RELIABILITY ANALYSIS OF TWO-STATE SERIES-CONSECUTIVE "*M* OUT OF *K*: F" SYSTEMS

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ABSTRACT

A non-stationary approach to reliability analysis of two-state series and consecutive "*m* out of *k*: F" systems is presented. Further, the series-consecutive "*m* out of *k*: F" system is defined and the recurrent formulae for its reliability function evaluation are proposed. Moreover, the application of the proposed formulae to reliability evaluation of the radar system composed of two-state components is illustrated.

1 INTRODUCTION

The basic analysis and diagnosis of systems reliability are often performed under the assumption that they are composed of two-state components. It allows us to consider two states of the system reliability. If the system works its reliability state is equal to 1 and if it is failed its reliability state is equal to 0. Reliability analysis of two-state consecutive "*k* out of *n*: F" systems can be done for stationary and non-stationary case. In the first case the system reliability is the independent of time probability that the system is in the reliability state 1. For this case the main results on the reliability evaluation and the algorithms for numerical approach to consecutive "*k* out of *n*: F" systems are given for instance in Antonopoulou & Papstavridis (1987), Barlow & Proschan (1975), Hwang (1982), Malinowski & Preuss (1995), Malinowski (2005). Transmitting stationary results to non-stationary time dependent case and the algorithms for numerical approach to evaluation of this reliability are presented in Guze (2007a), Guze (2007b). Other more complex two-state systems are discussed in Kołowrocki (2004). The paper is devoted to the combining the results on reliability of the two-state series and consecutive "*m* out of *n*: F" systems into the formulae for the reliability function of the series-consecutive "*m* out of *k*: F" systems with dependent of time reliability functions of system components Guze (2007a), Guze (2007b), Guze (2007c).

2 RELIABILITY OF A SERIES AND CONSECUTIVE "*M* OUT OF *N*: F" SYSTEMS

In the case of two-state reliability analysis of series systems and consecutive "*m* out of *n*: F" systems we assume that (Guze 2007c):

- n is the number of system components,
- E_i, $i = 1,2,...,n,$ are components of a system,
- T_i are independent random variables representing the lifetimes of components E_i, $i = 1,2,...,n,$
- $R_i(t) = P(T_i > t), t \in < 0,\infty),$ *is* a reliability function of a component E_i, $i = 1,2,...,n,$
- $F_i(t) = 1 - R_i(t) = P(T_i \le t), t \in < 0,\infty),$ is the distribution function of the component E_i lifetime T_i, $i = 1,2,...,n,$ also called an unreliability function of a component E_i, $i = 1,2,...,n.$

In further analysis we will use one of the simplest system structure, namely a series system.

Definition A two-state system is called series if its lifetime T is given by

$$T = \min_{1 \le i \le n}\{T_i\}.$$

The scheme of a series system is given in Figure 1.

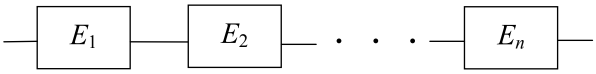

Figure 1. The scheme of a series system

The above definition means that the series system is not failed if and only if all its components are not failed or equivalently the system is failed if at least one of its components is failed. It is easy to motivate that the series system reliability function $\overline{\boldsymbol{R}}_n(t) = P(T > t), \ t \in <0, \infty),$ is given by

$$\overline{\boldsymbol{R}}_n(t) = \prod_{i=1}^{n} R_i(t), \ t \in <0, \infty). \tag{1}$$

Definition 2. A two-state series system is called homogeneous if its component lifetimes T_i have an identical distribution function

$$F(t) = P(T_i \le t), \ t \in <0, \infty), \ i = 1,2,...,n,$$

i.e. if its components E_i have the same reliability function

$$R(t) = 1 - F(t), \ t \in <0, \infty).$$

The above definition results in the following simplified formula

$$\overline{\boldsymbol{R}}_n(t) = [R(t)]^n, \ t \in <0, \infty), \tag{2}$$

for the reliability function of the homogeneous two-state series system.

Definition 3. A two-state system is called a two-state consecutive "*m* out of *n*: F" system if it is failed if and only if at least its *m* neighbouring components out of n its components arranged in a sequence of $E_1, E_2, \ldots, E_n$, are failed.

