ASYMPTOTIC APPROACH TO RELIABILITY EVALUATION OF LARGE "M OUT OF L"- SERIES SYSTEM IN VARIABLE OPERATION CONDITIONS

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ABSTRACT

The semi-markov model of the system operation process is proposed and its selected parameters are defined. There are found reliability and risk characteristics of the multi-state "m out of l"-series system. Next, the joint model of the semi-markov system operation process and the considered multi-state system reliability and risk is constructed. The asymptotic approach to reliability and risk evaluation of this system in its operation process is proposed as well.

1 INTRODUCTION

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and complicated operating processes. This complexity very often causes evaluation of systems reliability to become difficult. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet these systems, for instance, in piping transportation of water, gas, oil and various chemical substances or in transport using belt conveyers and elevators.

Taking into account the importance of safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis (Kolowrocki 2004). The assumption that the systems are composed of multi-state components with reliability state degrading in time without repair gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability characteristics is often very difficult to fix and to analyse. A convenient tool for solving this problem is semi-markov modelling (Grabski 2002, Kolowrocki & Soszynska 2005, Soszynska 2006 a, b, Soszynska 2007 a, b, c) of the systems operation processes which is proposed in the paper. In this model, the variability of system components reliability characteristics is pointed by introducing the components' conditional reliability functions determined by the system operation states. Therefore, the common usage of the multi-state system's limit reliability functions in their reliability evaluation and the semi-markov model for system's operation process modelling in order to construct the joint general system reliability model related to its operation process is proposed. On the basis of that joint model, in the case, when components have exponential reliability functions, unconditional multi-state limit reliability functions of the "*m* out l_n"-series system are determined.

2 SYSTEM OPERATION PROESS

We assume that the system during its operation is operating in $v, v \in N,$ different operation states. After this assumption we can define the system operation process $Z(t),\ t \in < 0,+\infty >,$ with discrete states from the set of states

$$Z = \{z_1, z_2, \ldots, z_v\}.$$

In practice a convenient assumption is that *Z*(*t*) is a semi-markov process (Grabski 2002, Kolowrocki & Soszynska 2005, Soszynska 2006 a, b, Soszynska 2006 a, b, c) with its conditional sojourn times θ_{bl} at the operation state z_b when its next operation state is $z_l,\ b, l = 1,2,...,v,\ b \neq l.$ In this case this process may be described by:
- the vector of probabilities of the initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of its transitions between the states $[p_{bl}]_{vxv}$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the sojourn times $\theta_{bl},\ b \neq l.$

If the sojourn times $\theta_{bl},\ b,\ l = 1,2,...,v,\ b \neq l,$ have Weibull distributions with parameters $\alpha_{bl},\ \beta_{bl}$, i.e., if for $b, l = 1,2,...,v,\ b \neq l,$

$$H_{bl}(t) = P(\theta_{bl} < t) = 1 - \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0,$$

then their mean values are determined by

$$M_{bl} = E[\theta_{bl}] = \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}),\ b, l = 1,2,...,v,\ b \neq l. \quad (1)$$

The unconditional distribution functions of the process $Z(t)$ sojourn times θ_b at the operation states $z_b,\ b = 1,2,...,v,$ are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl}[\,1 \text{ - } \exp[\text{-}\,\alpha_{bl} t^{\beta_{bl}}\ t]],\ = 1 - \sum_{l=1}^{v} p_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0,\ b = 1,2,...,v, \quad (2)$$

and, considering (1), their mean values are

$$M_b = \mathrm{E}[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl} = \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}),\ b = 1,2,...,\mathrm{v}, \quad (3)$$

and variances are

$$D_b = D[\theta_b] = E[(\theta_b)^2] - (M_b)^2, \quad (4)$$

where, according to (2),

$$E[(\theta_b)^2] = \int_0^\infty t^2 dH_b(t) = \sum_{l=1}^{v} p_{bl} \int_0^\infty t^2 \alpha_{bl}\beta_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}] t^{\beta_{bl}-1} dt = \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{2}{\beta_{bl}}} \Gamma(1+\frac{2}{\beta_{bl}}),\ b = 1,2,...,v.$$

Limit values of the transient probabilities

$$p_b(t) = P(Z(t) = z_b),\ \ t \ge 0,\ \ b = 1,2,...,v,$$

at the operation states z_b are given by

$$p_b = \lim_{t\to\infty} p_b(t) = \pi_b M_b / \sum_{l=1}^{v} \pi_l M_l,\ \ b = 1,2,...,v, \qquad (5)$$

where M_b are given by (3) and the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases}$$

3 MULTI STATE " M OUT OF L"- SERIES SYSTEM

In the multi-state reliability analysis to define systems with degrading components we assume that all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \ge 1$, the reliability states are ordered, the state 0 is the worst and the state z is the best and the component and the system reliability states degrade with time t without repair. The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse ones. The way in which the components and system states change is illustrated in Figure 1.
One of multi-state reliability structures with components degrading in time (Kolowrocki 2004, Kolowrocki et. al 2005) are "m out of l_n"- series systems.

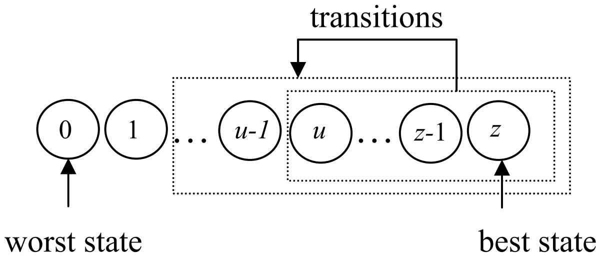

Figure 1. Illustration of states changing in system with ageing components.

