DEAR THEORY IN SYSTEM DYNAMIC ANALYSIS

R. Guo
•
University of Cape Town, Cape Town, South Africa

e-mail: RenkuanGuo@uct.ac.za

D. Guo
•
South African National Biodiversity Institute, Cape Town, South Africa

e-mail: guo@sanbi.org

ABSTRACT

In this paper, we introduce our newly created DEAR (an abbreviation of Differential Equation Associated Regression) theory, which merges differential equation theory, regression theory and random fuzzy variable theory into a new rigorous small sample based inferential theoretical foundation. We first explain the underlying idea of DEAR modelling, its classification, and then the *M*-estimation of DEAR model. Furthermore, we explore the applicability of DEAR theory in the analysis in system dynamics, for example, repairable system analysis, quality dynamics analysis, stock market analysis, and ecosystem analysis, etc.

1 INTRODUCTION

In real world, many phenomena can be abstracted into mathematical dynamic systems. Differential equation theory provides many effective models for system dynamics. The focus of a system dynamics should be the characteristics intrinsic to the system and its evolving or developing patterns. To achieve this goal, the investigation on the system ought to base on the data extracted from the system itself. In other words, it is critical to utilize the sample data to test and validate hypothesized system model.

However, it is a well known fact that sampling from a system is usually a difficult task and an expensive exercise. Therefore, inference on the system dynamics based on small sample becomes an urgent and elementary task. Small sample inference has already obtained attention to many researchers, for example, in probability theory, the small sample asymptotics (Field and Ronchetti, 1990, 1991), the Bayesian inference, in fuzzy set theory proposed by Zadeh (1965, 1978), the plausible inference, and particularly, in the grey system theory proposed by Deng (1985), small sample inference is its flashing feature.

In this paper, to address the dilemma of using differential equation for describing continuous system dynamics, while only a small discrete data sequence sampled from the system is available, we propose Differential Equation Associated Regression, abbreviated as DEAR, model. DEAR theory couples differential equation and regression together (Guo et al., 2006) with delicate approximation schemes. However, these approximations introduce additional errors, which are identified as fuzzy error terms in nature. Thus, the coupled regression in DEAR theory is a random fuzzy regression.

2 NONLINEAR THINKING OF DEAR

Without loss of generality, a simple linear differential equation:

$$\frac{dx}{dt}=\alpha+\beta x \tag{1}$$

will be used in this paper for illustrative purpose. Let $\hat{x}_i^{(1)}$ denote an approximation to the primitive function $x(t)$ at t_i, and let $\Delta x_i/\Delta t_i$ be an approximation to the derivative function dx/dt at t_i, with $\Delta t_i = t_i - t_{i-1}$, and $\Delta x_i = x(t_i) - x(t_{i-1})$.

Definition 2.1: If a dynamic system governed by Equation 1 is sampled at its derivative level, denoted by $X^{(0)}=\left\{x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)}\right\}$, the coupled equation system

$$\begin{cases}\dfrac{dx}{dt}=\alpha+\beta x\\ x_i^{(0)}=\alpha+\beta\hat{x}_i^{(1)}+e_i,\ i=1,2,\ldots,n\end{cases} \tag{2}$$

is called Type I DEAR model.

Definition 2.2: If a dynamic system governed by Equation 1 is sampled at its primitive level, denoted by $X^{(1)}=\left\{x(t_1),x(t_2),\ldots,x(t_n)\right\}$, the coupled equation system

$$\begin{cases}\dfrac{dx}{dt}=\alpha+\beta x\\ \dfrac{\Delta x_i}{\Delta t_i}=\alpha+\beta x(t_i)+e_i,\ i=1,2,\ldots,n\end{cases} \tag{3}$$

is called Type II DEAR model.

Note that the second equation in the paired equation system like Equation 2 and 3 is called coupled regression, while the first one, i.e., the differential equation is called the associated differential equation.

Now, Let us examine Type I DEAR model first. The system dynamics is governed by the linear differential equation $dx/dt=\alpha+\beta x$, or equivalently, nonlinear functional $x(t)=f(t;\alpha,\beta)$. If the sample could be very large, it is possible to perform a non-linear statistical modelling in term of standard maximum likelihood procedure to estimate system parameter $\underline{\theta}=(\alpha,\beta)$. However, if only small sample observations are available, the "best" modelling exercise is to fit a simple regression model $\hat{x}(t)=\hat{\gamma}_0+\hat{\gamma}_1 t$, called primitive regression, for approximating the system dynamics $x(t)=f(t;\alpha,\beta)$. Figure 1 shows that the blue-dot straight line $\hat{x}(t)=\hat{\gamma}_0+\hat{\gamma}_1 t$ will poorly approximate nonlinear curve $x(t)=f(t;\alpha,\beta)$ in the (t,x) space (or (t,x)-coordinate system).

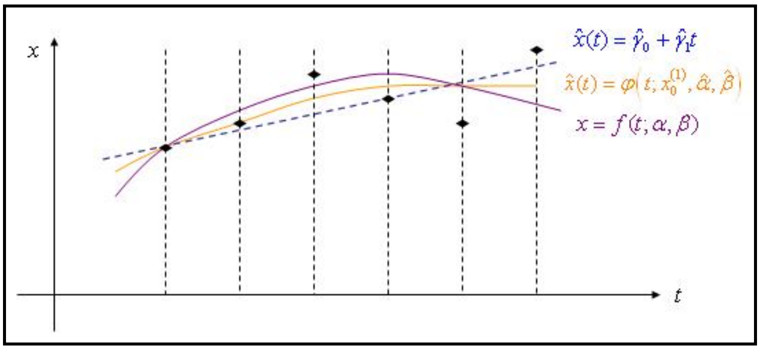

Figure 1. Two approximations to nonlinear curve $x(t)=f(t;\alpha,\beta)$ in (t,x) space.

Let us consider the case where sampling observations, $X^{(0)} = \{x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}\}$, are collected at derivative level. By a linear transformation, approximations to primitive function level observations are obtained, denoted by $\{x(t_1), \hat{x}(t_2), \cdots, \hat{x}(t_n)\}$, say, by partial sum. In terms of Type I DEAR model thinking, we first fit the coupled regression, i.e., the second equation in DEAR equation system in Equation 2 in the (x, x') space (or (x, x')-coordinate system), where x' denotes the derivative of x with respect to t, i.e., $x' = dx/dt$.

