MAINTENANCE POLICY FOR DETERIORATING SYSTEM WITH EXPLANATORY VARIABLES

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ABSTRACT

This paper discusses the problem of the optimization of maintenance threshold and inspection period for a continuously deteriorating system with the influence of covariates. The deterioration is modeled by an increasing stochastic process. The process of covariates is assumed to be a temporally homogeneous finite-state Markov chain. A model similar to the proportional hazards model is used to represent the influence of the covariates. Parametric estimators of the unknown parameters are obtained by using Least Square Method. The optimal maintenance threshold and the optimal inspection interval are derived to minimize the expected average cost. Comparisons of the expected average costs under different conditions of covariates and different maintenance policies are given by numerical results of Monte Carlo simulation.

1 INTRODUCTION

Optimal replacement problems for deteriorating systems have been intensively studied in the past decades by a number of researchers (for instance, Aven & Jensen (1999), Wang (2002) and Wang & Pham (2006), van Noortwijk (2009)). As far as continuously deteriorating systems are considered, most of the attention has been focused on static environment and on monotonic increasing deterioration systems, with periodic or non-periodic inspection. Various stochastic processes have been proposed to represent the degradation or wear process (e.g. Grall et al. (2002), Bérenguer et al. (2003) and van Noortwijk (2009)). Recently more interest and attention has been given to two approaches. One approach is to deal with degradation models including explanatory variables (covariates). These variables describe the dynamic environment; in the experiments of life science and engineering, they are often expressed by the proportional hazards model (Newby (1994), Singpurwalla (1995), Meeker & Escobar (1998) and Lawless & Crowder (2004)). Bagdonavičius & Nikulin (2000) propose a method to model an increasing degradation by a gamma process which includes time-dependent covariates. Makis & Jardine (1992) consider an optimal replacement problem for a system with stochastic deterioration which depends on its age and also on the value of covariates. Kharoufeh & Cox (2005) deal with a degradation-based procedure to estimate lifetime distribution, where the single-unit system is exposed to a stochastically evolving environment characterized by a stationary continuous-time Markov chain. Meeker et al. (1998)

describe a degradation reliability model, where the dynamical temperature is represented by an accelerated model. The other approach is to consider a non-monotonic deteriorating system with increasing tendency (Newby & Dagg (2002), Newby & Dagg (2003), Newby & Barker (2006), Barker & Newby (2009)). Barker and Newby (2009) study the problem of optimal inspection and maintenance policy for a non-monotonic system. They use the last exiting time from a critical set instead of the first hitting time to determine the optimal policy.

In this paper we focus on the optimal policy of periodic inspection/replacement for a monotonic deteriorating system with explanatory variables (covariates), in which the covariate process is supposed to be a temporally homogeneous Markov chain. The influence of the covariates on degradation is considered by a multiplicative exponential function. The system is supposed to be failed when the system state crosses a fixed threshold known as failure threshold. The purpose is to propose an optimal maintenance policy for the considered system in order to minimize the global long-run expected average maintenance cost per time unit.

The other particularity of this paper is that we compare the maintenance cost under following cases: (1) the optimization when the covariates are defined as a Markov chain; (2) the optimization when the covariates $Z_n = i$ ($i = 1,2,3$) are fixed; (3) the weighted mean of the optimal costs for each $Z_n = i$ ($i = 1,2,3$) weighted by the steady-state probabilities. All results are illustrated by a Monte Carlo study.

The structure of the paper is as follows. In Section 2 we model the degradation process by a stochastic process, where the influence of the covariates is modeled by a multiplicative exponential function. In Section 3 we study the maintenance optimization problem. Finally, we compare the expected average maintenance costs per unit time for the different cases mentioned above.

2. STOCHASTIC DETERIORATION PROCESS

In this section, we consider a single-unit replaceable system in which an item is replaced with a new one, either at failure or at preventive replacement.

