R_1\ R_2\ R_3]$ is given. Then, correspondingly, we have $\overline{R_1} = X \backslash R_1$; $\overline{R_2} = Y \backslash R_2$; and $\overline{R_3} = Z / R_3$.

The following theorem can be proved based on the properties of the Cartesian product [1].

Theorem 1. The complement of the *C*-*n*-tuple $T = [R_1\ R_2\ \ldots\ R_n]$ is the C-system

$C = \begin{bmatrix} \overline{R_1} & * & \dots & * \\ * & \overline{R_2} & \dots & * \\ \dots & \dots & \dots & \dots \\ * & * & \dots & \overline{R_n} \end{bmatrix}$ of dimension $n \times n$, in which each diagonal component is the complement of the corresponding component of C-n-tuple T, and the other components are dummy.

Consider an example. Assume that in the space $S = X \times Y \times Z$ mentioned above a C-n-tuple $T = [\{b, d\}\ \{f, h\}\ \{a, b\}]$ is given. By Theorem 1, its complement is the C-system

$$\overline{T} = \begin{bmatrix} X \setminus \{b,d\} & * & * \\ * & Y \setminus \{f,h\} & * \\ * & * & Z \setminus \{a,b\} \end{bmatrix} = \begin{bmatrix} \{a,c\} & * & * \\ * & \{g\} & * \\ * & * & \{c\} \end{bmatrix}.$$

By Theorem 1, the C-systems that represent the complement of a C-n-tuple can be represented as a single n-tuple of sets using for designation inverted square brackets. Then, we obtain the equality

$$\overline{T} = \begin{bmatrix} \{a,c\} & * & * \\ * & \{g\} & * \\ * & * & \{c\} \end{bmatrix} =]\{a, c\}\ \{g\}\ \{c\}[.$$

This brief representation of the diagonal C-system generates a new structure of the N-tuple algebra, which is called a *D-n-tuple*. It turns out that this structure not only allows one to represent briefly diagonal C-systems, but is also used independently in certain operations and retrieval requests. The terms "C-n-tuple" and "D-n-tuple" are not chosen randomly. In the simplest case, C-n-tuple and D-n-tuple correspond to conjunction and disjunction of one-place predicates with different variables. Using D-n-tuples, we can formulate one more (the fifth) structure of the N-tuple algebra, a D-system.

A D-system is a structure similar to a matrix whose rows contain similar D-n-tuples, which is interpreted as the intersection of the sets of elementary n-tuples belonging to these D-n-tuples.

The representation of a D-system is similar to the representation of a C-system, but, instead of square brackets, we use inverted ones. For example, the complement to the C-system

$$F[XYZ] = \begin{bmatrix} \{a,b,d\} & \{f,h\} & \{b\} \\ \{b,c\} & * & \{a,c\} \end{bmatrix},$$

given in the space S can be represented as the D-system

$$\overline{F} = \begin{bmatrix} X \setminus \{a,b,d\} & Y \setminus \{f,h\} & Z \setminus \{b\} \\ X \setminus \{b,c\} & Y \setminus * & Z \setminus \{a,c\} \end{bmatrix} = \begin{bmatrix} \{c\} & \{g\} & \{a,c\} \\ \{a,d\} & \varnothing & \{b\} \end{bmatrix}.$$

Thus, the D-system is *dual* (according to the de Morgan laws) to the C-system and is the *intersection* of two C-n-tuples. The algorithms for converting D-n-tuples and D-systems into C-systems equal to them have been developed.

The complete analogy between the structures of the N-tuple algebra and formulas of predicate calculus can easily be established. Consider the main ideas. In the predicate calculus, *conjunction* of one-place predicate with different variables corresponds to the C-n-tuples in the trivial case (when particular attributes are not related to multiplace relations). For example, the C-n-tuple $P[XYZ] = [P_1\ P_2\ P_3]$, where $P_1 \subseteq X$; $P_2 \subseteq Y$; and $P_3 \subseteq Z$, corresponds to the logical formula $H = P_1(x) \wedge P_2(y)\ \wedge P_3(z)$. The negation of the formula H (*disjunction* of one-place predicates) $\neg H = \neg P_1(x) \vee \neg P_2(y)\ \vee \neg P_3(z)$ corresponds to the D-n-tuple $\overline{P} =]\ \overline{P_1}\ \ \overline{P_2}\ \ \overline{P_3}\ [$. An empty object of the N-tuple algebra corresponds to an *identically false formula*.

An object of the N-tuple algebra corresponds to a *satisfiable formula*. An elementary *n*-tuple that is involved in the composition of a nonempty object of the N-tuple algebra corresponds to a *satisfying substitution* of the logical formula.

If a dummy attribute is introduced in a *C*-*n*-tuple or in a *C*-system, then this procedure corresponds to the inference rule known in the predicate calculus which is called the *generalization rule*. For example, if the object of the N-tuple algebra $G[XZ] = \begin{bmatrix} \{a,c\} & * \\ \{a,c,d\} & \{b,c\} \end{bmatrix}$ corresponds to a formula $F(x, z)$ in the predicate calculus, then, appending to this object a dummy attribute Y, we obtain the object of the N-tuple algebra $G_1[XYZ] = \begin{bmatrix} \{a,c\} & * & * \\ \{a,c,d\} & * & \{b,c\} \end{bmatrix}$, which corresponds to the formula $\forall yF(x, z)$ obtained from $F(x, z)$ by the generalization formula.

Together with operations of the algebra of sets on objects of the N-tuple algebra, the following three additional operations on attributes are introduced:

(1) the transposition of attributes and the their corresponding columns of the matrix of the object of the N-tuple algebra;

(2) addition of a new dummy attribute; and

(3) attribute elimination.

We also use two quantifier operations $\exists x(P)$ and $\forall x(P)$, which not only recognize identical falsity or satisfiability of the corresponding structure, but also, in the case of satisfiability, allows one to obtain the object of the N-tuple algebra corresponding to this expression. Certain combinations of operations with attributes and operations of the algebra of sets give the opportunity to perform negation, composition, and join of relations, and logical inference and operations with quantifiers. In detail, see this in [4–6].

S. RESOURSE SAVING IN COMPUTER IMPLEMENTATION OF STRUCTURES OF THE N-TUPLE ALGEBRA

Computational complexity of operations of the algebra of sets and testing inclusion depends on the class of structures the employed objects of the N-tuple algebra belong to. For example, an inclusion of a *C*-*n*-tuple into a *C*-system is tested in general with the help of an algorithm of exponential computational complexity, while the algorithm for testing of an inclusion of a *C*-*n*-tuple and even *C*-system into a *D*-system has a polynomial complexity. To fulfill certain operations and tests, it is required to transform an object of the N-tuple algebra into an object of the *alternative class* that is equivalent to it (for example, a *C*-system into *D*-system and vice versa), which is achieved for the *C*-system or the *D*-system by algorithms of exponential complexity. The operation of complement of an object of the N-tuple algebra in all cases is fulfilled by an algorithm of polynomial computational complexity, but, in this case, the system is transformed into the alternative class. The operations of intersection and union of objects of the N-tuple algebra that belong to the same class are fulfilled by algorithms of polynomial complexity, but if they belong to different classes, then, to fulfill these operations, it is necessary to transform one of them into another class.

In problems that are known in logic as problems of deductive inference, frequently, it is required to test that one object of the N-tuple algebra is included into another, as well as to fulfill quantifier operations. Table 1 presents various combinations of objects of the N-tuple algebra and the sign "+" label the combinations for which the execution algorithms of the corresponding operations are polynomial under the condition that all domains of attributes are simple sets (i.e., not multiplace relations). Note that, in all cases, to test whether a given elementary *n*-tuple belongs to any structure, we need an algorithm of polynomial complexity.

Table 1

Action	*C*-n-tuple	*C-system*	*D*-n-tuple	*D*-system
Testing of inclusion of a C-n-tuple into	+		+	+
Testing of inclusion of a C-system into	+		+	+
Testing of inclusion of a D-n-tuple into	+		+	+
Testing of inclusion of a D-system into				
Quantifier operation $\forall x$	+		+	+
Quantifier operation $\exists x$	+	+	+	

However, within the framework of the N-tuple algebra, methods for reducing complexity of computationally exponential algorithms, as well as methods for recognizing particular cases of structures were developed that allow one to fulfill the corresponding operations, transformations, and testing for polynomial time. Even in the cases when it is not possible to use an algorithm of polynomial computational complexity, the required computational resources can be reduced by using natural parallelism inherent in objects of the N-tuple algebra.

In contrast to conventional data structures applied in the computer implementation of logical and logical-probabilistic analysis, the structures of the N-tuple algebra are *matrixwise*, which, using the corresponding software–hardware implementation, makes it possible to reduce relatively easily the computational resources by paralleling the operations.

The computer implementation of objects of the N-tuple algebra employs parallelism at the levels of (1) of components; (2) rows; and (3) matrices. *At the level of components*, we can represent domains and their subsets in the form of a totality of logical vectors. To implement operations of the algebra of sets and of tests of inclusion, we can apply logical operations with integer vectors. *At the level of rows*, we are able to fulfill simultaneously operations or tests of inclusion with all pairs of components of *C*-*n*-tuples and *D*-*n*-tuples. *At the level of matrices*, we can fulfill simultaneously operations of the algebra of sets and the test of inclusion for a set of pairs elements of which are rows (*C*-*n*-tuples and *D*-*n*-tuples from different objects of N-tuple algebra. For example, in the computation of the intersection of two *C*-systems (see algorithm 2), all operations of intersection of *C*-*n*-tuples employed in this algorithm can be executed in parallel.

T. LOGIC AND PROBABILITY

The term "probabilistic logic" has been widely applicable in AI since the publication of work [7] by Nilsson. His idea was extended and developed by other researchers [8, 9]. In these and other publications on probabilistic logic, the following problem was posed: given estimates of the probabilities of a certain set of event represented by formulas of propositional calculus, it is necessary to find a probabilistic estimate of the event represented by a logical formula different from the initial ones. Another aspect of the combination of probability and logic, e.g., the aspect that was implemented in logical-probabilistic methods (LPM) [10], where the probability of formulas is calculated based on the probabilistic values of logical variables have not been considered in those works. Moreover, the analysis of papers [7–9] has shown that the combination of classical concepts of "probability" and "logic" results in certain nonclassical logics. However, in this paper, we consider the concept of probabilistic logic within the framework of the N-tuple algebra.

The combination of concepts of "logic" and "probability" is rather difficult. At the first glance, it is completely simple if we take as a ground the system of axioms proposed by A.N. Kolmogorov [11], in which the algebra of events embedded into a probabilistic measure corresponds to the algebra of sets. For example, for events represented by sets A and B, the probabilistic measure of their union can be calculated as

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

Thus, together with the probabilities $p(A)$ and p(B), to compute the probabilities of the event $A \cup B$, it is necessary to know the probability $p(A \cap B)$, which, within the limits of certain constraints, in particular $p(A \cap B) \leq \min(p(A), p(B))$, does not depend on $p(A)$ and $p(B)$. If in addition, we have $A \cap B = \varnothing$, then the events A and B are dependent. However, if we assume that A and B are different logical variables rather than sets, then the probability of disjunction of these events can be calculated by the formula

$$p(A \vee B) = p(A) + p(B) - p(A)pB).$$

To calculate this formula, it is sufficient to set only the probability of the events A and B.

The question is why, for logical relations, another methodology takes place for computing probability, although it seems reasonable that the algebra of sets and the Boolean algebra are isomorphic. The answer to this question is a key point in the combination of the concepts of "logic" and "probability". It is the matter of fact that, in classical logic, elementary events corresponding to different logical variables are inconsistent; therefore, any logical formula of n free variables is isomorphic to a certain n-place relation, and the events that correspond to the discriminate variables belong to different attributes. In other words, logical variables may be dependent but not initially and only by the fact that they are involved into a certain logical formula, which determines the dependence between them.

Absurdity (from the point of view of mathematical logic) of another approach can be seen from the following example, which is presented sometimes in papers on probabilistic logic: for logical variables (but not for formulas!) X and Y, the probabilities $p(X)$, $p(Y)$, and $p(X \wedge Y)$ are given, and the last probability is not necessarily equal to the product of the preceding ones.

This clearly implies that *in the embedding of logical systems into the probabilistic space, it is necessary to take into account that we deal with the system that is isomorphic to the algebra of sets according to the Kolmogorov system of axioms, but, in structure, the sets themselves are sets of n-tuples involved in multiplace relations.*

It is this circumstance, which is taken into account in the N-tuple algebra explicitly or implicitly, is not taken into account in different versions of "probabilistic logic." The assumptions that events that correspond to different logical variables can be dependent in themselves, i.e., without taking into account the logical formula that relates them, means that the laws of mathematical logic are violated. It is not the same when we deal with the formulas, in which the dependence between different variables is established or with different logical formulas, which can be dependent only under the assumption that they contain at least one free variable that is common for them.

4. N-TUPLE ALGEBRA-BASED PROBABILISTIC ANALYSIS OF SYSTEMS

Consider methods of probabilistic simulation that use the N-tuple algebra in greater detail. The basic cross-linking concept of the N-tuple algebra is the concept of C-n-tuple. If we know the probabilistic measures of components of the C-n-tuple, then the measure of the C-n-tuple can be calculated as the product of the measures of its components. For example, when the C-n-tuple $R = [A\ B\ C]$ is given in measurable attributes and the measures of its components are equal to $\mu(A)$, $\mu(B)$, and $\mu(C)$, respectively, then we have

$\mu I = \mu(A) \bullet \mu(B) \bullet \mu(C)$.

If we deal with the embedding of logical formulas in the probabilistic space, then all attributes of the space in which the totality of objects of the N-tuple algebra is given have measure one, and all objects of the N-tuple algebra have measures that do not exceed one. This corresponds to the probabilistic measure not only in numerical relations, but also by the fact that the system of events simulated by the N-tuple algebra is isomorphic to the algebra of sets.

To compute the measures of objects from the N-tuple algebra that are different from *C*-*n*-tuples, it is necessary to *orthogonalize* them, i.e., to transform into an equivalent *C*-system in which the intersection of any pair of *C*-*n*-tuples is an empty set. The methods of orthogonalization of arbitrary objects of the N-tuple algebra have been developed; the results in detail can be found in [2–6]. Note that the measure of an orthogonal *C*-system is equal exactly to the sum of the measures of *C*-*n*-tuples that belong to it. In addition, the following regularity has been established: *the orthogonalization not only allows one to prepare an object of the N-tuple algebra for calculating its probability, but also, in many cases, reduces the computational cost substantially in solving other problems* (*e.g., in solving the satisfiability problem*).

If an object of the N-tuple algebra is a representation of formulas of propositional calculus, then it is given in the universe $\{0, 1\}^n$, where n is the number of logical variables of the formula. Each column of a *C*-*n*-tuple or a *C*-system is related to a certain logical variable. The variable x_k corresponds to the kth column, the state 1 in the object of the N-tuple algebra corresponds to the literal x_k, and the state 0 corresponds to the literal $\bar{x}_k$. Any row (*C*-*n*-tuple) in a *C*-system corresponds to the conjunction of the formula expressed as the disjunctive normal form (DNF). If some clause misses the variables that are involved in the composition of formula, then, instead of them, the corresponding dummy variable "$*$" is inserted in the *C*-*n*-tuple.

Example 1. Assume that the formula of propositional calculus

$$F_Q = (x_1 \wedge \neg x_3) \vee (\neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2). \qquad (4.1)$$

is given. Since there are three logical variables here, this formula can be represented as an object of the N-tuple algebra Q in the universe $\{0, 1\}^3$

$$Q = \begin{bmatrix} \{1\} & * & \{0\} \\ * & \{0\} & \{1\} \\ \{0\} & \{1\} & * \end{bmatrix}.$$

This formula and the object of the N-tuple algebra that corresponds to it are orthogonal; therefore Q can be expressed directly in terms of the probabilistic measure. Assume that, in the object Q of the N-tuple algebra, the probabilities of events are given as follows: pi is the probability of the event 1 in the ith attribute, and $1–p_i$ is the probability of the event 0 in the ith attribute. Taking into account that the measure of a *C*-*n*-tuple is equal to the product of the measures of its components, and the measure of an orthogonal *C*-system is the sum of the measures of *C*-*n*-tuples belonging to it, we obtain the formula

$$p(Q) = p_1(1–p_3)+(1–p_2)p_3+(1–p_1)p_2. \qquad (4.2)$$

In the LPM, formula (4.2) is called the probabilistic function (PF) of formula (4.1). This function can also be derived from the orthogonal *C*-system using the change of elements of components by the probabilities corresponding to them and by the transformation of the system into a polynomial. At the first glance, it seems that the structures of the N-tuple algebra provide only a different way for expressing logical formulas. However, when the models get complicated (in particular, in the transition to many-state systems), using the N-tuple algebra, it turns out to be possible to simplify essentially the algorithms for solving a number of problems considered in LPM. In addition, in the embedding into the probabilistic space, the concept of "regression equation" is introduced in the N-tuple algebra, which allows one to pose and solve the problem of probabilistic logic in accordance with the Nilsson statement. *If in the probabilistic functions of type* (4.2)*, we suppose that p_i are variables rather than fixed numbers, then these formulas are an exact*

regression equation of the corresponding logical formula. The proof of this proposition can be found in [5]. Consider an example of many-state system.

Example 2. Let $R = \begin{bmatrix} \{a_1, a_2\} & \{b_1, b_3\} \\ \{a_3\} & \{b_1, b_2\} \end{bmatrix}$ be an orthogonal *C*-system with three states given in the space $\{a_1, a_2, a_3\} \times \{b_1, b_2, b_3\}$ with probabilities $p(a_i)$ and $p(b_i)$, and $p(a_3) = 1 - p(a_1) - p(a_2)$ and $p(b_3) = 1 - p(b_1) - p(b_2)$. Then, the probability of the event expressed by the object *R* of the *N*-tuple algebra if the required probabilities are substituted is

$$p\mathrm{I} = (p(a_1) + p(a_2))(1 - p(b_2)) + (1 - p(a_1) - p(a_2))(p(b_1) + p(b_2)).$$

The presented approach corresponds to the direct problem of logical–probabilistic analysis when, for given probabilities of elementary events, the probability of a complex event is calculated. In the *inverse problem*, the statement is different. The problem is, based on the data on the probabilities of certain complex events, we should calculate the probabilities of elementary events. After this, we can calculate the probabilities of other complex events. Problems solved in probabilistic logic are of this type. Consider the example presented in the paper by Nilsson [7].

Example 3. Given a totality of events specified by formulas *A* and $A \supset B$ of propositional calculus, and $p(A) = p_1$ and $p(A \supset B) = p_2$. It is necessary to estimate the probability $p(B)$ of the event *B*.

We solve this problem by the methods of the N-tuple algebra. There are only two logical variables *A* and *B* that are also elementary events in this case. Assume that the probabilities of these events are $p(A)$ and $p(B)$, respectively. The conditions of the problem imply that $p(A) = p_1$. Let us express the given formulas in the structures of the N-tuple algebra using the universe $\{0, 1\}^2$:

$$A = [\{1\}\ *];\ B = [*\ \{1\}];\ \ A \supset B = \overline{A} \vee B =]\{0\}\ \{1\}[= \begin{bmatrix} \{0\} & * \\ \{1\} & \{1\} \end{bmatrix}$$

(here the *D*-*n*-tuple corresponding to formula $A \supset B$ is transformed into an orthogonal *C*-system).

On this ground, we write the probabilistic formulas for the events *A* and $A \supset B$

$$P(A) = p_1; \qquad P(A \supset B) = (1 - p(A)) + p(A)p(B) = p_2.$$

We obtain the system of two equations

$$p(A) = p_1;$$
$$(1 - p(A)) + p(A)\,p(B) = p_2.$$

This easily implies that

$$p(B) = \frac{p_1 + p_2 - 1}{p_1}.$$

In this problem, we obtain an exact solution, while in [7] the solution is obtained as the inequality $p_2 + p_1 - 1 \le p(B) \le p_2$.

In general, the algorithm for solving problems of probabilistic logic is as follows. Assume that we have initial logical formulas F_i with given probabilities $p(F_i)$ and a formula G whose probability $p(G)$ has to be calculated. Then, it is necessary to fulfill the following sequence of operations:

(1) formulas F_i and *G* are transformed into orthogonal *C*-systems;

(2) for each of these systems, the regression equations $E(F_i)$ and $E(G)$ are derived;

(3) the system of equations $\{ E(F_i)\}$ is formed and solved;

(4) if the system of equations $\{ E(F_i)\}$ has a unique solution, then the obtained values of variables are substituted into the formula $E(G)$ and an exact solution is found.

In the Nilsson problem, an exact solution was obtained by the methods of the N-tuple algebra. However, this situation is not possible in all cases. Consider an example.

Example 4. Given probabilities of the events described by the logical formulas

$$p(A \vee B) = a; \quad p(A \wedge B) = b.$$

Find estimates of $p(A)$ and $p(B)$. Let us express the given events in the system as orthogonal C-systems

$$A \vee B \Leftrightarrow]\{1\} \quad \{1\}[= \begin{bmatrix} \{1\} & * \\ \{0\} & \{1\} \end{bmatrix};$$

$$A \wedge B \Leftrightarrow [\{1\} \quad \{1\}].$$

We derive the system of equations

$p(A) + (1 - p(A))\, p(B) = a$;

$p(A)\, p(B) = b$.

Solving this system, we obtain

$$p(A) = \frac{a + b \pm \sqrt{(a+b)^2 - 4b}}{2}; \quad p(B) = \frac{a + b \mp \sqrt{(a+b)^2 - 4b}}{2}.$$

It is clear that the obtained solutions do not give a unique solution in the cases when the radicand is not equal to 0 (this is possible for $p(A) \neq p(B)$). If we take into account that both initial formulas are symmetric, then this uncertainty was caused by the conditions of the problem.

For the presented examples, we can test the calculation numerically, if the probabilistic models corresponding to them are constructed. For example, for example 4, the probabilistic model is as follows: assume that two coins are tossed, note that the probability that at least once heads have occurred (the formula A .B corresponds to this event), and the probability that heads have occurred in two tosses (formula $A \vee B$) are known. If we know the probability of the event that heads have occurred (for correct coins, it is equal to 0.5), by the laws of probability theory, we can calculate the probability of these complex events $p(A \vee B) = 0.75$ and $p(A \wedge B) = 0.25$. The substitution of these values into the formulas for $p(A)$ and $p(B)$ presented above gives a correct answer. A similar test can be performed for incorrect coins, when the probability that heads occur differs from 0.5.