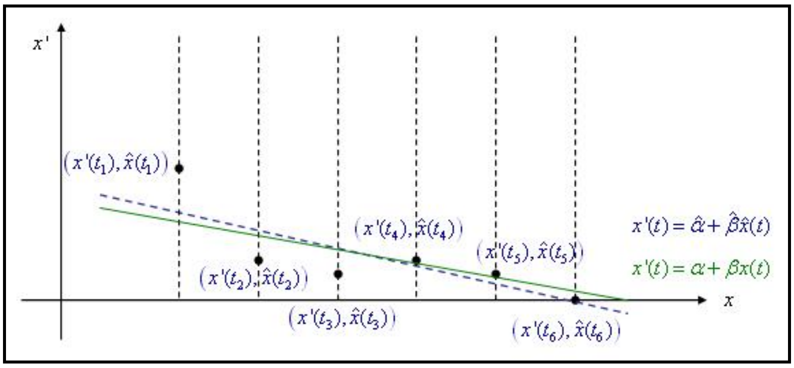

Figure 2. Type I approximation in (x, x') space.

From the fitting of the coupled regression, $x_i^{(0)} = a + b\hat{x}_i^{(1)} + e_i$, the estimator of parameter $\underline{\theta} = (\alpha, \beta)$, denoted by $\hat{\underline{\theta}} = (\hat{\alpha}, \hat{\beta})$ is obtained. Now, in the (x, x') space, we fit straight line $\hat{x}' = \hat{\alpha} + \hat{\beta}\hat{x}$ to approximate the straight line $x' = a + bx$. It is obvious this model goodness-of-fit could be very good even with small sample.

Once the parameter $\underline{\theta} = (\alpha, \beta)$ is obtained, by solving the approximated linear differential equation $dx/dt = \hat{\alpha} + \hat{\beta}x$, we will obtain an approximated nonlinear curve $x' = \varphi\left(t; x_0^{(1)}, \hat{\alpha}, \hat{\beta}\right)$, (yellow-colored curve in Figure 1), which is expected to approximate the primitive curve in relatively high accuracy.

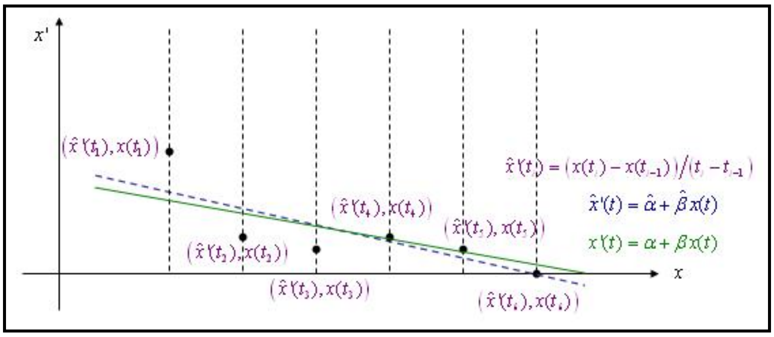

Figure 3. Type II approximation in (x, x') space.

Let us consider the case in which the sampling observations are collected at primitive function level, denoted as $X^{(1)} = \{x(t_1), x(t_2), \cdots, x(t_n)\}$. Then in terms of DEAR Type II model thinking, the derivatives could be approximated, for example, by the divided difference, i.e., $\Delta x_i / \Delta t_i$, or other approaches available. Just as shown in Figure 3, fitting $\hat{x}' = \Delta x / \Delta t = \hat{\alpha} + \hat{\beta}x$ for approximating line $x' = a + bx$. Similarly, the estimated parameter $\hat{\underline{\theta}} = \left(\hat{a}, \hat{b}\right)$ will lead the nonlinear approximation $x' = \varphi\left(t; x_0^{(1)}, \hat{a}, \hat{b}\right)$ to the primitive function $x(t) = f(t;a,b)$ in (t,x) space (shown in Figure 1).

It is necessary to emphasize here that DEAR model is often starting with hypothesized differential equation model for a system dynamics and then obtaining the corresponding coupled regression. The converse direction is also possible. In other words, after a regression model is established based on the small sample data extracted from an unknown system dynamics, an appropriate differential equation is selected according to the Coupling Principle stated in Guo et al. (2006) and then the DEAR model is built up. For example, a set of system data $\{x(t_i), i = 1,2,\cdots,n\}$ is collected and a fitted regression model takes the form

$$\hat{x}^{(0)}(t_i) = \frac{\Delta x_i}{\Delta t_i} = a\,x(t_i) + b\,x^m(t_i) \tag{4}$$

Then, the associated differential equation is a Bernoulli equation of the form:

$$\frac{dx}{dt} + p(t)x^2 = q(t)x^m \tag{5}$$

Then a Type II DEAR model is established

$$\begin{cases} \dfrac{dx}{dt} = a\,x + b\,x^m \\ \dfrac{\Delta x_i}{\Delta t_i} = a\,x(t_i) + b\,x^m(t_i) + e_i \end{cases} \tag{6}$$

It should be fully aware that the solution to the estimated Bernoulli equation

$$\frac{dx}{dt} = \hat{a}x + \hat{b}x^m \tag{7}$$

which results in a solution

$$\varphi\left(t;\hat{a},\hat{b}\right) = \sqrt[1-m]{\frac{\hat{b}}{\hat{a}}\left(e^{\hat{a}(1-m)(t-t_1)} - 1\right) + c(t_1)}, \quad (m \neq 0,1) \tag{8}$$

for facilitating the nonlinear approximation to the true system dynamics $x = f(t;a,b,g)$.

3 RANDOM FUZZY VARIABLE FOUNDATION FOR DEAR

In order to achieve the target of nonlinear modeling with small sample, DEAR utilizes various approximations. Type I DEAR model utilizes the approximation of an integral (i.e., primitive function) by partial sum, $\hat{x}(t_i) = \sum_{j=2}^{i} x'(t_i)\left(t_i - t_{j-1}\right)$ and Type II DEAR model relies on the approximation of a derivative, $x'(t_i)$, by divided difference, $\left(x(t_i) - x(t_{i-1})\right)/\left(t_i - t_{i-1}\right)$.