To define them, we additionally assume that E_{ij}, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, l_2,..., l_{k_n}, $n \in N$, are components of a system, $T_{ij}(u)$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, l_2,..., l_{k_n}, $n \in N$, are independent random variables representing the lifetimes of components E_{ij} in the state subset $\{u, u+1,..., z\}$, while they were in the state z at the moment $t = 0$, $e_{ij}(t)$ are components E_{ij} states at the moment t, $t \in <0,\infty)$, $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u,u+1,...,z\}$ while it was in the reliability state z at the moment $t = 0$ and $s(t)$ is the system reliability state at the moment t, $t \in < 0,\infty)$.

Definition 1. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1),..., R_{ij}(t,z)],\ t \in < 0,\infty),$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t)$$

for $t \in < 0,\infty)$, $u = 0,1,...,z$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, is the probability that the component E_{ij} is in the reliability state subset $\{u,u+1,...,z\}$ at the moment t, $t \in < 0,\infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

Definition 2. A vector

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1, \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,0), \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1),..., \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)],$$

where

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t)$$

for $t \in < 0,\infty)$, $u = 0,1,...,z$, is the probability that the system is in the reliability state subset $\{u,u+1,...,z\}$ at the moment t, $t \in < 0,\infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a system.

It is clear that from *Definition 1* and *Definition 2*, for $u = 0$, we have $R_{ij}(t,0) = 1$ and $\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,0) = 1$.

Definition 3. A multi-state system is called "m out of l_n"- series if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \min_{1\leq i\leq k_n} T_{(l_i-m_i+1)}(u), \quad m_i \leq l_i\,,\ ,\ u = 1,2,...,z,$$

where $T_{(l_i-m_i+1)}(u)$ is m_i-th maximal statistics in the random variables set

$$T_{i1}(u), T_{i2}(u), ..., T_{il_i}(u), \quad i = 1,2,...,k_n\,,\ u = 1,2,...,z.$$

Definition 4. A multi-state "m out of l_n"- series system is called regular if $l_1 = l_2 = \ldots = l_{k_n} = l_n$ and $m_1 = m_2 = ... = m_{k_n} = m$, $l_n, m \in N$, $m \leq l_n$.

Definition 5. A multi-state "m out of l_n"- series system is called homogeneous if its component lifetimes $T_{ij}(u)$ have an identical distribution function, i.e.

$$F(t,u) = P(T_{ij}(u) \le t),\ t \in <0,\infty),\ \ u = 1,2,...,z,\ i = 1,2,...,k_n,\ \ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function, i.e.

$$R(t,u) = 1 - F(t,u),\ t \in <0,\infty),\ u = 1,2,...,z.$$

From the above definitions it follows that the reliability function of the homogeneous and regular "m out of l_n"- series system is given by (Kolowrocki 2004, Kolowrocki et al 2005)

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1, \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)],$$

(6)

where

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = [1 - \sum_{i=0}^{m-1}\binom{l_n}{i}[R(t,u)]^i\ \ [1 - R(t,u)]^{l_n - i}]^{k_n},\ t\in<0,\infty),\ u = 1,2,...,z, \tag{7}$$

or by

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot) = [1, \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)], \tag{8}$$

where

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = [\sum_{i=0}^{l_n - m}\binom{l_n}{i}[1 - R(t,u)]^i [R(t,u)]^{l_n - i}]^{k_n},\qquad t\in<0,\infty),\qquad u = 1,2,...,z,$$

(9)

where k_n is the number of "m out of l_n" subsystems connected series and l_n is the number of components of the "m out of l_n" subsystems.

Under these definitions, if $\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z$, or $\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z$, then

$$M(u) = \int_0^\infty \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)dt,\ u = 1,2,..., z, \tag{10}$$

or

$$M(u) = \int_0^\infty \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)dt,\ u = 1,2,..., z, \tag{11}$$

is the mean lifetime of the multi-state non-homogeneous regular "m out of l_n"- series system in the reliability state subset $\{u, u+1,..., z\}$, and the variance is given by

$$D[T(u)] = 2\int_0^\infty t\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)dt - E^2[T(u)], \tag{12}$$

or by

$$D[T(u)] = 2\int_0^\infty t \ \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)dt - E^2[T(u)]. \tag{13}$$

The mean lifetime $\overline{M}(u)$, $u = 1,2,...,z$, of this system in the particular states can be determined from the following relationships

$$\overline{M}(u) = M(u) - M(u+1), \ u = 1,2,...,z-1, \ \overline{M}(z) = M(z). \tag{14}$$

Definition 6. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t), \ t \in < 0,\infty),$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the reliability state z at the moment t = 0 is called a risk function of the multi-state homogeneous regular "m out of l_n"- series system.