2.1. Deterioration model without covariates

The degradation of the system is represented by a continuous-state stochastic process $D(t)$ with initial degradation level $D(0) = 0$. We also suppose that the increment of the system can be modeled by a continuous nonnegative-valued process $X(t)$ with exponential distribution, that is, the random increment $D(s) - D(t)$ subjects to an exponential distribution with mean $\lambda(s-t)$.

Suppose that the deterioration can be observed at each time unit t_k ($k = 1,2,\cdots$), the discrete observed stochastic processes are defined as follows: $D_k = D(t_k)$ and $X_k = X(t_k)$.
The process D_n is defined as:

$$D_n = D_{n-1} + X_n \quad , \tag{1}$$

where X_n are random variables of exponential distribution with mean μ_n, denoted by $X_n \sim \varepsilon(1/\mu_n)$.

Denote $\lambda_i = \dfrac{1}{\mu_i}$, it can be proved (see Appendix) that if $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \varepsilon(\lambda_i)$ are independent, then:

(1) If $\lambda_i = \lambda$ are the same parameters, then X will be an Erlang distributed variable with parameters (n, λ) (Soong (2004));

(2) If $\lambda_i \neq \lambda_j$ for $i \neq j$, when $y > 0$, the density probability function will be:

$$f_X(y) = \left(\prod_{i=1}^{n} \lambda_i\right) \sum_{i=1}^{n-1} \frac{\exp(-\lambda_i y) - \exp(-\lambda_n y)}{\prod_{1 \le j \le n;\, j \ne i} (\lambda_j - \lambda_i)} .$$

Since the degradation is calculated as $D_n = \sum_{i=1}^{n} X_i$ with independent exponentially distributed increment X_i, then we can obtain the distribution function, the density function of the deteriorating process D_n by the above results.

2.2. Modeling the influence of covariates on degradation

The covariate process $Z = \{Z(t), t \ge 0\}$ is assumed to be a temporally homogeneous discrete Markov process with finite states $S = \{1, 2, \cdots, K\}$, here S describes the states of the dynamic environment. Suppose that covariates are available only at each time unit t_k $(k = 1, 2, \cdots)$, and the covariates at time t_k are defined by $Z_k = Z(t_k)$

Let $P_{ij}(k) = P(Z_{k+1} = j \mid Z_k = i)$ be the transition probabilities of process $\{Z_k, k = 1, 2, \cdots\}$. The filtration $\Im_t = \sigma\{Z_s : s \le t\}$ denotes the history of the covariates. Since the process Z is a finite temporally homogeneous Markov process, so $P_{ij}(k) = P_{ij}$ does not depend on k for all $i, j \in S$. We denote by $P = (P_{ij})$ the transition matrix.

We assume that the increment of the degradation at time t_n depends only on the covariates at that time. We shall denote by D_n the observed process at time t_n, defined as:

$$D_n = D_{n-1} + X_n(Z_n), \tag{2}$$

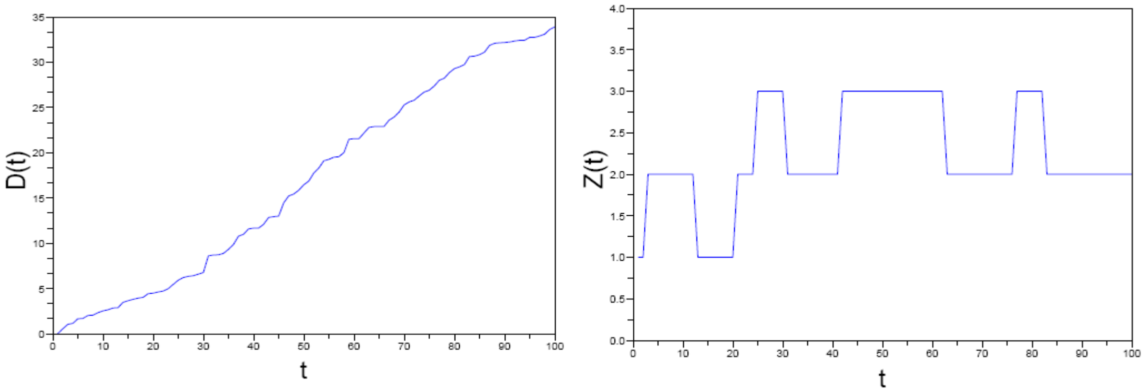
where $X_n(Z_n)$ are exponential distributed with mean parameters $\mu_n(Z_n)$. So $\{D_n, Z_n\}$ is a non-homogeneous Markov process in the sense that the transition probabilities satisfy the following equality:

$$\begin{aligned} &P(D_n \le y, Z_n = j \mid D_{n-1} = x, D_{n-2} = x_{n-2}, \cdots, \quad D_1 = 0; Z_{n-1} = i, Z_{n-2} = z_{n-2}, \cdots, Z_1 = z_1) \\ &= P(D_n \le y, Z_n = j \mid D_{n-1} = x, Z_{n-1} = i) . \end{aligned}$$

To describe precisely the influence of the covariates $Z_n = z_n$ on X_n, similar to the proportional hazards model proposed by Cox (1972), we suppose that the parameters $\mu_n(Z_n)$ depend on Z_n as follows:

$$\mu_n(Z_n) = \mu_0 \exp(\beta_1 1_{\{Z_n = 1\}} + \cdots + \beta_k 1_{\{Z_n = K\}}) = \mu_0 \exp(\beta_{Z_n}), \tag{3}$$

where $\beta = (\beta_1, \cdots, \beta_K)$ is a regression parameter. Considering the symmetrical property of β, without loss of generality, in what follows, we assume that $\beta_1 \le \cdots \le \beta_K$.



(a) (b)

Fig.1 An example of the non-maintained degradation process (a) and the corresponding covariates process (b)

The distribution function and the density function of the increment under the condition of $Z_n = z_n$ are calculated in the same way as before. Then the distribution F_n of $D_n = \sum_{i=1}^{n} X_i(Z_i)$ can be derived using the method of convolution and the total probability formula.

Example 1 An example of degradation for 100 days is given in Figure 1, where Z_n is a 3-state Markov chain with transition matrix $P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0.02 & 0.95 & 0.03 \\ 0.00 & 0.05 & 0.95 \end{pmatrix}$ (corresponds to a steady-state distribution (0.3, 0.5, 0.2)), initial state $Z_0 = 1$, $\beta = (0.2, 0.5, 1)$, the baseline mean parameters $\mu_0 = 0.2$.

For the covariates with initial state $Z_0 = 1$, denoted by $\pi^n = (\pi_1^n, \pi_2^n, \cdots, \pi_K^n)$ the distribution of the covariates Z_n with $\pi_i^n = P(Z_n = i \mid Z_0 = 1)$ the conditional distribution of Z_n under the condition of $Z_0 = 1$. We have

$$(\pi_1^n, \pi_2^n, \cdots, \pi_K^n) = (1, 0, \cdots, 0) P^n ,$$

and $\lim_{n \to +\infty} \pi_i^n = \pi_i$, where π_i is the steady-state distribution of the Markov chain.

In this case, the distribution F_n of $D_n = \sum_{i=1}^{n} X_i(Z_i)$ will be:

$$F_{n+1}(x) = \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} P\left(\sum_{k=1}^{n+1} X_k(i_{k-1}) \le x \right) P_{1 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n} .$$

When the covariates form a steady-state Markov chain, each replacement makes the system restart from its new state $D_0 = 0$ and the covariates Z_n follow their trajectory. Let us denote T_n the instant of replacement (preventive or corrective), then the variables (D_t, Z_t) and (D_{t+T_n}, Z_{t+T_n}) have the same distribution, therefore the trajectory of the degradation does not depend on the history before the replacement. Henceforth, the deterioration process is a renewal process.

2.3 Parametric estimation using least square method

In this section, we use the least square method to estimate the unknown parameters. The data sample is all the degradation data observed before failure, i.e., before the beyond of the critical threshold L.

