ASYMPTOTIC DEPENDENCE OF AVERAGE FAILURE RATE AND MTTF FOR A RECURSIVE, MESHED NETWORK ARCHITECTURE

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ABSTRACT

The paper is concerned with the exact and asymptotic calculations of the availability, average failure rate and MTTF (Mean Time To Failure) for a recursive, meshed architecture proposed by Beichelt and Spross. It shows that the asymptotic size dependences of average failure rate and MTTF are different, but not inverse of each other, as is unfortunately assumed too frequently. Besides, the asymptotic limit is reached for rather small networks.

1 INTRODUCTION

Network availability and reliability have long been a practical issue in telecommunication networks, among others. Quality of Service (QoS) requirements imply high availabilities A, but also a good knowledge of the failure frequency $\overline{\nu}$ – and of the average failure rate $\overline{\lambda}=\overline{\nu}\,/\,A$ – of (for instance) point-to-point connections, when the system is repairable. When the system is not repairable, an important parameter is the MTTF (Mean Time To Failure). As explained in many textbooks (Shooman 1968, Singh & Billinton 1977, Kuo & Zuo 2003), a system whose failure rate λ is constant over time has a reliability described by the exponential distribution $R(t)$ = exp(-λ t), so that the MTTF, defined by

$$\text{MTTF} = \langle t \rangle = \int_0^\infty t\ (-R'(t)\,dt\,) = \int_0^\infty R(t)\,dt\,, \tag{1}$$

is in this case $\text{MTTF}_{\text{exp}} = 1/\lambda$. This may lead to confusions in repairable systems, where it may still be legitimate to consider constant failure rates for each element of the system, and yet obtain an average failure rate $\overline{\lambda}$. If A is the availability of a system made of m elements – whose failures are assumed to be statistically independent – having individual availabilities p_i ($1 \le i \le m$) and constant failure rates λ_i, then (Buzacott 1967, Singh & Billinton 1974, Schneeweiss 1981, 1983, Shi 1981, Hayashi 1991, Druault-Vicard & Tanguy 2006)

$$\overline{\lambda} = \frac{1}{A}\sum_{i=1}^{m} \lambda_i\ p_i\,\frac{\partial A}{\partial p_i}\,. \tag{2}$$

Most results of the literature are devoted to series-parallel systems, where all components are identical, with the same (constant) failure rate λ. For n components in series, $A(p) = p^n$ so that the aggregate failure rate is equal to $\overline{\lambda} = \lambda\,\frac{p}{A}\frac{\partial A}{\partial p} = n\,\lambda$. The reliability of n components in series is $R(t)$ =

$[\exp(-\lambda\ t)]^n = \exp(-\ n\ \lambda\ t)$, which gives MTTF = $1/(n\ \lambda)$. For n components in parallel however, $R(t) = 1-(1-\exp(-\lambda\ t))^n$, which leads to

$$\mathrm{MTTF}_{parallel} = \frac{1}{\lambda}\sum_{i=1}^{m}\frac{1}{i} = \frac{1}{\lambda}\left(\ln n + \mathrm{C} + \frac{1}{2n} + \ldots\right) \tag{3}$$

for n large (Shooman 1968, Kuo 2003) and where C = 0.577216... is the Euler constant. Quite generally, it is therefore important to estimate the reliability and related parameters of large systems in order to get a better understanding of key issues (Kołowrocki 2004).
In this work, we consider a recursive, meshed – not series-parallel – network configuration first considered by Beichelt and Spross (Beichelt & Spross 1989) as well as Prékopa and collaborators (Prékopa et al. 1991). For the repairable case we shall use the availability $A(p)$, and in the non-repairable case the reliability $R(p(t))$, even though $A(p)$ and $R(p)$ are formally identical for the same network made of identical components. We show in detail that when such a system is large, knowledge of the generating function of the reliability/availability allows us find the analytic, asymptotic expressions for $\bar{\lambda}$ and MTTF. These expressions, which both have simple n-dependences, are *not* the inverse of each other: while for $\bar{\lambda}$, we find again a linear dependence in n (Druault-Vicard & Tanguy 2006), we obtain a $n^{-1/5}$ dependence for the MTTF. Besides, they are in very good agreement with the exact values even when n remains relatively small.