Considering *Definition 6* and *Definition 2*, we have

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,r), \ t \in < 0,\infty), \tag{15}$$

and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{16}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

4 MULTI STATE " M OUT OF L"- SERIES SYSTEM IN ITS OPERATION PROCESS

We assume that the changes of the process $Z(t)$ states have an influence on the system components E_{ij} reliability and the system reliability structure as well. Thus, we denote the conditional reliability function of the system component E_{ij} while the system is at the operational state z_b, $b = 1,2,...,v$, by

$$[R^{(i,j)}(t,\cdot)]^{(b)} = [1, \ [R^{(i,j)}(t,1)]^{(b)}, ..., \ [R^{(i,j)}(t,z)]^{(b)}],$$

where for $t \in < 0,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

$$[R^{(i,j)}(t,u)]^{(b)} = P(T_{ij}^{(b)}(u) > t \mid Z(t) = z_b)$$

and the conditional reliability function of the system while the system is at the operational state z_b, $b = 1,2,...,v$, by

$$[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot)]^{(b)} = [1,\ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1)]^{(b)},...,\ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)]^{(b)}] \text{ for } t \in <0,\infty),\ u=1,2,...,z,\ \ b=1,2,...,\nu,$$

where according to (7), we have

$$[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} = P(T^{(b)}(u) > t | Z(t) = z_b) = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(t,u)]^{(b)}]^i \ [1-[R(t,u)]^{(b)}]^{l_n - i}]^{k_n}$$

$$\text{for } t \in <0,\infty),\ \ u=1,2,...,z,\ \ b=1,2,...,\nu,$$

or by

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot)]^{(b)} = [1,\ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1)]^{(b)},...,\ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)]^{(b)} \text{ for } t \in <0,\infty),\ u=1,2,...,z,\ \ b=1,2,...,\nu,$$

where according to (9), we have

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} = P(T^{(b)}(u) > t | Z(t) = z_b) = [\sum_{i=0}^{l_n-m} \binom{l_n}{i} [1-[R(t,u)]^{(b)}]^i \ [[R(t,u)]^{(b)}]^{l_n - i}]^{k_n}$$

$$\text{for } t \in <0,\infty),\ \ u=1,2,...,z,\ \ b=1,2,...,\nu.$$

The reliability function $[R^{(i,j)}(t,u)]^{(b)}$ is the conditional probability that the component E_{ij} lifetime $T^{(b)}_{ij}(u)$ in the reliability state subset $\{u,u+1,...,z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. Similarly, the reliability function $[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)}$ or $[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u,u+1,...,z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. In the case when the system operation time is large enough, the unconditional reliability function of the system

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1),...,\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)],$$

where

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = P(T(u) > t) \text{ for } u=1,2,...,z,$$

or

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1),...,\ \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)],$$

where

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = P(T(u) > t) \text{ for } u=1,2,...,z,$$

and $T(u)$ is the unconditional lifetime of the system in the reliability state subset $\{u, u+1, ..., z\}$, is given by

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p_b \ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)},$$

(17)

or

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} \cong \sum_{b=1}^{v} p_b \ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} \quad (18)$$

for $t \geq 0$ and the mean values and variances of the system lifetimes in the reliability state subset $\{u, u+1, ..., z\}$ are

$$M(u) \cong \sum_{b=1}^{v} p_b M_b(u) \quad \text{for} \quad u = 1, 2, ..., z,$$

(19)

where

$$M_b(u) = \int_0^{\infty} [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt, \quad (20)$$

or

$$M_b(u) = \int_0^{\infty} [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt, \quad (21)$$

and

$$D[T^{(b)}(u)] = 2\int_0^{\infty} t \ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)], \quad (22)$$

or

$$D[T^{(b)}(u)] = 2\int_0^{\infty} t \ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)] \quad (23)$$

for $b = 1, 2, ..., \nu$, $t \geq 0$, and p_b are given by (5).
The mean values of the system lifetimes in the particular reliability states u, by (14), are

$$\overline{M}(u) = M(u) - M(u+1), \ u = 1, 2, ..., z-1, \ \overline{M}(z) = M(z). \quad (24)$$

5 LARGE MULTI STATE " M OUT OF L"- SERIES SYSTEM IN ITS OPERATION PROCESS

Definition 7. A reliability function

$$\mathfrak{R}(t,\cdot) = [1, \mathfrak{R}(t,1), ..., \mathfrak{R}(t,z)],\ t \in (-\infty,\infty),$$

where

$$\mathfrak{R}(t,u) = \sum_{b=1}^{v} p_b [\mathfrak{R}(t,u)]^{(b)},$$

is called a limit reliability function of a multi-state homogeneous regular "m out of l_n"- series system in its operation process with reliability function

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)],$$

or

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z),$$

where $\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)$, $\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)$, $u = 1,2,...,z$, are given by (17) and (18) if there exist normalising constants

$$a_n^{(b)}(u) > 0,\ \ b_n^{(b)}(u) \in (-\infty,\infty),\ \ b = 1,2,...,v,\ u = 1,2,...,z,$$

such that for $t \in C_{[\mathfrak{R}(u)]^{(b)}}$, $u = 1,2,...,z,\ b = 1,2,...,v,$

$$\lim_{n\to\infty} [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = [\mathfrak{R}(t,u)]^{(b)},$$

or

$$\lim_{n\to\infty} [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = [\mathfrak{R}(t,u)]^{(b)}.$$

Hence, the following approximate formulae are valid

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p^b [\mathfrak{R}(\frac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)},u)]^{(b)},\ u = 1,2,...,z, \tag{25}$$

or

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p^b [\mathfrak{R}(\frac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)},u)]^{(b)},\ u = 1,2,...,z. \tag{26}$$

The following auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 1. If

(i) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 0$, $m = \text{const}$, as $n \to \infty$,

