SHOCK MODELS UNDER POLICY N

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ABSTRACT

We present the life distribution of a device subject to shocks governed by phase-type distributions. The probability of failures after shocks follows discrete phase-type distribution. Lifetimes between shocks are affected by the number of cumulated shocks and they follow continuous phase-type distributions. The device can support a maximum of N shocks. We calculate the distribution of the lifetime of the device and illustrate the calculations by means of a numerical application. Computational aspects are introduced. This model extends other previously considered in the literature.

1 INTRODUCTION AND BACKGROUND

The classical shocks model of Esary et al. [2] study the lifetime of a device subject to shocks that arrive randomly following a Poisson process $\{N(t), t \geq 0\}$. The device has a probability $\overline{P}_k$ of survival to k shocks. Then, the survival function of the model, H(t), is given by:

$$\overline{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\} \overline{P}_k \tag{1}$$

This model has been studied under non-parametric methodology by different authors, considering reliability classes. Neuts et al. (1981), introduced phase-type distributions and calculated explicitly the lifetime distribution of the device. Manoharan et al. (1992) considered a finite mixture of homogeneous Poisson process as arrival process. In these previous papers, the number of shocks that arrives to the device is unlimited. In Neuts et al. (2000) phase-distributions are used to study a model subject to a limited number of failures.

We present a model limiting the number of shocks that the device can stand. The probability of failure due to the shocks follow a discrete phase-type distribution. The interarrival times between shocks depend on the number of cumulated shocks.

The process that governs the system is a Markov one with vectorial state space. We calculate the lifetime distribution of the device and present a numerical example illustrating the calculations.

In Section 2 the shock model is presented. In Section 3 the Markov model that governs the system is constructed, and the lifetime distribution of the device determined. In Section 4 a numerical application is performed.

Given that the phase-type distributions play a fundamental role throughout the paper, we define them in the discrete and continuous cases. More details about these distributions can be seen in Neuts (1981).

2 DEFINITIONS

Definition 1. The distribution H(·) on [o,∞[is a phase-type distribution (PH-distribution) with representation (α,T) if it is the distribution of the time untill the absorption in a Markov process on the states {1,...,m,m+1} generator

$$\begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}$$

and initial row probability vector α of order *m*. We assume that the states {1,...,m} are all transient. Throughout this paper *e* denotes a column vector with all components equal to one the dimension of which is determined by the context. The matrix T of order *m* is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $-Te=T^0\geq 0$. The distribution of H(·) is given by,

$$H(x)=1-\alpha\exp(Tx)e,\ x\geq 0$$

It will be denoted that H(·) follows PH(α,T) distribution.

Definition 2. A density $\{p_k\}$ of the nonnegative integers is of phase type if and only if there exists a finite Markov chain with transition probability matrix P of order n+1 of the form

$$\begin{pmatrix} S & S^0 \\ 0 & 1 \end{pmatrix}$$

and initial probability vector (β,β_{n+1}), where β is a row n-vector. Here S is a subestochastic matrix such that $Se+S^0=e$, and (I-S) is non-singular. The density of the time until absorption is given by

$$p_0=\beta_{n+1},$$
$$p_k=\beta S^{k-1}S^0,\ \text{for k}\geq 1$$

It will be denoted that $\{p_k\}$ follows a $PH_d(\beta,S)$.
We use the Kronecker product (see [1]).

3 SHOCK MODEL

Suppose that a device is subjected to shocks according to the following assumptions.

1. Let $X^{(k)}$ be the interarrival times between the shocks kth and (k+1)th, k=0,1,... These random times follow distributions $PH(\beta^{(k)},S^{(k)})$ of order n_k.
2. The device can accumulate a maximum of N shocks, in such a way that it is replaced by a new one to the arrival of the N+1 shock. We denote by p_k the probability of failure of the device due to the arrival of the kth shock, k≥1. We assume that p_k follows a distribution $PH_d(\gamma,L)$ of order N+1.

This representation is given by γ=(1,0,…,0) and

$$L = \begin{pmatrix} 0 & l_1 & 0 & \dots & 0 \\ 0 & 0 & l_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & l_N \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}; L^0 = \begin{pmatrix} 1-l_1 \\ 1-l_2 \\ \vdots \\ 1-l_N \\ 1 \end{pmatrix}$$

The entries l_k, k=1,...,N, denote the conditional probability that the system will survive to the kth shock, given that it has survived to the (k-1)th shock. These l_k are useful to find a representation to the distribution $\{p_k\}$. It is clear that

$$p_k = \gamma L^{k-1} L^0, k \geq 1$$

The survival probability to the kth shock is

$$\overline{P}_k = \sum_{k=\nu+1}^{N+1} p_\nu = \gamma\left(L^k - L^{N+1}\right)e, k = 1,\dots,N; \overline{P}_0 = 1 \tag{2}$$

Under the assumptions the survival function (1) follows a PH-distribution, that will be calculated.