The approximation brings error, which is fuzzy in nature according to nonclassical mathematical analysis. The total error term, $\varepsilon_i = \zeta_i + e_i$, in coupled regression will come from two error sources: random sampling error, denoted by e, and the approximation-caused fuzzy error, denoted by ζ. Therefore, the coupled regression is a random fuzzy variable regression. Therefore, we need to have some knowledge of random fuzzy variable theory.

Random fuzzy variable is a special case of hybrid variable defined in a chance space proposed by Liu (2004), which is a Cartesian product of a probability space and a credibility space for describing hybrid events in which randomness and fuzziness coexist.

Definition 3.1: (Liu, 2007) Let $(\Theta, 2^{\Theta}, \mathrm{Cr})$ be a credibility measure space and $(\Omega, \mathcal{A}, \Pr)$ a probability space. The product $(\Theta, 2^{\Theta}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space.

Typically, the product $(\Theta, 2^{\Theta}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \Pr)$ may be written as $(\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Cr} \times \Pr)$. The Cartesian product space $\Theta \times \Omega$ is typically defined by $\Theta \times \Omega = \{(\theta, \omega) : \theta \in \Theta, \omega \in \Omega\}$ and the Cartesian product σ-algebra $2^{\Theta} \times \mathcal{A} = \{A \times B : A \in 2^{\Theta}, B \in \mathcal{A}\}$, which is a special σ-algebra constituted by events with product form $A \times B,\ A \in 2^{\Theta}, B \in \mathcal{A}$. Note here that 2^{Θ} is the power set of space Θ, which is the largest σ-algebra of set Θ, while $\mathcal{A}$ is just a σ-algebra of set Ω. Therefore, $2^{\Theta} \times \mathcal{A}$ is a σ-algebra of set $\Theta \times \Omega$, but a very special one. As to $\mathrm{Cr} \times \Pr$ which is a product measure of the two essential uncertain measures: credibility measure and probability measure. Nevertheless, the product measure may take different forms. One of them, which satisfies the requirements of uncertainty measure proposed by Liu (2004), is called the chance measure, denoted as $\mathrm{Ch}\{\ \}$, which is composed of the two essential measures: credibility measure and probability measure.

Definition 3.2: (Liu, 2007) Let $(\Theta, 2^{\Theta}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space and an (measurable) event of form $Z = X \times Y$ such that $Z = \{\theta : \theta \in X \subset \Theta\} \times \{\omega : \omega \in Y \subset \Omega\} \in 2^{\Theta} \times \mathcal{A}$, then a chance measure is defined as:

$$\mathrm{Ch}\{Z\} = \begin{cases} \sup_{\theta \in \Theta} \left(\mathrm{Cr}\{\theta\} \wedge \Pr\{Z(\theta)\} \right) & \text{if } \sup_{\theta \in \Theta} \left(\mathrm{Cr}\{\theta\} \wedge \Pr\{Z(\theta)\} \right) < 0.5 \\ 1 - \sup_{\theta \in \Theta} \left(\mathrm{Cr}\{\theta\} \wedge \Pr\{Z^{c}(\theta)\} \right) & \text{if } \sup_{\theta \in \Theta} \left(\mathrm{Cr}\{\theta\} \wedge \Pr\{Z(\theta)\} \right) \geq 0.5 \end{cases} \tag{9}$$

If the product measure $\mathrm{Cr} \times \Pr$ is defined by the chance measure defined in Definition 2.9, i.e., $\mathrm{Cr} \times \Pr\{\cdot\} = \mathrm{Ch}\{\cdot\}$, then the chance measure space $(\Theta, 2^{\Theta}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \Pr)$ may be written as $(\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch})$.

Definition 3.3: (Liu, 2007) Let $(\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch})$ be a chance space. A hybrid variable $\xi : (\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch}) \to \mathrm{R}$ is a measurable function from the chance space into a set of real numbers. In other words, for any Borel set of real numbers, $B \in \mathcal{B}(\mathrm{R})$, event $\{(\theta, \omega) \in \Theta \times \Omega : \xi(\theta, \omega) \in B\} \in 2^{\Theta} \times \mathcal{A}$.

The typical examples of hybrid variables are fuzzy random variable and random fuzzy variable. Liu (2004, 2007) defines a random fuzzy variable as a measurable mapping from the credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to a set of random variables. Again, we should be aware that a random fuzzy variable here takes real numbers as its values, which behaves very similar to a random variable.

Definition 3.4: Let $(\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch})$ be a chance space and ξ be a hybrid variable. Then the chance distribution $\Upsilon : (\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch}) \to [0,1]$ for ξ if and only if:

$$\Upsilon(x) = \mathrm{Ch}\{(\theta, \omega) \in \Theta \times \Omega : \xi(\theta, \omega) \leq x\} \tag{10}$$

Theorem 3.5: (Liu, 2007) Let $(\Theta \times \Omega, 2^{\Theta} \times \mathcal{A}, \mathrm{Ch})$ be a chance space. A function $\Upsilon : \mathrm{R} \to [0,1]$ is a chance distribution for a hybrid variable ξ if and only if:

$$\lim_{x\to-\infty}\Upsilon(x)\le 0.5\le\lim_{x\to+\infty}\Upsilon(x)$$
$$\lim_{y\downarrow x}\Upsilon(y)=\Upsilon(x)\ \text{if}\ \lim_{y\downarrow x}\Upsilon(y)>0.5\ \text{or}\ \Upsilon(x)\ge 0.5 \tag{11}$$

Definition 3.6: (Liu, 2007) Let $(\mathrm{Q}r\ \mathrm{W},2^{\mathrm{Q}}r\ A,\mathrm{Ch})$ be a chance space and $\Upsilon(\cdot)$ be the chance distribution for a hybrid variable ξ, a function $\varphi:\mathrm{R}\to[0,+\infty)$ is called as a chance density if and only if:

$$\Upsilon(x)=\int_{-\infty}^{x}\varphi(y)\mathrm{d}y$$
$$\int_{-\infty}^{+\infty}\varphi(y)\mathrm{d}y=1 \tag{12}$$

Definition 3.7: Let $(\mathrm{Q}r\ \mathrm{W},2^{\mathrm{Q}}r\ A,\mathrm{Ch})$ be a chance space and $\Upsilon(\cdot)$ be the chance distribution for a hybrid variable ξ. The chance distribution $\Upsilon(\cdot)$ is absolutely continuous if and only if the chance density $\varphi(\cdot)$ is continuous.