2 NETWORK ARCHITECTURE: A CASE STUDY

2.1 Description

The network configuration defined by Beichelt and Spross (Beichelt & Spross 1989) is represented in Figure 1. They wanted to estimate the two-terminal reliability between the endpoints of the structure (in the original paper, the destination point was S_6).

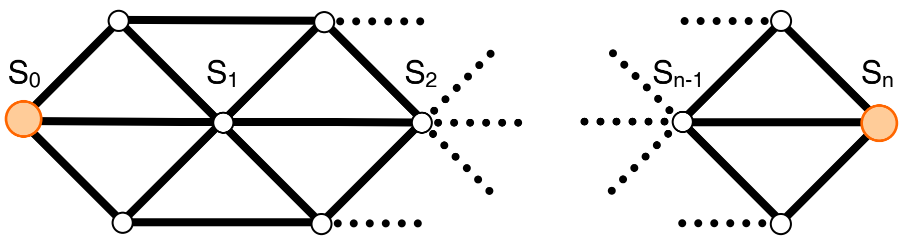

Figure 1. Recursive network architecture (Beichelt and Spross 1989). The source is S_0 and the destination is S_n.

Following the method developed in (Tanguy 2007), we have been able to show that the two-terminal reliability between S_0 and S_n may be expressed as a product of transfer matrices, in which each edge or link probability of functioning is arbitrary. It turns out that this transfer matrix is 15×15. However, if nodes are perfect and if links have the same reliability/availability p, things are much simpler, because a single transfer matrix needs be considered, the successive powers of which are to be calculated. Fortunately, these necessarily obey a recursion relation of finite order stemming from the characteristic polynomial of the transfer matrix. When dealing with $\mathrm{Rel}_2(S_0 \rightarrow S_n) \equiv \mathrm{Rel}_2^{(n)}$, a very useful tool is the generating function formalism (Stanley 1997), since it encodes the exact result in a very concise manner.

2.2 Generating function of the reliability/availability

The generating function $G(z) = \Sigma_n \mathrm{Rel}_2^{(n)}(p)\, z^n$ may eventually be written as $G(z) = N(z)/D(z)$ (Tanguy 2007), where

$$
\begin{aligned}
N(z) =\;& 1 - (1-p)^2\, p\,(1+4p+8p^2-20p^3+9p^4)\, z \\
& - (1-p)^3\, p^3\,(-2-7p+13p^2+26p^3-74p^4+38p^5+29p^6-34p^7+9p^8)\, z^2 \\
& - (1-p)^6\, p^5\,(1+6p-17p^2+4p^3+3p^4+5p^5-14p^6+8p^7+5p^8-7p^9+2p^{10})\, z^3 \\
& - (1-p)^9\, p^8\,(-2+16p^2-4p^3-36p^4+34p^5-7p^6-3p^7+p^8)\, z^4 \\
& + (1-p)^{13}\, p^{11}\,(-1+3p-p^2-3p^3+p^4)\, z^5,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
D(z) =\;& 1 - p\,(2+4p-p^2-33p^3+58p^4-38p^5+9p^6)\, z \\
& + (1-p)^2\, p^2\,(1+6p+11p^2-31p^3-44p^4+168p^5-158p^6+20p^7+63p^8 \\
& \qquad -43p^9+9p^{10})\, z^2 \\
& - (1-p)^4\, p^4\,(2+10p-2p^2-73p^3+138p^4-105p^5+41p^6-40p^7+64p^8-41p^9-5p^{10} \\
& \qquad +21p^{11}-11p^{12}+2p^{13})\, z^3 \\
& + (1-p)^8\, p^6\,(1+8p+2p^2-4p^3-30p^4+23p^5+43p^6-76p^7+47p^8-10p^9 \\
& \qquad -2p^{10}+p^{11})\, z^4 \\
& - (1-p)^{12}\, p^9\,(2+6p+p^2-18p^3+13p^4+2p^5-4p^6+p^7)\, z^5 + (1-p)^{16}\, p^{12}\, z^6.
\end{aligned}
\tag{5}
$$