(ii) $\overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,u) = \sum_{b=1}^{v} p_b [1 - \sum_{i=0}^{m-1} \frac{[[V(t,u)]^{(b)}]^i}{i!} \exp[-[V(t,u)]^{(b)}]]^k$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \ [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)}, \ t \in (-\infty,\infty),$$

where

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(t,u)]^{(b)}]^i \ [1 - R(t,u)]^{(b)}]^{l_n - i}]^{k_n}, \ t \in (-\infty,\infty), \ u = 1,2,...,z, \ b = 1,2,...,v, \quad (27)$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,z)], \ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if (Kolowrocki et al 2005)

$$\lim_{n\to\infty} n[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}] = [V(t,u)]^{(b)}, \ t \in C_{[V(u)]^{(b)}}, \ u = 1,2,...,z, \ b = 1,2,...,v. \quad (28)$$

Proposition 1. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (29)

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 0$, $m = \text{const}$, as $n \to \infty$,

(iii) $a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}$, $b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)} \log n$, $u = 1,2,...,z$, $b = 1,2,...,v$, (30)

then

$$\overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,z)], \quad t \in (-\infty,\infty), \quad (31)$$

where

$$\overline{\mathfrak{R}_3^{(m)}}(t,u) = \sum_{b=1}^{v} p_b \big[1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]]\big]^k \quad \text{for } t \in (-\infty,\infty) \ u = 1,2,...,z, \tag{32}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \big[1 - \sum_{i=0}^{m-1} \frac{\exp[-it\lambda^{(b)}(u) + i\log n]}{i!} \ \exp[\exp[-t\lambda^{(b)}(u) - \log n]]\big]^k \tag{33}$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t + \log n}{\lambda^{(b)}(u)} \to \infty \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),\ b = 1,2,...,v,\ u = 1,2,...,z,$$

then, according to (29) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-t - \log n] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ \ b = 1,2,...,v.$$

Hence, considering (28), it appears that

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \lim_{n\to\infty} n\exp[-t - \log n] = \exp[-t]$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

which means that according to *Lemma 1* the limit reliability function of that system is given by (31)-(32). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 2. If

(i) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(ii) $\overline{\mathfrak{R}^{(\eta)}}(t,u) = \sum_{b=1}^{v} p_b \big[-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-[v(t,u)]^{(b)}} e^{-\frac{x^2}{2}} dx\big]^k$ is a non-degenerate reliability function, where $[v(t,u)]^{(b)}$ is a non-increasing function

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \ [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1-\sum_{i=0}^{m-1}\binom{l_n}{i}[[R(t,u)]^{(b)}]^i\ [1-[R(t,u)]^{(b)}]^{l_n-i}]^{k_n} \quad t\in(-\infty,\infty), b=1,2,...,v,\ \ u=1,2,...,z, \quad (12)$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,z)],\ t\in(-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty}\frac{(n+1)[R(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}-m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = [v(t,u)]^{(b)} \quad \text{for} \quad t\in C_{[v(u)]^{(b)}}, \quad u=1,2,...,z, \quad b=1,2,...,v.$$

(35)

Proposition 2. If components of the multi-state homogeneous, regular "*m* out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$$[R(t,u)]^{(b)} = 1 \text{ for } t<0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t] \text{ for } t\geq 0,\ u=1,2,...,z,\ b=1,2,...,v, \quad (36)$$

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to \eta$, $0<\eta<1$, as $n\to\infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}\sqrt{\frac{n-m+1}{(n+1)m}}, \quad (37)$$

$$b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}\log\frac{n+1}{m}, \quad u=1,2,...,z,\ b=1,2,...,v, \quad (38)$$

then

$$\overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,z)],\ t\in(-\infty,\infty), \quad (39)$$

where

$$\overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,u) = \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp[-\frac{x^2}{2}]\,dx\big]^k \text{ for } t\in(-\infty,\infty), u=1,2,...,z, \quad (40)$$

is the multi-state limit reliability function of that system , i.e. for *n* large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}}\exp[-\frac{x^2}{2}]\,dx\big]^k \cong \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{t\lambda^{(b)}(u)-\log\frac{n+1}{m}}{\sqrt{\frac{n-m+1}{(n+1)m}}}}\exp[-\frac{x^2}{2}]\,dx\big]^k \quad (41)$$

for $t\in(-\infty,\infty)$, $u=1,2,...,z$.

Proof. For n large enough we have

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)}\sqrt{\frac{n-m+1}{(n+1)m}} + \frac{1}{\lambda^{(b)}(u)}\log\frac{n+1}{m} > 0 \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Therefore, according to (37)-(38) for n large enough we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-t\sqrt{\frac{n-m+1}{(n+1)m}} - \log\frac{n+1}{m}] = [1 - t\sqrt{\frac{n-m+1}{(n+1)m}} + o(\frac{1}{\sqrt{n}})]\frac{m}{n+1}$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (35), it appears that

$$[v(t,u)]^{(b)} = \lim_{n\to\infty}\frac{(n+1)[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} - m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = \lim_{n\to\infty}[-t + o(\frac{1}{\sqrt{n}})\sqrt{\frac{(n+1)(n-m+1)}{m}}] = -t$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$

which means that according to *Lemma 2* the limit reliability function of that system is given by (39)-(40). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 3. If

(i) $k_n \to k = \text{const},\ l_n = n,\ m/n \to 1,\ (n-m) \to \overline{m} = \text{const},$ as $n \to \infty$,