The discussions of the chance distribution $\Upsilon(\cdot)$ will be limited in the class of absolutely continuous chance distributions.

Theorem 3.8: Let $(\mathrm{Q}r\ \mathrm{W},2^{\mathrm{Q}}r\ A,\mathrm{Ch})$ be a chance space and $\Upsilon(\cdot)$ be the chance distribution for a hybrid variable ξ, which is absolutely continuous. Then:

$$\Upsilon(-\infty)=0,\ \Upsilon(+\infty)=1$$
$$\Upsilon(x)<\Upsilon(y)\ \text{if}\ x<y,\ \forall x,y\in\mathrm{R} \tag{13}$$

Furthermore, the inverse function of $\Upsilon\{\cdot\}$ exists and is denoted as $\Upsilon^{-1}(\cdot)$.

Definition 3.9: (Liu, 2004) Let $(\mathrm{Q}r\ \mathrm{W},2^{\mathrm{Q}}r\ A,\mathrm{Ch})$ be a chance space and ξ be a hybrid variable. Then the expected value of ξ is defined by:

$$\mathrm{E}_{\Upsilon}[\xi]=\int_{0}^{+\infty}\mathrm{Ch}\{\xi\ge r\}\mathrm{d}r-\int_{-\infty}^{0}\mathrm{Ch}\{\xi\le r\}\mathrm{d}r \tag{14}$$

Let $e=\mathrm{E}_{\Upsilon}[\xi]$, then the variance is defined as $V_{\Upsilon}[\xi]=\mathrm{E}_{\Upsilon}\left[(\xi-e)^2\right]$.

Finally, let us discuss the average hance measure concept given by Liu (2007).

Definition 3.10: Let $(\mathrm{Q},2^{\mathrm{Q}},\mathrm{Cr})r\ (\mathrm{W},A,\mathrm{Pr})$ be a chance space and ξ be a random fuzzy variable, then the average chance distribution is

$$\Psi(x)=\mathrm{ch}\{\xi\le x\}=\int_{0}^{1}\mathrm{Cr}\left\{\theta:\mathrm{Pr}\{\xi(\omega,\theta)\le x\}\ge\beta\right\}\mathrm{d}\beta \tag{15}$$

and the average chance density is a positive function $\psi:\mathrm{R}\to\mathrm{R}^{+}$ such that

$$\Psi(x)=\int_{-\infty}^{x}\psi(u)\mathrm{d}u \tag{16}$$

If the product measure $\mathrm{Cr}\times\mathrm{Pr}$ is defined by the average chance measure defined in Equation 10, i.e., $\mathrm{Cr}\times\mathrm{Pr}\{\cdot\}=\mathrm{ch}\{\cdot\}$, then the average chance measure space $\left(\Theta, 2^{\Theta}, \mathrm{Cr}\right)\times(\Omega, \mathcal{A}, \mathrm{Pr})$ may be written as $\left(\Theta\times\Omega, 2^{\Theta}\times\mathcal{A}, \mathrm{ch}\right)$.

The error structure in the dear modelling theory is assumed to be random fuzzy

$$\varepsilon = e + \tau \tag{17}$$

in which e is the fuzzy approximation error to the derivative and τ is the random error term.
For inference purposes, similar to statistical linear model theory, it is typically assumed that the random error is normal variable with zero-mean and constant variance, i.e., $\tau \sim N\left(0,\sigma^2\right)$.

However, the fuzzy error e is intrinsically dependent upon point x, the difference on x when using divided difference to approximate derivative at point x. Let e be assumed to be a triangular fuzzy variable with a membership having parameter $(x-a,x,x+b)$, $a>0,b>0$,

$$\mu_e(u)=\begin{cases}\dfrac{u-x+a}{a} & x-a<u\le x\\ \dfrac{b-u+x}{b} & x<u\le x+b\\ 0 & \text{otherwise}\end{cases} \tag{18}$$

Accordingly, the credibility distribution function of fuzzy error e at point x is

$$\Lambda(e)=\begin{cases}0 & \text{if } e\le x-a\\ \dfrac{e-(x-a)}{2a} & \text{if } x-a\le e<x\\ \dfrac{e+b-x}{2b} & \text{if } x\le e<x+b\\ 1 & \text{if } e\ge x+b\end{cases} \tag{19}$$

Then the average chance distribution of normal random fuzzy error term ε at point x takes a form

$$\begin{aligned}\Psi(\varepsilon) &= \frac{\varepsilon-(x-a)}{2a}\left(\Phi\left(\frac{\varepsilon-(x-a)}{\sigma}\right)-\Phi\left(\frac{\varepsilon-x}{\sigma}\right)\right)\\ &+\frac{\varepsilon+b-x}{2b}\left(\Phi\left(\frac{\varepsilon-x}{\sigma}\right)-\Phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\right)+\Phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\\ &-\frac{\sigma}{2a}\int_{\frac{\varepsilon-x}{\sigma}}^{\frac{\varepsilon-(x-a)}{\sigma}} u\phi(u)\mathrm{d}u-\frac{\sigma}{2b}\int_{\frac{\varepsilon-(x+b)}{\sigma}}^{\frac{\varepsilon-x}{\sigma}} u\phi(u)\mathrm{d}u\end{aligned} \tag{20}$$

and the average chance density is

$$\begin{aligned}\psi(\varepsilon) &= \frac{1}{2a}\left(\Phi\left(\frac{\varepsilon-(x-a)}{\sigma}\right)-\Phi\left(\frac{\varepsilon-x}{\sigma}\right)\right)\\ &+\frac{\varepsilon-(x-a)}{2a\sigma}\left(\phi\left(\frac{\varepsilon-(x-a)}{\sigma}\right)-\phi\left(\frac{\varepsilon-x}{\sigma}\right)\right)\\ &+\frac{1}{2b}\left(\Phi\left(\frac{\varepsilon-x}{\sigma}\right)-\Phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\right)\\ &+\frac{\varepsilon+b-x}{2b\sigma}\left(\phi\left(\frac{\varepsilon-x}{\sigma}\right)-\phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\right)+\frac{1}{\sigma}\phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\\ &-\frac{1}{2a}\left(\frac{\varepsilon-(x-a)}{\sigma}\phi(\frac{\varepsilon-(x-a)}{\sigma})-\frac{\varepsilon-x}{\sigma}\phi\left(\frac{\varepsilon-x}{\sigma}\right)\right)\\ &-\frac{1}{2b}\left(\frac{\varepsilon-x}{\sigma}\phi\left(\frac{\varepsilon-x}{\sigma}\right)-\frac{\varepsilon-(x+b)}{\sigma}\phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\right)\end{aligned} \tag{21}$$