We deduce for $n = 6$

$$
\begin{aligned}
\frac{\mathrm{Rel}_2^{(6)}(p)}{p^6} =\;& 1+42p+328p^2+826p^3-1473p^4-11400p^5-9975p^6+61060p^7+160918p^8 \\
& -153606p^9-1203380p^{10}-101102p^{11}+6957668p^{12}+2134306p^{13}-33913956p^{14} \\
& -11462384p^{15}+179889959p^{16}-49002916p^{17}-965490222p^{18}+2056136956p^{19} \\
& -213511696p^{20}-7360834390p^{21}+19329198282p^{22}-29836117826p^{23} \\
& +33105011509p^{24}-28179232812p^{25}+18911540288p^{26}-10111211062p^{27} \\
& +4305721566p^{28}-1446762862p^{29}+376155108p^{30}-73146582p^{31} \\
& +10029258p^{32}-865872p^{33}+35442p^{34},
\end{aligned}
\tag{6}
$$

so that $\mathrm{Rel}_2^{(6)}(0.9)$ is equal to 0.9974544308852755355007942390030310588362, which is close to the upper bound given by Beichelt and Spross (Beichelt & Spross 1989). A partial fraction decomposition of G(z) gives

$$
G(z) = \sum_{i=1}^{6} \frac{\alpha_i}{1-\varsigma_i\, z}\,. \tag{7}
$$

There are six eigenvalues ζ_i; a few of them may be pairs of complex conjugate values for some values of p. When the ζ_i's are distinct, (7) immediately gives

$$\mathrm{Rel}_2^{(\mathrm{n})}(p) = \sum_{i=1}^{6} \alpha_i \, \varsigma_i^{\,n} . \tag{8}$$

2.3 Asymptotic reliability/availability

In the limit $n \to \infty$, a single eigenvalue will prevail in the sum of (8), that of largest modulus. In the following, we shall name it ζ_+. It is real for the whole range $0 \le p \le 1$ (see Fig. 2), and necessarily goes to 1 when $p \to 1$ because $\mathrm{Rel}_2^{(\infty)}(p = 1) = 1$; all other eigenvalues tend to zero in that limit.

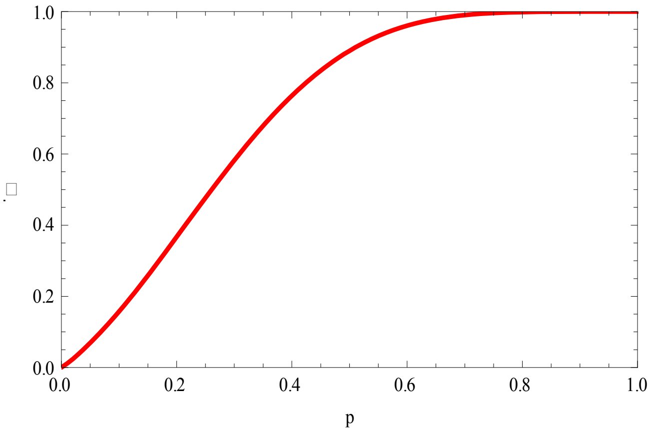

Figure 2. Variation of ζ_+ with p; $\zeta_+(0.9) = 0.9999596999379792$.

Even though it is not possible to get an analytic expression for ζ_+ as a function of p (D(z) is of degree 6 in z), we may compute it numerically very effectively, and also derive the expansion of ζ_+ as a function of $q = 1 - p$ for small q's. Using symbolic software, we deduce from the constraint $D(1/\zeta_+) = 0$

$$\varsigma_+ \to 1 - 4q^5 - 4q^7 + 9q^8 + 9q^9 + 13q^{10} + \ldots \tag{9}$$

When p is close to zero, we have instead:

$$\varsigma_+ \to p + \sqrt{2}\, p^{3/2} + p^2 + \frac{5\sqrt{2}}{4} p^{5/2} + \ldots \tag{10}$$

The prefactor α_+ is deduced from p and ζ_+ because it is closely related to the residue of $G(z)$ at $z = 1/\zeta_+$. The general result is in fact

$$\alpha_+ = \frac{-\varsigma_+ N(1/\varsigma_+)}{D'_z(1/\varsigma_+)} \tag{11}$$

where $D'_z = \partial D(z)/\partial z$. From the knowledge of p and the numerical value of $\zeta_+(p)$, we simply obtain $\alpha_+(p)$, which is plotted in Figure 3.