(ii) $\overline{\boldsymbol{\mathfrak{R}}}^{(\overline{m})}(t,u) = \sum_{b=1}^{v} p_b [\sum_{i=0}^{\overline{m}} \frac{[[\overline{V}(t,u)]^{(b)}]^i}{i!}\exp[-[\overline{V}(t,u)]^{(b)}]]^k$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,1), ..., \overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u)]^{(b)} = [\sum_{i=0}^{l_n-m}\binom{l_n}{i}[1-[R(t,u)]^{(b)}]^i\ [1-[R(t,u)]^{(b)}]^{l_n-i}]^{k_n},\ t\in(-\infty,\infty), u = 1,2,...,z,\ b = 1,2,...,v, \quad (42)$$

is its reliability function at the operational state z_b,

then

$\overline{\overline{\mathfrak{R}}}^{(\overline{m})}(t,\cdot) = [1, \overline{\overline{\mathfrak{R}}}^{(\overline{m})}(t,1), ..., \overline{\overline{\mathfrak{R}}}^{(\overline{m})}(t,z)]$, $t \in (-\infty,\infty)$,

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = [\overline{V}(t,u)]^{(b)} \text{ for } t \in C_{[\overline{V}(u)]^{(b)}},\ u = 1,2,...,z,\ b = 1,2,...,v. \quad (43)$$

Proposition 3. If components of the multi-state homogeneous, regular "*m* out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,
$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (44)

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 1$, $(n-m) \to \overline{m} = \text{const}$, as $n \to \infty$

(iii) $a_n^{(b)}(u) = \dfrac{1}{n\lambda^{(b)}(u)}$, $b_n^{(b)}(u) = 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (45)

then

$$\overline{\overline{\mathfrak{R}}}_2^{(\overline{m})}(t,\cdot) = [1, \overline{\overline{\mathfrak{R}}}_2^{(\overline{m})}(t,1), ..., \overline{\overline{\mathfrak{R}}}_2^{(\overline{m})}(t,z)],\ t \in (-\infty,\infty), \quad (46)$$

where

$$\overline{\overline{\mathfrak{R}}}_2^{(\overline{m})}(t,u) = \begin{cases} 1 & \text{for } t < 0, \\ \sum_{b=1}^{v} p_b\, [\sum_{i=0}^{\overline{m}} \frac{t^i}{i!} \exp[-t]]^k & \text{for } t \geq 0,\ u = 1,2,...,z, \end{cases} \quad (47)$$

is the multi-state limit reliability function of that system , i.e. for *n* large enough we have

$$\overline{\overline{\mathbf{R}}}_{k_n,l_n}^{(m)}(t,u) \cong \begin{cases} 1 & \text{for } t < 0, \\ \sum_{b=1}^{v} p_b\, [\sum_{i=0}^{\overline{m}} \frac{(n\lambda^{(b)}(u)t)^i}{i!} & \text{for } t \geq 0, \\ \exp[-n\lambda^{(b)}(u)t]]^k & . \end{cases} \quad (48)$$

$u = 1,2,...,z$.

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{n\lambda^{(b)}(u)} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{n\lambda^{(b)}(u)} \geq 0 \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then according to (44) we obtain

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$
$$= 1 - \exp[-\frac{t}{n}] \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Hence, considering (43), it appears that

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \lim_{n\to\infty} n[1 - \exp[-\frac{t}{n}]] = \lim_{n\to\infty} n[\frac{t}{n} + o(\frac{1}{n})] = t$$

for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

which means that according to *Lemma 3* the limit reliability function of that system is given by (46)-(47). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).
Lemma 4. If

(i) $k_n = n$, $l_n \to l = \text{const},\ m \leq l_n$, as $n \to \infty$

(ii) $\overline{\boldsymbol{\mathfrak{R}}}(t,u) = \sum_{b=1}^{v} p_b \exp[-\overline{[V}(t,u)]^{(b)}]$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(t,u)]^{(b)}]^i\ [1 - [R(t,u)]^{(b)}]^{l_n - i}]^{k_n}\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$
(49)

is its reliability function at the operational state z_b,

then

$$\overline{\boldsymbol{\mathfrak{R}}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^i \ [[F(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = \overline{[V}(t,u)]^{(b)} \tag{50}$$

for $t \in C_{[\overline{V}(u)]^{(b)}}$, $u = 1,2,...,z,\ b = 1,2,...,v.$

Proposition 4. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$$[R(t,u)]^{(b)} = 1 \text{ for } t<0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t] \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v, \tag{51}$$

(ii) $k_n = n,\ l_n \to l,\ l \in (0,\infty), m = \text{const},\ m \le l_n,$

(iii) $$a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}}, \tag{52}$$

$$b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v, \tag{53}$$

then

$$\overline{\mathfrak{R}}_2(t,\cdot) = [1, \overline{\mathfrak{R}}_2(t,1), ..., \overline{\mathfrak{R}}_2(t,z)],\ t \in (-\infty,\infty), \tag{54}$$

where

$$\mathfrak{R}_2(t,u) = 1 \text{ for } t < 0,\ u = 1,2,...,z, \tag{55}$$

$$\mathfrak{R}_2(t,u) = \sum_{b=1}^{v} p_b \exp[-t^{l-m+1}] \text{ for } t \ge 0,\ u = 1,2,...,z, \tag{56}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) = 1 \text{ for } t < 0,\ u = 1,2,...,z, \tag{57}$$

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}]^{l-m+1}] \text{ for } t \ge 0,\ u = 1,2,...,z. \tag{58}$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} < 0 \text{ for } t<0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} \geq 0 \text{ for } t \geq 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v,$$

then, according to (51), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= \exp[-\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}}] = 1 - o(1),$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)] = 1 - \exp[-\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}}]$$

$$= \frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} - o(\frac{1}{n^{1/(l_n-m+1)}}) \text{ for } t \geq 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v.$$