After assumption that:

- T is a random variable representing the lifetime of the consecutive "*m* out of *n*: F" system,
- $\boldsymbol{CR}_n^{(m)}(t) = P(T > t), t \in <0, \infty),$ is the reliability function of a non-homogeneous consecutive "*m* out of *n*: F" system,
- $\boldsymbol{CF}_n^{(m)}(t) = 1 - \boldsymbol{CR}_n^{(m)}(t) = P(T \le t), t \in <0, \infty),$ is the distribution function of a consecutive "*m* out of *n*: F" system lifetime T,

we can formulate the following auxiliary theorem (Guze 2007c).

Lemma 1. The reliability function of the two-state consecutive "*m* out of *n*: F" system is given by the following recurrent formula

$$\boldsymbol{CR}_n^{(m)}(t) = \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n} F_i(t) & \text{for } n = m, \\ R_n(t)\boldsymbol{CR}_{n-1}^{(m)}(t) & \\ + \sum_{j=1}^{m-1} R_{n-j}(t)\boldsymbol{CR}_{n-j-1}^{(m)}(t) & \\ \cdot \prod_{i=n-j+1}^{n} F_i(t) & \text{for } n > m, \end{cases} \quad (3)$$

for $t \in <0, \infty)$.

Definition 4. The consecutive "*m* out of *n*: F" system is called homogeneous if its components lifetimes T_i have an identical distribution function

$F(t) = P(T_i \le t),\ i = 1,2,\dots,n,\ t \in <0,\infty)$,

i.e. if its components E_i have the same reliability function

$R(t) = 1 - F(t),\ t \in <0,\infty)$.

Lemma 1 simplified form for homogeneous systems takes the following form.

Lemma 2. The reliability function of the homogeneous two-state consecutive "*m* out of *n*: F" system is given by the following recurrent formula

$$\boldsymbol{CR}_n^{(m)}(t) = \begin{cases} 1 & \text{for n} < m, \\ 1 - [F(t)]^n & \text{for } n = m, \\ R(t)\boldsymbol{CR}_{n-1}^{(m)}(t) & \\ + R(t)\sum_{j=1}^{m-1} F^{j-1}(t) & \\ \cdot \boldsymbol{CR}_{n-j-1}^{(m)}(t) & \text{for n} > m, \end{cases} \quad (4)$$

for $t \in <0, \infty)$.

3 RELIABILITY OF TWO-STATE SERIES-CONSECUTIVE "M OUT OF K: F" SYSTEM

To define a two-state series-consecutive "*m* out of *k*: F" systems, we assume that

E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$,

are two-state components of the system having reliability functions

$R_{ij}(t) = P(T_{ij} > t),\ t \in <0,\infty),$

where

T_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$,

are independent random variables representing the lifetimes of components E_{ij} with distribution functions

$F_{ij}(t) = P(T_{ij} \le t),\ t \in <0,\infty).$

Moreover, we assume that components E_{i1}, E_{i2}, …, E_{il_i}, i = 1, 2, …, k, create a series subsystem S_i, i = 1, 2, …, k, and that these subsystems are arranged in a sequence S_1, S_2, …, S_k.

Definition 5. A two-state system is called a series-consecutive "*m* out of *k*: F" system if it is failed if and only if at least its *m* neighbouring series subsystems out of *k* its series subsystems arranged in a sequence of S_1, S_2, …, S_k, are failed.

According to the above definition and formula (1) the reliability function of the subsystem S_i is given by

$$\overline{\boldsymbol{R}}_{il_i}(t) = \prod_{j=1}^{l_i} R_{ij}(t) \tag{5}$$

and its lifetime distribution function is given by

$$\overline{\boldsymbol{F}}_{il_i}(t) = 1 - \overline{\boldsymbol{R}}_{il_i}(t) = 1 - \prod_{j=1}^{l_i} R_{ij}(t), \tag{6}$$

for $i = 1,2,...,k$, $t \in <0,\infty)$.