Here again, we may consider two limits. For $p \to 1$,

$$\alpha_+ \to 1 - 2q^3 - 4q^4 + 10q^5 - 7q^6 + \ldots \tag{12}$$

while when $p \to 0$,

$$\alpha_{+} \rightarrow \frac{1}{2}+\frac{\sqrt{2}}{4}\, p^{1/2} - \frac{11\sqrt{2}}{16}\, p^{3/2} + \ldots \tag{13}$$

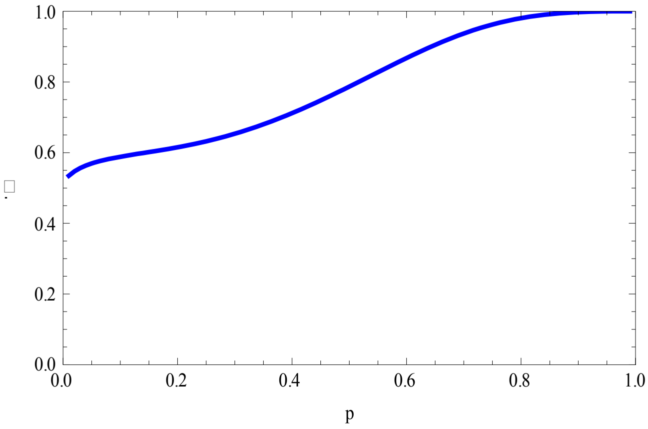

Figure 3. Variation of ζ_+ with p; ζ_+ (0.9) = 09976956497611774972.

The essential result is that, when n is large,

$$\mathrm{Rel}_2^{(n)}(p) = \alpha_+\, \varsigma_+^{\,n} + (\text{neglig. terms}) \tag{14}$$

Basically, it looks as if the recursive network is made of n elements in series, each of which having the reliability/availability ζ_+. The two asymptotic expressions of $\bar{\lambda}$ and MTTF we shall derive as functions of n in the following section are a mere consequence of (14).

3 AVERAGE FAILURE RATE

3.1 Exact expression

In the case of identical links with constant failure rate λ, (2) gives

$$\bar{\lambda}_n = \frac{\bar{\nu}_n}{A_n(p)} = \lambda \frac{p}{A_n(p)} \frac{\partial A_n(p)}{\partial p} \tag{15}$$

Knowing $A_n \equiv \mathrm{Rel}_2^{(n)}(p)$ by recursion (using (4)-(5)), the derivative is easily obtained for arbitrary values $0 \le p \le 1$.

3.2 Asymptotic expression

Because $\mathrm{Rel}_2^{(n)} \approx \alpha_+ \zeta_+^{\,n}$ for n large, we get (Druault-Vicard & Tanguy 2006)

$$\bar{\lambda}_n \approx \lambda \left[n \frac{d \ln \varsigma_+}{d \ln p} + \frac{d \ln \alpha_+}{d \ln p} \right]. \tag{16}$$

Of course, it would be easier to get $d \ln \zeta_+ / d \ln p$ and $d \ln \alpha_+ / d \ln p$ if ζ_+ were known analytically. Still, as in the formal calculation of α_+, $D(1/\zeta_+) = 0$ implies that

$$\frac{\partial D}{\partial p}(1/\varsigma_+)+\left(-\frac{\varsigma_+'(p)}{\varsigma_+(p)^2}\right)\frac{\partial D}{\partial z}(1/\varsigma_+)=0\ , \tag{17}$$

from which we deduce $\zeta_+'(p)$ and then

$$\frac{d\ln\varsigma_+}{d\ln p}=p\,\varsigma_+\,\frac{D'_p\,(1/\varsigma_+)}{D'_z\,(1/\varsigma_+)}\ , \tag{18}$$

$$\frac{d\ln\alpha_+}{d\ln p}=\frac{d\ln\varsigma_+}{d\ln p}\left[1-\frac{N'_z\,(1/\varsigma_+)}{\varsigma_+\,N(1/\varsigma_+)}+\frac{D''_{zz}\,(1/\varsigma_+)}{\varsigma_+\,D'_z\,(1/\varsigma_+)}\right]+p\,\frac{N'_p\,(1/\varsigma_+)}{N(1/\varsigma_+)}-p\,\frac{D''_{zp}\,(1/\varsigma_+)}{D'_z\,(1/\varsigma_+)}\ . \tag{19}$$

Their variations for $0\le p\le 1$ are displayed in Figures 4-5.

