ACCURACY SOLUTION OF A.A. NOVIKOV PROBLEM

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Introduction

In this paper we consider the Laplas model described by the following autoregressive random sequence

$$X_k = RX_{k-1} + \eta_k,\ X_0 = 0,\ \tau = \inf(k : X_k \ge X),$$

$$P(\eta_k > t) = \frac{\exp(-\lambda t)}{2},\ t > 0,\ P(\eta_k \le t) = \frac{\exp(\lambda t)}{2},\ t \le 0,. \tag{1}$$

Our problem is to calculate a distribution of a reaching moment τ. This problem was put before the authors of this paper by A.A. Novikov. The problem origins in the risk theory and in the reliability theory. M. Jacobson found approximate solution of this problem by martingale technique. V.V. Mazalov suggested to solve the problem for $R<1$ approximately by some recurrent procedure which includes a compressing operator. In this paper we apply some recurrent integral equalities to find accuracy solution represented by mixtures of exponentials. This solution is illustrated by numerical calculations. Our solution may be used for an arbitrary R and when as R so X depend on k.

1. Main results

Denote $X_k^k = 0$, $X_k^{k-s} = XR^{k-s}$, $s = 1,...,k,$ and designate for $k \ge 1$

$$T_k(x) = P(X_k > x,\ \tau \ge k),\ x \ge 0,\ S_k(x) = P(X_k \le x, \tau \ge k),\ x < 0,$$

$$P(\tau = k) = T_k(X).$$

Denote

$$B_1(k,s,j) = \exp\left(-\lambda X_{k+1}^{k+2-s}\left(\frac{1}{R^{j+1}} - 1\right)\right),\ B_2(k,s,j) = \exp\left(\lambda X_{k+1}^{k+2-s}\left(\frac{1}{R^{j+1}} + 1\right)\right),$$

$$B_3(k,s,j) = \exp\left(-\lambda X_{k+1}^{k+1-s}\left(\frac{1}{R^{j+1}} + 1\right)\right),\ B_4(k,s,j) = \exp\left(\lambda X_{k+1}^{k+1-s}\left(\frac{1}{R^{j+1}} - 1\right)\right),$$

$$A_1(k,s,j) = B_1(k,s,j) - B_4^{-1}(k,s,j),\quad A_2(k,s,j) = B_2(k,s,j) - B_3^{-1}(k,s,j),$$

$$A_3(k,s,j) = B_2^{-1}(k,s,j) - B_3(k,s,j),\quad A_4(k,s,j) = B_1^{-1}(k,s,j) - B_4(k,s,j).$$

Theorem 1. *The following formulas are true for* $k \ge 0$, *and for* $x \ge 0$

$$T_k(x) = \sum_{j=0}^{k-s} a_{k\,k-s\,j} \exp\left(-\frac{\lambda x}{R^j}\right) + \sum_{j=0}^{k-s} b_{k\,k-s\,j} \exp\left(\frac{\lambda x}{R^j}\right) + c_{k\,k-s} \tag{2}$$

with $X_k^{k-s+1} \le x \le X_k^{k-s}$ *for some* $s \in \{1,...,k\}$ *and for* $x < 0$

$$S_k(x) = \sum_{j=0}^{k-1} d_{k\,j} \exp\left(\frac{\lambda x}{R^j}\right) \tag{3}$$

and

$$P(\tau = k) = a_{k\,0\,0} \exp(-\lambda X). \tag{4}$$

Here

$$a_{1\,0\,0}=\frac{1}{2},\ b_{1\,0\,0}=0,\ c_{1\,0}=0,\ d_{1\,0}=\frac{1}{2},\ c_{1\,-1}=0 \tag{5}$$

and

$$a_{k+1\,k+1-s\,j}=-a_{k\,k-s\,j-1}\frac{R^{2j}}{1-R^{2j}},\ 0<j\le k-s+1,\ 1\le s\le k, \tag{6}$$

$$b_{k+1\,k+1-s\,j}=-b_{k\,k-s\,j-1}\frac{R^{2j}}{1-R^{2j}},\ 0<j\le k-s+1,\ 1\le s\le k, \tag{7}$$