The probabilistic relations obtained based on the ntuple algebra allow one not only to estimate the probability of complex events for given distribution functions of events in each attribute, but also to solve the inverse problem of probabilistic analysis, i.e., to estimate the types and parameters of marginal distributions (distributions in attributes and certain projections).

First, we consider the following statement of the problem: a system is represented either in structures of the N-tuple algebra or in the form of a system of logical functions, and, for any variable, the probability distribution is known. In a multidimensional space, a distribution in either attributes or in certain projections of this space is called marginal distribution. It is necessary to calculate the probability distribution of the system and to estimate the stability of this distribution. The problems of this type arise in evaluating the reliability and safety of systems with complex structure and logical–probabilistic risk management in business and industry [12]. Using the relations derived above, logical systems in which these problems are solved can be conversed into measurable systems in the N-tuple algebra.

Assume that every attribute of a certain set of objects of the N-tuple algebra is represented by a finite system of events. The following two variants of specifying a system of events in the attribute X_i:

(1) in the form of an elementary system (i.e., a system of pairwise inconsistent events);

(2) on a continuous probability distribution $p(x_i)$ in the form of a finite set of intervals (a_i, b_i) nonoverlapping in general, where a_i and b_i are the values of the parameter x_i and $a_i < b_i$.

For the first variant, it is sufficient to assign to each event its probability. The second variant can be reduced to the first one by the following procedure:

(1) in the system $\{(a_i, b_i)$ of intervals of the parameter x_i, the system of events is split into the set of pairwise nonoverlapping intervals–quanta;

(2) the initial system of events is transformed into a discrete one assigning to each initial event a certain set of quanta such that their union is equal to this event;

(3) for each quantifier e_r, compute the value of the probability p_r using as the lower and upper integration limits for $p(x_i)$ the endpoints of this quantum equal to the values of the parameter x_i.

If this procedure has been performed for each attribute of the object of the N-tuple algebra, then its probability is computed in the following order:

(1) the object of the N-tuple algebra is orthogonalized;

(2) the object of the N-tuple algebra is transformed into a polynomial in which we assign to each quantum the corresponding value of the probability.

Example 5. The system is given in the space $X \times Y$, where the attributes are represented in the form of intervals, and X= [0, 7] and Y= [0, 5]. The densities of probability distributions $f_1(x, d_1, e_1)$ and $f_2(y, d_2, e_2)$ on attributes are also known, where d_1, e_1, d_2, and e_2 are the parameters of the distributions. In this system a certain event is given in the form of an object of the N-tuple algebra

$$R[XY] = \begin{bmatrix} \{a_1\} & \{b_1\} \\ \{a_2, a_4\} & \{b_2\} \\ \{a_3\} & \{b_3\} \end{bmatrix},$$

where a_i and b_j are intervals given in Table 2. It is necessary to determine the order of calculations for computing the probability of the event R.

It is sufficient to use only open intervals in order to solve this problem. To simplify the system, we construct increasing series of endpoints of intervals in the attributes, for X, 0; 1.7; 2.8; 3.4; 4.3; 5.5; 6.4; and 7; for Y, 0; 1.4; 2.3; 3.2; and 5. Then, we obtain the following sets of elementary intervals for the attributes X (Table 3) and Y (Table 4).

Table 2

a_1	a_2	a_3	a_4	b_1	b_2	b_3
[0, 2.8]	[1.7, 3.4]	[3.4, 5.5]	[4.3, 6.4]	[0, 2.3]	[1.4, 3.2]	[2.3, 5.0]

Table 3

r_1	r_2	r_3	r_4	r_5	r_6	r_7
(0, 1.7)	(1.7, 2.8)	(2.8, 3.4)	(3.4, 4.3)	(4.3, 5.5)	(5.5, 6.4)	(6.4, 7)

Table 4

q_1	q_2	q_3	q_4
(0, 1.4)	(1.4, 2.3)	(2.3, 3.2)	(3.2, 5)

When we replace the intervals with the corresponding sets of quanta, we obtain

$a_1=\{r_1, r_2\}$; $a_2=\{r_2, r_3\}$; $a_3=\{r_4, r_5\}$; $a_4=\{r_5, r_6\}$;

$b_1=\{q_1, q_2\}$; $b_2=\{q_2, q_3\}$; $b_3=\{q_3, q_4\}$.

After the substitution into the initial C-system, we obtain

$$R = \begin{bmatrix} \{r_1, r_2\} & \{q_1, q_2\} \\ \{r_2, r_3, r_5, r_6\} & \{q_2, q_3\} \\ \{r_4, r_5\} & \{q_3, q_4\} \end{bmatrix}.$$

For each quantum r_i or q_j, we calculate the corresponding probability. For example,

$$p(r_3) = \int_{2,8}^{3,4} f_1(x, d_1, e_1)dx.$$

Now, we can orthogonalize the corresponding complex events. For the event R, we compute $\overline{R}$, and after the transformation of $\overline{R}$ into an orthogonal C-system, we find $p\mathrm{I} = 1 - p(\overline{R})$. Then, we obtain (the intermediate calculations are eliminated)

$$\overline{R} = \begin{bmatrix} \{r_3, r_4, r_5, r_6, r_7\} & \{q_1\} \\ \{r_4, r_7\} & \{q_2\} \\ \{r_1, r_7\} & \{q_3\} \\ \{r_1, r_2, r_3, r_6, r_7\} & \{q_4\} \end{bmatrix}.$$

Next, substituting the probabilities of quanta and using the theorems of the N-tuple algebra, we arrive at the expression

$$p\mathrm{I} = 1 - p(\overline{R}) = 1 - ((p(r_3)+p(r_4)+p(r_5)+p(r_6)+p(r_7))p(q_1)+(p(r_4)+p(r_7))p(q_2)+$$
$$+(p(r_1)+p(r_7))p(q_3) +(p(r_1)+p(r_2)+p(r_3)+p(r_6)+p(r_7))p(q_4)).$$

When solving the inverse problem for systems with many states, it is not always possible to solve the system of equations exactly since the number of variables in the regression equations is comparable with the number of all quanta and may exceed the number of equations. For example, in example 5, the number of quanta in the attribute *X* is seven; therefore, the number of unknown parameters of just this attribute is smaller by one, i.e., six. However, the problem can be solved approximately, if it is represented as an approximation problem. Assume that the attribute X is split into k_i quanta ($k_i > 2$). Then, we take as unknowns the types and parameters of continuous distributions for each attribute, rather than the magnitudes of quanta. Usually, the number of parameters of distributions does not exceed two — they will be unknown quantities. To estimate them, we can use optimization methods, in which the control actions are the types and parameters of marginal distributions, and the goal function is a generalized parameter, e.g., the mean value of the absolute deviations of the calculated values of the probabilities of the investigated complex events from the actual ones.

CONCLUSIONS

The application of the N-tuple algebra allows one to solve the direct and inverse problems of probabilistic analysis of logical systems in a multidimensional space not restricting ourselves to a particular class of distributions. As marginal distributions in solving the direct and inverse problems, we can use not only a normal distribution, but also any other distribution.

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TECHNIQUE FOR FINDING SAMPLING DISTRIBUTIONS FOR TRUNCATED LAWS WITH SOME APPLICATIONS TO RELIABILITY ESTIMATION

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ABSTRACT

In this paper, the problem of finding sampling distributions for truncated laws is considered. This problem concerns the very important area of information processing in Industrial Engineering. It remains today perhaps the most difficult and important of all the problems of mathematical statistics that require considerable efforts and great skill for investigation. The technique discussed here is based on use of the unbiasedness equivalence principle, the idea of which belongs to the authors, and often provides a neat method for finding sampling distributions. It avoids explicit integration over the sample space and the attendant Jacobian but at the expense of verifying completeness of the recognized family of densities. Fortunately, general results on completeness obviate the need for this verification in many problems involving exponential families. The proposed technique allows one to obtain results for truncated laws via the results obtained for non-truncated laws. It is much simpler than the known techniques. The examples are given to illustrate that in many situations this technique allows one to find the results for truncated laws and to estimate system reliability in a simple way.

KEYWORDS

Truncated law, Sampling distribution, Unbiasedness equivalence principle, Reliability estimation

1. INTRODUCTION

The truncated distributions have found many applications. Several examples have been given employing the truncated distributions in fitting rainfall data and animal population studies where observations usually begin after migration has commenced or concluded before it has stopped [1-2]. Similar situations arise with regard to aiming errors (range, deflection, etc.) in gunnery and other bombing accuracy studies. For example, in gun camera missions, the view angle of the camera defines a known truncation point for an exponentially distributed random variable, observable as some function of the radial error or the distance from the aiming point to the point of impact [3]. A situation for the truncated Poisson distribution would occur when one wishes to fit a distribution to Poisson-like data consisting of numbers of individuals in certain groups which posses a given attribute, but in which a group cannot be sampled unless at least a specified number of its members have the attribute. For example, the group may be a household of people, and the attribute measles; the specified number would then be one. Other examples arise in life testing and

reliability problems, where if failure is caused by a wear-out mechanism or is a consequence of accumulated wear, then the length-of-life of a system can be expected to be of finite dimension. The object of the present paper is to obtain a sampling distribution for truncated law with a known truncation point and a minimum variance unbiased estimator of the reliability function for this model using the results obtained for non-truncated law. It is known that a sampling distribution for truncated law may be derived using, namely, the method based on characteristic functions [4], the method based on generating functions [5], or the combinatorial method [6]. In this paper, a much simpler technique than the above ones is proposed. It allows one to obtain the results for truncated laws more easily.

U. UNBIASEDNESS EQUIVALENCE PRINCIPLE

Suppose an experiment yields data sample $X^n = (X_1, \dots, X_n)$ relevant to the value of a parameter $\boldsymbol{\theta}$ (in general, vector). Let $L_X(x^n;\boldsymbol{\theta})$ denote the probability or probability density of X^n when the parameter assumes the value $\boldsymbol{\theta}$. Considered as a function of $\boldsymbol{\theta}$ for given $X^n=x^n$, $L_X(x^n;\boldsymbol{\theta})$ is the likelihood function. If the data sample X^n can be summarized by a sufficient statistic $\mathbf{S}$, one can write $L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta}) \propto L_X(x^n;\boldsymbol{\theta})$. Further, for any non-negative function $\omega(\mathbf{s})$, $\omega(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta})$ is also a likelihood function equivalent to $L_X(x^n;\boldsymbol{\theta})$. Suppose we recognize a function $\omega(\mathbf{s})$ such that $\omega(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta})$, regarded as a function of $\mathbf{s}$ for a given $\boldsymbol{\theta}$, is a density function. It can be shown that this is the sampling density of $\mathbf{S}$ if the family of recognized densities is complete.

The unbiasedness equivalence principle consists in the following. If

$$L_X(x^n;\boldsymbol{\theta},\vartheta)=[w(\boldsymbol{\theta},\vartheta)]^n L_X(x^n;\boldsymbol{\theta}), \tag{1}$$

represents the likelihood function for the truncated law, where $w(\boldsymbol{\theta},\vartheta)$ is some function of a parameter $(\boldsymbol{\theta},\vartheta)$ associated with truncation, ϑ is a known truncation point (in general, vector), then a sampling density for the truncated law is determined by

$$g_\vartheta(\mathbf{s};\boldsymbol{\theta}) = \hat{w}(\mathbf{s})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{s};\boldsymbol{\theta}), \quad \mathbf{s}\in \mathsf{S}_\vartheta, \tag{2}$$

where

$$\hat{w}(\mathbf{s})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{s};\boldsymbol{\theta}) = \varphi(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta},\vartheta) \propto L_X(x^n;\boldsymbol{\theta},\vartheta), \tag{3}$$

$g(\mathbf{s};\boldsymbol{\theta})$ is a sampling density of a sufficient statistic $\mathbf{s}(X^n)$ (for a family of densities $\{f(x;\boldsymbol{\theta})\}$) determined on the basis of $L_X(X^n;\boldsymbol{\theta})$, $\hat{w}(\mathbf{S})$ is *an unbiased estimator* of $1/[w(\boldsymbol{\theta},\vartheta)]^n$ with respect to $g(\mathbf{s};\boldsymbol{\theta})$, $\mathbf{s}\in\mathsf{S}$ (a sample space of a non-truncated sufficient statistic $\mathbf{S}$), $\varphi(\mathbf{S})$ is a function of $\mathbf{S}$ for a given $\boldsymbol{\theta}$, which is equivalent to unbiased estimator $\hat{w}(\mathbf{S})$ of $1/[w(\boldsymbol{\theta},\vartheta)]^n$, i.e.,

$$\varphi(\mathbf{S}) \propto \hat{w}(\mathbf{S}) \tag{4}$$

or

$$\varphi(\mathbf{S}) = \hat{w}(\mathbf{S})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{S};\boldsymbol{\theta}) / L_{\mathbf{S}}(\mathbf{S};\boldsymbol{\theta},\vartheta), \tag{5}$$

$g_\vartheta(\mathbf{s};\boldsymbol{\theta})$ is the sampling density of a sufficient statistic $\mathbf{S}$ (for a family of densities $\{f_\vartheta(x;\boldsymbol{\theta})\}$) when the truncation parameter ϑ is known, S_ϑ is a sample space of a truncated sufficient statistic $\mathbf{S}$.

V. EXAMPLES OF APPLICATIONS OF THE UNBIASEDNESS EQUIVALENCE PRINCIPLE TO FINDING SAMPLING DISTRIBUTIONS FOR TRUNCATED LAWS

Example 3.1 (*Sampling distribution for the left-truncated Poisson law*). Let the Poisson probability function be denoted by

$$f(x;\theta)=\frac{\theta^x}{x!}e^{-\theta}, \quad x=0,1,2,\dots . \tag{6}$$

The probability function of the restricted random variable, which is truncated away from some $\vartheta\geq 0$, is then

$$f_\vartheta(x;\theta)=w(\theta,\vartheta)f(x;\theta), \quad x=\vartheta+1,\vartheta+2,\dots , \tag{7}$$

where

$$w(\theta,\vartheta)=\left(\sum_{j=\vartheta+1}^{\infty}\frac{\theta^j}{j!}e^{-\theta}\right)^{-1}=\left(1-\sum_{j=0}^{\vartheta}\frac{\theta^j}{j!}e^{-\theta}\right)^{-1}. \tag{8}$$

Consider a sample of n independent observations $X_1, X_2, \dots, X_n$, each with probability function $f_\vartheta(x;\theta)$, where the likelihood function is defined as

$$L_X(x^n;\theta,\vartheta)=\prod_{i=1}^{n}f_\vartheta(x_i;\theta)=\left[w(\theta,\vartheta)\right]^n L_X(x^n;\theta)=\left[w(\theta,\vartheta)\right]^n\prod_{i=1}^{n}f(x_i;\theta)$$

$$=\left[w(\theta,\vartheta)\right]^n e^{-n\theta}\frac{\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}, \tag{9}$$

and let

$$S=\sum_{i=1}^{n}X_i, \quad s=n(\vartheta+1),n(\vartheta+1)+1,\dots . \tag{10}$$

It is well known that

$$S=\sum_{i=1}^{n}X_i, \quad s=0,1,\dots . \tag{11}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. A result of [7] states that sufficiency is preserved under truncation away from any Borel set in the range of X. Hence, in the case at hand S is sufficient for $\{f_\vartheta(x;\theta)\}$. It can be verified that S is also complete.

For the sake of simplicity but without loss of generality, consider the case $\vartheta=0$. This is at the same time the most important case for applications and the easiest with which to deal. It follows from (2) that

$$g_{\vartheta}(s;\theta)=\hat{w}(s)\left[w(\theta,\vartheta)\right]^{n} g(s;\theta)=\frac{\theta^{s} n!}{(e^{\theta}-1)^{n} s!} C_{s}^{n}, \quad s=n, n+1, \ldots , \tag{12}$$

where

$$g(s;\theta)=\frac{(n\theta)^{s}}{s!} e^{-n\theta}, \quad s=0,1,\ldots, \tag{13}$$

$$[\mathrm{w}(\theta,\vartheta)]^{n}=\frac{1}{(1-e^{-\theta})^{n}}, \tag{14}$$

$$\hat{w}(s)=\frac{n!}{n^{s}} C_{s}^{n}, \tag{15}$$

C_s^n denotes the Stirling number of the second kind [8] defined by

$$C_{s}^{n}=\begin{cases}\dfrac{1}{n!}\displaystyle\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} j^{s}, & s=n, n+1, \ldots, \\ 0, & s<n,\end{cases} \tag{16}$$

$$\begin{aligned} E\{\hat{w}(s)\}&=\sum_{s=0}^{\infty}\hat{w}(s) g(s;\theta)=\sum_{s=0}^{\infty}\frac{n!}{n^{s}} C_{s}^{n}\frac{(n\theta)^{s}}{s!} e^{-n\theta} \\ &=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} e^{-(n-j)\theta}\sum_{s=0}^{\infty}\frac{(j\theta)^{s}}{s!} e^{-j\theta}=(1-e^{-\theta})^{n}, \end{aligned} \tag{17}$$

This is the same result that of Tate and Goen [9]. Their proof was based on characteristic functions.

Example 3.2 (*Sampling distribution for the right-truncated exponential law*). Let the probability density function of the right-truncated exponential distribution be denoted by

$$f_{\vartheta}(x;\theta)=w(\theta,\vartheta) f(x;\theta), \quad 0\le x\le \vartheta, \tag{18}$$

where

$$w(\theta,\vartheta)=\frac{1}{1-e^{-\vartheta/\theta}}, \tag{19}$$

$$f(x;\theta)=(1/\theta) e^{-x/\theta}, \quad x\in[0,\infty). \tag{20}$$

Consider a sample of n independent observations $X_1, X_2, \ldots, X_n$, each with density $f_{\vartheta}(x;\theta)$, where the likelihood function is determined as

$$L_X(x^n;\theta,\vartheta) = \prod_{i=1}^{n} f_{\vartheta}(x_i;\theta) = \left[w(\theta,\vartheta)\right]^n L_X(x^n;\theta)$$

$$= \left[w(\theta,\vartheta)\right]^n \prod_{i=1}^{n} f(x_i;\theta) = \left[w(\theta,\vartheta)\right]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{21}$$

It is well known that

$$S = \sum_{i=1}^{n} X_i, \quad s \in [0,\infty), \tag{22}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. It follows from (2) that

$$g_{\vartheta}(s;\theta) = \widehat{w}(s)\left[w(\theta,\vartheta)\right]^n g(s;\theta) = \frac{e^{-s/\theta}}{\Gamma(n)\theta^n(1-e^{-\vartheta/\theta})^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s-(n-j)\vartheta)_+]^{n-1}$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n(1-e^{-\vartheta/\theta})^n} \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} [(s-(n-j)\vartheta)_+]^{n-1}, \quad s \in [0, n\vartheta], \quad n \geq 1, \tag{23}$$

where $a_+ = \max(0,a)$,

$$g(s;\theta) = \frac{s^{n-1}}{\Gamma(n)\theta^{\mathrm{n}}} e^{-s/\theta}, \quad s \in [0,\infty), \tag{24}$$

$$[w(\theta,\vartheta)]^n = \frac{1}{(1-e^{-\vartheta/\theta})^n}, \tag{25}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-1}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s-(n-j)\vartheta)_+]^{n-1}, \tag{26}$$

$$E\{\widehat{w}(s)\} = \int_0^{\infty} \widehat{w}(s) g(s;\theta) ds$$

$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} e^{-(n-j)\vartheta/\theta} \int_0^{\infty} \frac{[(s-(n-j)\vartheta)_+]^{n-1}}{\Gamma(n)\theta^n} e^{-(s-(n-j)\vartheta)_+/\theta} d(s-(n-j)\vartheta)_+ = (1-e^{-\vartheta/\theta})^n. \tag{27}$$

This is the same result that of Bain and Weeks [4]. Their proof was based on characteristic functions.

Example 3.3 (*Sampling distribution for the doubly truncated exponential law*). Consider an exponential distribution (20) that is doubly truncated at a lower truncation point (ϑ_1) and an upper truncation point (ϑ_2). The probability density function of the doubly truncated exponential distribution is defined as

$$f_{\vartheta}(x;\theta) = w(\theta,\vartheta) f(x;\theta), \quad \vartheta_1 \le x \le \vartheta_2, \tag{28}$$

where $\vartheta=(\vartheta_1,\vartheta_2)$,

$$w(\theta,\vartheta) = \frac{1}{e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta}}. \tag{29}$$

Consider a sample of n independent observations $X_1, X_2, \ldots, X_n$, each with density $f_{\vartheta}(x;\theta)$, where the likelihood function is determined as

$$L_X(x^n;\theta,\vartheta) = \prod_{i=1}^{n} f_{\vartheta}(x_i;\theta) = [w(\theta,\vartheta)]^n L_X(x^n;\theta)$$

$$= [w(\theta,\vartheta)]^n \prod_{i=1}^{n} f(x_i;\theta) = [w(\theta,\vartheta)]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{30}$$

It is well known that

$$S = \sum_{i=1}^{n} X_i, \quad s \in [0,\infty), \tag{31}$$

is a complete sufficient statistic for the family $\{f(x;\theta)\}$. It follows from (2) that

$$g_{\vartheta}(s;\theta) = \hat{w}(s)[w(\theta,\vartheta)]^n g(s;\theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1)_+]^{n-1}$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n} \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1)_+]^{n-1}, \quad s \in [n\vartheta_1, n\vartheta_2],$$

$$n \ge 1, \tag{32}$$

where $a_+=\max(0,a)$, $g(s,\theta)$ is given by (24),

$$[w(\theta,\vartheta)]^n = \frac{1}{(e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n}, \tag{33}$$

$$\hat{w}(\mathrm{s}) = \frac{1}{s^{n-1}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1))_+]^{n-1}, \tag{34}$$

$$E\{\hat{w}(s)\} = \int_0^{\infty} \hat{w}(s) g(s;\theta) ds = (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n. \tag{35}$$

4. VALIDITY OF THE UNBIASEDNESS EQUIVALENCE PRINCIPLE

The theoretical results of this investigation into the validity of the proposed *unbiasedness equivalence principle* (UEP) for finding sampling distributions for truncated laws are largely contained in the theorem given below. We introduce the following notation and assumptions. Let X^n be a random variable taking on values x^n in a space X_ϑ, let A be a σ–field of subsets of X_ϑ, and let $(\boldsymbol{\theta}, \vartheta)$ be a parameter associated with truncation, where ϑ is a known truncation point. For all values of the parameter $\boldsymbol{\theta}$ in some parameter space $\boldsymbol{\Theta}$, let P_ϑ be a probability measure on A; i.e., for any set A in A, $P_\vartheta(A;\boldsymbol{\theta})$ is the probability that X^n will belong to A when the parameter has the value $\boldsymbol{\theta}$. Let $\mathbf{S}=\mathbf{s}(X^n)$ be a statistic on the measurable space $(\mathrm{X}_\vartheta, \mathrm{A})$ taking on values in a measurable space $(\mathrm{S}_\vartheta, \mathrm{B})$. For each $\boldsymbol{\theta}\in\boldsymbol{\Theta}$, let G_ϑ be the probability distribution of $\mathbf{S}$ when X^n has the distribution P_ϑ; i.e., for any $B\in\mathrm{B}$, $G_\vartheta(B;\boldsymbol{\theta}) = P_\vartheta(\mathbf{s}^{-1}(B);\boldsymbol{\theta})$, where $\mathbf{s}^{-1}(B)$ is the set of points x^n in X_ϑ for which $\mathbf{s}(x^n)\in B$.