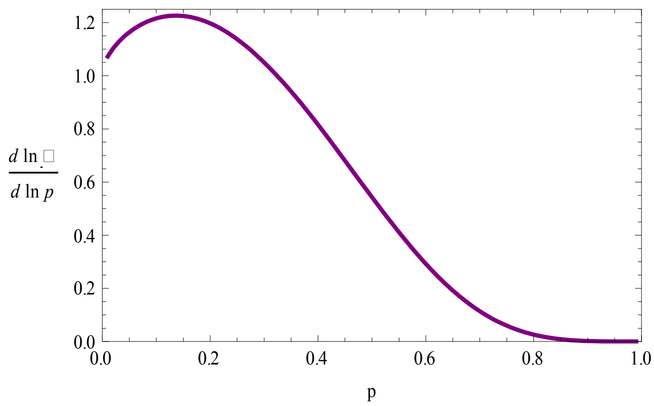

Figure 4. Variation of $d\ln\zeta_+/\mathrm{d}\ln p$ with p.

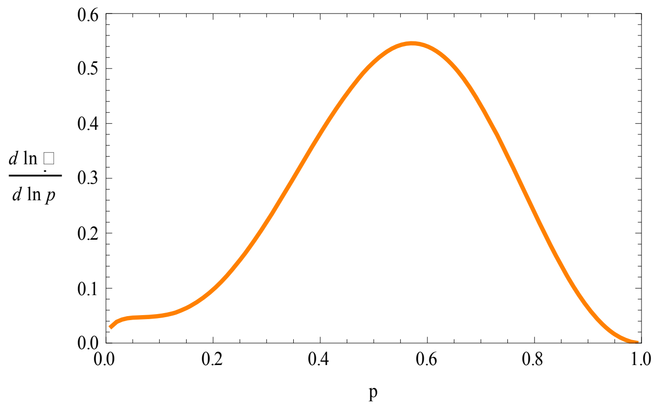

Figure 5. Variation of $d\ln\alpha_+/\mathrm{d}\ln p$ with p.

Note that, unsurprisingly, they exhibit singular behaviors in the vicinity of $p = 0$:

$$\frac{d\ln\varsigma_+}{d\ln p}\to 1+\frac{\sqrt{2}}{2}\,p^{1/2}+\frac{11\sqrt{2}}{8}\,p^{3/2}+\ldots \tag{20}$$

$$\frac{d\ln\alpha_+}{d\ln p}\to \frac{\sqrt{2}}{4}\,p^{1/2}-\frac{1}{4}\,p-\frac{31\sqrt{2}}{16}\,p^{3/2}+\ldots \tag{21}$$

Exact results as well as the linear approximation (see (16)) are displayed in Figure 6 for $p = 0.9$. We see that the agreement is excellent even for $n = 2$.

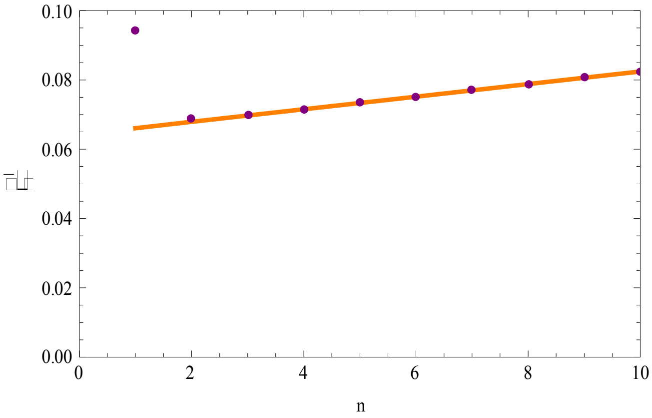

Figure 6. Comparison between exact results (purple) and asymptotic approximation 0.06426+0.0018180 n (orange) for $\overline{\lambda}/\lambda$ and $p = 0.9$.