i.e.,

$$\begin{aligned}\psi(\varepsilon) &= \frac{1}{2a}\left(\Phi\left(\frac{\varepsilon-(x-a)}{\sigma}\right)-\Phi\left(\frac{\varepsilon-x}{\sigma}\right)\right)\\ &+\frac{1}{2b}\left(\Phi\left(\frac{\varepsilon-x}{\sigma}\right)-\Phi\left(\frac{\varepsilon-(x+b)}{\sigma}\right)\right)\end{aligned} \tag{22}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and cumulative distribution function of standard normal random variable respectively.

4 *M*-ESTIMATOR FOR DEAR PARAMETERS

Assuming that the system is sampled at system primitive function level, $x(t)$, the n observation is denoted by $X^{(1)} = \left(x^{(1)}(t_1), x^{(1)}(t_2), \cdots, x^{(1)}(t_n)\right)$, then a Type II model of dear subfamily takes a form

$$\begin{cases} \dfrac{dx}{dt} + bx = q_0 + q_1 t + q_2 t^2 \\ y_k = \dfrac{\mathrm{D}x^{(1)}(t_k)}{\mathrm{D}t_k} = q_0 + q_1 t_k + q_2 t_k^2 + (-b)x^{(1)}(t_k) \end{cases} \tag{23}$$

According to Liu's Maximum Uncertainty Principle (Liu, 2007), for independent random fuzzy variables, the object function can be formed in the following way,

$$\begin{aligned} & J(q_0, q_1, q_2, \beta; a, b, \sigma) \\ &= \sum_{k=2}^{n}\left(\Psi_h\left(y_k - \left(q_0 + q_1 t_k + q_2 t_k^2 + (-\beta)x_k\right)\right) - 0.5\right)^2 \end{aligned} \tag{24}$$

Denote $\varepsilon_k = y_k - \left(q_0 + q_1 t_k + q_2 t_k^2 - \beta x_i\right)$

$$\begin{aligned} & \frac{\partial J(\theta_0, \theta_1, \theta_2, \theta_3; a, b, \sigma)}{\partial \theta_i} \\ &= 2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k) - 0.5\right)\frac{\partial}{\partial \theta_i}\Psi(\varepsilon_k) \\ &= 2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k) - 0.5\right)\psi(\varepsilon_k)\frac{\partial(\varepsilon_k)}{\partial \theta_i} = 0 \end{aligned} \tag{25}$$

Then *M*-functional equation system is then

$$\begin{cases} \dfrac{\partial J}{\partial q_0} = -2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)=0 \\ \dfrac{\partial J}{\partial q_1} = -2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)t_k=0 \\ \dfrac{\partial J}{\partial q_2} = -2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)t_k^2=0 \\ \dfrac{\partial J}{\partial \beta} = 2\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)x_k=0 \\ \dfrac{\partial J}{\partial a} = \sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\dfrac{x_k-\varepsilon_k}{2a^2}\left(\Phi\left(\dfrac{\varepsilon_k-(x_k-a)}{\sigma}\right)-\Phi\left(\dfrac{\varepsilon_k-x_k}{\sigma}\right)\right)+\dfrac{\sigma}{2a^2}\int_{\frac{\varepsilon_k-x_k}{\sigma}}^{\frac{\varepsilon_k-(x_k-a)}{\sigma}}u\phi(u)\mathrm{d}u\right)=0 \\ \dfrac{\partial J}{\partial b} = \sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\dfrac{x_k-\varepsilon_k}{2b^2}\left(\Phi\left(\dfrac{\varepsilon_k-x_k}{\sigma}\right)-\Phi\left(\dfrac{\varepsilon_k-(x_k+b)}{\sigma}\right)\right)+\dfrac{\sigma}{2b^2}\int_{\frac{\varepsilon_k-(x_k+b)}{\sigma}}^{\frac{\varepsilon_k-x_k}{\sigma}}u\phi(u)\mathrm{d}u\right)=0 \\ \dfrac{\partial J}{\partial \sigma}=0 \end{cases} \tag{26}$$

where

$$\begin{aligned} \frac{\partial J}{\partial \sigma} =& -\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\frac{\varepsilon_k-(x_k-a)}{2a\sigma^2}\left((\varepsilon_k-(x_k-a))\phi\left(\frac{\varepsilon_k-(x_k-a)}{\sigma}\right)-\phi\left(\frac{\varepsilon_k-x_k}{\sigma}\right)\right)\right) \\ &-\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\frac{\varepsilon_k+b-x_k}{2b\sigma^2}\left((\varepsilon_k-x_k)\phi\left(\frac{\varepsilon_k-x_k}{\sigma}\right)-(\varepsilon_k-(x_k+b))\phi\left(\frac{\varepsilon_k-(x_k+b)}{\sigma}\right)\right)\right) \\ &-\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\frac{\varepsilon_k-(x_k+b)}{\sigma^2}\phi\left(\frac{\varepsilon_k-(x_k+b)}{\sigma}\right) \\ &-\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\frac{1}{2a}\int_{\frac{\varepsilon_k-x_k}{\sigma}}^{\frac{\varepsilon_k-(x_k-a)}{\sigma}}u\phi(u)du-\frac{1}{2a\sigma}\left(\frac{(\varepsilon_k-(x_k-a))^2}{\sigma}\phi\left(\frac{\varepsilon_k-(x_k-a)}{\sigma}\right)-\frac{(\varepsilon_k-x_k)^2}{\sigma}\phi\left(\frac{\varepsilon_k-x_k}{\sigma}\right)\right)\right) \\ &-\sum_{k=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\left(\frac{1}{2b}\int_{\frac{\varepsilon_k-(x_k+b)}{\sigma}}^{\frac{\varepsilon_k-x_k}{\sigma}}u\phi(u)du-\frac{1}{2b\sigma}\left(\frac{(\varepsilon_k-x_k)^2}{\sigma}\phi\left(\frac{\varepsilon_k-x_k}{\sigma}\right)-\frac{(\varepsilon_k-(x_k+b))^2}{\sigma}\phi\left(\frac{\varepsilon_k-(x_k+b)}{\sigma}\right)\right)\right) \end{aligned} \tag{27}$$

