DIRECT CALCULATIONS OF A REACHING MOMENT DISTRIBUTION FOR AN AUTOREGRESSIVE RANDOM SEQUENCE BY RECCURENT INTEGRAL EQUALITIES

Tsitsiashvili G.Sh., Osipova M.A.

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Institute for Applied Mathematics, Far Eastern Branch of RAS
690041, Vladivostok, Radio str. 7,
e-mail: guram@iam.dvo.ru, mao1975@list.ru

In this paper we consider an autoregressive random sequence

$$X_k = RX_{k-1} + \eta_{k-1},\ X_0 = 0,$$

with $0 < R < 1$ and assume that η_k has exponentials mixture distribution with

$$P(\eta_k > t) = \sum_{r=1}^{l} p_r \exp(-\lambda_r t).$$

Our problem is to calculate a distribution of a reaching moment $\tau = \inf(k : X_k \geq X)$. This problem origins in the risk theory, in the financial mathematics, in the statistics of random processes and in the reliability theory. Interest to mixtures of exponentials as approximations of distributions with heavy tails is initiated by papers [1], [2]. At first look this problem may be solved by martingale technique. But in different applications when $R > 1$ or R, X depend on k or we need a distribution of a jump over X it is too complicated. In this paper we apply some recurrent integral equalities to get over these difficulties. We solve the considered problem in symbols and illustrate obtained solutions by numerical calculations.

Denote the deriving moment for this sequence reaching some boundary and designate

$$X_k^j = XR^j,\ k \geq 1,\ 0 \leq j \leq k-1,\ X_k^k = 0.$$

Theorem 1. *The following formula are true for* $k \geq 1$:

$$T_k(x) = P(X_k > x,\ \tau \geq k) = \sum_{r=1}^{l} \sum_{j=0}^{k-s} a_{k\,k-s\,j\,r} \exp\left(-\frac{\lambda_r x}{R^j}\right), \tag{1}$$

$X_k^{k-s+1} \leq x \leq X_k^{k-s}$, $s = 1,\dots,k$, *and*

$$P(\tau = k) = \sum_{r=1}^{l} a_{k\,0\,0\,r} \exp(-\lambda_r X). \tag{2}$$

Here for $1 \leq r, q \leq l$

$$a_{1\,0\,0\,r} = p_r, \tag{3}$$

$$a_{k+1\,k+1-s\,0\,q} = I(s \neq 1) \sum_{t=1}^{s-1} \sum_{j=0}^{k-t} \sum_{r=1}^{l} \frac{p_q \lambda_r a_{k\,k-t\,j\,r}}{\lambda_r - R^{j+1}\lambda_q} A(k,t,q,r,j) +$$

$$+ I(s \neq k+1) \sum_{j=1}^{k-s} \sum_{r=1}^{l} \frac{p_q \lambda_r a_{k\,k-s\,j\,r}}{\lambda_r - R^{j+1}\lambda_q} B(k,s,q,r,j),\ 1 \leq s \leq k+1, \tag{4}$$

$$a_{k+1\,k+1-s\,j\,r} = -\sum_{q=1}^{l} \frac{p_q \lambda_r a_{k\,k-s\,j-1\,r}}{\lambda_r - R^j \lambda_q},\ 0 < j \leq k+1-s,\ 1 \leq s \leq k, \tag{5}$$

with

$$B(k,s,q,r,j) = \exp\left(-X_{k+1}^{k+2-s}\left(\frac{\lambda_r}{R^{j+1}} - \lambda_q\right)\right),$$

$$A(k,t,q,r,j) = B(k,t,q,r,j) - \exp\left(-X_{k+1}^{k+1-t}\left(\frac{\lambda_r}{R^{j+1}} - \lambda_q\right)\right).$$

Proof. As $T_1(x)=\sum_{r=1}^{l} p_r \exp(-\lambda_r x)$, $0\le x\le X$, so we have an equality $P(\tau=1)=\sum_{r=1}^{l} p_r \exp(-\lambda_r X)$ and (1)-(3) are true.