W. Assume the family $\mathrm{P} = \{P_\vartheta: \boldsymbol{\theta}\in\boldsymbol{\Theta}\}$ of probability distributions of X^n is dominated by a totally σ–finite measure μ over $(\mathrm{X}_\vartheta, \mathrm{A})$, i.e., there exists, for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}$, a non-negative A – measurable function $p_\vartheta(x^n;\boldsymbol{\theta})$ such that

$$P_\vartheta(A;\boldsymbol{\theta}) = \int_A p_\vartheta(x^n;\boldsymbol{\theta})\, d\mu(x^n) \tag{36}$$

for all $\mathrm{A}\in\mathrm{A}$. (The integrand $p_\vartheta(x^n;\boldsymbol{\theta})$ is called the density of P_ϑ w.r.t. (with respect to) μ).

(ii) Assume that $\mathbf{s}(X^n)$ is sufficient for P. From the Halmos-Savage factorization theorem [10], $\mathbf{s}(X^n)$ is sufficient if and only if for each $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ there exists a non-negative B-measurable function $L_{\mathbf{S}}(\mathbf{s}(x^n);\boldsymbol{\theta},\vartheta)$ on S_ϑ and a non-negative A – measurable function v on X_ϑ such that

$$p_\vartheta(x^n;\boldsymbol{\theta}) = L_{\mathbf{S}}(\mathbf{s}(x^n);\boldsymbol{\theta},\vartheta)v(x^n) \quad (\mu). \tag{37}$$

(The symbol (μ) following a statement means that the statement holds except on a set of μ-measure zero). In (37), we will assume that $L_{\mathbf{S}}$ and v are finite (μ).

(iii) Assume we recognize some likelihood function $L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta},\vartheta)$ equivalent to likelihood function $L_X(x^n;\boldsymbol{\theta},\vartheta)$. Define a σ–finite measure ρ over $(\mathrm{X}_\vartheta, \mathrm{A})$ by

$$\rho(A) = \int_A v(x^n)\, d\mu(x^n), \quad \text{all } A\in\mathsf{A}. \tag{38}$$

Then, from (36), (37), and (8),

$$P_\vartheta(A;\boldsymbol{\theta}) = \int_A L_{\mathbf{S}}(\mathbf{s}(x^n);\boldsymbol{\theta},\vartheta)\, d\rho(x^n), \quad \text{all } A\in\mathsf{A}. \tag{39}$$

(iv) Assume we recognize a totally σ-finite measure η over $(\mathrm{S}_\vartheta, \mathrm{B})$ such that the measure $\rho\,\mathbf{s}^{-1}$ over $(\mathrm{S}_\vartheta, \mathrm{B})$ is absolutely continuous w.r.t. η; i.e., $\eta(B)=0$ implis that $\rho\,\mathbf{s}^{-1}(B)=0$, where $\rho\,\mathbf{s}^{-1}(B)$ denotes the ρ – measure of the inverse image of B.

(v) Assume we recognize a positive B-measurable function φ on S_ϑ such that

$$\int_{\mathrm{S}_\vartheta} L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta},\vartheta)\varphi(\mathbf{s})\, d\eta(\mathbf{s}) \equiv 1 \tag{40}$$

for all $\theta \in \Theta$. Assume further that for any measurable set B of positive η – measure, there exists a $\theta \in \Theta$ and a measurable subset B_1 of B of positive η – measure over which $L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})$ is positive.

From (40), $\{L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}):\theta \in \Theta\}$ is a family of densities w.r.t. η. For $B \in$ B, let

$$G_{\vartheta}(B;\theta) = \int_{B} L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s}). \tag{41}$$

Thus, (v) provides us with a family of densities, but at this stage we do not know if this recognized family is the family of sampling densities of $\mathbf{S}$.

(vi) Assume we recognize that the family $\{L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}):\theta \in \Theta\}$ is complete, i.e.,

$$\int_{\mathsf{S}_{\vartheta}} \phi(\mathbf{s})L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s}) \equiv 0 \quad \text{for all } \theta \in \Theta \tag{42}$$

implies

$$\phi(\mathbf{s}) \equiv 0 \tag{43}$$

except on a set D with $G_{\vartheta}(D;\theta) = 0$ for all $\theta \in \Theta$.

Theorem 1 (*Sampling distribution for truncated law*). Under assumptions (i) through (vi), G_{ϑ} has a density with respect to η and $L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})$ is a version of it, i.e.,

$$L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s}) = \widehat{w}(\mathbf{s})\left[w(\theta,\vartheta)\right]^{n} g(\mathbf{s};\theta) \tag{44}$$

is the sampling density, $g_{\vartheta}(\mathbf{s};\theta)$, of the sufficient statistic $\mathbf{s}(X^n)$.

Proof. We show first that (43) and the second part of (v) imply that $\phi(\mathbf{s}) \equiv 0$ (η). For suppose there exists a measurable B with $\eta(B)>0$ and $\phi(\mathbf{s})\neq 0$ over B. Then $B \subset D$, so $G_{\vartheta}(B;\theta)=0$ for all $\theta \in \Theta$. But, from (v), there exists a $B_1 \subset B$ for which $G_{\vartheta}(B_1;\theta)>0$ for some θ, contradicting $G_{\vartheta}(B;\theta)=0$ for all $\theta \in \Theta$. Now, by a theorem in [10], there exists a non-negative measurable function ψ on S_{ϑ} such that

$$\int Q_{\vartheta}(\mathbf{s}(x^n);\theta)d\rho(x^n) = \int Q_{\vartheta}(\mathbf{s};\theta)\psi(\mathbf{s})d\eta(\mathbf{s}) \tag{45}$$

for every measurable function Q_{ϑ}, in the sense that if either integral exists, then so does the other and the two are equal. In (45), let $Q_{\vartheta}(\mathbf{s},\theta) = \chi_B L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)$, where χ_B is the characteristic function of B ($B \in$ B). Then there exists a $\psi(\mathbf{s})$ such that

$$\int_{\mathbf{s}^{-1}(B)} L_{\mathbf{S}}(\mathbf{s}(x^n);\theta,\vartheta)d\rho(x^n) = \int_{B} L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\psi(\mathbf{s})d\eta(\mathbf{s}) \tag{46}$$

for all $B \in$ B. Note that the left side of (45) is $G_{\vartheta}(B;\theta)$.

In (42), let $\phi(\mathbf{s}) = 1-[\psi(\mathbf{s})/\varphi(\mathbf{s})]$. From (40) and (46),

$$\int_{\mathsf{S}_{\vartheta}} \left[1 - \frac{\psi(\mathbf{s})}{\varphi(\mathbf{s})}\right] L_{\mathbf{S}}(\mathbf{s};\theta,\vartheta)\varphi(\mathbf{s})d\eta(\mathbf{s}) = 0 \tag{47}$$

for all $\theta \in \Theta$. Thus, from (43), $\psi(\mathbf{s})=\varphi(\mathbf{s})$ almost everywhere (η), and, from (47),

$$L_{\mathbf{S}}(\mathbf{s};\boldsymbol{\theta},\vartheta)\varphi(\mathbf{s}) = \hat{w}(\mathbf{s})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{s};\boldsymbol{\theta}) \tag{48}$$

is a version of the density of G_ϑ with respect to η. □

X. FINDING RELIABILITY ESTIMATORS FOR TRUNCATED LAWS VIA THE UNBIASEDNESS EQUIVALENCE PRINCIPLE

Consider a system that is required to operate for a given 'mission time', t. The reliability of this system for the right-truncated distribution of time-to-failure with the probability density function $f_\vartheta(x;\boldsymbol{\theta})$ may be defined as

$$R(t) = \Pr(x \geq t) = \int_t^\vartheta f_\vartheta(x;\boldsymbol{\theta})dx. \tag{49}$$

Due to the Rao-Blackwell and Lehmann-Scheffé theorem [11] a minimum variance unbiased (MVU) estimator for R may be obtained as

$$\hat{R}(t) = \int_t^\vartheta f_\vartheta(x;\mathbf{s})dx, \tag{50}$$

where X may be any one of the observations $(X_1, \ldots, X_n)$ from $f_\vartheta(x;\boldsymbol{\theta})$, $\mathbf{S}$ is a complete sufficient statistic for $\{f_\vartheta(x;\boldsymbol{\theta})\}$, and $f_\vartheta(x;\mathbf{s})$ is the conditional distribution of X given $\mathbf{S}=\mathbf{s}$; $f_\vartheta(x;\mathbf{s})$ is obtained as

$$f_\vartheta(x;\mathbf{s}) = \frac{f_\vartheta(x,\mathbf{s};\boldsymbol{\theta})}{g_\vartheta(\mathbf{s};\boldsymbol{\theta})} = \frac{\hat{w}_f(x,\boldsymbol{\theta},\vartheta)}{\hat{w}(\mathbf{s})}, \tag{51}$$

where

$$f_\vartheta(x,\mathbf{s};\boldsymbol{\theta}) = \hat{w}_f(x,\boldsymbol{\theta},\vartheta)[w(\boldsymbol{\theta},\vartheta)]^n g(\mathbf{s};\boldsymbol{\theta}) \tag{52}$$

is the joint probability density of X and $\mathbf{S}$, $\hat{w}_f(x,\boldsymbol{\theta},\vartheta)$ is an unbiased estimator of

$$w_f(x,\boldsymbol{\theta},\vartheta) = \frac{f_\vartheta(x;\boldsymbol{\theta})}{[w(\boldsymbol{\theta},\vartheta)]^n}. \tag{53}$$

with respect to $g(\mathbf{s};\boldsymbol{\theta})$.

It should be noted that (50) can be obtained by different method as

$$\hat{R}(t) = \frac{\hat{w}_R(t,\boldsymbol{\theta},\vartheta)}{\hat{w}(\mathbf{s})}, \tag{54}$$

where $\hat{w}_R(t,\boldsymbol{\theta},\vartheta)$ is an unbiased estimator of

$$w_R(t,\boldsymbol{\theta},\vartheta)=\frac{R(t)}{[w(\boldsymbol{\theta},\vartheta)]^n} \tag{55}$$

with respect to $g(\mathbf{s};\boldsymbol{\theta})$.

Example 5.1 (*MVU estimator of reliability for the right truncated exponential distribution*). Let $X^n=(X_1, \ldots, X_n)$ be a random sample of size n from a population with density (18). Then it follows from (50) (or (54)) that the MVU estimator of $R(t)$ is obtained as

$$\widehat{R}(t)=\frac{\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\left([(S-(j\vartheta+t))_+]^{n-1}-[(S-(j+1)\vartheta)_+]^{n-1}\right)}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}[(S-j\vartheta)_+]^{n-1}}. \tag{56}$$

As a particular case, if $\vartheta \to \infty$ that is the variable X is assumed unrestricted, the corresponding MVU estimator of reliability reduces to

$$\widehat{R}(t)=[(1-t/S)_+]^{n-1}. \tag{57}$$

For instance, suppose that the following failure times, in hours, are available from a given system: 4.2, 9.8, 16, 20 and that the truncation point ϑ=25 hours and the mission time t=5 hours. Clearly s=50 hours. Substituting these values in (56), the estimate of reliability is obtained as $\widehat{R}(t)=0.824$. Had we assumed, however, that the observations are coming from the complete population, the estimate of reliability would have been, from (57), $\widehat{R}(t)=0.729$.

Example 5.2 *(MVU estimator of reliability for the right-truncated gamma distribution). Let* $X^n=(X_1, \ldots, X_n)$ be a random sample of size n from a population with density

$$f_\vartheta(x;\boldsymbol{\theta})=w(\boldsymbol{\theta},\vartheta)\frac{1}{\Gamma(\delta)}\sigma^{-\delta}x^{\delta-1}e^{-x/\sigma}, \quad 0<x\le\vartheta, \quad \sigma>0, \quad \delta>0, \tag{58}$$

where ϑ is point of truncation, $\boldsymbol{\theta}=(\sigma,\delta)$, and $w(\boldsymbol{\theta},\vartheta)$ is such that

$$w(\boldsymbol{\theta},\vartheta)\int_0^{\vartheta}\frac{1}{\Gamma(\delta)}\sigma^{-\delta}x^{\delta-1}e^{-x/\sigma}dx=1. \tag{59}$$

This distribution has found applications in a number of diverse fields, for instance, in fitting of length-of-life data under fatigue. Note that for δ=1, the right-truncated gamma distribution reduces to the right-truncated exponential distribution with parameter σ. Although, this distribution is a special case of gamma distribution and gives a good fit to length-of-life data in many situations, it is not suitable since its use carries the implication that at any time future life-length is independent of past history.

To find MVU estimator of $R(t)$ we apply the above technique. If the shape parameter δ in (58) is assumed to be known, then it is well known that

$$S=\sum_{i=1}^{n}X_i \tag{60}$$

is a complete sufficient statistic for σ. The probability density function of the sampling distribution of S is given by

$$g_{\vartheta}(s;\boldsymbol{\theta}) = \widehat{w}(s)\left[w(\boldsymbol{\theta},\vartheta)\right]^{n} g(s;\boldsymbol{\theta}) = \frac{[w(\boldsymbol{\theta},\vartheta)]^{n}\Gamma^{n}(\delta)}{\sigma^{n\delta}\Gamma(n\delta)} e^{-s/\sigma}$$

$$\times \sum_{r=0}^{n}(-1)^{r}\binom{n}{r}[(s-r\vartheta)_{+}]^{n\delta-1}\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right), \quad s\in(0,\, n\vartheta), \tag{61}$$

where

$$\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right) = \sum_{\substack{\{r_0,\,r_1,\ldots,\,r_{\delta-1}\}:\\ \sum_{j=0}^{\delta-1} r_j = r}} \frac{r!\varpi!}{\prod_{j=0}^{\delta-1}(r_j!)\sum_{j=0}^{\delta-1}(j!)^{r_j}}\binom{n\delta-1}{\varpi}\left(\frac{\vartheta}{(s-r\vartheta)_{+}}\right)^{\varpi}, \tag{62}$$

$$\varpi = \sum_{j=0}^{\delta-1} j r_j. \tag{63}$$

The joint distribution of X and S is given by

$$f_{\vartheta}(x,s;\boldsymbol{\theta}) = \widehat{w}_{f}(x,\boldsymbol{\theta},\vartheta)[w(\boldsymbol{\theta},\vartheta)]^{n} g(s;\boldsymbol{\theta}) = \frac{[w(\boldsymbol{\theta},\vartheta)]^{n}\Gamma^{(n-1)}(\delta)}{\sigma^{n\delta}\Gamma((n-1)\delta)} x^{\delta-1} e^{-s/\sigma}$$

$$\times \sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}(s-r\vartheta-x)^{(n-1)\delta-1}\,\Delta\left((n-1)\delta-1,\frac{\vartheta}{(s-r\vartheta-x)_{+}}\right). \tag{64}$$

Thus the conditional distribution of X given S is

$$f_{\vartheta}(x;s) = \frac{f_{\vartheta}(x,s;\boldsymbol{\theta})}{g_{\vartheta}(s;\boldsymbol{\theta})} = \frac{\widehat{w}_{f}(x,\boldsymbol{\theta},\vartheta)}{\widehat{w}(s)} = \frac{\Gamma(n\delta)}{\Gamma(\delta)\Gamma((n-1)\delta)}$$

$$\times \sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}x^{\delta-1}[(s-r\vartheta-x)_{+}]^{(n-1)\delta-1}\,\Delta\left((n-1)\delta-1,\frac{\vartheta}{(s-r\vartheta-x)_{+}}\right)$$

$$\times\left(\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}(s-r\vartheta)^{n\delta-1}\Delta\left(n\delta-1,\frac{\vartheta}{(s-r\vartheta)_{+}}\right)\right)^{-1}. \tag{65}$$

Hence the MVU estimator of $R(t)$ at time t is given by

$$\widehat{R}(t) = \int_{t}^{\vartheta} f_{\vartheta}(x;s)dx = \frac{\widehat{w}_{R}(t,\boldsymbol{\theta},\vartheta)}{\widehat{w}(s)} = \frac{\Gamma(n\delta)}{\Gamma(\delta)\Gamma((n-1)\delta)}\sum_{r=0}^{n-1}(-1)^{r}\binom{n-1}{r}$$

$$\times \int_{t}^{\vartheta} x^{\delta-1}[(s-r\vartheta-x)_{+}]^{(n-1)\delta-1}\Delta\left((n-1)\delta-1, \frac{\vartheta}{(s-r\vartheta-x)_{+}}\right)dx$$

$$\times\left(\sum_{r=0}^{n}(-1)^{r}\binom{n}{r}(s-r\vartheta)^{n\delta-1}\Delta\left(n\delta-1, \frac{\vartheta}{(s-r\vartheta)_{+}}\right)\right)^{-1}. \tag{66}$$

It may be remarked that the result (66) though at the first look appears quite unwieldy is not so in practical applications, particularly when the sample size is small.

As a particular case, if $\vartheta \to \infty$ that is the random variable X is assumed unrestricted, the distribution of the sufficient statistics from equation (61) reduces to

$$g(s;\boldsymbol{\theta}) = \frac{1}{\Gamma(n\delta)\sigma^{n\delta}} s^{n\delta-1} e^{-s/\sigma}, \quad s \in (0,\infty) \tag{67}$$

and the corresponding MVU estimator of reliability at time t is given by

$$\widehat{R}(t) = \frac{\Gamma(n\delta)}{s^{\delta-1}} \sum_{j=0}^{\delta-1} \frac{s^{j} t^{\delta-1-j}(1-t/s)^{(n-1)\delta+j}}{(\delta-1-j)![(n-1)\delta+j]!}, \tag{68}$$

which corresponds to Basu's [12] equation (9).

Y. CONCLUSIONS

The authors hope that this work will stimulate further investigation using the approach on specific applications to see whether obtained results with it are feasible for realistic applications.

It will be noted that the similar approach also can be used to find the sampling distribution for truncated law when some or all of its truncation parameters are left unspecified.

For instance, consider Example 3.3, where it is assumed that the truncation parameter $\boldsymbol{\vartheta}=(\vartheta_1, \vartheta_2)$ is unknown. It is known that the statistic $(X_{(1)}, X_{(n)}, S)$, where

$$X_{(1)} = \min_{1\le i\le n} X_i, \tag{69}$$

$$X_{(n)} = \max_{1\le i\le n} X_i, \tag{70}$$

and

$$S = \sum_{i=2}^{n-1} X_i, \tag{71}$$

is a complete sufficient statistic for a set of parameters $(\vartheta_1, \vartheta_2, \theta)$. In this case, the likelihood function of a sample is determined as

$$L_X(x_{(1)}, x_{(n)}, x_2, ..., x_{n-1}; \theta, \vartheta) = n(n-1) f_\vartheta(x_{(1)}; \theta) f_\vartheta(x_{(n)}; \theta) \prod_{i=2}^{n-1} f_\vartheta(x_i; \theta)$$

$$= h_\vartheta(x_{(1)}, x_{(n)}; \theta) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} \prod_{i=2}^{n-1} f(x_i; \theta) = h_\vartheta(x_{(1)} x_{(n)}; \theta) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} \frac{1}{\theta^{n-2}} e^{-\sum_{i=2}^{n-1} x_i / \theta}, \tag{72}$$

where

$$h_\vartheta(x_{(1)}, x_{(n)}; \theta) = n(n-1)[F_\vartheta(x_{(n)}; \theta) - F_\vartheta(x_{(n)}; \theta)]^{n-2} f_\vartheta(x_{(1)}; \theta) f_\vartheta(x_{(n)}; \theta),$$

$$x_{(1)} < x_{(n)}, \quad x_{(1)}, x_{(n)} \in [\vartheta_1, \vartheta_2], \tag{73}$$

is the joint probability density function of the order statistics $x_{(1)}$ and $x_{(n)}$), $F_\vartheta(\cdot)$ is the probability distribution function. It is well known that

$$S = \sum_{i=2}^{n-1} X_i, \quad s \in [0, \infty), \tag{74}$$

is a complete sufficient statistic for the family $\{f(x; \theta)\}$. It follows from (2) and (72) that

$$g_\vartheta(s; \theta) = \widehat{w}(s) \left[w(\theta, x_{(1)}, x_{(n)}) \right]^{n-2} g(s; \theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n-2)\theta^{n-2}[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s - (n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_+]^{n-3}$$

$$= \frac{e^{-s/\theta}}{\Gamma(n-2)\theta^{n-2}[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}} \sum_{j=1}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s - (n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_+]^{n-3},$$

$$s \in [(n\text{-}2)x_{(1)}, (n\text{-}2)x_{(n)}], \quad n \geq 3, \tag{75}$$

where

$$g(s; \theta) = \frac{s^{n-3}}{\Gamma(n-2)\theta^{n\text{-}2}} e^{-s/\theta}, \quad s \in [0, \infty), \tag{76}$$

$$\left[w(\theta, x_{(1)}, x_{(n)})\right]^{n-2} = \frac{1}{[e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}}, \tag{77}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-3}} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^{n-2-j} [(s-(n-2)x_{(1)} - (n-2-j)(x_{(2)} - x_{(1)}))_+]^{n-3}, \tag{78}$$

$$E\{\widehat{w}(s)\} = \int_0^\infty \widehat{w}(s) g(s;\theta) ds = [e^{-x_{(1)}/\theta} - e^{-x_{(2)}/\theta}]^{n-2}. \tag{79}$$

Thus, the sampling distribution of the sufficient statistic ($X_{(1)}$, $X_{(n)}$, S) for ($\vartheta_1, \vartheta_2, \theta$) is given by

$$g_{\vartheta}(x_{(1)}, x_{(n)}, s;\theta) = h_{\vartheta}(x_{(1)}, x_{(n)};\theta) g_{\vartheta}(s;\theta). \tag{80}$$

In other words, we have the following results.