$$a_{k+1\,k+1-s\,0}=\frac{1}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\,k-t\,j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\,k-t\,j}}{1+R^{j+1}}\right]+\right.$$
$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\,k-s\,j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\,k-s\,j}}{1+R^{j+1}}\right]\right),\ 1\le s\le k+1, \tag{8}$$

$$b_{k+1\,k+1-s\,0}=-\frac{1}{2}\left(\sum_{t=s+1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,t,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{A_4(k,t,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]+\right.$$
$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_3(k,s,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{B_4(k,s,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]\right),\ 1\le s\le k,\ b_{k+1\,0\,0}=0, \tag{9}$$

$$c_{k+1\,k+1-s}=c_{k\,k-s}-a_{k\,0\,0}\exp(-\lambda X),\ 1\le s\le k,\ \ c_{k+1\,0}=c_{k+1\,-1}=0, \tag{10}$$

$$d_{k+1\,0}=\frac{1}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1-R^{j+1}}+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\,k-s\,j}}{1+R^{j+1}}A_3(k,s,j)+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\,k-s\,j}}{1-R^{j+1}}A_4(k,s,j)\right), \tag{11}$$

$$d_{k+1\,j+1}=-d_{k\,j}\frac{R^{2(j+1)}}{1-R^{2(j+1)}},\ 0\le j\le k-1. \tag{12}$$

2. Theorem 1 proof

It is obvious that

$$T_1(x)=\frac{\exp(-\lambda x)}{2},\ \ x>0,\ \ S_1(x)=\frac{\exp(\lambda x)}{2},\ \ x\le 0, \tag{13}$$

and

$$P(\tau=1)=\frac{\exp(-\lambda X)}{2}.$$

Calculate now for $x\ge 0$

$$T_2(x)=\left(\int_{-\infty}^{0}dS_1\left(\frac{u}{R}\right)-\int_{0}^{\min(x,XR)}dT_1\left(\frac{u}{R}\right)\right)\frac{\exp(-\lambda(x-u))}{2}\ -\int_{\min(x,XR)}^{XR}dT_1\left(\frac{u}{R}\right)\left(1-\frac{\exp(\lambda(x-u))}{2}\right).$$

As a result obtain

$$T_2(x)=\frac{1}{2\left(1-R^2\right)}\left(\exp(-\lambda x)-R^2\exp\left(-\frac{\lambda x}{R}\right)\right)-\frac{1}{2}\exp(-\lambda X)+$$
$$+\frac{\exp(\lambda x)\exp(-\lambda X(1+R))}{4(1+R)},\ 0\le x\le XR, \tag{14}$$

$$T_2(x)=\frac{\exp(-\lambda x)}{2}\left(\frac{1}{1-R^2}-\frac{\exp(\lambda(R-1)X)}{2(1-R)}\right),\ XR\le x\le X. \tag{15}$$

Calculate now for $x<0$

$$S_2(x)=\int_{-\infty}^{x}dS_1\left(\frac{u}{R}\right)\left(1-\frac{\exp(-\lambda(x-u))}{2}\right)+\left(\int_{x}^{0}dS_1\left(\frac{u}{R}\right)-\int_{0}^{XR}dT_1\left(\frac{u}{R}\right)\right)\frac{\exp(\lambda(x-u))}{2}.$$

As a result obtain

$$S_2(x)=\exp\left(\frac{\lambda x}{R}\right)\left(\frac{1}{2}-\frac{1}{4(1+R)}-\frac{1}{4(1-R)}\right)+$$
$$+\frac{\exp(\lambda x)}{2}\left(\frac{1}{2(1-R)}+\frac{1}{2(1+R)}(1-\exp(-\lambda X(1+R)))\right),\ x<0. \tag{16}$$

In an accordance with (14) - (16) assume by an induction that for $x\ge 0$ the formula (2) is true and for $x<0$ the formula (3) takes place. Then for $x\ge 0$

$$T_{k+1}(x)=J_1(x)+J_2(x)+J_3(x)+J_4(x). \tag{17}$$

Here

$$J_1(x)=\int_{-\infty}^{0} dS_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}. \tag{18}$$