Then for each $i = 0,1,\ldots,m-1$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = 1$$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = 0 \text{ for } t < 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - o(1)]^i \to 1 \text{ as } n \to \infty \text{ for } t \geq 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = [\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} - o(\frac{1}{n^{1/(l_n-m+1)}})]^{l_n-i}$$

$$= \frac{t^{l_n-i}}{[n\binom{l_n}{m-1}]^{(l_n-i)/(l_n-m+1)}} [1 - o(\frac{1}{[n\binom{l_n}{m-1}]^{(l_n-i)/(l_n-m+1)}})]^{l_n-i} \text{ for } t \geq 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = o(1) \text{ for } i = 0,1,\ldots,m-2, \; t \geq 0, \; u = 1,2,\ldots,z, \; b = 1,2,\ldots,v,$$

$$[[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = \frac{t^{l_n - m+1}}{n\binom{l_n}{m-1}}[1-o(1)] \text{ for } i = m-1,\ t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = \lim_{n\to\infty} n\cdot 0 = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = \lim_{n\to\infty} n\binom{l_n}{m-1}[1-o(1)]^{m-1}\frac{t^{l_n - m+1}}{n\binom{l_n}{m-1}}[1-o(1)] = t^{l-m+1}$$

for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

which means that according to *Lemma 4* the limit reliability function of that system is given by (54)-(56). □

Proposition 5. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (59)

(ii) $k_n = n$, $c << l_n$, $c \log n - l_n >> s$, $c > 0$, $s > 0$, m = constant ($m/l_n \to 0$, as $n \to \infty$) or $m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{[[n\binom{l_n}{m-1}]^{\frac{1}{l_n - m+1}} - 1]\lambda^{(b)}(u)(l_n - m+1)}, \quad (60)$$

$$b_n^{(b)}(u) = -\frac{1}{\lambda^{(b)}(u)}\log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n - m+1}}],\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (61)$$

then

$$\overline{\mathfrak{R}}_3(t,\cdot) = [1, \overline{\mathfrak{R}}_3(t,1), ..., \overline{\mathfrak{R}}_3(t,z)], \quad t \in (-\infty,\infty), \quad (62)$$

where

$$\mathfrak{R}_3(t,u) = \sum_{b=1}^{v} p_b \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty), u = 1,2,...,z, \quad (63)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[t[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\, \lambda^{(b)}(u)(l_n - m + 1)\,]$$

$$+ \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}\,]\,[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1](l_n - m + 1)] \quad \text{for} \quad t \in (-\infty,\infty), \quad u = 1,2,...,z.$$

(64)

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) > 0 \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) \to 0 \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (59) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty),$$

$$u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - o(\frac{1}{a_n^{(b)}(u)t + b_n^{(b)}(u)}) \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and considering

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),$$

we obtain

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Hence, for each $i = 0,1,...,m-1$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - o(\frac{1}{a_n^{(b)}(u)t + b_n^{(b)}(u)})]^i \to 1$$

as $n \to \infty$ for $t \in (-\infty,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+ \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})]^{l_n - i}$$

$$[1 + \frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n - i}$$

$$= [n\binom{l_n}{m-1} + o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n - i}{l_n - m + 1}} [1 + \frac{t}{(l_n - m + 1)[1 + o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n - i} \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

.

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = o(1) \text{ for } i = 0,1,...,m-2,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = n[n\binom{l_n}{m-1}]^{-1} [1 + \frac{t}{(l_n - m + 1)}]^{l_n - m + 1} [1 - o(1)]$$

for $i = m-1$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \quad [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} \, [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} \, [n\binom{l_n}{m-1}]^{-1} \, [1 + \frac{t}{(l_n - m + 1)}]^{l_n-m+1} = \exp[t] \ \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (62)-(64). □

Proposition 6. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$$[R(t,u)]^{(b)} = 1 \text{ for } t < 0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t] \quad \text{for} \quad t \geq 0, \quad u = 1,2,...,z,\ b = 1,2,...,v,$$
(65)

(ii) $k_n = n$, l_n - $c \log n \sim s$, $c > 0$, $s \in (-\infty,\infty)$, m = constant ($m/l_n \to 0$, as $n \to \infty$) or $m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\lambda^{(b)}(u)(l_n - m + 1)}, \quad (66)$$

$$b_n^{(b)}(u) = -\frac{1}{\lambda^{(b)}(u)} \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}],\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (67)$$

then

$$\overline{\mathfrak{R}}_3(t,\cdot) = [1, \overline{\mathfrak{R}}_3(t,1), ..., \overline{\mathfrak{R}}_3(t,z)], \qquad t \in (-\infty,\infty),$$
(68)

where

$$\overline{\mathfrak{R}}_3(t,u) = \sum_{b=1}^{v} p_b \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty), \quad (69)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[t[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\ \lambda^{(b)}(u)(l_n - m + 1) + \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}]$$

$$[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1](l_n - m + 1)] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z. \quad (70)$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) > 0 \ \text{ for } t \in (-\infty,\infty),\ b = 1,2,...,v,\ u = 1,2,...,z,$$

and

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty, \; b = 1,2,...,v, \;\; u = 1,2,...,z,$$

then, according to (65) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; b = 1,2,...,v, \; u = 1,2,...,z,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,..,z, \; b = 1,2,..,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})] \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] \text{for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n(u)})] \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v.$

Hence, for each $i = 0,1,...,m-1$ for n large enough, we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]^i \exp[-i\lambda^{(b)}(u)b_n^{(b)}(u)] \to 1$$

$$\text{as } n \to \infty \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+ \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n - i}$$

$$= [1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})]^{l_n-i}\ [1+\frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]+o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n-i}$$

$$= [n\binom{l_n}{m-1} + o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n-i}{l_n-m+1}}\ [1+\frac{t}{(l_n-m+1)[1+o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n-i}$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