Denote $Q_k(x)=P(RX_k>x,\ \tau\ge k+1)$ and calculate

$$Q_1(x)=\sum_{r=1}^{l} p_r \exp\left(-\frac{\lambda_r x}{R}\right),\ 0\le x\le XR .$$

Then we have

$$T_2(x)=-\int_0^{\min(x,XR)}\sum_{q=1}^{l} p_q \exp\left(-\lambda_q(x-u)\right)dQ_1(u)=$$

$$=\sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r \exp\left(-\lambda_q x\right)}{\lambda_r-R\lambda_q}\left(1-\exp\left(-\min(x,XR)\left(\frac{\lambda_r}{R}-\lambda_q\right)\right)\right)=$$

$$=\begin{cases}\sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q}\left(\exp\left(-\lambda_q x\right)-\exp\left(-\frac{\lambda_r x}{R}\right)\right), & 0\le x\le XR,\\ \sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r \exp\left(-\lambda_q x\right)}{\lambda_r-R\lambda_q}\left(1-\exp\left(-X\left(\lambda_r-R\lambda_q\right)\right)\right), & XR\le x\le X.\end{cases}$$

So

$$T_2(x)=\sum_{r=1}^{l}\sum_{j=0}^{2-s} a_{2\ 2-s\ j\ r}\exp\left(-\frac{\lambda_r x}{R^j}\right),\ X_2^{3-s}\le x\le X_2^{2-s},\ \ s=1,2,$$

with

$$a_{210q}=\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q},\ a_{211q}=-\sum_{q=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q},$$

$$a_{200q}=\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q}A(1,1,q,r,0),\ A(1,1,q,r,0)=\left(1-\exp\left(-X\left(\lambda_r-R\lambda_q\right)\right)\right).,$$

Then we have $P(\tau=2)=\sum_{r=1}^{l} a_{200r}\exp(-\lambda_r X)$. So for $k=1$ the formulas (4), (5) are true also.

Assume that the formulas (1), (2) are true for fixed k then

$$-dQ_k(u)=\sum_{r=1}^{l}\sum_{j=0}^{k-s} a_{k\ \ k-s\ \ j\ \ r}\frac{\lambda_r}{R^{j+1}}\exp\left(-\frac{\lambda_r u}{R^{j+1}}\right)du,\ X_{k+1}^{k+2-s}\le u\le X_{k+1}^{k+1-s},\ \ s=1,...,k .$$

So

$$T_{k+1}(x)=-\int_0^{\min\left(x,X_{k+1}^1\right)}\sum_{q=1}^{l} p_q \exp\left(-\lambda_q(x-u)\right)dQ_k(u)=$$

$$=\sum_{s=1}^{k}\int_{\min\left(x,X_{k+1}^{k+2-s}\right)}^{\min\left(x,X_{k+1}^{k+1-s}\right)}\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{a_{k\,k-s\,j\,r}\,p_q\lambda_r\exp\left(-\lambda_q x\right)}{R^{j+1}}\exp\left(-\frac{u\lambda_r}{R^{j+1}}+u\lambda_q\right)du=$$

$$=\sum_{s=1}^{k}\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp\left(-\lambda_q x\right)a_{k\,k-s\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}A_1(k,s,q,r,j),\ 0\le x\le X,$$

with

$$A_1(k,s,q,r,j)=\exp\left(-\min\left(x,X_{k+1}^{k+2-s}\right)\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right)-\exp\left(-\min\left(x,X_{k+1}^{k+1-s}\right)\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right).$$

From another side we search $T_{k+1}(x)$ as follows:

$$T_{k+1}(x)=\sum_{r=1}^{l}\sum_{j=1}^{k+1-s} a_{k+1\ k+1-s\ \ j\ r}\exp\left(-\frac{\lambda_r x}{R^j}\right),\ X_{k+1}^{k+2-s}\le x\le X_{k+1}^{k+1-s},\ s=1,...,k+1 .$$

So for $s=1$ we have

$$\sum_{r=1}^{l}\sum_{j=0}^{k} a_{k+1\,k\,j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right)=\sum_{j=0}^{k-1}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r a_{k\,k-1\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}\left(\exp(-\lambda_q x)-\exp\left(-\frac{\lambda_r x}{R^{j+1}}\right)\right)$$

and obtain (4), (5). For $2\le s\le k+1$ we have

$$\sum_{r=1}^{l}\sum_{j=0}^{k+1-s} a_{k+1\,k+1-s\,j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right)=\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp(-\lambda_q x)a_{k\,k-t\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}A(k,s,q,r,j)+$$

$$+I(s\ne k+1)\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp(-\lambda_q x)a_{k\,k-s\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}\left(B(k,s,q,r,j)-\exp\left(-x\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right)\right).$$

As a result obtain (2), (4), (5).

Denominators in (4), (5) may be small or even zero. This circumstance creates difficulties in calculations. These difficulties may be got over by a following statement.