Calculate now

$$J_2(x)=-\int_{0}^{\min\left(x,X_{k+1}^{1}\right)} dT_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=$$
$$=\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+ \tag{19}$$
$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\,k-t\,j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)},\ x\ge 0.$$

Then we have for $X_{k+1}^{k+2-s}\le x\le X_{k+1}^{k+1-s}$

$$J_3(x)=-\int_{\min\left(x,X_{k+1}^{1}\right)}^{X_{k+1}^{1}} dT_k\left(\frac{u}{R}\right)=-a_{k\,0\,0}\exp(-\lambda X)+$$
$$+\sum_{j=0}^{k-s}a_{k\,k-s\,j}\exp\left(-\frac{\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\,k-s\,j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\,k-s} \tag{20}$$

and

$$J_4(x)=\int_{\min\left(x,X_{k+1}^{1}\right)}^{X_{k+1}^{1}} dT_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=$$
$$=\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\,k-s\,j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{\min\left(\max\left(x,X_{k+1}^{k+2-s}\right),X_{k+1}^{1}\right)}^{\min\left(\max\left(x,X_{k+1}^{k+1-s}\right),X_{k+1}^{1}\right)}+ \tag{21}$$
$$+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\,k-s\,j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(\max\left(x,X_{k+1}^{k+2-s}\right),X_{k+1}^{1}\right)}^{\min\left(\max\left(x,X_{k+1}^{k+1-s}\right),X_{k+1}^{1}\right)}.$$

for $s\in\{1,...,k\}$,

$$J_3(x)=J_4(x)=0,\ s=k+1. \tag{22}$$

From (17) - (22) we have for $x\ge 0$, $s\in\{1,...,k\}$

$$T_{k+1}(x)=-a_{k\,0\,0}\exp(-\lambda X)+\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+ \tag{23}$$
$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\ k-t\ j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Big|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\sum_{j=0}^{k-s}a_{k\ k-s\ j}\exp\left(\frac{-\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\ k-s\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\ k-s}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\ k-t\ j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Big|_{\min\left(\max(x,X_{k+1}^{k+2-s}),X_{k+1}^{1}\right)}^{\min\left(\max(x,X_{k+1}^{k+1-s}),X_{k+1}^{1}\right)}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\ k-t\ j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Big|_{\min\left(\max(x,X_{k+1}^{k+2-s}),X_{k+1}^{1}\right)}^{\min\left(\max(x,X_{k+1}^{k+1-s}),X_{k+1}^{1}\right)}=$$

and for $s=k+1$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\ j}}{1+R^{j+1}}+ \tag{24}$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\ k-t\ j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Big|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\ k-t\ j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Big|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}.$$

Using the formulas (2) rewrite the formulas (23), (24) as follows: for $s\in\{1,...,k\}$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\ j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\ k-t\ j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\ k-t\ j}}{1+R^{j+1}}\right]+\right. \tag{25}$$

$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\ k-s\ j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\ k-s\ j}}{1+R^{j+1}}\right]\right)-$$

$$-\frac{1}{2}\sum_{j=0}^{k-s}\left[\frac{\exp\left(-\frac{\lambda x}{R^{j+1}}\right)a_{k\ k-s\ j}}{1-R^{j+1}}+\frac{\exp\left(\frac{\lambda x}{R^{j+1}}\right)b_{k\ k-s\ j}}{1+R^{j+1}}\right]+$$

$$+\sum_{j=0}^{k-s}a_{k\ k-s\ j}\exp\left(-\frac{\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\ k-s\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\ k-s}-a_{k\ 0\ 0}\exp(-\lambda X)-$$

$$-\frac{\exp(\lambda x)}{2}\left(\sum_{t=s+1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,t,j)a_{k\ k-s\ j}}{1+R^{j+1}}+\frac{A_4(k,t,j)b_{k\ k-s\ j}}{1-R^{j+1}}\right]-\right.$$

$$\left.-\sum_{j=0}^{k-s}\left[\frac{B_3(k,s,j)a_{k\ k-s\ j}}{1+R^{j+1}}+\frac{B_4(k,s,j)b_{k\ k-s\ j}}{1-R^{j+1}}\right]\right)-\frac{1}{2}\sum_{j=0}^{k-s}\left[\frac{\exp\left(-\frac{\lambda x}{R^{j+1}}\right)a_{k\ k-s\ j}}{1+R^{j+1}}+\frac{\exp\left(\frac{\lambda x}{R^{j+1}}\right)b_{k\ k-s\ j}}{1-R^{j+1}}\right]=$$