Then, the solution to the *M*-equation (non-linear) equation system, denoted as $\left(\hat{q}_0,\hat{q}_1,\hat{q}_2,\hat{\beta},\hat{a},\hat{b},\hat{\sigma}\right)$, is called an *M*-estimator of Subfamily A of dear model. Particularly, $\underline{\hat{\Gamma}}^T=\left(\hat{q}_0,\hat{q}_1,\hat{q}_2,\hat{\beta}\right)$ is the *M*-estimator for the coefficients defining the motivated differential equation.

$$\begin{cases} \sum_{i=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)=0 \\ \sum_{i=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)t_k=0 \\ \sum_{i=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)t_k^2=0 \\ \sum_{i=2}^{n}\left(\Psi(\varepsilon_k)-0.5\right)\psi(\varepsilon_k)x_k=0 \end{cases} \tag{28}$$

Define

$$\begin{aligned} g(\varepsilon_k) = & \frac{\sigma}{2a} \int_{\frac{\varepsilon_k - x_k}{\sigma}}^{\frac{\varepsilon_k-(x_k-a)}{\sigma}} u\phi(u)\mathrm{d}u + \frac{\sigma}{2b} \int_{\frac{\varepsilon_k-(x_k+b)}{\sigma}}^{\frac{\varepsilon_k - x_k}{\sigma}} u\phi(u)\mathrm{d}u \\ & + \frac{x_k - a}{2a}\left(\Phi\left(\frac{\varepsilon_k - (x_k - a)}{\sigma}\right) - \Phi\left(\frac{\varepsilon_k - x_k}{\sigma}\right)\right) \\ & + \frac{x_k - b}{2b}\left(\Phi\left(\frac{\varepsilon_k - x_k}{\sigma}\right) - \Phi\left(\frac{\varepsilon_k - (x_k + b)}{\sigma}\right)\right) \\ & + 0.5 - \Phi\left(\frac{\varepsilon_k - (x_k + b)}{\sigma}\right) \end{aligned} \tag{29}$$

then

$$\Psi(\varepsilon_k) - 0.5 = \varepsilon_k \psi(\varepsilon_k) - g(\varepsilon_k) \tag{30}$$

Further define

$$\hat{y}_k = \hat{\alpha}_0 + \hat{\alpha}_1 t_k + \hat{\alpha}_2 t_k^2 - \hat{\beta} x_k \tag{31}$$

Denote

$$\underline{y} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, X = \begin{bmatrix} 1 & t_2 & t_2^2 & -x_2 \\ 1 & t_3 & t_3^2 & -x_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & -x_n \end{bmatrix}, W^{-1} = \begin{bmatrix} \psi(\varepsilon_2) & 0 & \cdots & 0 \\ 0 & \psi(\varepsilon_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi(\varepsilon_n) \end{bmatrix}, \underline{g} = \begin{bmatrix} g(\varepsilon_2) \\ g(\varepsilon_3) \\ \vdots \\ g(\varepsilon_n) \end{bmatrix} \tag{32}$$

Equation 28 can be written as an adjusted weighted normal equation form

$$X^T W^{-2} X \underline{\Gamma} = X^T W^{-1} \underline{y} + X^T \underline{g} \tag{33}$$

Finally, the coefficient *M*-estimator $\hat{\Gamma}$ satisfies the adjusted weighted normal equation, Equation 33, which is expected to play critical roles in the variance-covariance estimation for the *M*-estimator $\hat{\Gamma}$ by noticing that

$$\underline{\hat{\Gamma}} = \left(X^T W^{-2} X\right)^{-1} X^T W^{-1} \underline{y} + \left(X^T W^{-2} X\right)^{-1} X^T \underline{g} \tag{34}$$

Remark 4.1: The *M*-estimator for coupled differential equation coefficients actually specify the dynamics fully. However, we need to be aware that $\underline{\hat{\Gamma}}$ itself is a random fuzzy vector because the random fuzzy nature of the "observations" $\{y_k\}$.

5 APPLICABILITY OF DEAR MODEL

DEAR model in nature revealing the intrinsic changing dynamics of a continuous system. The final mathematical structure is an estimated differential equation for approximating the true dynamics. Therefore DEAR model may apply to any system governed by differential equation(s).

5.1 Repair effect estimation

Repairable system analysis and maintenance optimization are a problem to reveal the law of the system functioning dynamics and the evaluation of repair effects in terms of system performance data in statistical sense. It is noticeable that another class of system maintenance optimization papers appeared in journals and conferences, however, most of them are seeking "system optimum" under mathematical assumptions without justifications in terms of actual system performances. It is obvious the later models are in mathematical sense.

The repairable system dynamics in DEAR platform assumes that a system is governed by differential equation (either single one or a set of equations), say $T = f(t;\underline{q})$. Due to various internal and external causes, system demonstrates a repeated pattern of functioning, stop, repairing, and resuming function again (Guo, 2007). As an illustration, let us assume the system dynamics is governed by

$$\frac{dT}{dt} = \alpha T^2 + \beta T + \gamma \tag{35}$$

Then from system functioning time records, denotes as $\{T(t_1), T(t_2), \ldots, T(t_n)\}$. Then the DEAR system is

$$\begin{cases} \dfrac{dT}{dt} = \alpha + \beta T \\ \dfrac{\Delta T_i}{\Delta t_i} = \alpha T^2(t_i) + \beta T(t_i) + \gamma + \varepsilon_i \end{cases} \tag{36}$$

Let $W^{-1} = \operatorname{diag}\left(\delta\left(\dfrac{\Delta T_i}{\Delta t_i} - \left(\alpha T^2(t_i) + \beta T(t_i) + \gamma\right)\right)^2\right)$ and

$$\underline{Y} = \begin{bmatrix} \Delta T_1/\Delta t_1 \\ \Delta T_2/\Delta t_2 \\ \vdots \\ \Delta T_n/\Delta t_n \end{bmatrix}, \ \underline{\Pi} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \ X = \begin{bmatrix} T^2(t_1) & T(t_1) & 1 \\ T^2(t_2) & T(t_2) & 1 \\ \vdots & \vdots & \vdots \\ T^2(t_n) & T(t_2) & 1 \end{bmatrix} \tag{37}$$