From last equation we obtain

$$[[F(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = o(1)$$

for $i = 0,1,...,m-2,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = n[n\binom{l_n}{m-1}]^{-1}\ [1+\frac{t}{(l_n-m+1)}]^{l_n-m+1}\ [1-o(1)]$$

for $i = m-1,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (50), it appears that

$$[\bar{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)l_n-i} = \lim_{n\to\infty} n\binom{l_n}{m-1}\ [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1}\ [n\binom{l_n}{m-1}]^{-1}\ [1+\frac{t}{(l_n-m+1)}]^{l_n-m+1} = \exp[t]\ \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (68)-(69). □

Proposition 7. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (71)

(ii) $k_n = n,\ \ l_n - c\log n >> s,\ c > 0,\ s > 0,\ m = \text{constant}$ ($m/l_n \to 0$, as $n \to \infty$) or

$m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)\log n\binom{l_n}{m-1}}$, (72)

$$b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)} \log(\frac{l_n - m + 1}{\log n \binom{l_n}{m-1}}, \; u = 1,2,...,z, \; b = 1,2,...,v, \tag{73}$$

then

$$\overline{\mathfrak{R}}_3(t,\cdot) = [1, \overline{\mathfrak{R}}_3(t,1), ..., \overline{\mathfrak{R}}_3(t,z)], \quad t \in (-\infty,\infty), \tag{74}$$

where

$$\overline{\mathfrak{R}}_3(t,u) = \sum_{b=1}^{v} p_b \; \exp[\text{-}\exp[t]] \text{ for } t \in (\text{-}\infty,\infty), \; u = 1,2,...,z, \tag{75}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[\lambda^{(b)}(u) t \log n \binom{l_n}{m-1} - \log[\frac{l_n - m + 1}{\log n \binom{l_n}{m-1}} \log n \binom{l_n}{m-1}]]] \tag{76}$$

for $t \in (\text{-}\infty,\infty), \; u = 1,2,...,z.$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) \to +\infty \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty), \; b = 1,2,...,v, \; u = 1,2,...,z,$$

and

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty,$$

then, according to (71) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]\exp[-\lambda^{(b)}(u)b_n^{(b)}] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})] \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, for each $i = 0,1,...,m-1$ for n large enough, we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]^i \exp[-i\lambda^{(b)}(u)b_n^{(b)}(u)] \to 1$$

$$\text{as } n \to \infty \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+ \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n - i}$$

$$= [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})]^{l_n - i}\ [1 + \frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n - i}$$

$$= [n\binom{l_n}{m-1} + o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n - i}{l_n - m + 1}}\ [1 + \frac{t}{(l_n - m + 1)[1 + o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n - i}$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = o(1) \text{ for } i = 0,1,...,m-2,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = n[n\binom{l_n}{m-1}]^{-1}\ [1 + \frac{t}{(l_n - m + 1)}]^{l_n - m + 1}\ [1 - o(1)]$$

for $i = m-1,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = \lim_{n\to\infty} n\binom{l_n}{m-1} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} [n\binom{l_n}{m-1}]^{-1} [1 + \frac{t}{(l_n - m + 1)}]^{l_n - m+1} = \exp[t] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (74)-(75). □

The next auxiliary theorem is proved in (Kolowrocki 2005).

Lemma 5. If

(i) $k_n = n$, $l_n \to l = \text{const}$, $m \le l_n$, as $n \to \infty$

(ii) $\overline{\boldsymbol{\mathfrak{R}}}(t) = \sum_{b=1}^{v} p_b \exp[-[\overline{V}(t)]^{(b)}]$ is a non-degenerate reliability function,

(iii) $\overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,\cdot) = [1, \overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,1), ..., \overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\overline{\boldsymbol{R}}}_{k_n,l_n}^{(m)}(t,u)]^{(b)} = [\sum_{i=0}^{l_n - m} \binom{l_n}{i} [1 - [R(t,u)]^{(b)}]^i [[R(t,u)]^{(b)}]^{l_n - i}]^{k_n},\ t\in(-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (77)$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} k_n [1 - \sum_{i=0}^{l_n - m} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}] = [\overline{V}(t,u)]^{(b)} \quad (78)$$

for $t \in C_{[\overline{V}(u)]^{(b)}}$, $u = 1,2,...,z,\ b = 1,2,...,v.$

Proposition 8. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$
(79)

(ii) $k_n = n$, $c << l_n$, $c \log n - l_n >> s$, $c > 0$, $s > 0$, $(l_n - m) = \overline{m}$ = const, ($m/l_n \to 1$ as $n \to \infty$),

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}$, $b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (80)

then

$$\overline{\mathfrak{R}}_2(t,\cdot) = [1, \overline{\mathfrak{R}}_2(t,1), ..., \overline{\mathfrak{R}}_2(t,z)],\ t \in (-\infty,\infty), \tag{81}$$

where

$$\overline{\mathfrak{R}}_2(t,u) = 1 \text{ for } t < 0, \tag{82}$$

$$\overline{\mathfrak{R}}_2(t,u) = \sum_{b=1}^{v} p_b \exp[-t^{\overline{m}+1}] \text{ for } t \geq 0,\ u = 1,2,...,z, \tag{83}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) = 1 \text{ for } t < 0, \tag{84}$$