4 MTTF CALCULATIONS

4.1 Exact expression

We are now considering a non-repairable system, and its reliability $R_n(t)$. Let us recall that

$$\mathrm{MTTF}_n = \int_0^\infty R_n(t)\,dt \ . \tag{22}$$

If each element has reliability $p(t) = \exp(-\lambda\, t)$, we can write $t = (-\,1/\,\lambda) \ln p(t)$ and then (22) as

$$\mathrm{MTTF}_n = \frac{1}{\lambda}\int_0^1 \frac{dp}{p}\,\mathrm{Rel}_2^{(n)}(p) \ . \tag{23}$$

We can reuse the results obtained in Section II. Clearly, the exact MTTF_n is obtained from (23), since such an integration is routinely performed by mathematical software.

4.2 Asymptotic expression

The calculation of the asymptotic expansion of MTTF_n is based again on $R_n \approx \alpha_+\,\zeta_+^n$ when n is large:

$$\mathrm{MTTF}_n \approx \frac{1}{\lambda}\int_0^1 \frac{dp}{p}\,\alpha_+(p)\,\varsigma_+^n(p) \ . \tag{24}$$

We have plotted ζ_+ and ζ_+^{40} in Figure 7. Because ζ_+ vanishes for $p \to 0$, the $1/p$ factor does not play a significant role in the integral. As n increases, the essential contribution to the integral will obviously come from the domain "p close to unity".

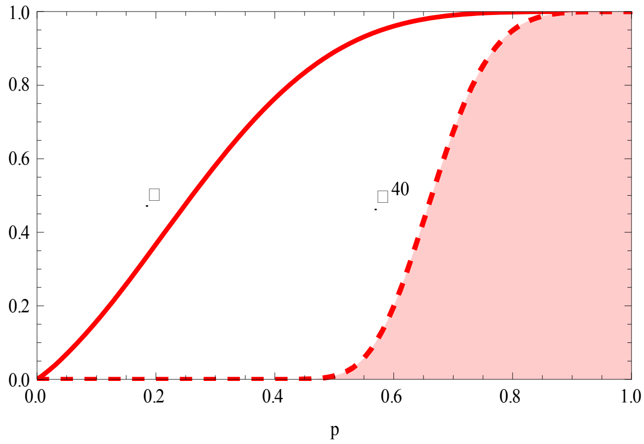

Figure 7. Variation with p of ζ_+ and ζ_+^{40}.

The best approach is therefore to use q as the variable of integration

$$\text{MTTF}_n \approx \frac{1}{\lambda}\int_0^1 \frac{dq}{1-q}\ \alpha_+(1-q)\ \varsigma_+^n(1-q)\ . \tag{25}$$

The gist of the calculation, quite standard in asymptotic expansions, is to extract the prevailing contribution of the integrand when $q \to 0$. We can write

$$\varsigma_+^n = \exp\left(-n\,(-\ln\,\varsigma_+)\right) \tag{26}$$

and derive the expansion of - ln ζ_+ in q from (9)

$$-\ln\,\varsigma_+ = 4q^5 + 4q^7 - 9q^8 - 9q^9 - 5q^{10} + \ldots\,, \tag{27}$$

so that

$$\varsigma_+^n = e^{-4nq^5}\ \exp\left(-n\,(4q^7 - 9q^8 - 9q^9 - 5q^{10} + \ldots)\right). \tag{28}$$

This manipulation may seem quite formal, but now we can use a rescaled variable $\tau = 4\ n\ q^5$, or, equivalently, set $q = \tau^{1/5}/(4\ n)^{1/5}$. Equation (28) then gives

$$\varsigma_+^n = e^{-\tau}\ \exp\left[-n\left(4\left(\frac{\tau}{4n}\right)^{7/5} - 9\left(\frac{\tau}{4n}\right)^{8/5} + \ldots\right)\right] = e^{-\tau}\ \exp\left[-\frac{4}{n^{2/5}}\left(\frac{\tau}{4}\right)^{7/5} + \frac{9}{n^{3/5}}\left(\frac{\tau}{4}\right)^{8/5} + \ldots\right]. \tag{29}$$