Then the *M*-estimator for $\underline{\Pi}$ is

$$\hat{\underline{\Pi}} = \left(X^T W^{-1} X\right)^{-1} X^T W^{-1} \underline{Y} \tag{38}$$

Hence the approximate Riccati equation takes the form

$$\frac{dT}{dt} = \hat{\alpha} T^2 + \hat{\beta} T + \hat{\gamma} \tag{39}$$

Denote the solution to Equation 33 by $\hat{T} = \varphi\left(t, \hat{\underline{\Pi}}\right)$, which will be used to approximate the true system functioning dynamics $T = f(t, \underline{\Pi})$. Also denote the "weighted" residual by $\hat{\varepsilon}_i^v$ resulting from Equation 36. Define the residual

$$\hat{\varepsilon}_i = T(t_i) - \left(\hat{\alpha} T(t_i) + \hat{\beta} T(t_i) + \hat{\gamma}\right) \tag{40}$$

The actual function time can be partitioned into three terms:

$$T(t_i) = \hat{T}(t_i) + \hat{\varepsilon}_i^v + \hat{\tau}_i \tag{41}$$

It is obvious that the fitted dynamics $\hat{T}(t_i)$ and weighted residual $\hat{\varepsilon}_i^v$ are DEAR-explained. Notice that the term $\hat{\tau}_i = \hat{\varepsilon}_i - \hat{\varepsilon}_i^v$ is DEAR-unexplained quantity. Therefore the logical interpretation of $\hat{\tau}_i$ is repair effect (accumulated at time t_i). In general, $\{\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_n\}$ are random fuzzy quantities so that the parameters for the average chance distribution of $\{\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_n\}$ can be obtained.

5.2 DEAR predictive quality control charts

Carvalho and Machado (2006) pointed, "In a global market, companies must deal with a high rate of changes in business environment. … The parameters, variables and restrictions of the production system are inherently vagueness." In other words, the shortening product life cycle and diversification have brought the vagueness and randomness together, which is a form of hybrid uncertainty, into manufacturing systems. Therefore, the traditionally continuous production and large sample based quality control schemes may not be suitable. Therefore establishing small sample oriented approximate quality index differential equation in terms of DEAR theory, which enjoys highly predictive power will help quality assurance in today's industries greatly.

Guo (2006) and Guo and Dunne (2006) have explored the predictive quality control schemes in terms of grey differential equation model. The DEAR-predictive control schemes will avoid the weakness in earlier work and offer a more rigorous development.

5.3 Climate Change Modelling

Climate changes have posed high risk on earth ecosystems. Environmental research communities now successfully convince governmental leaders worldwide and let the climate change become a hot topic. Biodiversity evolution is also a system dynamics governed by complicated differential equation systems. The critical issue is the parameter estimation for the differential equation systems. Biodiversity researchers have managed initial success in terms of multivariate version of DEAR model – PDEAR, for example, D. Guo et al. (2007, 2008), and R. Guo et al. (2008). Predictably, DEAR modeling in ecosystem will get more and more attention in the future.

6 CONCLUDING REMARKS AND OPEN QUESTION

In this paper we introduce a new small sample based continuous differential equation modeling theory. We use a simple linear equation in Equation 1 for illustrative purposes, however, as we pointed out that DEAR contains a collection of rich families. Table 1 offers a collection of partial families in Type II DEAR model.

Table 1. The richness of DEAR families

Family	Type II DEAR
Family 1	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$
Family 2	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 e^{\delta t} + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 e^{\delta t_k} + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$
Family 3	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 \sin(\omega t + \varpi) + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 \sin(\omega t_k + \varpi) + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$
Family 4	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 e^{\delta t} \sin(\omega t + \varpi) + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 e^{\delta t_k} \sin(\omega t_k + \varpi) + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$
Family 5	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 p_q(t) + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 p_q(t_k) + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$
Family 6	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 e^{\delta t} p_q(t) + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 e^{\delta t_k} p_q(t_k) + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$

Family 7	$\begin{cases} \dfrac{dx}{dt} = \alpha_0 p_q(t)\sin(\omega t + \varpi) + \alpha_1 x \\ \dfrac{\Delta x_k}{\Delta t_k} = \alpha_0 p_q(t_k)\sin(\omega t_k + \varpi) + \alpha_1 x(t_k) + \varepsilon_k \end{cases}$

However, there are many open questions and many challenges in future DEAR developments. The first one is model specification (or identification) problem. It is true that DEAR model starts with a hypothesized differential equation model. Given real-world dynamic system, it is often there is no priori knowledge on the system and thus many possible candidate differential equation models may be suitable for the hypothesized model. Which one would be the best? We can not guarantee anything, particularly under the small sample availability. The second question is the model validation problem, which can be considered from the two aspects: the filtering the existing data (or backward prediction) and the extrapolation (or forward prediction). In either case, model accuracy criteria are required to be investigated, particularly, we have to admit that the average chance of quadratic forms and ratio of them are undeveloped yet. The third open question is given a set of data, the DEAR model may just start with a set of subset regression models and then couple with corresponding differential equation models. For example, Table 2 lists a set of data from a system.

Table 2. A system state recording data

No.	Time t_k	Obs $x^{(1)}(t_k)$	Approx. Der. $\hat{x}^{(0)}(t_k)$	Der. $x^{(0)}(t_k)$
1	0.50	7.1788	N/A	
2	0.55	7.1236	-1.104573	-1.0198
3	0.60	7.0768	-0.935078	-0.8504
4	0.65	7.0385	-0.765817	-0.6813
5	0.70	7.0087	-0.596785	-0.5123
6	0.75	6.9873	-0.427976	-0.3436
7	0.80	6.9743	-0.259385	-0.1752

For this data set, Table 3 lists 10 sub-regression models with excellent R-square value and significant regression coefficients.