In a singly truncated case, when a truncation point on the left, ϑ_1, is unknown, a sampling distribution of the sufficient statistic ($X_{(1)}$, S) for (ϑ_1, θ) is given by

$$g_{\vartheta_1}(x_{(1)}, s;\theta) = h_{\vartheta_1}(x_{(1)};\theta) g_{\vartheta_1}(s;\theta), \tag{81}$$

where

$$X_i \sim f_{\vartheta_1}(x_i;\theta) = w(\theta, \vartheta_1) f(x_i;\theta) = \frac{1}{1 - F(\vartheta_1;\theta)} f(x_i;\theta), \quad x_i \geq \vartheta_1, \quad i = 1, \ldots, n, \tag{82}$$

$$h_{\vartheta_1}(x_{(1)};\theta) = n[1 - F_{\vartheta_1}(x_{(1)};\theta)]^{n-1} f_{\vartheta_1}(x_{(1)};\theta) \tag{83}$$

is the probability density function of the order statistic $X_{(1)}$,

$$g_{\vartheta_1}(s;\theta) = \widehat{w}(s)\left[w(\theta, x_{(1)})\right]^{n-1} g(s;\theta), \tag{84}$$

$s \equiv s(X_2, \ldots, X_n)$.

In a singly truncated case, when a truncation point on the right, ϑ_2, is unknown, a sampling distribution of the sufficient statistic ($X_{(n)}$, S) for (ϑ_2, θ) is given by

$$g_{\vartheta_2}(x_{(n)}, s;\theta) = h_{\vartheta_2}(x_{(n)};\theta) g_{\vartheta_2}(s;\theta), \tag{85}$$

where

$$X_i \sim f_{\vartheta_2}(x_i;\theta) = w(\theta, \vartheta_2) f(x_i;\theta) = \frac{1}{F(\vartheta_2;\theta)} f(x_i;\theta), \quad x_i \leq \vartheta_2, \quad i = 1, \ldots, n, \tag{86}$$

$$h_{\vartheta_2}(x_{(n)};\theta) = n[F_{\vartheta_2}(x_{(n)})]^{n-1} f_{\vartheta_2}(x_{(n)};\theta) \tag{87}$$

is the probability density function of the order statistic $X_{(n)}$,

$$g_{\vartheta_2}(s;\theta) = \widehat{w}(s)\left[w(\theta, x_{(n)})\right]^{n-1} g(s;\theta), \tag{88}$$

$s \equiv s(X_1, \ldots, X_{n-1})$.

In a doubly truncated case, when a lower truncation point, ϑ_1, and an upper truncation point, ϑ_2, are unknown, a sampling distribution of the sufficient statistic $(X_{(1)}, X_{(n)}, S)$ for $(\vartheta_1, \vartheta_2, \theta)$ is given by

$$g_{\vartheta}(x_{(1)}, x_{(n)}, s; \theta) = h_{\vartheta}(x_{(1)}, x_{(n)}; \theta) g_{\vartheta}(s; \theta). \tag{89}$$

where

$$X_i \sim f_{\vartheta}(x_i; \theta) = w(\theta, \vartheta_1, \vartheta_2) f(x_i; \theta) = \frac{1}{F(\vartheta_2; \theta) - F(\vartheta_1; \theta)} f(x_i; \theta), \quad \vartheta_1 \le x_i \le \vartheta_2, \; i = 1, \ldots, n, \tag{90}$$

$$h_{\vartheta}(x_{(1)}, x_{(n)}; \theta) = n[F_{\vartheta}(x_{(n)}; \theta) - F_{\vartheta}(x_{(1)}; \theta)]^{n-2} f_{\vartheta}(x_{(n)}; \theta) \tag{91}$$

is the joint probability density function of the order statistic $X_{(1)}$ and $X_{(n)}$,

$$g_{\vartheta}(s; \theta) = \widehat{w}(s)\left[w(\theta, x_{(1)}, x_{(n)})\right]^{n-2} g(s; \theta), \tag{92}$$

$s \equiv s(X_2, \ldots, X_{n-1})$.

If, say, we deal with a left-truncated exponential distribution,

$$f_{\vartheta_1}(x; \theta) = w(\theta, \vartheta_1) f(x; \theta), \quad \vartheta_1 \le x < \infty, \tag{93}$$

where

$$w(\theta, \vartheta_1) = \frac{1}{e^{-\vartheta_1/\theta}} \tag{94}$$

and a truncation point on the left, ϑ_1, is unknown, then it follows immediately from (81) that the sampling distribution of the sufficient statistic $(X_{(1)}, S)$, $S = \sum_{i=2}^{n} X_i$, for (ϑ_1, θ) is given by

$$g_{\vartheta_1}(x_{(1)}, s; \theta) = h_{\vartheta_1}(x_{(1)}; \theta) g_{\vartheta_1}(s; \theta) = \left[n\left(\frac{e^{-x_{(1)}/\theta}}{e^{-\vartheta_1/\theta}}\right)^{n-1} \frac{1}{e^{-\vartheta_1/\theta}} \frac{1}{\theta} e^{-x_{(1)}/\theta} \right]$$

$$\times \left[\frac{[s-(n-1)x_{(1)}]^{n-2}}{s^{n-2}} \frac{1}{[e^{-x_{(1)}/\theta}]^{n-1}} \frac{s^{n-2}}{\Gamma(n-1)\theta^{n-1}} e^{-s/\theta} \right] = \frac{n}{\theta} e^{-n(x_{(1)}-\vartheta_1)/\theta} \frac{[s-(n-1)x_{(1)}]^{n-2}}{\Gamma(n-1)\theta^{n-1}} e^{-[s-(n-1)x_{(1)}]/\theta}, \tag{95}$$

which corresponds to the well-known result [11].

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BOOTSTRAPPING TIME SERIES WITH APPLICATION TO RISK MANAGEMENT

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ABSTRACT: The bootstrap method is an extensive computational approach, based on Monte Carlo simulation, useful for understanding random samples and time series. It is a powerful tool, especially when only a small data set is used to predict the behaviour of systems or processes. This paper presents the results of an investigation on using bootstrap resampling (different types: uniform, importance based, block structured etc.) for time series appearing during software life cycle (mainly the software testing phase, and debugging), economics, and environment (air pollution generated by cement plants) in order to help the activity of staff working on risk management for software projects, risk management in finance, and those working on environment risk management.

z. INTRODUCTION

Risk in the sense of the possibility of losses is an important chapter for many organizations, not only financial markets, but also for industry. One important principle belonging to the general scientific knowledge in the area of risk claims that "it is impossible to manage the risk without quantitative measurement and analysis of risk", according to (Solojentsev, 2005).

There are various aspects concerning risk management, depending on the field under consideration. As a consequence, different methodologies (models) were developed (Aven 2003, Solojentsev 2005, Kontio 1997, Higuera & Haimes 1996, Entrop et al.2007, Todinov 2006 etc.)

This paper addresses the usage of the bootstrap approach for analysing time series appearing when modelling some measurements during the evolution of processes. Bootstrap proved to be a valuable approach for a large class of applications according to (Efron & Tibshirani 1993, Albeanu et al. 2007), and the references mentioned related to time series.

The remainder of the paper is organized as follows. A short introduction to general bootstrap approach is given in the second section. Algorithmic aspects concerning bootstrapping time series and challenging problems related to the model selections are presented in the third section. The fourth section discusses on three case studies covering different fields of economical activities: cement plants pollution, inflation rate and software risk management.

The concluding section establishes the most important challenges to deal with when using bootstrapping time series.

2. PROBLEM DEFINITION AND BACKGROUND INFORMATION

2.1. Bootstrap methodology

Bootstrap is a simple but powerful Monte-Carlo method to assess statistical accuracy or to estimate a distribution from sample's statistics. The methods are suitable for any level of odellin being useful for fully parametric, semi-parametric, and completely nonparametric analysis. These

approaches are not only in use by statisticians, but also are applied anywhere statistics can be used: life sciences, business, social sciences, econometrics, reliability etc. For the aim of this paper we outline the basic bootstrap principle (see Efron 1979, Efron & Tibshirani 1986), and the application of bootstrap sampling for time series in order to help the activity of staff working on risk management.

Let X be a random variable and F the cumulative distribution function of the variable X. The Bootstrap method, introduced by Efron (1979), is useful, at least, for the estimation of: a) the distribution function of a random variable $R(X, F)$; b) a functional relation $V(F)$, or c) the accuracy of a statistics s obtained from a sample $(X_1, X_2, ..., X_n)$ of size n from X (the *accuracy* describing the variability of s when independent estimations $s(1), s(2), \ldots$, of the statistics s, are obtained by resampling).

The bootstrap technique uses the sample $(X_1, X_2, ..., X_n)$ to obtain the sampling cumulative distribution function $F_n(x)$ in order to replace the true cumulative distribution function F: $F_n(x) = (1/n)$ *cardinal* $\{x_i \square x;\ 1 \square i \square n\}$. To repeatedly simulate bootstrap samples $X^* := (X_1^*, X_2^*, ..., X_n^*)$ from F_n , random number generators should be used according to the Monte-Carlo approaches. Then, for each bootstrap sample, it is recalculated: a) the distribution function of the random variable $R(X^*, F_n)$; b) the functional relation $V(F_n)$ or $V(F_n^*)$ and c) the statistics $s^*( )$. The accuracy of the statistics s can be derived under an appropriate statistical inference study on the sequence $s^*()$.

The bootstrap resampling can be realised in various ways. Uniform resampling and the importance resampling are the mostly used. As a common example of the usage of the uniform resampling, we refer to the bootstrap algorithm for estimating standard errors. However, when some observations are more important than others, the importance resampling can provide close to real conclusions. If resampling is based on importance resampling weights, then the bootstrap estimates are re-weighted as if uniform resampling is done.

2.2. Bootstrapping time series

Time series play an important role in modelling, analysing and forecasting the behaviour of systems (Cochrane 2005, Hamilton 1994, Burtschy 1997, Madsen 2007). There are numerous aspects concerning time series. In the following will be described only those models and algorithms required by our case studies.

Let $\{x_t;\ t = 1, 2, ..., T\}$ be a time series and L be the *lag operator*: $Lx_t = x_{t-1};\ t > 1$. The ARMA model having order (p, q) is given by:

$$\varphi(L)\, x_t = \theta(L)\, u_t, \qquad (1)$$

where $\varphi(L)x_t = (a_0L^0 + a_1L^1 + \ldots + a_pL^p)x_t = a_0x_t + a_1x_{t-1} + \ldots + a_px_{t-p}$, $p \geq 0$, $a_p \neq 0$, $\theta(L)u_t = (b_0L^0 + b_1L^1 + \ldots + b_qL^q)u_t = b_0u_t + b_1u_{t-1} + \ldots + b_pu_{t-q}$, $q \geq 0$, $b_q \neq 0$, and $\{u_t\}$ is an uncorrelated process with zero mean and finite variance.

The ARMA bootstrap algorithm proceeds as follows:

1 Determine the order of the ARMA(p,q) process.

2 Estimate the parameters: $\hat{\varphi}(L)$, $\hat{\theta}(L)$.

3 Resample from $\hat{u}_t = \hat{\theta}^{-1}(L)\hat{\varphi}(L)x_t$ (after re-centring the $\hat{u}_t$ around zero).

4 Choose a large positive integer τ, set $x_t^* = 0$ for $t < -\tau$, and generate *iid* draws for u_t^*, with $t = -\tau, ..., T$.

5 Generate pseudo-data: $x_t^* = \hat{\varphi}^{-1}(L)\hat{\theta}(L)u_t^*$ for $t = -\tau, \ldots, T$ and retain the last T values of x_t^* .

6 Calculate the bootstrap parameter estimates: $\hat{\varphi}^*(L)$, $\hat{\theta}^*(L)$.

7 Repeat steps 3-6 many times and built up the empirical distribution to obtain the functional relation or analyse the required statistics.

The ARMA parameters can be estimated using different methods including maximum likelihood (ML) algorithms, as presented by Boaz (1994) and others.

As Berkowitz & Kilian (2000) mentioned, the bootstrap can perform well when the parametric model provides a good approximation to the true model. In practice, for a sample of size T, the model and the order (p, q) are unknown. Different scenarios have to be considered and a selection procedure will be applied. The most used procedures are, according to (Alonso et al. 2004): the final prediction error, the Akaike information criterion, the Bayesian information criterion and the Akaike's Information Corrected Criterion (AICC). For the investigation presented in this paper, the AICC method was used:

$$AICC = -2\ln\Lambda(\hat{a},\hat{b},\hat{\sigma}^2) + \frac{2T(p+q+1)}{T-(p+q)-2} \quad ,(2)$$

where

$\hat{a}$ is the estimated AR parameters; $\hat{b}$ is the estimated MA parameters; $\hat{\sigma}^2$ is the variance of the white noise, and $\Lambda(\cdot,\cdot,\cdot)$ is the likelihood of the data under the Gaussian ARMA model.

The block bootstrap is the best-known method for implementing the bootstrap time series, as (Härdle et al. 2003) mentioned. The method "consists of dividing the data into blocks of observations and sampling the blocks randomly with replacement." For the time series considered above, with non-overlapping blocks of length l, the first block is composed by observations $\{x_j;\ j = 1, 2, ..., l\}$, the second block contains observations $\{x_{l+j};\ j = 1, 2, ..., l\}$, and so forth. When using the overlapping (*moving*) blocks of length l, the first block is composed by observations $\{x_j;\ j = 1, 2, ..., l\}$, the second block consists of observations $\{x_{j+1};\ j = 1, ..., l\}$, and so forth. The method of resampling is based on the replacement approach. The block bootstrap with random block length is a stationary bootstrap because a stationary data series is obtained.

Seasonal time series are a special class of time series, appearing in environmental risk management or the multi-version software testing. These time series are typically modelled by equation

$$x_t = \mu_t + u_t, \text{ and } \mu_t = \mu_{t-d},\ t > d, \qquad (3)$$

where d is the period (day, week, month etc.) of some deterministic (but unknown) function μ_t, and $\{u_t,\ t>0\}$ is a stationary process with mean zero. In general, if μ_t is not a constant, the seasonal model is not stationary, that is a "seasonal block bootstrap" method (denoted, in the following, by SBB) is necessary. In the following let us remember the Politis (2001) approach that proved a good behaviour for time series obtained when monitoring the pollution of cement plants.

The SBB algorithm considers that there exists an integer *n* such as $T = nd$, $b\ (< n)$ a given positive integer, $k = nb$, and works along the following steps:

1 Let $i_0, i_1, ..., i_{k-1}$ be drawn independent identically distributed uniform on the set $\{1, 2, ..., n-b+1\}$;

2 Build the bootstrap pseudo-series $X_1^*, X_2^*, ..., X_l^*$, where $l = kbd$, and

$$X^*_{mbd+j} := X_{i_m d+j-1} \qquad (4)$$

for $m = 0, 1, ..., k-1$, and $j = 1, 2, ..., bd$.

The estimation of seasonal component μ_i, $i = 1, 2, ..., d$, and the overall mean $\bar{\mu} = d^{-1}\sum_{i=1}^{d}\mu_i$ are realised by means of averages of the "sampled" series:

$$\hat{\mu}_i = n^{-1}\sum_{j=0}^{n-1} X_{i+jd} \text{ , and } \hat{\bar{\mu}} = d^{-1}\sum_{i=1}^{d} \bar{\mu}_i \text{ .} \qquad (5)$$

The usefulness of the SBB method consists of interval estimates obtaining for μ_i and μ by means of successfully approximating the distribution of $\hat{\mu}_i$ and $\hat{\bar{\mu}}$ by their bootstrap versions computed based on the bootstrap pseudo-series $X_1^*, X_2^*, ..., X_l^*$, by

$$\bar{\mu}_i^* = (kb)^{-1}\sum_{j=0}^{kb-1} X_{i+jd}^* \text{ , and } \hat{\bar{\mu}}^* = d^{-1}\sum_{i=1}^{d} \hat{\mu}_i^* \text{ .(6)}$$

This model is used in the place of "residual" block bootstrap obtainable by resampling of the residuals $\hat{Y}_t := X_t - \hat{\mu}_t$. The pseudo-series $\hat{Y}_1^*, \hat{Y}_2^*, \cdots, \hat{Y}_l^*$ is used to generate the bootstrap series $X_t^* := \hat{\mu}_t + \hat{Y}_t^*$, *t = 1, 2, ..., l.*

It was proved (see Politis, 2001) that overlapping plays an important role in bootstrap efficiency: "the maximum overlap leads to maximum efficiency". A data based adapted procedure for choosing the block size *l,* in finite samples, based on the (Berkowitz & Kilian, 2000) method, in order to maximize the average accuracy.

Given the stationary series $\{x_t; t = 1, 2, ..., T\}$, the bootstrap approach can be used to select the block size suitable for a maximum accuracy in estimating some statistic of interest, according to the following steps:

1 Approximate the given time series by a parametric ARMA(p, q), or AR(p) model, with order selected by AICC approach.

2 Generate B (≥ 512) Monte Carlo trials of length T from the model fitted above.

3 For each Monte Carlo trial generate overlapping blocks bootstrap data $\{X_t^*\}$ for different block sizes k.

4 Compute the statistics of interest $\{X_t^*(k)\}$.

5 Select the block size k* which, on average, produces the most accurate test statistics, point estimate, or confidence interval across Monte Carlo trials.

6 Use the block size k* to apply the Block bootstrap or SBB method for the original data $\{x_t; t = 1, 2, ..., T\}$.

There are available other methods for bootstrapping time series: (Berkowitz & Kilian 2000, Härdel et al. 2003) and Politis (2003) to mention only some references. The above selected approaches proved to be suitable (computing effort, accuracy) for the investigation on using time series for risk management in finance, environment and software reliability.

2.3. Risk management

There are many definitions of the term "risk", all of them including two important characteristics, namely uncertainty (an event may or not may occur) and loss (an event has undesired effects): risk being the possibility of suffering losses caused by an event that will probably occurs.

Generally speaking, risk management is a systematic process for identifying, analysing and controlling risks.

Multi-criteria decision aided, soft computing, and statistical analysis are some important approaches when speak about "Decide with minimum risk". Recently, time-series risk models were proposed, mainly for insurance business (Wan et al. 2005, Zhang et al. 2007). Also, other researchers proved that time-series analysis and forecasting play an important role in risk management. These progresses can be accompanied by bootstrap methodology in order to apply a risk preventive approach.

3. CASE STUDIES

3.1. Bootstrapping time series applied for software risk management

According to Kontio (1997), "software development is often plagued with unanticipated problems which cause projects to miss deadlines, exceed budgets, or deliver less than satisfactory products". Even if these problems cannot be eliminated completely some of them can be well controlled well by taking appropriate preventive action.

Practically, the software development organizations are exposed to a large plethora of risk factors. Some of them are: human resources quality, unrealistic schedule and budget, the mismatching of requirements and developed item, continuous alteration of requirements, outsourcing generated problems, overestimation of infrastructure capability etc. Software organizations may be able to avoid a large number of such problems if they use systematic risk management procedures and techniques early in projects. Any methodology has to monitor such resources and multivariate time series are obtained using a measurement methodology as provided by Fenton & Pfleeger (1996).

One approach to analysis time series for software reliability is based on soft computing techniques as shown by Albeanu & Popentiu-Vladicescu (2005). However, during this investigation we found that classical time series analysis methods when combined with bootstrap resampling provide valuable information even if the size of the sample is not large, when used for Software Risk Management (SRM).

When speak about software metrics for risk management, some metrics can be considered as critical, called SRM-critically, and will be analysed with time series methodologies. Other metrics will be analysed by graph methods, like in Risk. It methodology (Kontio & Basili 1996, Kontio 1997).

The SRM-critically metrics are: a) the difference between actual expenses and the initially declared project cost; b) the difference between actual expenses and the predicted values obtained using the COCOMO approach; c) the ratio between real project progress and the planned project progress (explained by the Gantt chart, see Figure 1); Faults received per week (critically per month), and the successful debug actions per week.

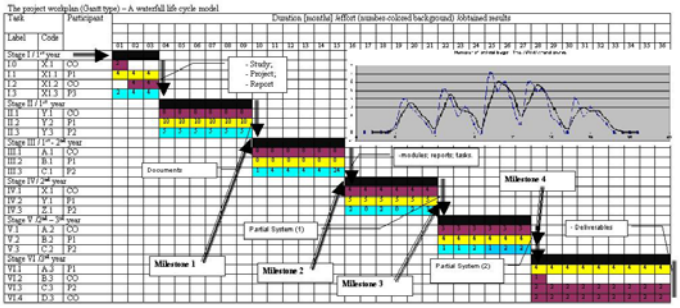

Figure 1. Waterfall model and the time series of critical bugs per month

Other metrics like internal complexity, code readability, or the portability are not as critical metrics, if these are not stated by requirements agreement.

For software project developed based on waterfall model having modular structures, but every module, except the first one, is dependent at least on previous model we experience a seasonal time series of critical bugs.

Applying the SBB approach we obtain the trend curve shown in Figure 2. This analyse was done before any moment of time indicated by Milestone (1 to 4, for the project under discussion).

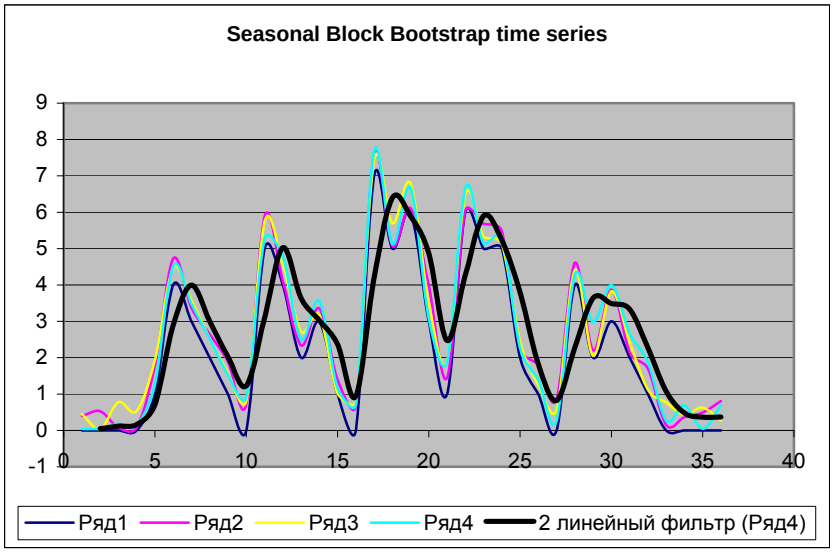

Figure 2. Seasonal Block Bootstrap (the last three time series, and the trend curve obtained by ARMA model for the last generated series).

Using this type of analysis important information was obtained not only for the staff involved in preventive risk management, but also for project manager, having opportunity to improve the structure of working teams (three main partners) and for rescheduling the financial resources before any milestone point.