Hence and by *Lemma 1* denoting by $\boldsymbol{CR}^{(m)}_{k,l_1,l_2,...,l_k}(t) = P(T>t),\ t \in <0,\infty),$ the reliability function of the series-consecutive "*m* out of *k*: F" system, we get the next result.
Lemma 3. The reliability function of the two-state series-consecutive "*m* out of *k*: F" system is given by the following recurrent formula

$$\boldsymbol{CR}^{(m)}_{k,l_1,l_2,...,l_k}(t) =$$

$$
= \begin{cases} 1 & \text{for } k < m, \\ 1 - \prod_{i=1}^{k} [\overline{\boldsymbol{F}}_{il_i}(t)] & \text{for } k = m, \\ [\overline{\boldsymbol{R}}_{kl_k}(t)] \boldsymbol{CR}^{(m)}_{k-1,l_1,l_2,\dots,l_k}(t) & \\ + \sum_{j=1}^{m-1} [\overline{\boldsymbol{R}}_{k-jl_{k-j}}(t)] \boldsymbol{CR}^{(m)}_{k-j-1,l_1,l_2,\dots,l_k}(t) & \\ \cdot \prod_{i=k-j+1}^{k} \overline{\boldsymbol{F}}_{il_i}(t) & \text{for } k > m, \end{cases} \tag{7}
$$

$$
= \begin{cases} 1 & \text{for } k < m, \\ 1 - \prod_{i=1}^{k} [1 - \prod_{j=1}^{l_i} R_{ij}(t)] & \text{for } k = m, \\ [\prod_{j=1}^{l_k} R_{kj}(t)] \boldsymbol{CR}^{(m)}_{k-1,l_1,l_2,\dots,l_k}(t) & \\ + \sum_{j=1}^{m-1} [\prod_{v=1}^{l_{k-j}} R_{k-jv}(t)] \boldsymbol{CR}^{(m)}_{k-j-1,l_1,l_2,\dots,l_k}(t) & \\ \cdot \prod_{i=k-j+1}^{k} [1 - \prod_{v=1}^{l_i} R_{iv}(t)] & \text{for } k > m, \end{cases} \tag{8}
$$

for $t \in <0, \infty)$.

Motivation. Assuming in (3) that $R_i(t) = \overline{\boldsymbol{R}}_{il_i}(t)$ and $F_i(t) = \overline{\boldsymbol{F}}_{il_i}(t) = 1 - \overline{\boldsymbol{R}}_{il_i}(t)$, we get formula (7) and next considering (5) and (6) we get (8).

Definition 6. The series-consecutive "*m* out of *k*: F" systems is called regular if

$$l_1 = l_2 = \ldots = l_k = l, \; l \in N.$$

Definition 7. The series-consecutive "*m* out of *k*: F" system is called homogeneous if its components lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t), \; i = 1,2,\dots,k, \; j = 1,2,\dots,l_i \quad t \in <0, \infty),$$

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t), \; i = 1,2,\dots,k, \; j = 1,2,\dots,l_i, \; t \in <0, \infty).$$

Under *Definition 6* and *Definition 7*, denoting by $\boldsymbol{CR}^{(m)}_{k,l}(t) = P(T > t),\ t \in < 0, \infty),$ the reliability function of a homogeneous and regular series-consecutive "*m* out of *k*: F" system, from *Lemma 3*, we get following result.

Lemma 4. The reliability function of the homogeneous and regular two-state series-consecutive "*m* out of *k*: F" system is given by

$$\boldsymbol{CR}^{(m)}_{k,l}(t) = \begin{cases} 1 & \text{for } k < m, \\ 1 - [1 - R^l(t)]^k & \text{for } k = m, \\ R^l(t)\boldsymbol{CR}^{(m)}_{k\text{-}1,l}(t) & \\ + \sum_{j=1}^{m-1} R^l(t)\boldsymbol{CR}^{(m)}_{k\text{-}j\text{-}1,l}(t) & \\ \cdot [1 - R^l(t)]^{j-1} & \text{for } k > m, \end{cases} \tag{9}$$

$t \in < 0, \infty).$

4 APPLICATION

Example 1. Let us consider the radar system. The system is composed of k = 5 radar towers. We assume that every radar tower is the series subsystem with components: a radar, an antenna, an emitter and a set. We assume that radar system is failed, if its two consecutive of five towers are failed. It means that we consider a regular two-state series-consecutive "2 out of 5: F" system.
Considering formula (8) and after assuming that $m = 2$, $k = 5$ and $l_1 = l_2 = l_3 = l_4 = l_5 = l = 4$, we get the following reliability function for radar system:

- for $m = 2$, $k = 1$:

$$\boldsymbol{CR}^{(2)}_{1,l_1,l_2,l_3,l_4,l_5}(t) = 1, \text{ for } t \in < 0, \infty). \tag{10}$$

- for $m = 2$, $k = 2$:

$$\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{4} R_{ij}(t)] = R_{11}(t)R_{12}(t)R_{13}(t)R_{14}(t) + R_{21}(t)R_{22}(t)R_{23}(t)R_{24}(t)$$

$$- R_{11}(t)R_{12}(t)R_{13}(t)R_{14}(t)R_{21}(t)R_{22}(t)R_{23}(t)R_{24}(t), \text{ for } t \in < 0, \infty). \tag{11}$$

- for $m = 2$, $k = 3$:

$$\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t) = [\prod_{j=1}^{4} R_{3j}(t)]\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t) + [\prod_{v=1}^{4} R_{2v}(t)]\boldsymbol{CR}^{(2)}_{1,l_1,l_2,l_3,l_4,l_5}(t) \cdot [1 - \prod_{v=1}^{4} R_{3v}(t)]$$

$$= [R_{31}(t)R_{32}(t)R_{33}(t)R_{34}(t)]\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t) + [R_{21}(t)R_{22}(t)R_{23}(t)R_{24}(t)]\boldsymbol{CR}^{(2)}_{1,l_1,l_2,l_3,l_4,l_5}(t)$$

$$\cdot[1 - R_{31}(t)R_{32}(t)R_{33}(t)R_{34}(t)], \text{ for } t \in <0,\infty). \tag{12}$$

- for $m = 2$ and $k = 4$ we get

$$\boldsymbol{CR}^{(2)}_{4,l_1,l_2,l_3,l_4,l_5}(t) = [\prod_{j=1}^{4} R_{4j}(t)]\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t) + [\prod_{\nu=1}^{4} R_{3\nu}(t)]\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t) \cdot [1 - \prod_{\nu=1}^{4} R_{4\nu}(t)]$$

$$= [R_{41}(t)R_{42}(t)R_{43}(t)R_{44}(t)]\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t) + [R_{31}(t)R_{32}(t)R_{33}(t)R_{34}(t)]\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t)$$

$$\cdot[1 - R_{41}(t)R_{42}(t)R_{43}(t)R_{44}(t)], \text{ for } t \in <0,\infty). \tag{13}$$

- for $m = 2$ and $k = 5$:

$$\boldsymbol{CR}^{(2)}_{5,l_1,l_2,l_3,l_4,l_5}(t) = [\prod_{j=1}^{4} R_{5j}(t)]\boldsymbol{CR}^{(2)}_{4,l_1,l_2,l_3,l_4,l_5}(t) + [\prod_{\nu=1}^{4} R_{4\nu}(t)]\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t) \cdot [1 - \prod_{\nu=1}^{4} R_{5\nu}(t)]$$

$$= [R_{51}(t)R_{52}(t)R_{53}(t)R_{54}(t)]\boldsymbol{CR}^{(2)}_{4,l_1,l_2,l_3,l_4,l_5}(t) + [R_{41}(t)R_{42}(t)R_{43}(t)R_{44}(t)]\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t)$$

$$\cdot[1 - R_{51}(t)R_{52}(t)R_{53}(t)R_{54}(t)], \text{ for } t \in <0,\infty). \tag{14}$$

In particular case when we assume arbitrarily that the lifetimes T_{ij} of the components E_{ij}, $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4$, of the radar towers S_i, $i = 1, 2, 3, 4, 5$, have an exponential distributions of the form

$$F_{11}(t) = F_{21}(t) = F_{31}(t) = F_{41}(t) = F_{51}(t) = F_1(t) = 1 - \exp\{-\lambda_1 t\}, \text{ for } t \in <0,\infty), \ \lambda_1 > 0, \tag{15}$$

$$F_{12}(t) = F_{22}(t) = F_{32}(t) = F_{42}(t) = F_{52}(t) = F_2(t) = 1 - \exp\{-\lambda_2 t\}, \text{ for } t \in <0,\infty), \ \lambda_2 > 0, \tag{16}$$