Table 3. 10 fitted sub-regression models

No.	Sub-Regression Fitted	R^2
1	$y = \underset{(0.001122)}{-2.9636} + \underset{(0.00165)}{3.380732}t$	0.99999
2	$y = \underset{(3.9293678)}{37.534371} - \underset{(0.5585406)}{5.432364}x$	0.95943
3	$y = \underset{(0.000129)}{-2.98385} + \underset{(0.000387)}{3.44172}t - \underset{(0.000286)}{0.01458}t^2$	1.00000
4	$y = \underset{(0.0011942)}{-2.76468} + \underset{(0.0000979)}{3.36477}t - \underset{(0.0001605)}{0.02674}x$	1.00000
5	$y = \underset{(0.0002931)}{-2.86031} + \underset{(0.0000456)}{3.36499}t - \underset{(0.0000053)}{0.00187}x^2$	1.00000
6	$y = \underset{(1.0591E-08)}{1.165377} + \underset{(3.7187E-09)}{1.984874}t + \underset{(2.1831E-09)}{0.810084}t^2 - \underset{(1.2924E-09)}{0.506302}x$	1.00000
7	$y = \underset{(0.01040532)}{-2.761762} + \underset{(0.00646298)}{3.303776}t + \underset{(0.00380509)}{0.036038}t^2 - \underset{(0.00015769)}{0.003366}x^2$	1.00000

8	$y = \underset{(0.01330144)}{7.0753757} + \underset{(0.00001955)}{1.9749804t^2} - \underset{(0.00373362)}{1.2682231x} + \underset{(0.00026212)}{0.0050625x^2}$	1.00000
9	$y = \underset{(0.0000347)}{-2.951019} - \underset{(0.0167259)}{0.626354t} + \underset{(0.0046466)}{1.180800tx} - \underset{(0.0003216)}{0.087259tx^2}$	1.00000
10	$y = \underset{(287.0542)}{1255.3080} - \underset{(81.4615)}{351.0223x} + \underset{(5.7791)}{24.5172x^2}$	0.99420

As a matter of fact, the true system dynamics is close to sub-regression 6 is

$$\frac{dx}{dt} = 1.20q_0 + 2.0q_1 t + 0.80t^2 - 0.50x \quad (42)$$

But the data-fitted differential equation is

$$\frac{dx}{dt} = \underset{(1.0591E-08)}{1.165377} + \underset{(3.7187E-09)}{1.984874t} + \underset{(2.1831E-09)}{0.810084t^2} - \underset{(1.2924E-09)}{0.506302x} \quad . \quad (43)$$

ACKNOWLEDGEMENTS

This research is partially supported by South African National Research Foundation Grant FA2006042800024. Authors deeply appreciate referee's invaluable comments on open question aspects.

REFERENCES

1. Carvalho, H. and Machado, V.C. (2006). Fuzzy set theory to establish resilient production systems. *IIE Annual Conference*, Orlando, USA, Maio 2006.
2. Deng, J.L. (1985). *Grey Systems (Social˻Economical)*. Beijing: the Publishing House of Defense Industry. (*in Chinese*)
3. Dubois, D. and Prade, H. (1980). *Theory and Applications, Fuzzy Sets and Systems*. New York: Academic Press.
4. Field, C.A. and Ronchetti, E. (1990). Small Sample Asymptotics. Institute of Mathematical Statistics, *Lecture Notes-Monographs Series 13*, USA.
5. Field, C.A. and Ronchetti, E. (1991). An Overview of Small Sample Asymptotics, in: W. Stahel, S. Weisberg (eds.), Directions in Robust Statistics and Diagnostics, Part I. New York: Springer-Verlag.
6. Guo, D., Guo, R., Midgley, G.F. and Rebelo, A.G. (2007). PDEMR Modelling of Protea Species in the Population Size of 1 to 10, in Cape Floristic Region from 1992 to 2002, South Africa. *Journal of Geographical Information Sciences*, Vol. 12, No. 2: pp. 67-78.
7. Guo, D., Guo, R., Midgley, G.F., and Rebelo, A.G. (2008). PDEAR Model Prediction of Rare Protea Species under Climate Change Effects. *Proceedings of GISRUK 2008*, Manchester Metropolitan University, UK, April 2-4, 2008.
8. Guo, R. (2006).Grey Envelop Quality Control Charts. Journal of Quality, Vol. 13, No. 4, pp 401-410.
9. Guo, R. (2007). Grey Differential Equation GM(1,1) Model in Reliability Engineering. Chapter 26, pp 387-413. "Computational Intelligence in Reliability Engineering – Vol. 2. New Metaheuristics, Neural and Fuzzy Techniques in Reliability", Springer, Gregory Levitin (Editor).
10. Guo, R. (2007). Modeling Imperfectly Repaired System Data Via Grey Differential Equations with Unequal-Gapped Times. *Reliability Engineering and Systems Safety*, Vol. 92, Issue 3, March, pp. 378-391.

11. Guo, R. (2007). An Univariate DEMR Modeling on Repair Effects. Reliability: Theory & Applications. Vol. 2, No. 3-4, pp 89-98, December 2007. (Electronic Journal of International Group on Reliability – Gnedenko E-Forum).
12. Guo, R., Guo, D., and Thiart, C. (2006). The Coupling of regression Modelling and Differential Equation Model in GM(1,1) Modelling and Extended GM(1,1) Models, *Journal of Grey System*, 9(2) pp143-154.
13. Guo, R. Guo, D., Midgley, G.F. , and Rebelo, A.G. (2008) "PDEMR Modelling of Protea Rare Species Spatial Patterns" *Journal of Uncertain Systems*, Vol. 2, No. 1, pp. 31-53. February, 2008.
14. Guo, R. and Dunne, T. (2006) Grey Predictive Control Charts. Communications in Statistics – Theory and Methods, Vol. 35, No. 10, pp. 1857-1868, 2006.
15. Liu, B.D. (2004).Uncertainty Theory: An Introduction to Its Axiomatic Foundations. Berlin: Springer-Verlag Heidelberg.
16. Liu, B.D. Uncertainty Theory: An Introduction to Its Axiomatic Foundations, 2nd Edition 2007; Berlin: Springer-Verlag Heidelberg.
17. Zadeh, L.A. Fuzzy Sets. Information and Control 1965; Vol. 8: 338-353.
18. Zadeh, L.A. Fuzzy sets as basis for a theory of possibility. *Fuzzy Sets and Systems* 1978; Vol. 1: 3-28.