3.2. Bootstrapping time series for inflation forecasting

Time series analysis started to be widely used in economics and finance since the discovering of the fact that "univariate ARIMA models often have far better forecasting and explanatory power than extremely complicated multivariate macroeconomic models" as Golub & Tilman (2000) mentioned. Also, these models proved a good behaviour in software reliability prediction (Popentiu-Vladicescu 2001).

The Bootstrap proved to be an important approach for analysing interest rates in financial risk management, as shown by Dette & Weissbach (2006).

In our study, the Consumer Price Index (CPI), which measures inflation, was studied by bootstrapping corresponding time series in order to forecast the rate of inflation. The standard bootstrap approach was used for the series *rate of inflation* (computed based on CPI) from 1992 to 2007. The initial time series is shown in Figure 3. The last five bootstrap time series from a set of 200 Monte Carlo trials and the trend curve modelled according to ARMA(0,2) is shown in Figure 4.

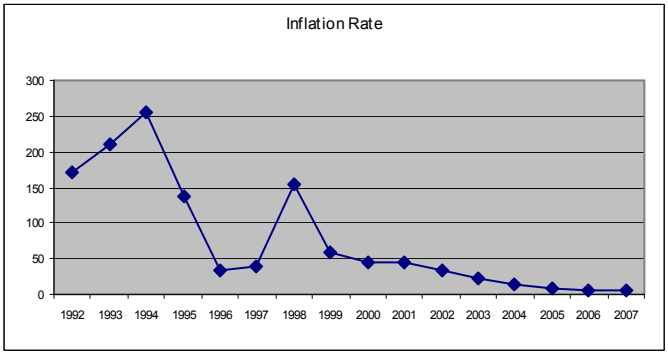

Figure 3. Inflation rate 1992-2007 (according to the Romanian Institute of Statistics: https://statistici.insse.ro/ipc/?lang=en)

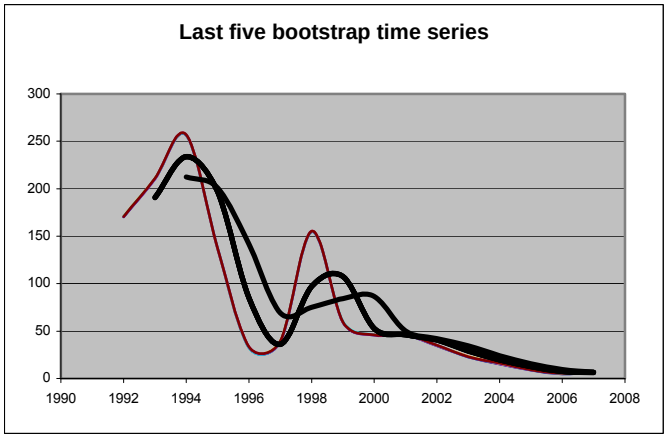

Figure 4. The trend curves estimated by the initial time series and the bootstrap time series of inflation rate

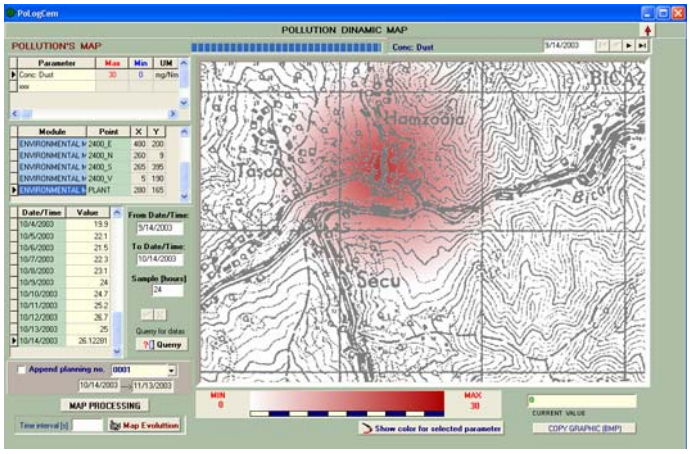

Figure 5. Pollution monitoring for increasing the health state in a cement plant region

When considered the CPI databases containing records at month level, some seasonal behaviour was identified. At global level, a moving average model was more suited.

The analyses checked the models parameters using the AICC formula given by (2). Other time series were used to investigate the bootstrap behaviour in order to provide confidence bands for dynamic financial analysis as in (Albeanu et al. 2007).

Other considerations on the accuracy of time series, interest rate and Survey forecast of inflation can be found on (Hafer & Hein 1984).

3.3. Bootstrapping pollution time series

Analysis of air pollution is important not only for meteorological point of view, but mainly for health (Gouveia & Fletcher 2000). This is the main reason that industrial pollution has to be monitored to keep the level of pollution in some limits according to the international regulations.

Environmental regulations for cement plants are becoming tougher and tougher, and cement manufacturers have to constantly review their anti-pollution measures. As presented in (Madsen et al. 2004) the best way to fight against pollution is to use computer-aided decision software being able to capture not only measurements for analysis, but also intelligent behaviour to provide information about the optimal configuration of the cement plant modules in order to keep some level of production under pollution regulations' constraints, which is similar to the Columbus approach of Solojentsev (2005).

For the time series analysed, using classical methods (Figure 5), we use, now, the bootstrap methodology to obtain information about accuracy estimation (Figure 6).

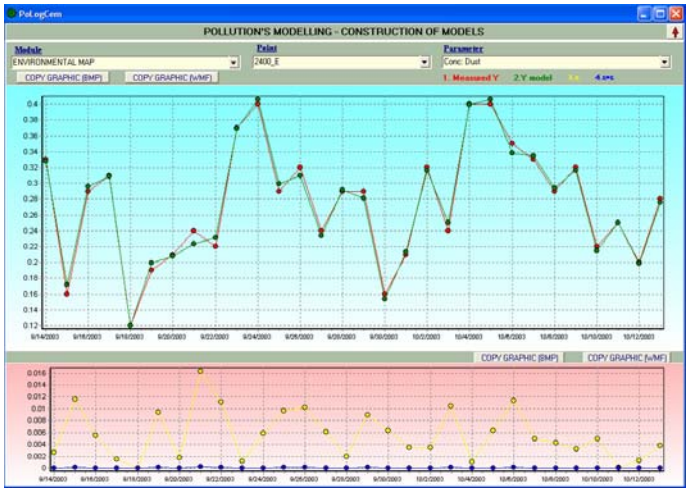

Figure 6. Bootstrapping time-series of air pollution by dust at 2400 m

We found that using AICC method is better than use the final prediction error approach as used in the initial software implementation.

4. CONCLUSIONS

Starting from idea that time-series represents an important approach in the prediction of the behaviour of some processes considered under risk management, this paper shows that bootstrap methodology is useful enough, but the researcher/manager has to choose the appropriate type of resampling.

The paper emphasizes on the utility of bootstrap resampling for different fields of practice considering three particular applications: software risk management, financial risk management and environment risk management.

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DISCRETE TIME MODELS OF FORWARD CONTRACTS INSURANCE

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Introduction

In this paper financial management model of forward contracts insurance suggested in [1,2] is considered by means of risk theory and heavy tailed technique. This model is based on a compensation principle. It attracted large interest and called active discussion among economists. So its mathematical analysis is initiated as economists so mathematicians.

Suppose that there are two insurance companies insuring both participants of some forward contract and working in discrete time with the net payouts $\xi+\eta,\ \xi-\eta$ during one step. Here ξ with $E\xi=a<0$ is common random summand for net payouts of these two companies. Then η with $E\eta=0$ is individual random claim of the first participant of the contract and $-\eta$ is individual claim of the second participant. The claim η (the claim $-\eta$) may be considered as a premium for $\eta<0$ (for $\eta>0$). Suppose that distribution functions (d.f.`s) $P(\xi\le x)=H(x)$, $P(\eta\le x)=S(x)$, $P(-\eta\le x)=L(x)$ and

$$\overline{H}(x)=o\left(\overline{S}(x)\right),\ \overline{H}(x)=o\left(\overline{L}(x)\right),\ x\to\infty \qquad (1)$$

with $\overline{F}(x)=1-F(x)$. Assume that ξ,η are independent random variables (r.v.`s) and $\xi$, $\eta$, $-\eta$ are subexponential r.v.`s.

Denote the one-step ruin probabilities of the companies with the initial capital x insuring the first and the second participants of the contract by

$$a_1(x)=P(\xi+\eta>x),\ a_2(x)=P(\xi-\eta>x)$$

and the one-step ruin probabilities of both companies and one of them by

$$a(x)=P\left[(\xi+\eta>x)\bigcap(\xi-\eta>x)\right].$$

Here $a_1(x),a_2(x)$ characterize individual risks of the insurance companies and $a(x)$ characterizes their group risk. Introduce $c(x)=P(\xi+\eta+\xi-\eta>x+x)=\overline{H}(x)$ one step ruin probability of these two companies aggregation. Here $c(x)$ characterizes as individual so group risks. The aggregation of these two companies allows to decrease individual risks $a_1(x),a_2(x)$ to small $\overline{H}(x)$ and to conserve the group risk $a(x)$ at small level $\overline{H}(x)$:

$$a_1(x)\sim\overline{S}(x),\ a_2(x)\sim\overline{L}(x),\ a(x)\sim c(x),\ \ x\to\infty. \qquad (2)$$

Main purpose of this paper is to obtain asymptotical comparisons analogous to (2) for individual and group risks in separate and aggregated insurance models. We speak about infinite horizon discrete time risk models without interest force with constant interest force and with stochastic interest force.

1. Preliminaries

Classes of distributions. Throughout, for a given r.v. X concentrated on $(-\infty,\infty)$ with a d.f. F then its right tail $\overline{F}(x)=P(X>x)$. For two d.f.'s F_1 and F_2 concentrated on $(-\infty,\infty)$ we write by $F_1*F_2(x)$ the convolution of F_1 and F_2 and write by $F_1^{*2}=F_1*F_1$

the convolution of F_1 with itself. All limiting relationships, unless otherwise stated, are for $x\to\infty$. Let $a(x)\ge 0$ and $b(x)>0$ be two infinitesimals, satisfying

$$l^-\le \liminf_{x\to\infty}\frac{a(x)}{b(x)}\le \limsup_{x\to\infty}\frac{a(x)}{b(x)}\le l^+.$$

We write $a(x)=O(b(x))$ if $l^+<\infty$, $a(x)=o(b(x))$ if $l^+=0$, and $a(x)\lesssim b(x)$ if $l^+=1$, $a(x)\gtrsim b(x)$ if $l^-=1$, and $a(x)\sim b(x)$ if both.

Introduce the following classes of d.f.`s concentrated on $[0,\infty)$:

$$\mathcal{S}=\left\{F(x):\ \lim_{x\to\infty}\frac{\overline{F^{*2}}(x)}{\overline{F}(x)}=2\right\},\quad \mathcal{L}=\left\{F(x):\forall t\ \lim_{x\to\infty}\frac{\overline{F}(x-t)}{\overline{F}(x)}=1\right\},$$

$$\mathcal{R}_{-\alpha}=\left\{F(x):\forall\theta>0\ \lim_{x\to\infty}\frac{\overline{F}(\theta x)}{\overline{F}(x)}=\theta^{-\alpha}\right\},\quad 0<\alpha<\infty,\quad \mathcal{R}=\bigcup_{0<\alpha<\infty}\mathcal{R}_{-\alpha},$$

$$\mathcal{R}_{-\infty}=\left\{F(x):\forall\theta>1\ \lim_{x\to\infty}\frac{\overline{F}(\theta x)}{\overline{F}(x)}=0\right\},$$

$$\mathcal{S}_*=\left\{F(x):\ \int_0^x\overline{F}(x-y)\overline{F}(y)dy\sim 2m_+\overline{F}(x),\ x\to\infty,\ m_+=\int_0^\infty\overline{F}(x)dx\right\}.$$

$\mathcal{S}$ is called the class of subexponential d.f.`s. $\mathcal{L}$ is called the class of long tailed d.f.`s. $\mathcal{R}$ (or $\mathcal{R}_{-\alpha}$) is called the class of regular varying d.f.`s (with index $\alpha$). $\mathcal{R}_{-\infty}$ is called the class of rapidly varying tailed d.f.`s.

Proposition 1. *The classes* $\mathcal{R}$, $\mathcal{S}$, $\mathcal{L}$ *satisfy the formula [3]* $\mathcal{R}\subset\mathcal{S}\subset\mathcal{L}$.

Proposition 2. *If* $F\in\mathcal{S}_*$ *then [4]* F, $1-\frac{1}{m_+}\overline{F_I}(x)\in\mathcal{S}$ *with* $\overline{F_I}(x)=\int_x^\infty\overline{F}(y)dy$.

More generally, d.f. F concentrated on $(-\infty,\infty)$ is also said to belong to these classes if its right-hand distribution $\tilde{F}(x)=F(x)\mathbb{I}(x>0)$ does.

Proposition 3. *Let* F_1 *and* F_2 *be two* d.f.`s concentrated on $(-\infty,\infty)$. *If* $F_1\in\mathcal{S}$, $F_2\in\mathcal{L}$ *and* $\overline{F_2}(x)=O(\overline{F_1}(x))$, *then [5,Lemma 3.2]* $F_1*F_2\in\mathcal{S}$ *and* $\overline{F_1*F_2}(x)\sim\overline{F_1}(x)+\overline{F_2}(x)$.

Proposition 4. *Suppose that* X,Y *are independent random variables with the d.f.`s* F_1,F_2 *concentrated on* $(-\infty,\infty)$ *and* $F_1\in\mathcal{L}, F_2(\infty=1)$ *then [6]* $P(X-Y>t)\sim\overline{F_1}(t)$.

Discrete time risk model under stochastic interest force. Consider a risk model with discrete time $n=1,2,...$ and denote X_n the insurer`s net loss - the total claim amount minus the total incoming premium within period n and Y_n the discount factor from time n to time $n-1$. Here X_n

is called insurance risk and Y_n is called financial risk. These random variables are independent with d.f.`s $F(t)$, $G(t)$ relatively.

Let $\{X_n, n=1,2,...\}$ be a sequence of independent and identically distributed (i.i.d.) r.v`s with generic random variable $X$, let $\{Y_n, n=1,2,...\}$ be another sequence of i.i.d. positive r.v.`s with generic random variable Y, and let the two sequences be mutually independent. Denote

$$\Psi(x)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}X_k\prod_{j=1}^{k}Y_j>x\right).$$

Then $\Psi(x)$ is infinite-time ruin probability of the risk model under stochastic interest force with initial capital x.

Proposition 5. *Suppose that* $F\in S_*$ *and* $EX=m<0$ *and* $P(Y=1)=1$ *then [7]*

$$\Psi(x)\sim\frac{1}{|m|}\overline{F_I}(x).$$

Proposition 6. *If* $F\in\mathcal{R}_{-\alpha}$, $0<\alpha<\infty$, *and* $E\max\left\{Y^{\alpha-\delta}, Y^{\alpha+\delta}\right\}<1$ *for some* $0<\delta<\alpha$ *then [8]*

$$\Psi(x)\sim\frac{EY^{\alpha}}{1-EY^{\alpha}}\overline{F}(x).$$

Proposition 7. *Suppose that* $P(Y=1+r)=1$ *for some* $0<r<\infty$. *If* $F\in\mathcal{R}_{-\alpha}$ *for some* $0<\alpha<\infty$ *then [9]*

$$\Psi(x)\sim\frac{\overline{F}(x)}{(1+r)^{\alpha}-1}.$$

If $F\in S\bigcap\mathcal{R}_{-\infty}$ *then* $\Psi(x)\sim\overline{F}((1+r)x)$.

2. Asymptotic comparison of individual and group risks for forward contracts insurance

Suppose that at the step k the net payouts of both participants of the step k forward contract are $\xi_k+\eta_k$, $\xi_k-\eta_k$. Here random sequences $\{\xi=\xi_0,\xi_1,...\}$ and $\{\eta=\eta_0,\eta_1,...\}$ are independent. Each of these two random sequences consists of i.i.d.r.v`s with their own common d.f.`s $P(\xi\le t)=H(t)$, $P(\eta\le t)=S(t)$ correspondingly and $H(t)$, $S(t)$, $L(t)\in S$ with $L(t)=P(-\eta\le x)$.

Define individual risks of the companies with initial capitals x insuring net payouts $\{\xi_1+\eta_1,\xi_2+\eta_2,...\}$, $\{\xi_1-\eta_1,\xi_2-\eta_2,...\}$ separately by

$$A_1(x)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k+\eta_k)\prod_{j=1}^{k}Y_j>x\right),\ A_2(x)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k-\eta_k)\prod_{j=1}^{k}Y_j>x\right).$$

Analogously define group risk of separately working insurance companies by

$$A(x)=P\left[\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k+\eta_k)\prod_{j=1}^{k}Y_j>x\right)\bigcap\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k-\eta_k)\prod_{j=1}^{k}Y_j>x\right)\right]$$

and common individual and group risk of aggregated company by

$$C(x)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k+\eta_k+\xi_k-\eta_k)\prod_{j=1}^{k}Y_j>2x\right)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}\xi_k\prod_{j=1}^{k}Y_j>x\right).$$

Lemma 1. *Suppose that the condition (1) is true then*

$$P(\xi+\eta>x)\sim\overline{S}(x),\ P(\xi-\eta>x)\sim\overline{L}(x).$$

Proof. This statement arises from the proposition 3.

Lemma 2. *The formula* $P(\xi-|\eta|>x)\sim\overline{H}(x)$ *is true.*

Proof. This statement arises from the propositions 1, 4.

Lemma 3. *The following inequality takes place*

$$A(x)\ge R(x),\ R(x)=P\left(\sup_{1\le i<\infty}\sum_{k=1}^{i}(\xi_k-|\eta_k|)\prod_{j=1}^{k}Y_j>x\right).$$

Proof. This statement arises from the inequalities $a\ge-|a|$, $-a\ge-|a|$, which are true for all real a.

Theorem 1. *Suppose that the condition (1) is true and* $H(t)$, $S(t)$, $L(t)\in S_*$, $P(Y=1)=1$ then

$$A_1(x)\sim\frac{\overline{S_I}(x)}{|a|},\ A_2(x)\sim\frac{\overline{L_I}(x)}{|a|},\ C(x)\sim\frac{\overline{H_I}(x)}{|a|},\ R(x)\sim\frac{\overline{H_I}(x)}{|a-E|\eta||}.$$

Proof. This statement arises from the propositions 5 and from the lemmas 1, 2.

Theorem 2. *Suppose that the condition (1) is true and for some* $\alpha_1,\alpha_2,\alpha_3$, $0<\alpha_1,\alpha_2<\alpha_3$, *d.f.`s* $S(x)\in\mathcal{R}_{-\alpha_1}$, $L(x)\in\mathcal{R}_{-\alpha_2}$, $H(x)\in\mathcal{R}_{-\alpha_3}$, $E\max\left\{Y^{\alpha_i-\delta(\alpha_i)},Y^{\alpha_i+\delta(\alpha_i)}\right\}<1$ *for some* $0<\delta(\alpha_i)<\alpha_i$, $1\le i\le 3$, *then*

$$A_1(x)\sim\frac{EY^{\alpha_1}}{1-EY^{\alpha_1}}\overline{S}(x),\ A_2(x)\sim\frac{EY^{\alpha_2}}{1-EY^{\alpha_2}}\overline{L}(x),\ C(x)\sim R(x)\sim\frac{EY^{\alpha_3}}{1-EY^{\alpha_3}}\overline{H}(x).$$

Proof. This statement arises from the propositions 6 and from the lemmas 1, 2.

Theorem 3. *Suppose that the condition (1) is true and* $P(Y=1+r)=1$ *for some* $0<r<\infty$

If for some $\alpha_1,\alpha_2,\alpha_3$, $0<\alpha_1,\alpha_2<\alpha_3$, *d.f.`s* $S(x)\in\mathcal{R}_{-\alpha_1}$, $L(x)\in\mathcal{R}_{-\alpha_2}$, $H(x)\in\mathcal{R}_{-\alpha_3}$ *then*

$$A_1(x)\sim\frac{\overline{S}(x)}{(1+r)^{\alpha_1}-1},\ A_2(x)\sim\frac{\overline{L}(x)}{(1+r)^{\alpha_2}-1},\ C(x)\sim R(x)\sim\frac{\overline{H}(x)}{(1+r)^{\alpha_3}-1}.$$

If d.f.`s $S(x)$, $L(x)$, $H(x)\in S\cap\mathcal{R}_{-\infty}$ *then*

$$A_1(x)\sim\overline{S}((1+r)x),\ A_2(x)\sim\overline{L}((1+r)x),\ C(x)\sim R(x)\sim\overline{H}((1+r)x).$$

Proof. This statement arises from the propositions 7 and from the lemmas 1, 2.

Theorem 4. *If the conditions of the theorem 1 (the theorem 2 or the theorem 3) are true then for some* $0<k<\infty$

$$A(x)\gtrsim kC(x),\ C(x)=o(A_1(x)),\ C(x)=o(A_2(x)).$$

Proof. This statement arises from the lemma 3 and from the theorems 1-3.

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ANALYSIS OF PORTS RELIABILITIES

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Introduction

This paper is devoted to algorithms of a calculation of ports reliabilities. A port is a no oriented graph with fixed initial and final nodes. As accuracy so asymptotic formulas are considered. Suggested algorithms have minimal numbers of arithmetical operations.

Using the article [1] results, in this paper algorithms of a calculation of asymptotic constants for ports reliabilities are constructed. These algorithms allow to estimate an influence of some arc reliability on a port reliability and to obtain an invariance condition when this influence is absent.

In solid state physics, surface physics and in nanotechnologies recursively defined ports are of large interest. An example of such a structure is in the monograph [2, fig. 7.7]: where each arc of a bridge scheme Γ is replaced by Γ. In this paper linear upper bounds of arithmetical operations numbers necessary to calculate as a reliability so its asymptotic constants are obtained. For a comparison it is worthy to say that a number of arithmetical operations necessary to calculate a reliability of a port increases as a geometrical progression of port arcs number.

1. Preliminaries

Consider a port Γ with a final number U of nodes, a set $W=\{w=(u,v),u,v\in U\}$ of arcs and fixed initial u_0 and final u^0 nodes. Denote $\mathcal{R}$ a set of all ways R in the port Γ, which connect the nodes u_0 and u^0. Suppose that $\mathcal{R}\neq\varnothing$. Consider the sets

$$\mathcal{A}=\{A\subset U, u_0\in A, u^0\notin A\},\ L=L(A)=\{(u,u'): u\in A, u'\notin A\}$$

and $\mathcal{L}=\{L(A), A\in\mathcal{A}\}$ - the set of all sections in Γ. Correspond for each arc $w\in W$ a logic variable $\alpha(w)=I$ (the arc w works), where $I(B)$ is an indicator function of an event B. Denote a quantity which characterizes a connectivity between the nodes u_0, u^0 in Γ by

$$\beta=\bigvee_{R\in\mathcal{R}}\bigwedge_{w\in R}\alpha(w). \qquad (1)$$

Suppose that $\alpha(w), w\in W$, are independent random variables,

$P(\alpha(w)=1)=p_w(h)$, $q_w(h)=1-p_w(h)$,

where h is some small parameter : $h\to 0$. In [1] the following statements are proved.