Since in general case, the mean degradation at time t_k is equal to

$$E(D_n)=\sum_{i=1}^{n}E(\Delta D_i)=\sum_{i=1}^{n}\sum_{j=1}^{K}E(\Delta D_i|Z_i=j)P(Z_i=j)=\sum_{i=1}^{n}\sum_{j=1}^{K}E(\Delta D_i|Z_i=j)\pi_j^i$$

Because of the difficulty of calculating the distribution π_j^i of Z at time t_i, and the case that π_j^i can be approximated by π_j when i is large enough, we can approximate the degradation mean as follows:

$$E(D_n)=\sum_{i=1}^{n}\sum_{j=1}^{K}E(\Delta D_i|Z_i=j)\pi_j=\sum_{i=1}^{n}\sum_{j=1}^{K}\Delta m_i\exp(\beta_j)\pi_j=\left(\sum_{j=1}^{K}\exp(\beta_j)\pi_j\right)\sum_{i=1}^{n}\Delta m_i$$

Therefore the Least Square Estimator $\hat{\theta}=(\mu_0,\beta_1,\beta_2,\beta_3)$ is defined by

$$\hat{\theta}=\arg\min_{\theta}Q_n(\theta), \tag{4}$$

where $Q_n(\theta)=\frac{1}{n}\sum_{i=1}^{n}(D_i-E(D_i))^2$.

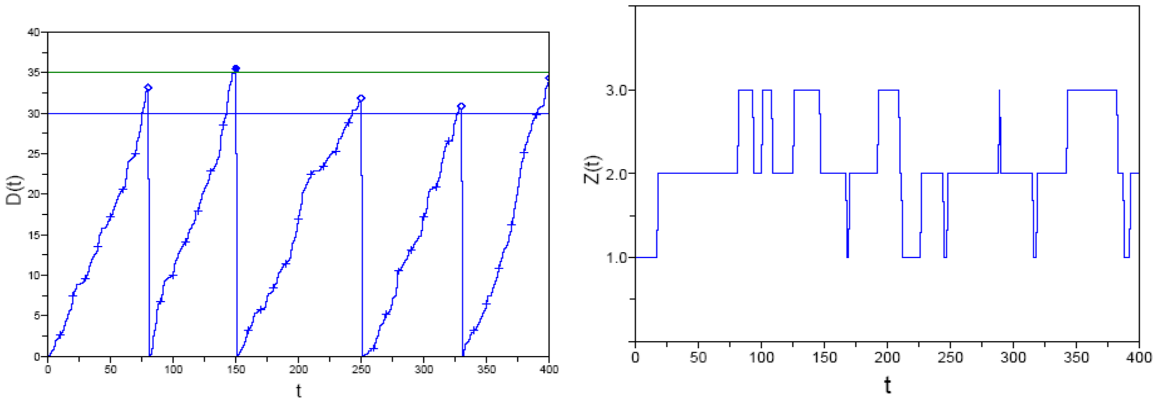
3. CONDITION-BASED PERIODIC MAINTENANCE MODEL

In this section, we study the optimal periodic maintenance policy for the deteriorating system described in Section 2.

Suppose that the system is a monotonically deteriorating stochastic system with initial state $D_0=0$, and the state can exclusively be monitored by inspections at the periodic times $T_k=k\tau$, where $\tau\in\aleph$ is the inspection interval. We now give some assumptions under which the model is studied.

(1) Inspections are perfect in the sense that they reveal the true state of the system and the explanatory variables.
(2) The system state is only known at inspection times and all the maintenance actions take place only at inspection times and they are instantaneous.
(3) Two maintenance operations are available only at the inspection time: preventive replacement and corrective replacement.
(4) The maintenance actions have no influence on the covariate process.

3.1 Maintenance decision



(a) (b)

Fig.2 An example of maintained deterioration system (a) and the corresponding covariate process (b)

Suppose that the system starts with $D_0 = 0$, and is perfectly inspected at periodic times $\Pi = \{\tau, 2\tau, \cdots\}$, ($\tau \in \aleph$), the states are only known at inspection times, and maintenance actions are instantaneous. We define a failure threshold L and a preventive maintenance threshold L_p ($L_p \le L$).