Equation (25) leads to

$$\text{MTTF}_n \approx \frac{1}{\lambda}\int_0^{4n} \frac{\tau^{-4/5}\,d\tau}{5(4n)^{1/5}}\ \frac{\alpha_+\left(1-\left(\frac{\tau}{4n}\right)^{1/5}\right)}{1-\left(\frac{\tau}{4n}\right)^{1/5}}\ e^{-\tau}\times\exp\left(-\frac{4}{n^{2/5}}\left(\frac{\tau}{4}\right)^{7/5} + \frac{9}{n^{3/5}}\left(\frac{\tau}{4}\right)^{8/5} + \ldots\right). \tag{30}$$

The upper bound of the integral depends on n. However, because of the $e^{-\tau}$ factor, the error made by replacing this upper bound by $+\infty$ vanishes exponentially with n (as also do the already discarded contributions of the eigenvalues different from ζ_+). Consequently, we can merely integrate $\tau^{-4/5}\ e^{-\tau}$

multiplied by an expression admittedly depending on τ and n, but which can be easily expanded in the $n \to \infty$ limit, assuming τ remains finite. For instance, the leading term of MTTF_n is (see (12))

$$\mathrm{MTTF}_n \to \frac{1}{\lambda}\int_0^{\infty} \frac{\tau^{-4/5}\, d\tau}{5(4n)^{1/5}}\, e^{-\tau} = \frac{1}{\lambda}\,\frac{\Gamma(1/5)}{5(4n)^{1/5}} = \frac{1}{\lambda}\,\frac{\Gamma(6/5)}{(4n)^{1/5}}\,, \tag{31}$$

where Γ(x) is the Euler gamma function. Using (30), going beyond the leading term is not difficult, and we find

$$\lambda\,\mathrm{MTTF}_n \to \frac{\Gamma(6/5)}{(4n)^{1/5}} + \frac{1}{5}\frac{\Gamma(2/5)}{(4n)^{2/5}} + \frac{2}{25}\frac{\Gamma(3/5)}{(4n)^{3/5}} - \frac{1}{40n} + \ldots \tag{32}$$

$$\approx \frac{0.6958417869}{n^{1/5}} + \frac{0.2547996219}{n^{2/5}} + \frac{0.05185668604}{n^{3/5}} - \frac{0.025}{n} + \ldots \tag{33}$$

By contrast to the series or parallel cases, the leading term in the asymptotic expansion of the MTTF has a behavior in $n^{-1/5}$, which slowly decreases with n. Each of the following terms of the expansion adds another $n^{-1/5}$ factor.

4.3 Comparison of exact and asymptotic results

We can now compare (33) with the exact values. The results are displayed in Figure 8. Even for $n \approx 10$, the asymptotic expansion gives a very satisfying agreement, despite the limited number of used terms (four).

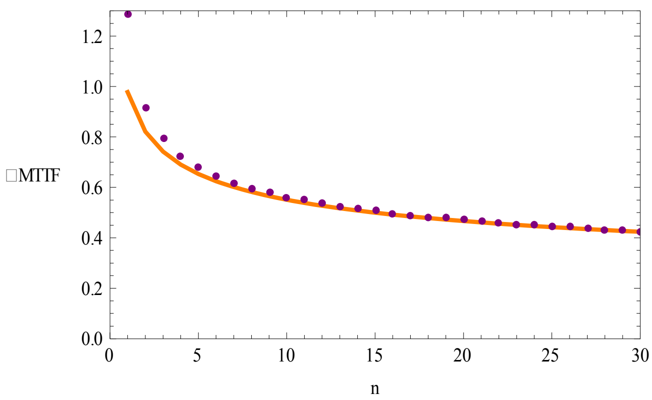

Figure 8. Comparison between exact values (purple) and the four-term asymptotic expansion of (33) (orange) of the MTTF.

5 CONCLUSION AND OUTLOOK

We have calculated the availability of the architecture studied by Beichelt and Spross, and shown that for perfect nodes and identical links with constant failure rate, the asymptotic expansions of the associated average failure rate and MTTF obey quite different power-law behaviors in n (the extension of the network). It could be useful as a reminder that average failure rate and MTTF are not necessarily the inverse of each other.

The present study may be easily generalized to various recursive networks. Actually, it is possible to find the asymptotic expansion of the MTTF for different classes of large, arbitrary recursive networks, even though the exact generating function is not known (Tanguy 2008).

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