Theorem 1. *Suppose that* $p_w(h)\sim\exp\left(-h^{-d(w)}\right)$, $h\to 0$, *where* $d(w)>0, w\in W$. *Then* $-\ln P(\beta=1)\sim h^{-D}$ *and* $D=D(\Gamma)=\min\limits_{R\in\mathcal{R}}\max\limits_{w\in\mathcal{R}} d(w)$.

Theorem 2. *Suppose that* $q_w(h)\sim\exp\left(-h^{-d_1(w)}\right)$, $h\to 0$, *where* $d_1(w)>0, w\in W$. *Then* $-\ln P(\beta=0)\sim h^{-D_1}$ *and* $D_1=D_1(\Gamma)=\max\limits_{L\in\mathcal{L}}\min\limits_{w\in\mathcal{L}} d_1(w)$.

Theorem 3. *Suppose that* $p_w(h) \sim h^{g(w)}, h \to 0$, *where* $g(w) > 0, w \in W$. *Then* $\ln P(\beta = 1) \sim G \ln h$ *and* $G = G(\Gamma) = \min\limits_{R \in \mathcal{R}} \sum\limits_{w \in \mathcal{R}} g(w)$.

Theorem 4. *Suppose that* $q_w(h) \sim h^{g_1(w)}, h \to 0$, *where* $g_1(w) > 0, w \in W$, *then* $\ln P(\beta = 0) \sim G_1 \ln h$ *and* $G_1 = G_1(\Gamma) = \min\limits_{L \in \mathcal{L}} \sum\limits_{w \in \mathcal{L}} g_1(w)$.

The constants G, G_1 [3] may be interpreted as a length of a shortest way or a minimal ability to handle of cross-sections in the port Γ correspondingly. In a definition of the constants D, D_1 a summation is replaced by a maximization. So the constants D, D_1 may be interpreted as a pseudo-length of the shortest way or a minimal pseudo-ability to handle in the port Γ.

Remark 1. Suppose that $\tau(w)$ are independent random variables which characterize life times of the arcs $w \in W$. Denote $P(\tau(w) > t) = p_w(h)$ and put the graph Γ life time equal to $\tau(\Gamma) = \min\limits_{R \in \mathcal{R}} \max\limits_{w \in R} \tau(w)$. If $h = h(t)$ is monotonically decreasing and continuous function and $h \to 0$, $t \to \infty$, then the theorems 1, 3 remain true if $P(\beta = 1)$ is replaced by $P(\tau(\Gamma) > t)$. If $h = h(t)$ is monotonically increasing and continuous function and $h \to 0$, $t \to 0$, then the theorems 2, 4 remain true if $P(\beta = 0)$ is replaced by $P(\tau(\Gamma) \le t)$. So it is possible to consider widely used in the reliability theory the exponential and the Weibull distributions of arcs life times.

Denote $\overline{\Gamma}$ a port with the nodes set $U = \{u_0, u_1, u_2, u_3\}$ and with the arcs set (fig.1)

$$W = \{w_1 = (u_0, u_1), w_2 = (u_0, u_2), w_3 = (u_1, u_3), w_4 = (u_2, u_3), w_5 = (u_1, u_2)\},.$$

The node u_0 is initial and the node u_3 is final. The scheme $\overline{\Gamma}$ [4] is called the bridge scheme and the arc w_5 – the bridge element in this scheme

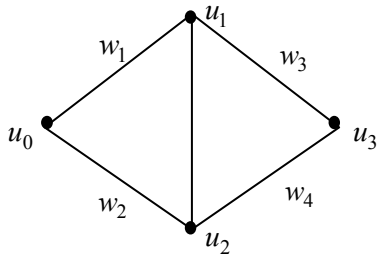

Fig. 1. Bridge scheme $\overline{\Gamma}$.

The scheme $\overline{\Gamma}$ reliability $P = P(p_1, ..., p_5)$ in a suggestion that the arcs $w_1, ..., w_5$ work independently with the probabilities $p_1, ..., p_5$ is calculated by the formula

$$P = p_5\left[1-(1-p_1)(1-p_2)\right]\left[1-(1-p_3)(1-p_4)\right] + (1-p_5)\left[1-(1-p_1p_3)(1-p_2p_4)\right] \quad (2)$$

To make these calculations it is necessary $n(\overline{\Gamma}) = 14$ arithmetical operations.

2. Element wise analysis

Remark that in an accordance with the formula (1) the logical function $\beta = \beta(\alpha(w), w \in W)$ has all properties of the monotone structure [2, гл.7]:

a) $\beta(\alpha(w) = 1, w \in W) = 1$, b) $\beta(\alpha(w) = 0, w \in W) = 0$,

c) $\beta(\alpha_1(w), w \in W) \le \beta(\alpha_2(w), w \in W)$, if $\alpha_1(w) \le \alpha_2(w), w \in W$.

Fix an arc $v \in W$ and using the complete probability formula [2, §7.4] obtain the following formulas:

$$P(\beta = 1) = P(\alpha(v) = 1) F_v^1 + P(\alpha(v) = 0) F_v^0, \tag{3}$$

$$F_v^\delta = P\left(\beta\left(\alpha(w), w \in W, w \notin v; \alpha(v) = \delta\right) = 1\right), \delta = 0,1,$$

and

$$F_v^0 \le F_v^1. \tag{4}$$

Define the graph Γ_v^0 by an exclusion of the arc $v = (u, u')$ from the graph Γ and the graph Γ_v^1 by a gluing of the nodes u, u' in the graph Γ_v^0. Using the previous section results and the formulas (3), (4) obtain the following statements.

Theorem 5. *Suppose that* $p_w(h) \sim \exp\left(-h^{-d(w)}\right)$, $h \to 0$, *where* $d(w) > 0, w \in W$. *Then* $-\ln P(\beta = 1) \sim h^{-D}$ *where* $D = \min\left[\max\left(d(v), D\left(\Gamma_v^1\right)\right), D\left(\Gamma_v^0\right)\right]$, $D\left(\Gamma_v^1\right) \le D\left(\Gamma_v^0\right)$.

Theorem 6. *Suppose that* $q_w(h) \sim \exp\left(-h^{-d_1(w)}\right)$, $h \to 0$, *where* $d_1(w) > 0, w \in W$. *Then* $-\ln P(\beta = 0) \sim h^{-D_1}$ *where* $D = \min\left[\max\left(d_1(v), D_1\left(\Gamma_v^1\right)\right), D_1\left(\Gamma_v^0\right)\right]$, $D_1\left(\Gamma_v^1\right) \le D_1\left(\Gamma_v^0\right)$.

Theorem 7. *If* $p_w(h) \sim h^{g(w)}$, $h \to 0$, *where* $g(w) > 0, w \in W$. *Then* $\ln P(\beta = 1) \sim G \ln h$ *and* $G = \min\left[g(v) + G\left(\Gamma_v^1\right), G\left(\Gamma_v^0\right)\right]$, $G\left(\Gamma_v^1\right) \le G\left(\Gamma_v^0\right)$.

Theorem 8. *If* $q_w(h) \sim h^{g_1(w)}$, $h \to 0$, *where* $g_1(w) > 0, w \in W$. *Then* $\ln P(\beta = 0) \sim G_1 \ln h$ *and* $G_1 = \min\left[g_1(v) + G_1\left(\Gamma_v^1\right), G_1\left(\Gamma_v^0\right)\right]$, $G_1\left(\Gamma_v^1\right) \le G_1\left(\Gamma_v^0\right)$.

Remark 2. *The constants* D, D_1, G, G_1 *do not depend on* $d(v), d_1(v), g(v), g_1(v)$ *correspondingly if and only if* $D\left(\Gamma_v^1\right) = D\left(\Gamma_v^0\right), D_1\left(\Gamma_v^1\right) = D_1\left(\Gamma_v^0\right), G\left(\Gamma_v^1\right) = G\left(\Gamma_v^0\right), G_1\left(\Gamma_v^1\right) = G_1\left(\Gamma_v^0\right)$. The fig. 2, 3 show how the parameters $d(v), g(v)$ influence on the constants $D(\Gamma), G(\Gamma)$.

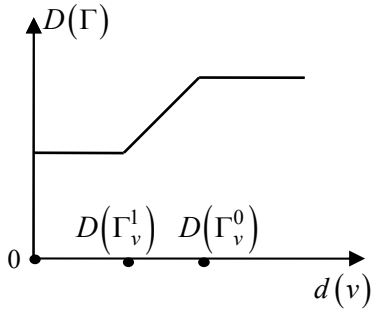

Fig. 2.

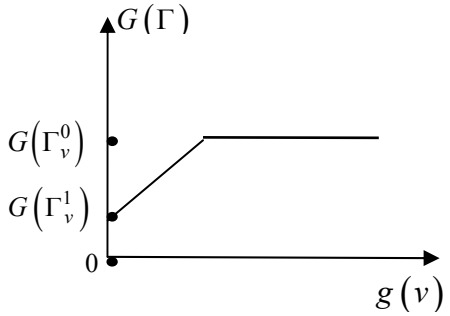

Fig. 3.

Example. Consider the port $\overline{\Gamma}$ (fig. 1) with independently working arcs $w_1, \ldots, w_5$ and show how the element w_5 reliability influences on the port reliability on an example of the constants $D\left(\overline{\Gamma}\right), G\left(\overline{\Gamma}\right)$ from the theorems 5, 7. Define the port $\overline{\Gamma}_{w_5}^0$ by an exclusion of the arc w_5 from the graph $\overline{\Gamma}$ and the port $\overline{\Gamma}_{w_5}^1$ by a gluing of the nodes u_1, u_2 in the graph $\overline{\Gamma}_{w_5}^0$.

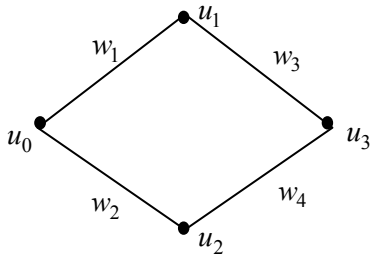

Fig. 4. Port $\overline{\Gamma}^0_{w_5}$

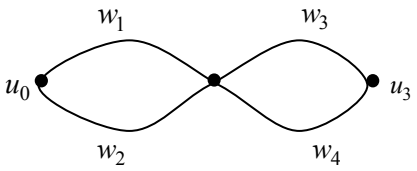

Fig.5. Port $\overline{\Gamma}^1_{w_5}$

If $p_{w_i}(h) \sim \exp\left(-h^{-d_i}\right), h \to 0$, with $d_i = d(w_i) > 0$, then it is easy to obtain the formulas

$$D\left(\overline{\Gamma}^0_{w_5}\right) = \min\left(\max\left(d_1, d_3\right), \max\left(d_2, d_4\right)\right), \; D\left(\overline{\Gamma}^1_{w_5}\right) = \max\left(\min\left(d_1, d_2\right), \min\left(d_3, d_4\right)\right),$$

$$D\left(\overline{\Gamma}\right) = \min\left[\max\left(d_5, D\left(\Gamma^1_{w_5}\right)\right), D\left(\Gamma^0_{w_5}\right)\right].$$

Here the equality $D\left(\overline{\Gamma}^0_{w_5}\right) = D\left(\overline{\Gamma}^1_{w_5}\right)$ is true in one of the following eight conditions:

$$1)\ d_3 \ge d_1 > d_2,\ 2)\ d_3 \ge d_1 = d_2,\ 3)\ d_4 \ge d_1 = d_2,\ 4)\ d_4 \ge d_2 > d_1,$$
$$5)\ d_1 \ge d_3 > d_4,\ 6)\ d_1 \ge d_3 = d_4,\ 7)\ d_2 \ge d_3 = d_4,\ 8)\ d_2 \ge d_4 > d_3.$$

If $p_{w_i}(h) \sim h^{g_i}, h \to 0$, with $g_i = g(w_i) > 0, i = 1,...,5$, then it is easy to obtain the formulas

$$G\left(\Gamma^0_{w_5}\right) = \min\left(\left(g_1 + g_3\right), \left(g_2 + g_4\right)\right), \; G\left(\Gamma^1_{w_5}\right) = \min\left(g_1, g_2\right) + \min\left(g_3, g_4\right),$$

$$G\left(\overline{\Gamma}\right) = \min\left[g_5 + G\left(\Gamma^1_{w_5}\right), G\left(\Gamma^0_{w_5}\right)\right].$$

Here the equality $G\left(\Gamma^0_{w_5}\right) = G\left(\Gamma^1_{w_5}\right)$ is true in one of the following two conditions:

$$1)\ g_4 \ge g_3, g_2 \ge g_1,\ 2)\ g_3 \ge g_4, g_1 \ge g_2.$$

Remark 3. If the graph $\overline{\Gamma'}$ is constructed by an addition of the arc $w_6 = (u_0, u_3)$ to the port $\overline{\Gamma}$ then $D(\overline{\Gamma'}) = \min(d_6, D(\overline{\Gamma})), G(\overline{\Gamma'}) = \min(g_6, G(\overline{\Gamma}))$ where d_6, g_6 are appropriate parameters of the arc w_6. As the graph $\overline{\Gamma'}$ is complete (each two its nodes is connected by some arc) so these formulas may be spread to a case when we take interest to a connectivity of each two nodes of the graph $\overline{\Gamma'}$ (this scheme is an analog of a transformer electrical scheme). For this purpose it is necessary to renumber the graph $\overline{\Gamma'}$ nodes.

3. Ports superposition

Define recursively a class of bridge schemes $\mathcal{B}$:

1) the arcs $w_1, w_2, ...,$ working independently with the probabilities $p_1, p_2, ...,$ belong to $\mathcal{B}$,

2) if the ports $\Gamma_1, ..., \Gamma_5 \in \mathcal{B}$ consist of nonintersecting sets of arcs then their superposition $\Gamma' = \overline{\Gamma}(\Gamma_1, ..., \Gamma_5)$ belongs to $\mathcal{B}$.

A number of arcs in the superposition Γ' is $m(\Gamma') = m(\Gamma_1) + ... + m(\Gamma_5)$ where $m(\Gamma_i)$ is a number of arcs in the port Γ_i. The reliability of the superposition Γ' equals to $P(P_1, ..., P_5)$ and is calculated by the formula (2) and needs

$$n_p(\Gamma') = n_p\left(\overline{\Gamma}\right) + n_p(\Gamma_1) + ... + n_p(\Gamma_5)$$

arithmetical operations where $n(\Gamma_i)$ is a number of arithmetical operations necessary to calculate the

reliability P_i.

If $n_p(\Gamma_i) \le n_p(\overline{\Gamma})(m(\Gamma_i)-1), 1 \le i \le 5$, then

$$n_p(\Gamma') \le n_p(\overline{\Gamma})(m(\Gamma')-1). \quad (5)$$

So a number of arithmetical operations necessary to calculate the reliability of the port $\Gamma' \in \mathcal{B}$ has a bound which is linear increasing by a number of the port Γ' arcs.

For the superposition $\Gamma' = \Gamma(\Gamma_1,...\Gamma_5)$ of the ports $\Gamma_1,...\Gamma_5 \in \mathcal{B}$ it is easy to obtain the recurrent formulas

$$D(\Gamma') = \min_{R \in \mathcal{R}} \max_{i: w_i \in R} D(\Gamma_i), D_1(\Gamma') = \max_{L \in \mathcal{L}} \min_{i: w_i \in L} D_1(\Gamma_i) \quad (6)$$

$$G(\Gamma') = \min_{R \in \mathcal{R}} \sum_{i: w_i \in R} G(\Gamma_i), G_1(\Gamma') = \max_{L \in \mathcal{L}} \sum_{i: w_i \in L} G_1(\Gamma_i) \quad (7)$$

Here $\mathcal{R}$, $\mathcal{L}$ are the sets of ways and cross sections in the graph $\overline{\Gamma}$. The constants

$$D(\Gamma_i), D_1(\Gamma_i), G(\Gamma_i), G_1(\Gamma_i), i = 1,...,5,$$

are calculated by the theorems 1-4 formulas. The formulas (6), (7) allow analogously to (5) to construct linear by $m(\Gamma')$ upper bounds for numbers of arithmetical operations $n_D(\Gamma'), n_{D_1}(\Gamma'), n_G(\Gamma'), n_{G_1}(\Gamma')$ which are necessary to calculate the constants $D(\Gamma'), D_1(\Gamma'), G(\Gamma'), G_1(\Gamma')$:

$$n_D(\Gamma') \le n_D(\overline{\Gamma})(m(\Gamma')-1),\ n_{D_1}(\Gamma') \le n_{D_1}(\overline{\Gamma})(m(\Gamma')-1),$$

$$n_G(\Gamma') \le n_G(\overline{\Gamma})(m(\Gamma')-1),\ n_{G_1}(\Gamma') \le n_{G_1}(\overline{\Gamma})(m(\Gamma')-1).$$

For a comparison remark that a number of arithmetical operations necessary to define the shortest way length or the minimal cross sections ability to handle in general type graphs [5] is significantly larger.

Remark 4. The constructed algorithm of a recursive definition of a port reliability for the class $\mathcal{B}$ with the generating scheme $\overline{\Gamma}$ and the upper bound (5) may be spread to a case of a finite set $\mathcal{G} = \{\Gamma\}$ of generating schemes with a replacement $n(\overline{\Gamma})$ in the formula (5) by $\max_{\Gamma \in \mathcal{G}} n(\Gamma)$. For example it is possible to construct $\mathcal{G}$ by the graphs with two arcs which are connected parallel and sequentially.

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OBJECT ORIENTED COMMONALITIES IN UNIVERSAL GENERATING FUNCTION FOR RELIABILITY AND IN C++.

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Abstract. The main idea of Universal Generating Function is exposed in reliability applications. Some commonalities in this approach and the C++ language are discussed.

Keywords: *Universal Generating function (UGF), C++, reliability*.

Introduction.

Usually, binary systems are considered in the reliability theory. However, this approach does not describe systems with several levels of performance sufficiently. Analysis of multi-state systems forms now a special branch of the reliability theory.

For analysis of such systems consisting of multi-state subsystems/elements, one can use the method of Universal Generating Functions (UGF), which is described below.

1. Generating function.

One frequently uses an effective tool in probabilistic combinatorial analysis: the method of generating functions. For a distribution function of a discrete random variable ξ such that $\Pr\{\xi = k\} = p_k$ for any natural k, the generating function has the form

$$\varphi(x) = \sum_k p_k x^k$$

Advantages of using a generating function are well established in this field, and we list a few of those:

(1) For many discrete distributions (e.g., binomial, geometrical, Poisson), there are compact forms of generating functions, which allows one to get analytical solutions quickly and easily.

(2) Moments of statistical distributions can be written in convenient forms. For example, the mathematical expectation of random variable ξ can be found as

$$E\{\xi\} = \frac{\partial}{\partial x}\varphi(x)\bigg|_{x=1}.$$

(3) If there are n independent random variables ξ_1, ξ_2, ..., ξ_n with the respective generating functions $\varphi_1(x)$, $\varphi_2(x),...,\varphi_n(x)$, then the following generation function can be written for the convolution of these distributions:

$$\varphi(x) = \prod_{j=1}^{n} \varphi_j(x).$$

where $\varphi_j(x) = \sum_k p_{jk} x^k$, and p_{jk} is the probability that j-th random variable takes value k.

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2. Computer alorithm for calculation product of GF's.

Let us present a generating function as a set of *objects*. Each object corresponds to a term in the generating function polynomial. It means that object is a pair of two values: the first is the coefficient, i.e. probability, *p*, and the second is the power of the argument, *a*, i.e. the corresponding random variable.

Consider a computational algorithm for calculation of the convolution of two distributions. One makes the following formal operations.

♦ Take two sets of objects: set $\{(p_{11}, a_{11}), (p_{12}, a_{12}),...,$ $(p_{1k}, a_{1k})\}$ for generating function $\varphi_1(x)$, and set $\{(p_{21}, a_{21}), (p_{22}, a_{22}),..., (p_{2m}, a_{2m})\}$ for generating function $\varphi_2(x)$.

♦ Find all cross "interactions" of objects of the first set with all objects of the second set, using the following rule:

[Interacting objects: (p_{1k}, a_{1k}) and (p_{2m}, a_{2m})] →
[Resulting object: $(p_{1k} p_{2m}; a_{1k} + a_{2m})$].

♦ For all resulting objects with different a_{1k_1} for object-1 and a_{2m_2} for object-2, but such that $a_{1k_1} + a_{2m_2} = a$, one forms a new final resulting object: $(\sum p_{1k_1} p_{2m_2};\ a)$. The total set of such final resulting objects gives us the needed solution: from here we can get probabilities for any *a*.

3. Universal generating function.

We have described a formalized procedure on sets of objects interaction coresponding to product of polynomials. But in practice, we meet a number of situations when this operation is not enough. Consider the following simple examples.

Example 1. Assume that there is a series connection of two (statistically independent) capacitors (Fig. 1).

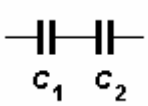

Fig. 1. Series connection of two capacitors.

Assume that c_1 and c_2 are random with discrete distributions: $p_{1k}=\Pr\{c_1=k\}$ and $p_{2j}=\Pr\{c_2=j\}$. One is interested in distribution of total capacity. It is impossible to find the solution with the help of a common generating function. However, there is a possibility to use formal algorithm, described above with the use of corresponding operations over the elements of the objects. The following procedure can be suggested:

♦ Take two sets of objects, S_1 and S_2:

$$S_1 = \{(p_{11}, c_{11}), (p_{12}, c_{12}),..., (p_{1k}, c_{1k})\}$$

and

$$S_2=\{(p_{21},c_{11}),(p_{22},c_{22}),...,(p_{2m},c_{2m})\},$$

where k is the number of discrete values of the first capacitor, and m is the same for the second one. Here the first element of the object is the probability and the second element is the respective capacity.

♦Find all cross "interactions", Ω, of objects of set S_1 with all objects of set S_2, using the following rule:

$$\Omega\{(p_{1i},c_{1i}),(p_{2j},c_{2j})\}=(p_{ij}^{*};c_{ij}^{*}).$$

Here p_{ij}^{*} is the resulting probability calculated in accordance with the multiplication rule (under assumption of independence) as

$$p_{ij}^{*}=\Omega_{(p)}\{p_{1i},p_{2j}\}=p_{1i}p_{2j},$$

where $\Omega_{(p)}$ is the rule of interaction of parameters p, which in this particular case is multiplication.