4 MARKOV MODEL

Under these assumptions, the probabilistic model that governs the system is a Markov process. The occupied exponential states by the device will be denoted by (k,i), k being the number of cumulated shocks, i the phase of the random variable $X^{(k)}$. We group these states in sets, named macro-states, that will be denoted by k, k=0,1,...,N. The number of exponential states of the macro-state k is n_{k+1}. The infinitesimal generator of the Markov process is built in terms of the transition between macro-states, and, consequently, it will be a generator formed by blocks.

We denote by T_k the lifetime of the device when the failure occurs at the arrival of the (k+1)th shock. It is clear that

$$T_k = \sum_{i=0}^{k} X^{(i)}, 0 \leq k \leq N \tag{3}$$

This random variable is the sum of independent random variables PH-distributed and follows a distribution PH($g^{(k)}$,$G^{(k)}$). We calculate this representation. If the device fails at the (k+1)th shock, it has survived to the first k shocks. Thus, the transitions between the up macro-states to the occurrence of the failure are 0→1→2→· · · →k. These transitions j→j+1, j= 0,1,...,k, occur when a non-fatal shock arrives being the device in the macro-state j, and these are governed by the absorption vector $S^{0(j)}$. Then, the new interarrival period initiates following the initial vector $\beta^{(j)}$. Thus, the matrix $G^{(k)}$ is

$$G^{(k)} = \begin{pmatrix} S^{(0)} & S^{0(0)}\beta^{(1)} & & \\ & S^{(1)} & S^{0(1)}\beta^{(2)} & \\ & & \ddots & \\ & & & S^{(k)} \end{pmatrix}$$

We assume that the device initiates with 0 shocks, so the initial vector $g^{(k)}$ is given by

$$g^{(k)} = \left(\beta^{(0)}, 0, \ldots, 0\right), k = 0, 1, \ldots, N$$

Denoting by T the lifetime of the system, we have

$$R(t) = P(T > t) = \sum_{k=0}^{N} p_{k+1} P(T_k > t), t \geq 0 \tag{4}$$

We determine the distribution of the random variable T as follows. This is the distribution of a finite mixture of PH-distributions, it is a PH-distribution with well-known representation (see [4]) given by (v,V), with,

$$v = \left(p_1 g^{(0)}, \quad p_2 g^{(1)}, \quad \ldots, \quad p_{N+1} g^{(N)}\right), V = \begin{pmatrix} G^{(0)} & & & \\ & G^{(1)} & & \\ & & \ddots & \\ & & & G^{(N)} \end{pmatrix} \tag{5}$$

The analytic expression of the survival function of T is:

$$P(T > t) = v \exp\{Vt\} e = \sum_{k=0}^{N} p_{k+1} g^{(k)} \exp\left(G^{(k)} t\right) e, t \geq 0 \tag{6}$$

5 NUMERICAL APPLICATION

We assume a system that can stand a maximum of 4 shocks, then the system is replaced to the arrival of the 4th shock. The probabilities of failure to the arrival of shocks are $p_0=0, p_1=0.2, p_2=0.2, p_3=0.3, p_4=0.3$. The shocks arrival time follow Erlang distributions with the following PH representations:

- $X^{(0)}$ ~>PH($\beta^{(0)}$,$S^{(0)}$), with

$\beta^{(0)} = (1,0), S^{(0)} = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$, sum of two identical exponentials with parameter $\lambda=1$.

- $X^{(1)}$ ~>PH($\beta^{(1)}$,$S^{(1)}$), with

$\beta^{(0)} = (1,0), S^{(0)} = \begin{pmatrix} -3 & 3 \\ & -3 \end{pmatrix}$, sum of two identical exponentials with parameter λ=3.

– $X^{(2)}$ ~>PH($\beta^{(2)}$,$S^{(2)}$), with

$\beta^{(0)} = (1,0,0), S^{(0)} = \begin{pmatrix} -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{pmatrix}$, sum of three identical exponentials with parameter λ=1.

– $X^{(3)}$ ~>PH($\beta^{(3)}$,$S^{(3)}$), with

$\beta^{(0)} = (1,0,0), S^{(0)} = \begin{pmatrix} -2 & 2 & \\ & -2 & 2 \\ & & -2 \end{pmatrix}$, sum of three identical exponentials with parameter λ=2.