If at inspection time $T_k = k\tau$ we have $D_{k\tau} < L_p$, then three exclusive events may occur at time T_{k+1}:

E1: $D_{(k+1)\tau} \ge L$: which means that the system fails at time $t \in (k\tau, (k+1)\tau]$ and it will be correctively replaced at time $(k+1)\tau$. Costs of corrective replacement C_F as well as a cumulative cost $C_d \times d$ corresponding to the 'inactivity' time have to be considered, where $d = (k+1)\tau - t$ is the cumulated 'inactivity' time.

E2: $D_{(k+1)\tau} \in [L_p, L)$: means that there is no failure in interval $t \in [k\tau, (k+1)\tau]$, however the degradation level is greater than the preventive threshold L_p at time $(k+1)\tau$. So a preventive replacement action takes place at $(k+1)\tau$ which induces a preventive maintenance cost.

E3: $D_{(k+1)\tau} < L_p$: means that the degradation level is always lower than L_p, so there is no replacement action at $(k+1)\tau$, we only have to take into account an inspection cost and the decision time is postponed to $(k+1)\tau$.

An example of a maintained system is given in Figure 2, where the preventive threshold $L_p = 30$, the corrective threshold $L = 35$, and $\tau = 5$, other parameters are the same as in Example 1.

3.2 Calculation of the maintenance cost

Each action of inspection and replacement results in a unit cost. Let C_i, C_p, C_F denote respectively the unit cost of inspection, preventive replacement and corrective replacement. We also consider the cost for 'inactivity' with per unit time cost C_d.

Then the cumulative maintenance cost in $(0, t]$ is:

$$C(t) = C_i N_i(t) + C_p N_p(t) + C_F N_F(t) + C_d d(t)\,, \tag{5}$$

where $N_i(t)$ (respectively $N_p(t), N_F(t)$) is the number of inspections (respectively number of preventive replacements, number of corrective replacements) from 0 to t.

The expected average cost is calculated as follows:

$$EC_\infty = \lim_{t\to\infty} \frac{EC(t)}{t} = \lim_{t\to\infty}\left(\frac{C_i E(N_i(t)) + C_p E(N_p(t)) + C_F E(N_F(t)) + C_d E(D(t))}{t}\right). \tag{6}$$

When the stochastic process (D,Z) is a regenerative process as we stated above, we can calculate the expected cost per time unit as follows:

$$EC_\infty(Z) = \frac{E(V(L(Z)))}{E(L(Z))}\ , \tag{7}$$

where $E(V((Z))$ and $E(L((Z))$ are respectively the expected cost and expected length of a renewal cycle.

Considering the three above exclusive events E1, E2, E3, denote by V_k (respectively L_k) the total cost (respectively length) from time T_k to the time when the system is replaced.

Since the total cost V_k (respectively the total length L_k) is a combination of the cost (respectively the length) in time interval $[T_k, T_{k+1})$ and the cost (respectively the length) after T_{k+1}, we calculate the total maintenance cost $V = V(Z)$ and the length of a renewal cycle $L = L(Z)$ by following iterate method:

$$V_k = (C_i + C_F + C_d d(t))1_{\{E_1\}} + (C_i + C_p)1_{\{E_2\}} + (C_i + V_{k+1})1_{\{E_3\}}, \tag{8}$$

$$L_k = \tau 1_{\{E_1\}} + \tau 1_{\{E_2\}} + (\tau + L_{k+1})1_{\{E_3\}}, \tag{9}$$

and the expectation will be

$$v_k = E(V_k) = (C_i + C_F)P(E_1) + C_d E(d(t)1_{\{E_1\}}) + (C_i + C_p)P(E_2) + C_i P(E_3) + E(V_{k+1}1_{\{E_3\}}), \tag{10}$$

$$l_k = \tau\,(P(E_1) + P(E_2) + P(E_3)) + E(L_{k+1}1_{\{E_3\}}). \tag{11}$$

The optimization problem is to find the value of $\tau*$ and L_p^* minimizing the expected long-run average maintenance cost:

$$(L_p^*, \tau*) = \underset{(L_p,\tau)}{\arg\min}\, EC_\infty(Z). \tag{12}$$

3.3 Description of the optimization procedurec

We now give a formal description of the optimization procedure.