Lemma 1. *Suppose that* $\lambda_1,...,\lambda_l$ *are different positive numbers and* $R=m/n$ *where* m,n *are integers and mutually simple. Then for any* $\varepsilon>0$ *satisfying inequalities* $|\lambda_i-\lambda_j|>2\varepsilon$, $1\le i\ne j\le l$, *there are rational numbers* $\tilde\lambda_1,...,\tilde\lambda_l$ *so that*

$$|\lambda_i-\tilde\lambda_i|<\varepsilon,\ \tilde\lambda_i\ne R^k\tilde\lambda_j,\ 1\le i\ne j\le l,\ k\ge 0. \tag{6}$$

Proof. Denote $L=mn$ for any $\varepsilon>0$ there are integers $N,k_1,...,k_l$ so that

$$\frac{1}{NL}<\frac{\varepsilon}{2},\ \left|\lambda_i-\frac{k_i}{N}\right|<\frac{\varepsilon}{2},\ 1\le i\le l,$$

then

$$|\lambda_i-\tilde\lambda_i|<\varepsilon,\ \text{with}\ \tilde\lambda_i=\frac{k_iL+1}{NL},\ 1\le i\le l.$$

As $|\lambda_i-\lambda_j|>2\varepsilon$ so $\tilde\lambda_i\ne\tilde\lambda_j$, $1\le i\ne j\le l$. Each pair of integers (k_iL+1,L), $1\le i\le l$, has not joint divisors larger 1 so $(k_iL+1)n^k\ne(k_jL+1)m^k$, $1\le i\ne j\le l$, $k\ge 0$.

Remark 1. *Suppose that* $X=1, R=0.9, l=13$ *and* λ_i, $p_i, 1\le i\le l,$ *are described by Table 1 then in an accordance with Theorem 1 we obtain Table 2.*

i	λ_i	p_i
1	4.491	0.193963
2	1.422	0.651199
3	0.371	0.147817
4	0.076	0.006832
5	0.014	1.88×10^{-4}
6	0.03	4.61×10^{-6}
7	5×10^{-4}	1.11×10^{-7}
8	8.8×10^{-5}	2.65×10^{-9}
9	1.6×10^{-5}	6.35×10^{-11}
10	2.9×10^{-6}	1.52×10^{-12}
11	5.4×10^{-7}	3.63×10^{-14}
12	9.7×10^{-8}	8.61×10^{-16}
13	1.5×10^{-8}	1.72×10^{-17}

Table 1.

k	$P(\tau=k)$
3	0.267786
6	0.214032
9	0.001387
12	0.000051
15	1.621×10^{-6}
18	4.747×10^{-8}
21	1.345×10^{-9}
24	3.755×10^{-11}
27	1.042×10^{-12}
30	2.9×10^{-14}

Table 2.

Remark 2. *The results of Theorem 1 remain valid for variable boundary*

$$\tau = \inf\left(k : X_k \geq X\left(k\right)\right)$$

with the replacement of the designations

$$X_k^j = \min\left(X_{k-1}^{j-1}R, X\left(k\right)\right),\ j = 1,...,k-1,\ X_k^k = 0,\ X_k^0 = X\left(k\right),\ k \geq 1.$$

Remark 3. *Obtained formulas are true for variable* R*:*

$$X_k = R_{k-1}X_{k-1} + \eta_{k-1},\ 0 \leq R_{k-1} \leq 1\ ,$$

with the replacement of the designations

$$X_k^j = XR_k^j,\ R_k^j = R_{k-1}^{j-1}R_{k-1},\ 0 \leq j \leq k-1\ ,\ R_k^0 = 1,\ X_k^k = 0.$$

Remark 4. *In an accordance with (2) we have that a jump of* X_k*,* $k \geq 1$*, over a level* X *may be characterized by the following formula:*

$$P\left(X_k > X + y / \tau = k\right) = \frac{\sum_{r=1}^{l} a_{k00r} \exp\left(-\lambda_r \left(X + y\right)\right)}{\sum_{r=1}^{l} a_{k00r} \exp\left(-\lambda_r X\right)}, y > 0\ .$$

The authors thank A.A. Novikov for useful discussions.

References

[1] Asmussen S. *Ruin Probabilitie*s. Singapore: World Scientific, 2000.
[2] Feldmann A., Whitt W. *Fitting mixtures of exponentials to long-tailed distributions to analyze network perfomance models* //Perfomance Evaluation. 1998. Vol. 31. Pp. 245-279.