The values of these distributions are simulated following these steps. We have performed a total of one hundred simulated values of the exponential distributions corresponding to the different parameters above assigned. Adding these values we calculated simulated values of the random variables X´s, and using these values we calculated simulated values of the device lifetime to kth shock T_k, k=1,2,3. We do not define the unit time.

– $T^{(0)}$~>PH($g^{(0)}$,$G^{(0)}$), with $g^{(0)}= \beta^{(0)}$,$G^{(0)}= S^{(0)}$

– $T^{(1)}$ ~>PH($g^{(1)}$,$G^{(1)}$), with $g^{(1)} = (\beta^{(0)},0,0), G^{(1)} = \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -3 & 3 \\ & & & -3 \end{pmatrix}$

– $T^{(2)}$ ~>PH($g^{(2)}$,$G^{(2)}$), with $g^{(2)} = (\beta^{(0)},0,0,0,0,0), G^{(1)} = \begin{pmatrix} -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & -3 & 3 & & & \\ & & & -3 & 3 & & \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \\ & & & & & & -1 \end{pmatrix}$

– $T^{(3)}$ ~>PH($g^{(3)}$,$G^{(3)}$), with $g^{(3)} = (\beta^{(0)},0,0,0,0,0,0,0,0)$, and,

$$G^{(3)} = \begin{pmatrix} -1 & 1 & & & & & & & & \\ & -1 & 1 & & & & & & & \\ & & -3 & 3 & & & & & & \\ & & & -3 & 3 & & & & & \\ & & & & -1 & 1 & & & & \\ & & & & & -1 & 1 & & & \\ & & & & & & -1 & 1 & & \\ & & & & & & & -2 & 2 & \\ & & & & & & & & -2 & 2 \\ & & & & & & & & & -2 \end{pmatrix}$$

From the total of one hundred of simulations, in Table 1 are presented the simulated values for the random variables T_k, k=1,2,3 for ten of these simulations in increasing order. The graphics of the survival functions to the shocks and the survival function of the system are plotted in Figures 1, 2, respectively, and they have been constructed using the total of one hundred simulations.

Table 1. Some lifetime simulated values to the different shocks

T_0	T_1	T_2	T_2
0.01403	0.01717	0.02573	0.03514
0.18643	0.24627	0.50829	0.66666
0.36890	0.54202	1.14390	1.49570
0.67094	0.93929	1.87180	2.44490
0.97299	1.33820	2.75260	3.4970
1.39050	1.88220	3.79240	4.81320
2.23760	3.07560	6.92490	8.68410
4.46070	6.30390	13.6980	17.3150
9.02980	12.8690	28.543	36.7110

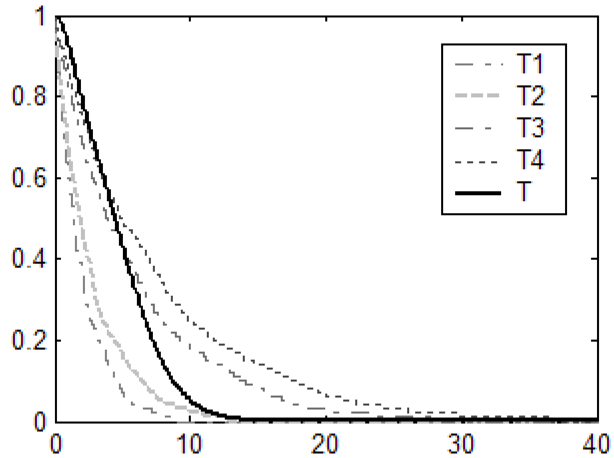

Figure 1. Survival functions to the shocks and Survival function of the system.

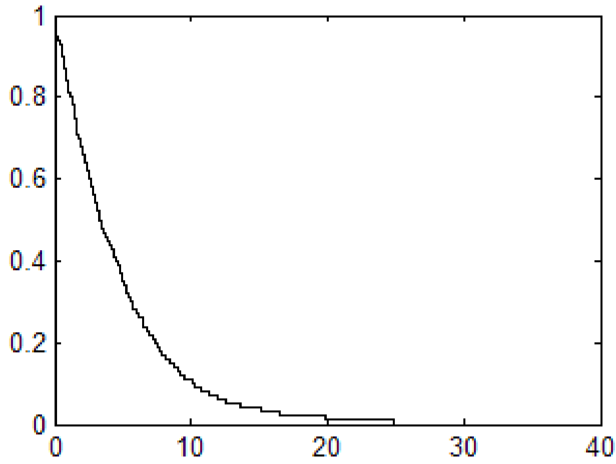

Figure 2. Simulation of the Survival function of the system.

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