$$=\sum_{j=0}^{k+1-s}a_{k+1\ k+1-s\ j}\exp\left(-\frac{\lambda x}{R^{j}}\right)+\sum_{j=0}^{k+1-s}b_{k+1\ k+1-s\ j}\exp\left(\frac{\lambda x}{R^{j}}\right)+c_{k+1\ k+1-s},$$

for $s=k+1$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\ j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\ k-t\ j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\ k-t\ j}}{1+R^{j+1}}\right]+\right.$$

$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\ k-s\ j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\ k-s\ j}}{1+R^{j+1}}\right]\right)=a_{k+1\,0\,0}\exp(-\lambda x). \tag{26}$$

Calculate now for $x<0$

$$S_{k+1}(x)=\int_{-\infty}^{x} dS_k\left(\frac{u}{R}\right)\left(1-\frac{\exp(-\lambda(x-u))}{2}\right)+\left(\int_{x}^{0} dS_k\left(\frac{u}{R}\right)-\int_{0}^{X_{k+1}^{1}} dT_k\left(\frac{u}{R}\right)\right)\frac{\exp(\lambda(x-u))}{2}= \tag{27}$$

$$=\sum_{j=0}^{k-1} d_{k\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)\left(1-\frac{1}{2\left(1+R^{j+1}\right)}-\frac{1}{2\left(1-R^{j+1}\right)}\right)+$$

$$+\frac{1}{2}\sum_{j=0}^{k-1}\frac{d_{k\ j}\exp(\lambda x)}{1-R^{j+1}}+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\ k-s\ j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{X_{k+1}^{k+1-s}}^{X_{k+1}^{k+2-s}}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\ k-s\ j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{X_{k+1}^{k+1-s}}^{X_{k+1}^{k+2-s}}=$$

$$=\sum_{j=0}^{k-1} d_{k\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)\left(1-\frac{1}{2\left(1+R^{j+1}\right)}-\frac{1}{2\left(1-R^{j+1}\right)}\right)+$$

$$+\frac{\exp(\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\ j}}{1-R^{j+1}}+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,s,j)a_{k\ k-s\ j}}{1+R^{j+1}}+\frac{A_4(k,s,j)b_{k\ k-s\ j}}{1-R^{j+1}}\right]\right)=\sum_{j=0}^{k} d_{k+1\ j}\exp\left(\frac{\lambda x}{R^{j}}\right).$$

From the equalities (13) - (16), (25) - (27) we have the formulas (4) - (12). Theorem 1 is proved.

Remark 1. *Obtained formulas remain true in a case of variable boundary and interest rate:*

$$X_k = R_{k-1}X_{k-1}+\eta_{k-1},\ \tau=\inf\left(k : X_k \ge X(k)\right)$$

then we rewrite

$$X_k^k=0,\ X_k^0=X(k),\ X_k^j=\min\left(X_{k-1}^{j-1}R_{k-1},X(k)\right),\ R_k^0=1,\ R_k^j=R_{k-1}^{j-1}R_{k-1},$$

and replace R^{j+1} *by* R_{k+1}^{j+1} *and* R^j *by* R_{k+1}^{j} *in previous formulas,* $1\le j\le k-1$, $k\ge 1$, *without assumption* $R_{k-1}<1$.

Remark 2. *The proof of Theorem 1 contains sufficiently complicated and long symbol transformations. The transformations create manifold mistakes. To avoid these mistakes we examined the transformations by numerical calculations.*

Remark 3. *Suppose that X=1, R=0.5, λ=0.4491 then in an accordance with Theorem 1 we obtain Table 1.*

k	$P(\tau=k)$
10	0.03052
20	0.00512968
30	0.000861841
40	0.000144798
50	0.0000243276
60	4.08729×10^{-6}
70	6.86708×10^{-7}
80	1.15374×10^{-7}
90	1.93841×10^{-8}
100	3.25672×10^{-9}

Table 1.

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