$$F_{13}(t) = F_{23}(t) = F_{33}(t) = F_{43}(t) = F_{53}(t) = F_3(t) = 1 - \exp\{-\lambda_3 t\}, \text{ for } t \in <0,\infty), \ \lambda_3 > 0, \tag{17}$$

$$F_{14}(t) = F_{24}(t) = F_{34}(t) = F_{44}(t) = F_{54}(t) = F_4(t) = 1 - \exp\{-\lambda_4 t\}, \text{ for } t \in <0,\infty), \ \lambda_4 > 0, \tag{18}$$

i.e. if the ralibility functions of the components E_{ij}, $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4$, of the radar towers S_i, $i = 1, 2, 3, 4, 5$, are given by

$$R_{11}(t) = R_{21}(t) = R_{31}(t) = R_{41}(t) = R_{51}(t) = R_1(t) = \exp\{-\lambda_1 t\}, \text{ for } t \in <0,\infty), \ \lambda_1 > 0, \tag{19}$$

$$R_{12}(t) = R_{22}(t) = R_{32}(t) = R_{42}(t) = R_{52}(t) = R_2(t) = \exp\{-\lambda_2 t\}, \text{ for } t \in <0,\infty), \ \lambda_2 > 0, \tag{20}$$

$$R_{13}(t) = R_{23}(t) = R_{33}(t) = R_{43}(t) = R_{53}(t) = R_3(t) = \exp\{-\lambda_3 t\}, \text{ for } t \in <0,\infty), \ \lambda_3 > 0, \tag{21}$$

$$R_{14}(t) = R_{24}(t) = R_{34}(t) = R_{44}(t) = R_{54}(t) = R_4(t) = \exp\{-\lambda_4 t\}, \text{ for } t \in <0,\infty), \ \lambda_4 > 0, \tag{22}$$

considering (10)-(14) and (15)-(22) we get the following recurrent formula for the reliability $\boldsymbol{CR}^{(2)}_{5,l_1,l_2,l_3,l_4,l_5}(t)$ of a regular and non-homogeneous radar system

$$\boldsymbol{CR}^{(2)}_{1,l_1,l_2,l_3,l_4,l_5}(t) = 1, \text{ for } t \in < 0, \infty).$$

$$\boldsymbol{CR}^{(2)}_{2,l_1,l_2,l_3,l_4,l_5}(t) = 2\exp\{-2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} - \exp\{-2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\}, \text{ for } t \in < 0, \infty).$$

$$\boldsymbol{CR}^{(2)}_{3,l_1,l_2,l_3,l_4,l_5}(t) = \exp\{-2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} - \exp\{-3(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} + \exp\{-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\}, \text{ for } t \in < 0, \infty).$$

$$\boldsymbol{CR}^{(2)}_{4,l_1,l_2,l_3,l_4,l_5}(t) = 3\exp\{-2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} - \exp\{-4(\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4)t\}, \text{ for } t \in < 0, \infty).$$

$$\boldsymbol{CR}^{(2)}_{5,l_1,l_2,l_3,l_4,l_5}(t) = \exp\{-5(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} - 4\exp\{-4(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} + 3\exp\{-3(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\} + \exp\{-2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)t\}, \text{ for } t \in < 0, \infty).$$

5 CONCLUSIONS

The paper is devoted to a non-stationary approach to reliability analysis of two-state series and consecutive "*m* out of *k*: F" systems. Two recurrent formulae for two-state reliability functions, a general one for non-homogeneous and its simplified form for regular and homogeneous two-state series-consecutive "*m* out of *k*: F" systems have been proposed. The formulae for a regular and non-homogeneous two-state series-consecutive "*m* out of *k*: F" has been applied to reliability evaluation for radar system. The considered radar system was a regular and non-homogeneous two-state series-consecutive "2 out of 5: F" system.

The input and structural reliability data of considered radar system have been assumed arbitrarily and therefore the obtained its reliability function evaluation should be treated as an illustration of the possibilities of the proposed methods and solutions only.

The proposed methods and solutions may be applied to any two-state series-consecutive "*m* out of *k*: F" systems.

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