Value of c_{ij}^{*} is the resulting capacity calculated in accordance with the harmonic sum rule for capacities:

$$c_{ij}^{*}=\Omega_{(c)}\{c_{1i},c_{2j}\}=\left(c_{1i}^{-1}+c_{2j}^{-1}\right)^{-1},$$

where $\Omega_{(c)}$ is the rule of interaction of parameters c.

♦ Assume that in result we obtain all $R=km$ possible resulting objects of kind $(p^{*};c^{*})$. Let us order all these resulting pairs in increase of value of c^{*}: $(p_1^{*};c_1^{*})$, ..., $(p_R^{*};c_R^{*})$. For some resulting pairs with numbers, say, i, i+1,... , i+j values of c^{*} can be the same and equal some C. We converge such objects into a single aggregated object with parameters: $(\sum_{i\le s\le i+j}p_s^{*};C)$. The total set of such final resulting objects gives us the needed solution.

The procedure can be easily expanded on a series connection of several independent capacitors.

$$\Omega_{(p)}^{SER}\{p_{1i},\ p_{2j},\ ...,\ p_{nr}\}=p_{1i}\cdot p_{2j}\cdot...\cdot p_{nr},$$

and

$$\Omega_{(c)}^{SER}\{c_{1i},\ c_{2j},\ ...,c_{nr}\}=\left[c_{1i}^{-1}+c_{2j}^{-1}+...+c_{nr}^{-1}\right]^{-1}.$$

Example 2. Pipeline consists of n series sections (pipes). Section j is characterized by random capacity, for which each value v is realized with some probability p. In this case,

$$\Omega_{(p)}^{PAR}\{p_{1i},\ p_{2j},\ ...,\ p_{nr}\}=p_{1i}\cdot p_{2j}\cdot...\cdot p_{nr},$$

and

$$\Omega_{(c)}^{SER}\{v_{1i},v_{2j},...,v_{nr}\}=\min\{v_{1i},\ v_{2j},\ ...,\ v_{nr}\},$$

Example 3. One measures a sum of values, each summand of which is random. With probability p_{js} value j is measured with standard deviation (STD) equal to σ_{js}. In this case, using notation similar to above, one has:

$$\Omega_{(p)}^{PAR}\{p_{1i},\ p_{2j},\ ...,\ p_{nr}\} = p_{1i}\cdot p_{2j}\cdot ...\cdot p_{nr},$$

and

$$\Omega_{(c)}\{\sigma_{1i},\ \sigma_{2j},\ ...,\sigma_{nk_n}\} = \sqrt{\sigma_{1i}^2+\sigma_{2j}^2+...+\sigma_{nr}^2}\ .$$

Examples can be continued and not necessarily with probabilistic parameters.

4. Formal description of the Method of Universal Generating Functions.

After these simple examples, let us begin with formal description of the Method of Universal Generating Function (UGF[3]). For a more vivid presentation, let us use special terminology to distinguish the UGF from the common generation function. This will relieve us from using traditional terms in a new sense, which may lead to some confusion. Moreover, we hope that this new terminology can help us, in a mnemonic sense, to remember and perhaps even to explain some operations.

In the ancient Roman army, a *cohort* (*C*) was the main combat unit. Each cohort consisted of *maniples* (*M*), which were independent and sometimes specialized combat units with several soldiers of different profiles. Several cohorts composed a *legion* (*L*). The use of this essentially military terminology appears to be convenient in this essentially peaceful mathematical application. A legion is close by its sense to a generating function, a cohort is close to a term of the generating function written in the form of expanded polynomial, and a maniple is close to a parameter of each term.

Starting with polynomial multiplication, in our approach, we will consider less restrictive operations (not only multiplication of terms) and more general parameters. For instance, multiplication of polynomials assumes getting products of coefficients and summation of powers. In our case, we will expand on such restrictive limits on operations.

Let's denote legion j by L_j. This legion includes v_j different cohorts, C_{jk}:

$$L_j = \left(C_{j1}, C_{j2}, ..., C_{jv_j}\right).$$

The number of cohorts within different legions might be different. However, in our approach, maniples, which consist of a cohort, must be similar by its structure.

Each cohort C_{jk} is composed of some maniples, M, each of which represents different parameters, special characteristics, and auxiliary attributes. Each cohort consists of the same set of maniples:

$$C_{jk} = \left(M_{jk}^{(1)}, M_{jk}^{(2)}, ..., M_{jk}^{(s)}\right).$$

To make description of the method more transparent, let us start with the examples of two legions, L_1 and L_2: each of which consists of the following cohorts, L_1=(C_{12},C_{12},C_{13}) and L_2=(C_{21},C_{22}), and each cohort C_{jk} includes two maniples $M_{jk}^{(1)}$ and $M_{jk}^{(2)}$,i.e. C_{jk}=($M_{jk}^{(1)}$, $M_{jk}^{(2)}$).

[3] UGF might be also read as Ushakov's Generating Function ☺.

Denote the operation of legion interaction by Ω_L. This operator is used to obtain the resulting legion L_{RES}. In this simple case, one can write:

$$L_{RES} = \Omega_L \{L_1, L_2\}. \qquad (1)$$

This interaction of legions produces six pairs of interactions between different cohorts, which generate the following resulting cohorts:

$$C_{RES-1} = \Omega_C \{C_{11}, C_{21}\}, C_{RES-2} = \Omega_C \{C_{11}, C_{22}\},$$

$$C_{RES-3} = \Omega_C \{C_{12}, C_{21}\}, \; C_{RES-4} = \Omega_C \{C_{12}, C_{22}\},$$

$$C_{RES-5} = \Omega_C \{C_{13}, C_{21}\}, \; C_{RES-6} = \Omega_C \{C_{13}, C_{22}\}.$$

Here $\Omega_C \{\bullet\}$ denotes the interaction of cohorts.

Interaction of cohorts consists of interaction between its costituent maniples. All cohorts contain maniples of the same types though with individual values of parameters. Let us take, for instance, resulting cohort $C_{RES\text{-}5}$, which is obtained as interaction of cohorts C_{13} and C_{21}. In turn, interaction of these particular cohorts consists in interaction of their corresponding maniples:

$$M^{(1)}_{RES-5} = \Omega^{(1)}_M \{M^{(1)}_{13}, M^{(1)}_{21}\}$$

$$M^{(2)}_{RES-5} = \Omega^{(2)}_M \{M^{(2)}_{13}, M^{(2)}_{21}\}$$

The rules of interaction between maniples of different types, i.e. $\Omega^{(1)}_M \{M^{(1)}_{1i}, M^{(1)}_{2j}\}$ and $\Omega^{(2)}_M \{M^{(2)}_{1i}, M^{(2)}_{2j}\}$ are (or might be) different.

Interaction of n legions can be written as:

$$L = \Omega_L (L_1, L_2, ..., L_n).$$

Operator Ω_L denotes a kind of "n-dimensional Cartesian product" of legions and special final "reformatting" of the resulting cohorts (like converging polynomial terms with the equal power for a common generating function). Since each legion j consists of v_j cohort, the total number of resulting cohorts in the final legion (after all legion interaction) is equal to

$$v = \prod_{1 \le j \le n} v_j .$$

Number v corresponds to the total number of cohorts' interactions.

5. Implementing UGF philosophy in computer language C++.

We would like use the UGF (Universal Generating Function) philosophy in an analysis tool and perform reliability calculations for real-world systems. Because we are talking about an (reliability) engineering discipline, all philosophies present the need to be converted into numerical

results and predictions. Thus, the UGF philosophy begs an implementation! The implementation task is to identify objects (maniple, cohort, legion) and program all interactions between them. Unfortunately, we run into a combinatoric explosion of possible interactions for a sysem consisting of a large number of (atomic) units. Even moderm computers are not able to enumerate astronomically large (2^{1000}) number of interaction states in system consisting of 1000 binary atomic units. Fortunately, for a class of frequently occuring practical systems, the situation is not as hopeless as it may first appear. For a system to be useful in engineering, it may only fail very infrequently. In a highly reliable system, the failure probability of all atomic units much smaller that the system failure probability. This fact makes most of the interactions exceedingly rare and they can be systematically ignored in an approximation scheme that retains only the dominant contributions.

Let us proceed to find an approximate implementation of the UGF philosophy for highly relaible systems in a system simulator. It should be reasonably easy to identify an atomic unit in reliability theory as a maniple. Independence of the maniples corresponds to statistical independence of the atomic units. A cohort is defined to be a collection of maniples. The same definition holds in the context of reliability theory, where the collection is defined by a failure criterion. In a series system, each atomic unit is assumed to provide distinct and critical functionality. This maps on to the notion of specialized combat units. In a parallel system, all atomic units are statistically identical. This improves survival probability during operation, either in the military or in system reliability! Thus, we may identify a subsystem in reliability engineering as a cohort in UGF formalism.

Interactions between the objects are identified in the simulator by their natural reliability names. *k-out-of-n* combinations are of primary interest. But this class includes the two most frequently appearing reliability structures: series (*n-out-of-n*) and parallel (1-*out-of-n*). In fact, probability of failure of a parallel system is negligible (higher order in numerical smallness) with an additional assumption of high availability of the atomic units. Obviously a series system can be made up of distinct units providing separate functionality to the system.

As an illustration let us consider a system S of two subsystems *A* and *B* in series. Let *A* be atomic and *B* be composed of two atomic units *X* and *Y* in parallel. One possible C++ coding for this (simple) system is

B=*Parallel*(X,*Y*); S = *Series*(A,*B*);

Properties (MTBF, MTTR etc.) of all atomic units are specified at the start of analysis. Operations like Series and Parallel are C++ member functions for the instances of class “unit”. We will not specify unit composition rules in this work. Most of these rules can be found in standard textbooks on reliability engineering. Interested readers may find the remaining ones (involving switching time and PEI) in Chakravarty and Ushakov (2000, 2002).

It remains to identify the “legion”. The preceding paragraphs almost suggest that a legion be identified with the entire system in reliability theory, where the system is further assumed to be represented by its generating function. We would like to note that that this analogy cannot be taken literally sometimes. It is common for a real world reliability system to have deeper hierarchies (e.g., system, equipment shelves, equipment racks, electornic cards) like modern day militaries. In such an elaborate system, we still identify the atomic units as maniples. At the other end, we identify the entire system as a “legion”! All intermediate stages in the hierarchy are considered generalized “cohorts”.

In Chakravarty and Ushakov (2000) implementation, any subsystem can be composed from other subsystems at the next lower level of hierarchy (or atomic units which are always at the lowest level). A newly formed subsystem provides an effective reliability description of all units

that compose this subsystem. This composition can be continued indefinitely to obtain an effectiveness measure for the entire system. They have shown that this can be recast as an approximation from a system generating function when all atomic units satisfy binary failure criteria (on/off) they are statistically independent, the system itself is highly reliable and reliability design of the system consists of hierarchical blocks.

6. Reliability analysis of Globalstar™ Gateways.

Globalstar is a low-earth-orbit (LEO) based telephony system with global coverage. The gateways make its ground segment that connect to the orbiting satellites. The gateways are cpmlex systems with more than a thousand components (e.g., electronic cards). Ushakov (1998), Chakravarty and Ushakov (2002) used the UGF approach for the reliability (performance) analysis of Globalstar™ gateways (fixed ground segment of a low earth orbit satellite communications system). Given the prominence of object oriented abstractions and operations in Globalstar design, it should not be surprising that the reliability analysis naturally fits into the UGF philosophy. Further, these ideas can be naturally implemented in the computer using an object oriented language.

Because of the object oriented nature of system reliability design in Globalstar (interaction between objects like system, racks, shelves, cards are triggered by failure, switching of failed units and changing user demand), Ushakov (1998) proposed that a system reliability simulator should be coded in an object oriented computer language like C++. Later, Chakravarty and Ushakov (2002) implemented a simulator for the Globalstar™ Gateway in C++.

In Chakravarty and Ushakov implementation for Globalstar, C++ objects are in one-to-one correspondence with reliability objects. An object is specified by mean time between failures (MTBF), mean time to repair/replace (MTTR) and an effectiveness weight (partial effectiveness index: PEI). By definition, PEI=1 for binary atomic units. All failure distributions are implicitly assumed to be Exponential. If failed units were to be automatically swapped, a switching time was also assigned by Chakravarty and Ushakov (2000). Even small switching time is important because it changes a parallel system “on paper” to a series system with small MTTR. This may have dramatic effect overall on system reliability.

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METHOD OF OPTIMAL SPARE ALLOCATION FOR MOBILE REPAIR STATION

Igor Ushakov [4]

Summary.

Method of finding optimal spare stock for Mobile Repair Station is suggested. Numerical calculations are performed with use of real field data. It showed significant improvement: probability of first fix for suggested variant is 0.967 in comparison with 0.534 for existing variant.

1. Introduction

There is a Service Base (SB) that serves clients' equipment within some zone. A client sends a request for repair to the SB when his equipment has failed. Immediately after a request an available Mobile Repair Station (MRS) is directed to the client. One of he most important index of quality of service is the so-called "fast fix" (FF) that takes just several minutes. FF is possible if there a needed Field Replacement Units (FRU) at the MRS spare stock is available. Otherwise, a special request is sent by MRS to its SB and the needed FRU is delivered to the client only in several hours. (In addition, it involves extra spending of money for restoration client's equipment.)

Equipment of clients can differ by configuration though consists of the same set of components, number of which exceeds several hundreds. Due to natural restrictions, the stock room is nor enough for keeping FRU of all possible types. Thus, the problem of optimal list of spares at MRS stock arises that provides maximum probability of FF under given restriction on the available room for spares.

2. Formulation of the problem.

Denote available space of MRS stock V^*. Let client j , $j = \overline{1, M}$, has equipment with n_k^j components of type k (let's call it component-k). s. Denote failure rate of a component of type k by λ_k, $k = \overline{1, N}$. Then the flow of requests formed by components- k , Λ_k, arriving at the SB can be written as

$$\Lambda_k = \lambda_k \sum_{1 \le j \le M} n_k^j \tag{1}$$

The total flow of requests, Λ, is equal to

$$\Lambda = \sum_{1 \le j \le N} \Lambda_k \tag{2}$$

It is clear that a current failure occurs due to a failure of component-k occurs with the probability

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$$p_k = \frac{\Lambda_k}{\Lambda} \tag{3}$$

Denote available space of MRS stock by V and physical volume of component-k by v_k. If one assumes that there are no multiple instantaneous failures and the probability that the second failure of the same equipment during FRS travel time is negligibly small, than the solution of the problem is very simple: one calculates values

$$w_k = \frac{p_k}{v_k} \tag{4}$$

and then takes first S components that satisfy the following condition:

$$\sum_{1 \le k \le S} v_k \le V < \sum_{1 \le k \le S+1} v_k \quad . \tag{5}$$

In practice, FRU of different types are approximately of the same volume, i.e. $v_k=v$. It means that instead of ordering values w_k , it is enough to order values p_k .

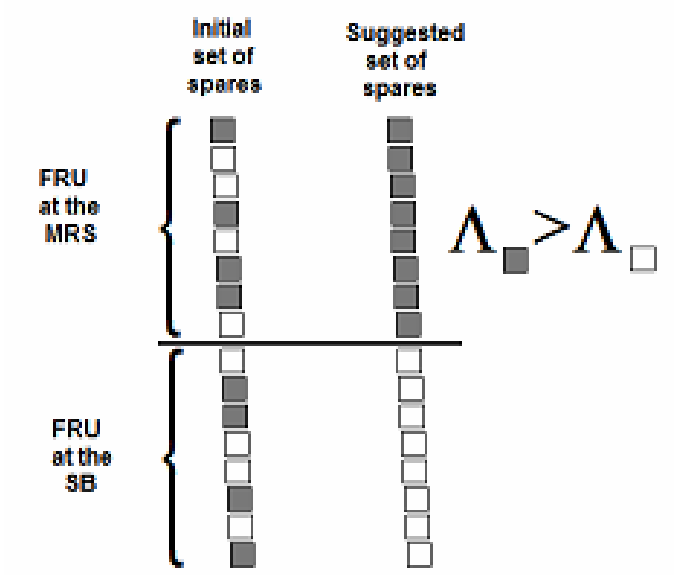

Figure 1. Explanation of the solution.

3. Case study

The following solution has been performed by contract with Hughes Network Systems (Germantown, Maryland, USA) for a maintenance service for ground clients of a global telecommunication system.

In this particular case, the volumes of FRU are approximately the same, so the limitation is for the total number of FRUs that is equal to 51.

Values of Λ_k for various components are given in Table 1. In this table, column "O" (for "old") contains the number of spares in the initial list and column "N" (for "new") contains the number of spares in the final list obtained by suggested method. For the sake of shortness, we omitted those types of equipment components, for which both spare lists (initial and suggested) have zero spare units at the MRS stock.

Table 1.

#	Part No.	Rate (per day)	O	N	x	#	Part No	Rate (per day)	O	N	x	#	Part No	Rate (per day)	O	N
1	30514-2	0.0149	0	1	x	25	30290-1	0.0038	0	1	x	49	20079-1	4.9 E-4	1	1
2	15338-1	0.0118	1	1	x	26	14364-1	0.0035	1	1	x	50	3043-1	3.9 E-4	1	1
3	14364-9	0.0114	0	1	x	27	30812-1	0.0032	0	1	x	51	3511-1	3.6 E-4	1	1
4	17668-1	0.0107	1	1	x	28	15901-1	0.0031	0	1	x	52	15233-1	2.9 E-4	2	0
5	3066-1	0.0103	2	1	x	29	12171-2	0.0025	1	1	x	53	15187-2	2.9 E-4	1	0
6	19847-1	0.0096	1	1	x	30	11836-4	0.0023	1	1	x	54	30491-1	2.9 E-4	1	0
7	11836-1	0.0095	1	1	x	31	19552-2	0.0023	0	1	x	55	12256-1	2.4 E-4	1	0
8	13847-2	0.0093	0	1	x	32	17668-2	0.0023	0	1	x	56	17114-1	2.3 E-4	1	0
9	3514-5	0.0088	1	1	x	33	10111-1	0.0022	0	1	x	57	30510-1	2.2 E-4	1	0
10	12076-1	0.0086	0	1	x	34	92132-4	0.0022	1	1	x	58	93634-8	2.2 E-4	1	0
11	17512-1	0.0077	1	1	x	35	70275-1	0.0021	0	1	x	59	10306-1	1.9 E-4	1	0
12	30038-2	0.0071	1	1	x	36	124364-6	0.0020	1	1	x	60	17132-1	1.9 E-4	1	0
13	16174-1	0.0069	1	1	x	37	110228-1	0.0018	1	1	x	61	30470-1	1.9 E-4	1	0
14	11836-9	0.0057	0	1	x	38	124871-1	0.0018	1	1	x	62	30066-2	1.4 E-4	2	0
15	30290-2	0.0056	1	1	x	39	113061-1	0.0017	0	1	x	63	30206-1	1.4 E-4	1	0
16	17960-9	0.0053	0	1	x	40	110119-1	0.0016	1	1	x	64	30626-1	1.2 E-4	1	0
17	92486-2	0.0053	0	1	x	41	200260-4	0.0015	1	1	x	65	92513-1	1.2 E-4	1	0
18	13847-1	0.0052	0	1	x	42	30330-1	0.0015	1	1	x	66	11667-1	8.6E-5	1	0
19	17960-1	0.0050	0	1	x	43	30467-1	0.0014	1	1	x	67	20228-3	6.7E-5	1	0
20	19847-2	0.0045	1	1	X	44	92428-2	0.0012	1	1	x	68	11485-1	4.2E-5	1	0
21	3727-2	0.0042	0	1	X	45	90096-2	8.6E-4	1	1	x	69	3512-4	1.8E-5	1	0
22	15901-2	0.0041	0	1	x	46	30279-1	8.4 E-4	1	1	x	70	11836-5	1.2E-5	1	0
23	1836-2	0.0041	1	1	x	47	30140-1	6.8 E-4	1	1	x	69				
24	1939-1	0.0039	1	1	x	48	111998-1	5.9 E-4	1	1	x	70				

From the complete list of equipment components (it is not presented), one can find that the total failure rate in the chosen service zone is equal to Λ=0.254 [1/day], i.e. approximately 1 failure in every 4 days. Failures covered by the initial set of spares form a failure flow with rate Λ_k = 0.136, and for suggested set of spares the analogous value equal to Λ_k^{opt} = 0.229.

It means that the probability of FF has been increased from $p_k = \frac{\Lambda_k}{\Lambda} = \frac{0.136}{0.254} \approx 0.535$ to $p_k^{opt} = \frac{\Lambda_k^{opt}}{\Lambda} = \frac{0.229}{0.254} \approx 0.902$.

Approximate evaluation of expected gain is the following. The entire service system spreading over the USA gets on the average about 44,000 calls a year. MRSs with initial spare stocks made about 44,000×(1-0.535)≈20,500 extra deliveries due to lack of needed spares. The suggested spare stock leads only to 44,000×(1-0.902)≈4,300 extra deliveries, i.e. about 16,500 extra deliveries less. Each visit takes on the average about 4 hrs (round trip) and about 0.5 hr for equipment inspection at the client site. Each visit costs at least $150, so the total gain is about 24.7 million a year.

MATHEMATICAL MODELS AND SOFTWARE TOOLS FOR QUALITY AND RISK MANAGEMENT ACCORDING STANDARD REQUIREMENTS

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Abstract

The offered mathematical models and supporting them software tools complexes (M&STC) are purposed for a systems analysts from customers, designers, developers, users, experts of testing laboratories and certification bodies, as well as a staff of quality maintenance for any complex system etc. M&STC are focused on providing system standard requirements on the base of modeling random processes that exist for the life cycle of any complex system. Models implement original author's mathematical methodology based on probability theory, theory for regenerating processes and methods for system analysis. M&STC may be also used in training and education for specializations "System engineering", "Software engineering", "System safety and security", "Information systems".

1. Introduction

According to standard ISO/IEC 15288 system is defined as a combination of interacting elements organized to achieve one or more stated purposes. An application of offered methodology uses to evaluate probabilities of "success", cost, time and quality risks and related profitability and expenses. This helps to solve on the scientific basis the next practical problems in system life cycle: analysis of quality management systems for enterprises, substantiation of quantitative system requirements to hardware, software, users, staff, technologies; requirements analysis, the evaluation of project engineering decisions; investigation of problems concerning potential threats to system operation including information security and protection against terrorists; evaluation of system operation quality, substantiation of recommendations for rational system use and optimization etc.