$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\left(\dfrac{l_n}{\overline{m}+1}\right)^{1/\overline{m}+1}]^{\overline{m}+1}]$ for $t \geq 0$, $u = 1,2,...,z.$
(85)

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} \geq 0 \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (79), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}) \text{ for } t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}) \text{ for } t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v.$$

Next, for each $i = \bar{m}+1, \bar{m}+2, ..., l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = 1 \text{ for } t < 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = 0 \text{ for } t < 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}})]^{l_n - i} \to 1 \quad \text{as } n \to \infty$$

for $t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = [\frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n(\frac{l_n}{\bar{m}+1})]^{i/(\bar{m}+1)}} [1 - o(1)]^i \text{ for } t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = o(\frac{1}{n(\frac{l_n}{\bar{m}+1})}) \text{ for } i = \bar{m}+2, \bar{m}+3, ..., l_n, \; t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = \frac{t^{\bar{m}+1}}{n(\frac{l_n}{\bar{m}+1})} [1 - o(1)] \text{ for } i = \bar{m}+1, \; t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\bar{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} \; [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} \; [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$+ \sum_{i=\bar{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} \; [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$u = 1,2,...,z,\ b = 1,2,...,v,$

then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \ \text{ for } \ t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n\binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \ \text{ for } \ t \geq 0,\ \ u = 1,2,...,z,\ b = 1,2,...,v.$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (81)-(83). □

Proposition. 9 If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (86)

(ii) $k_n = n,\ l_n - c \log n \sim s,\ c > 0,\ s \in (-\infty,\infty), (n-m) = \overline{m} = \text{const}, (m/l_n \to 1 \text{ as } n \to \infty)$

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}},\ b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (87)

then

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}_2(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}_2(t,z)],\ t \in (-\infty,\infty), \tag{88}$$

where

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = 1 \text{ for } t < 0, \tag{89}$$

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = \sum_{b=1}^{v} p_b \ \exp[-t^{\overline{m}+1}] \text{ for } t \ge 0,\ \ u = 1,2,...,z, \tag{90}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = 1 \text{ for } t < 0, \tag{91}$$

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\left(\frac{l_n}{\overline{m}+1}\right)^{1/\overline{m}+1}]^{\overline{m}+1}] \text{ for } t \ge 0,\ u = 1,2,...,z. \tag{92}$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} \ge 0 \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (86), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})$$

for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$

Next, for each $i = \overline{m}+1, \overline{m}+2, ..., l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}})]^{l_n - i} \to 1 \text{ as } n \to \infty$$

$$\text{for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = [\frac{t}{[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}} - o(\frac{1}{[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n\binom{l_n}{\overline{m}+1}]^{i/(\overline{m}+1)}}[1 - o(1)]^i \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = o(\frac{1}{n\binom{l_n}{\overline{m}+1}}) \text{ for } i = \overline{m}+2, \overline{m}+3, ..., l_n,\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}}[1 - o(1)] \text{ for } i = \overline{m}+1,\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{b)}]\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$u = 1,2,...,z,\ b = 1,2,...,v,$ then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \ \text{ for } \ t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \ \text{ for } \ t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (88)-(90). □

Proposition 10. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (93)

(ii) $k_n = n,\ \ l_n$ - $c \log n >> s,\ c > 0,\ s > 0,\ (n - m) = \overline{m} = \text{const},\ (m/l_n \to 1$ as $n \to \infty)$

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}},\ b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (94)

then

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}_2(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}_2(t,z)],\ t \in (-\infty,\infty), \tag{95}$$

where

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = 1 \text{ for } t < 0, \tag{96}$$

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = \sum_{b=1}^{v} p_b \ \exp[-t^{\overline{m}+1}] \text{ for } t \ge 0, \tag{97}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}(t,u)} = 1 \text{ for } t < 0, \tag{98}$$

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}(t,u)} \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\left(\frac{l_n}{\overline{m}+1}\right)^{1/\overline{m}+1}]^{\overline{m}+1}] \quad \text{for} \quad t \quad \geq \quad 0, \quad u = 1,2,...,z.$$

(99)

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} \geq 0 \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (93), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Next, for each $i = \overline{m}+1, \overline{m}+2,...,l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{l_n - i} \to 1 \quad \text{as} \quad n \to \infty$$

for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = [\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n(\frac{l_n}{\overline{m}+1})]^{i/(\overline{m}+1)}}\ [1 - o(1)]^i \ \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = o(\frac{1}{n(\frac{l_n}{\overline{m}+1})}) \ \text{ for } i = \overline{m}+2, \overline{m}+3,...,l_n,\ t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = \frac{t^{\overline{m}+1}}{n(\frac{l_n}{\overline{m}+1})}\ [1 - o(1)] \ \text{ for } i = \overline{m}+1,\ t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}],\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \text{ for } t < 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \;\; [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \;\; [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \text{ for } t \geq 0, \; u = 1,2,...,z, \; b = 1,2,...,v,$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (95)-(97). □

6 CONCLUSION

The purpose of this paper is to give the method of reliability analysis of multi-state "*m* out of *l*"- series systems in variable operation conditions. Their exact and limit reliability functions, in constant and in varying operation conditions, are determined. The paper proposes an approach to the solution of practically very important problem of linking the systems' reliability and their operation processes. To involve the interactions between the systems' operation processes and their varying in time reliability structures a semi-markov model of the systems' operation processes and the multi-state system reliability functions are applied. This approach gives practically important in everyday usage tool for reliability evaluation of the large systems with changing their reliability structures and components reliability characteristic during their operation processes. The results can be applied to the reliability evaluation of real technical systems.

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