2. Focusing on rational management

All complexes are offered for providing rational management. Management is a purposeful changing of an object state, a process or a system. Management is based on choosing one among a set of alternatives. Rational management is a management leading to the objective achievement according to the criterion of a chosen parameter extreme (minimum or maximum) under the set limitations. Classical examples of rational management are usually either maximization of a profit (an income, a degree of quality or security, etc.) under limitations on expenses or expenses minimization under limitations on an admissible quality and/or security level. It is clear that criterion and limitations may vary throughout the system life.

For rational management of processes it is necessary to know and plan their behaviour at various influences. For this purpose we offer for using about 100 the mathematical models [1,2]. As criterion parameters there are used the quantity measures (objective functions) characterizing a possibility of object achievement at different stages of a system life cycle. For example, an investor's criterion is the maximum income from the project implementation under limitations on

the production process and product quality. For the enterprise it is important to organize a quality management system properly – so that in the form of criterion it can choose the probability of qualified work performance, i.e. in time and without defects or the maximum probability of success for quality management policy concerning work complexes. A security service must provide safety of an object, a process or a system up to the mark. In this case there may be used the criterion of expenses minimum under limitations on the admissible level of dangerous influence risk taking countermeasures into account or minimum of a dangerous influence risk at limitations on expenses. The customer and the developer are interested in the final result – in this case as an integrated parameter there may be used such criterion as the maximum part of functional operations carried out with the admissible quality or the relative degree of customer satisfaction with the limitations on quality or expenses.

The first from the offered models is Complex for Evaluation of Information Systems Operation Quality (CEISOQ) [1-2,5-6]. The development of CEISOQ was based on the general purpose for all information systems (see Fig. 1 and 2).

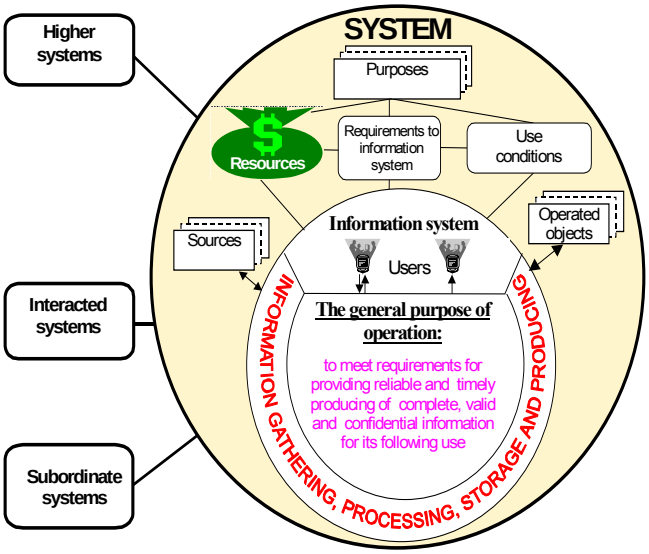

Fig.1 The purpose of information system in a SYSTEM

Fig. 2 The main CEISOQ window for choosing the model

The problems connected with usual computation of parameters are called direct operation research problems or analysis problems. The problems directed on a choice of variants maximizing

or minimizing values of objective functions under limitations are called inverse operation research problems or synthesis problems. The offered models and supporting software tools allow to solve both direct and inverse operation research problems.

3. Abstract formalization

This formalization is used to building a probabilistic space (Ω, B, P), where:

Ω - is a limited space of elementary events;
B – a class of all subspace of Ω-space, satisfied to the properties of σ-algebra [3];
P – a probability measure on a space of elementary events Ω.

Because, $\Omega=\{\omega_k\}$ is limited, there is enough to establish a reflection $\omega_k \rightarrow p_k = P(\omega_k)$ like that $p_k \geq 0$ and $\sum_k p_k = 1$. Such space (Ω, B, P) is built by the limited theorems for regenerative processes [3-4] and also by using principal propositions of probability theory and well famous results for single and multi-units queuing systems. This probabilistic space (Ω, B, P) is the essence of mathematical models to support an assessment of standard system processes.

4. Example of created mathematical model

Nowadays at system development and utilization an essential part of funds is spent on providing system protection from various dangerous influences able to violate system integrity. Under system integrity it means such system state when system purposes are achieved with the required quality under specified conditions of use. Such examples of dangerous influences are terrorists attacks, viruses or 'violators' influences, software defects events etc. As this problem wasn't studied carefully dangerous influences often reach their aims.

There are examined three typical technologies of providing protection from dangerous influences. In this paper it is illustrated only technology 1 that is based on preventive diagnostics of system integrity. Diagnostics is carried out periodically. It is assumed that except diagnostics means there are also included means of necessary integrity recovery after revealing of danger sources penetration into a system or consequences of negative influences. Integrity violations detecting is possible only as a result of diagnostics, after which system recovery is started. Dangerous influences on a system are acted step-by step: at first a danger source penetrates into a system and then after its activation begins to influence. System integrity is considered to be violated only after a danger source has influenced on a system. If to compare a system with a man technology 1 reminds a periodical diagnostics of a man's health state. If diagnostics results have revealed symptoms of health worsening a man is cured (and integrity is considered as recovered). Between diagnostics an infection penetrated into a man's body brings a man into an unhealthy state (a dangerous influence is realized and integrity is violated)–see Fig. 3.

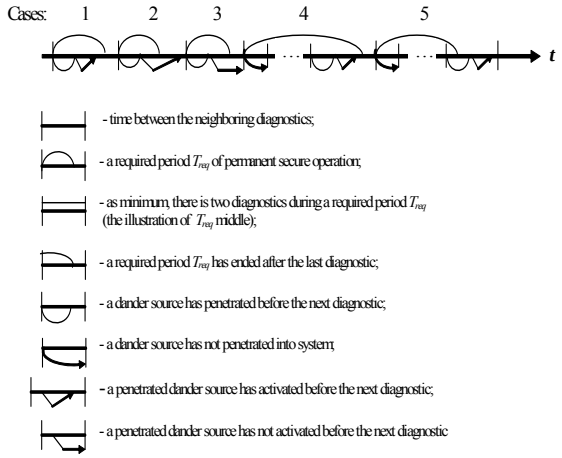

Fig. 3 The illustration of system resources protection against dangerous influences by technology 1

The availability of means of danger sources total-lot detecting and existence of ways of violated system integrity total-lot recovery is obligatory requirement. The offered models are supported by software tools CEISOQ.

Te system protection from dangerous influences may be evaluated if the next characteristics are known: frequency of influences for penetrating a danger source into a system (σ) ; mean activation time of a penetrated danger source (β); time between the end of diagnostic and the beginning of the next one ($T_{betw.}$); diagnostic time including the time of system integrity recovery ($T_{diag.}$); a required period of system operation ($T_{req.}$) for investigation. There are possible the next variants:

variant 1 – the assigned period T_{req} is less than established period between neighboring diagnostics ($T_{req} < T_{betw.}+T_{diag}$);

variant 2 – the assigned period T_{req} is more than or equals to established period between neighboring diagnostics ($T_{req} \geq T_{betw.}+T_{diag}$).

Statement 1. Under the condition of independence of considered characteristics the probability of dangerous influence absence for variant 1 is equal to

$$P_{infl.(1)}(T_{req}) = 1 - B_{penetr} * B_{activ}(T_{req}), \quad (1)$$

where $*$ - is convolution sign, $B_{penetr}(t)$ is the probability distribution function (PDF) of time between neighboring influences for penetrating a danger source, $B_{activ}(t)$ is the PDF of activation time of a penetrated danger source, for modeling $B_{penetr.}(t)=1-e^{-\sigma t}$, $B_{activ}(t)=1-e^{-t/\beta.}$

Note. This formula (1) is used also for the evaluation of security for system operation without diagnostics. There is supposed that before the beginning of period T_{req} system integrity is provided.

Statement 2. Under the condition of independence for considered characteristics the probability of dangerous influence absence for variant 2 is equal to

$$P_{infl.(2)}=P_{mdl}+P_{end} \quad (2),$$

where P_{mdl} – is the probability of dangerous influence absence within the period T_{req} since beginning to the last diagnostic, P_{end} – is the probability of dangerous influence absence within the period T_{req} after the last diagnostic, i.e. in the last remainder $T_{rmn}=T_{req} -N(T_{betw}+ T_{diag})$, N is the number of periods between diagnostics placed wholly within assigned period T_{req},

$N = [T_{req}/(T_{betw} + T_{diag})]$ – is integer part;

$P_{wholly(1)} =P_{infl.(1)}(T_{betw} +T_{diag.})$, is calculated by formula (1),

$$P_{mdl} = \frac{N\left(T_{betw.} + T_{diag.}\right)}{T_{req.}} \cdot P_{wholly(1)}^{N}, \qquad P_{end} = \frac{T_{rmn}}{T_{req.}} \cdot P_{\mathbf{inf}\, l.(1)}(T_{rmn}). \quad (3)$$

The mathematical proof is in [1,2]. This is one from more than 100 mathematical models offered to support an assessment of standard system processes.

5. Software tools to support an assessment of standard system processes

35 created software tools complexes implementing original mathematical models [1-2,5-6] consist complexes created in 2001-2005: for the Evaluation of Information Systems Operations Quality (CEISOQ, CEISOQ+); for Evaluation of System Vulnerability including Conditions of Terrorist Threats ("VULNERABILITY"); for complex analysis of system security ("ANALYSIS OF SYSTEM SECURITY"), for Modeling of System Life Cycle Processes "MODELING OF PROCESSES". The last complex "MODELING OF PROCESSES" includes multi-functional complexes for evaluation of Agreement (models and software tools "ACQUISITION", "SUPPLY"), Enterprise (models and software tools "ENVIRONMENT MANAGEMENT", "INVESTMENT MANAGEMENT", "LIFE CYCLE MANAGEMENT", "RESOURCE MANAGEMENT", "QUALITY MANAGEMENT") and Project (models and software tools "PROJECT PLANNING", "PROJECT ASSESSMENT", "PROJECT CONTROL", "DECISION-MAKING", "RISK MANAGEMENT", "CONFIGURATION MANAGEMENT", "INFORMATION MANAGEMENT") Processes Modeling and also for Technical Processes Modeling (models and software tools "REQUIREMENTS DEFINITION", "REQUIREMENTS ANALYSIS", "ARCHITECTURAL DESIGN", "HUMAN FACTOR", "IMPLEMENTATION", "INTEGRATION", "VERIFICATION", "TRANSITION", "VALIDATION", "OPERATION", "MAINTENANCE", "DISPOSAL").

The models created have undergone extensive testing in an operational environment and the results have been compared with the results of other independent models (if such exist). This comparison has provided documented evidence that the models implemented in these tool suites are realistic, including the reality of the calculations and the time&probabilistic characteristics. Created software tools are an original Russian creation patented by Rospatent, certified, have been presented at seminars, working groups, symposiums, conferences and forums since 2000 in Russia, Australia , the USA, Canada, France, Germany, Kuwait. In 2001 the CEISOQ [1-2,5-6] was awarded be the Golden Medal of the International Innovation and Investment Salon, in 2004-2005 the software tools "RISK MANAGEMENT", "HUMAN FACTOR" and "ARCHITECTURAL DESIGN" also were awarded by the Golden Medal of the International Exhibition "Intellectual Robots" and acknowledged as the software products of the year.

How these models adequacy may be conformed? Though any answer to these questions won't be irrefragable for a certain system we shall try to formulate our arguments.

Argument 1. The M&STC uses mathematical models formalizing standard processes on time line. Majority of dependencies gives upper and lower estimations. The fact is that while shaping models all mathematical results are initially drawn in the integral form. As input data are somehow connected with time after choosing distribution functions characterizing these data there were selected the gamma – distribution and the Erlang's distribution. Mathematicians know that these distributions approximate sums of positively distributed random variables well. Every temporary data are as a matter of fact such a sum of compound time expenses. Studies of regularities have shown that extremes are achieved on bounds of these distributions, i.e. of exponential and deterministic (discrete) distributions. Thus, real values will be somewhere between lower and upper estimations calculated by the software tools.

Argument 2. As a basis of models there is used the probability theory and the theory of regenerative processes. Proofs of basic theoretical results are cited in [1-2]. If to return in the 70-s of the last century we may remember the boom of mathematical modeling, defining calls flow reliable and time-probabilistic characteristics. The boom passed and appeared the reliability theory,

the queuing theory and a variety of models, which proved themselves to be effective. There are created standards and other normative documents regulating system methodical evaluations on the basis of these models. Nowadays these models are widely used and trusted because they produce reliable results confirmed in the course of time. It is worth to remind that these created theories and models are based on the probability theory and the theory of regenerative processes. The several offered models "The model of functions performance by a system in conditions of unreliability of its components", "The models complex of calls processing", "The model of entering into system data current concerning new objects of application domain" are the classical adapted models of the 80-s improved to meet the requirements of the present time. The other models are created on the basis of the limit theorem for regenerative processes developed in the 70-80-s in Moscow State University on the faculty of computing mathematics and cybernetics by professor Klimov's school [4]. Three-year testing of M&STC including beta-testing by fifty different companies raise confidence in models algorithmic correctness.

Argument 3. Skilled analysts know that if a probabilistic analytical model is incorrect then if input data are changed in the range from $-\infty$ to $+\infty$ there are always errors appearing either in infraction the probability theory laws or in illogic of dependencies behavior (most probably on the bounds of possible values) or in impossibility of obtained effects physical explanation. Bounds of input data in the M&STC are assigned in the range from $-\infty$ to $+\infty$ or to be more exact from 10^{-8} milliseconds to 10^{8} years.

Argument 4. As far as possible any designer tends to use several models of different authors. If results of different models use are not divergent a designer begins to trust not only to results but also to the models. Comparison of results of the M&STC use with results of other models use proved its high adequacy (concerning computations of reliability and time-probabilistic characteristics, the other models don't have analogues).

6. Examples of software tools application

The offered M&STC have been and are applied for solving the problems of:

information security and reliability for banks, transport systems, protected and military objects etc.;

rational protection for oil and gas systems in conditions of terrorist threats;

quality and reliability for cosmic robot systems and heat supply etc.;

risk analysis for dangerous coal mine and manufactures;

system certification;

education in the field of system analysis.

Below there are demonstrated some capabilities of the software tools "RISK MANAGEMENT".

Example 1 for demonstration the capabilities of subsystem "Evaluation of counteraction measures effectiveness". Let 10 barriers be installed in order to protect valuable resources of a system from unauthorized access. In table there are shown prospective characteristics of barriers (as counteraction measures) and the mean time of their possible overcoming by a specially prepared violator (as the time of keeping measure effectiveness). Real values of similar characteristics may be received as a result of natural experiments or application of other special models.

It is required to evaluate the risk of dangerous influence on a system in spite of counteraction measures during a week. The minimal admissible risk shouldn't be more than 0.0001. The initial data for calculations are shown in the Table.

Table

Characteristics of the threat scenario and the protection system

Barrier	***Change frequency of the barrier parameter value as time to the next strenthening of the measure***	***Mean time of barrier overco-ming by a violator***	***Possible way of barrier overcoming***
1. External guards	*Change of guards every 24 hours*	*10 hours*	*Latent penetration*
2. System of passes to the system with a change of security services	*Change of guards every 24 hours*	*10 minutes*	*Documents falsification, conspiracy, fraud*
3.The electronic key to get to the control unit	*5 years (time between changes)*	*1 week*	*Theft, forcible key withdrawal, conspiracy*
4. The password to enter the automated system	*1 month*	*10 days*	*Spying, compulsory questioning, conspiracy, selection of a password*
5. The password to get access to software devices	*1 month*	*10 days*	*—II—*
6. The password to get access to the required information	*1 month.*	*10 days*	*—II—*
7. The registered external information carrier with write access	*1 year*	*24 hours*	*Theft, forced registration, conspiracy*
8. Confirmation of a user identity, during a session of work with the computer	*1 month.*	*24 hours*	*Spying, compulsory questioning, conspiracy*
9. Telemonitoring	*Time between changes of software devices – 5 years*	*1 month*	*Simulation of a failure, false films, dressing up as employees, conspiracy*
10. Encoding of the most important information	*Change of keys every month*	*1 year*	*Decoding, conspiracy*

Solution. The analysis of the withdrawn calculated dependences has shown the following (see Fig. 4).

The first 3 barriers as counteraction measures are overcome with the probability about 0.34. Use of alternating passwords once a month for the 4th, 5th and 6th barriers allows to decrease the risk from 0.34 to 0.14. However, the general system protection after the introduction of the first six barriers remains rather weak. The 7th and 8th counteraction measures are practically useless. Use of telemonitoring means allows to decrease risk of dangerous influence on a system in spite of counteraction measures to 0.002 what also doesn't meet the stated requirements. The use of all 10 counteraction measures provides the required system protection.

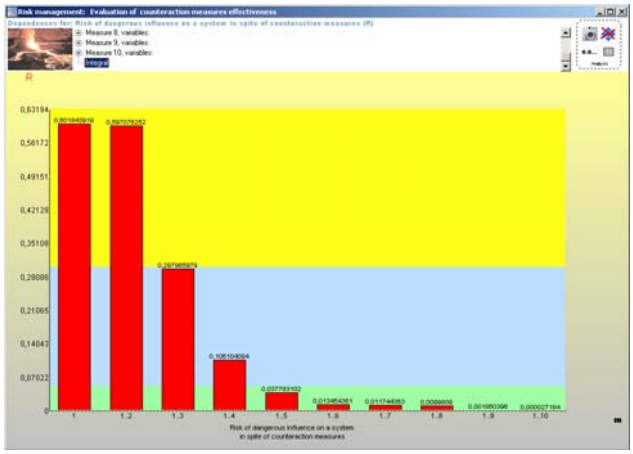

Fig. 4 *Results of computation for example 1*

The use of subsystem "Evaluation of expenses for risk retention" allows to define expenses against risks (see calculation results on Fig. 5). It is the capability to optimizing by criterion "risk-expenses".

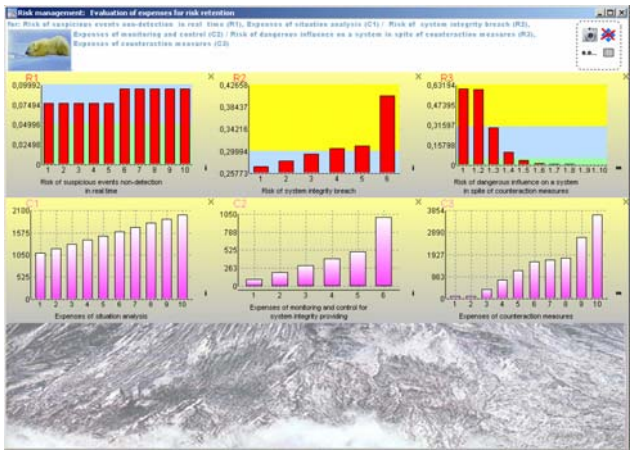

Fig. 5 *Results of computation by the subsystem "Evaluation of expenses for risk retention"*

The use of subsystem "Substantiation of counteraction strategy against risks" allows to evaluate different damages and expenses against risks for the given scenario.

The last complex "ANALYSIS OF SYSTEM SECURITY" allows to evaluate the integral security for the system consisting of any number of components. The condition is the components are united in parallel and/or consecutive order. The structure may be any degree of complexity. The strength of every measure (component) is approximated by exponential low.

Example 2 for demonstration the capabilities of subsystem "Analysis of integral security". Let the system contains 3 level of interacted subsystems: higher subsystem, interim subsystem and subordinate subsystem (see Fig.1). It may be territorially distributed enterprise or bank with the branches etc. Every subsystem have the valuable resources protected by the counteraction measures from the table (see example 1). The system structure, constructed by software tools for modeling, is on Fig.6. The frequency of threats source appearance is 10 times in a year by qualified violator, the mean time for system recovery is 1 hour.

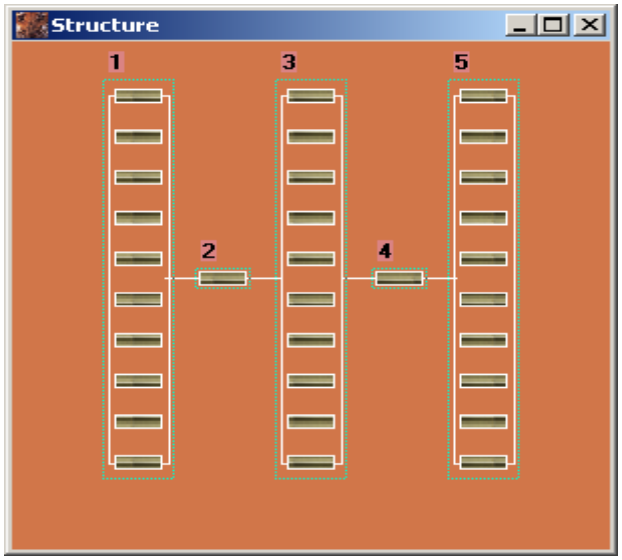

Fig. 6 System structure for modeling

It is required to evaluate the probability of providing system security in a year for the given scenario of threats.

Solution. The integral analysis of the withdrawn calculated dependences has shown the following (see Fig. 7). The probability of providing system security without control and monitoring is about 0.19, with periodic control (without permanent monitoring for the 9-th and 10-th barriers) - 0.39. The use of all 10 counteraction measures (including permanent monitoring for the 9-th and 10-th barriers) provides system security with the probability more than 0.95 against general expenses 104000 conditional units .

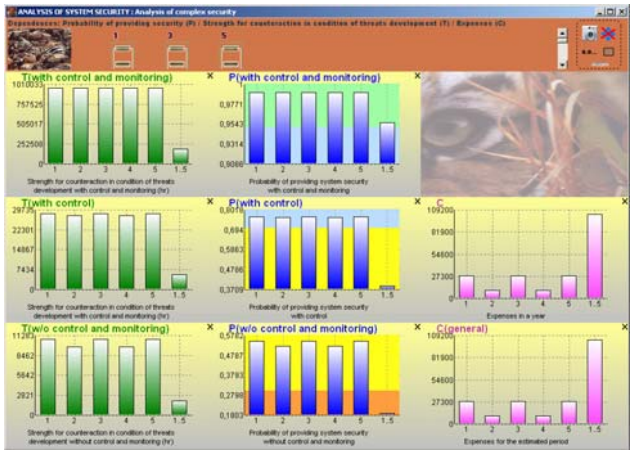

Fig. 7 Results of computation by the subsystem "Analysis of integral security"

7. Conclusion

Expected pragmatic effect from application is the next. It is possible to provide essential system quality and security rise and/or avoid wasted expenses in system life cycle on the base of processes modeling by the offered mathematical models and software tools complexes.

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