For a given L_p and τ, we estimate the expected maintenance cost as follows.

Step 0: Initialization.

At time $t_0 \equiv 0$, let $D_0 \equiv 0, Z_0 \equiv 1$.

Step 1: Generation of the trajectory of the degradation process

Table1. Estimation of the parameters: mean and standard deviation (within parentheses)

Sample size n	$\hat{\mu}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
100	0.180 (0.046)	0.186 (0.031)	0.480 (0.048)	0.985 (0.058)
200	0.182 (0.039)	0.189 (0.022)	0.485 (0.040)	0.989 (0.039)
500	0.183 (0.030)	0.190 (0.011)	0.488 (0.038)	0.993 (0.040)
1000	0.184 (0.031)	0.196 (0.012)	0.492 (0.043)	0.996 (0.042)

(1) Simulate a trajectory of the covariate process $\{Z_n\}$ with the initial state $Z_0 \equiv 1$ and transition matrix P.
(2) Generate a trajectory of the degradation process conditional upon the trajectory $\{Z_n\}$.

Step 2: Estimation of the maintenance cost conditional upon covariates above

Estimate the total maintenance cost and the total length based on N renewal cycles (N large enough). In each renewal cycle, the maintenance decision is taken according to the three exclusive events (E1)-(E3) mentioned above, the maintenance cost and the maintenance length are calculated as (8) and (9).

Step 3: Estimation of the expected average cost for a stationary Markov chain.

Repeat Step 0-Step 2 to derive the total maintenance cost and the total length for a stationary Markov chain, then calculate the expected average maintenance cost as (6) or (7) indicated. The repetition does not be stoped until the convergence of the expected average maintenance cost.

After the calculation of the expected average maintenance cost by the procedure above for each L_p and τ, we obtain a maintenance cost matrix with respect to L_p and τ, then the optimal decision $\left(L_p^*, \tau^*\right)$ can be derived based on the criteria (12)

4. NUMERICAL RESULTS

4.1 Numerical results for parametric estimators

We apply the least square estimator for a degradation sample described in Section 2. The estimator is defined by (4).

We simulate $N = 1000$ samples with various sample size n. For each sample we give the estimator of the unknown parameter $\theta = (\mu_0, \beta_1, \beta_2, \beta_3)$ for $\theta_0 = (0.2, 0.2, 0.5, 1)$. In Table 1 we summarized the results for Least Square Estimation. For each estimator we give the empirical mean and the empirical standard deviation based on the N estimators we obtained.

The results in Table 1 show that the least square method has a good behavior to estimate the unknown parameters.

4.2 Numerical results for optimal periodic maintenance

In this section we give numerical results of our maintenance optimization problem. The deteriorating system is the system defined in Example 1. We consider four different cases of unit maintenance cost:

Table 2. The optimal preventive threshold, the optimal inspection period and the expected average maintenance cost with periodical inspection

Covariates	(L_p^*, τ^*, C^*) (Case 1)	(L_p^*, τ^*, C^*) (Case 2)	(L_p^*, τ^*, C^*) (Case 3)	(L_p^*, τ^*, C^*) (Case 4)
Z general	(12, 60, 1.0607)	(12, 54, 1.1238)	(19, 63, 1. 5923)	(11, 80, 2. 6891)
Z=1	(21, 120, 0.5158)	(21, 114, 0. 5263)	(23, 123, 0. 9292)	(21, 120, 1. 3016)
Z=2	(20, 87, 0.7183)	(18, 81, 0.7901)	(19, 90, 1. 2955)	(19, 90, 1.7511)
Z=3	(18, 51, 1.2509)	(16, 48, 1.3437)	(17, 51, 2.2431)	(19, 54, 2. 9740)
Mean cost	0.8376	0.90344	1.50657	2.028111

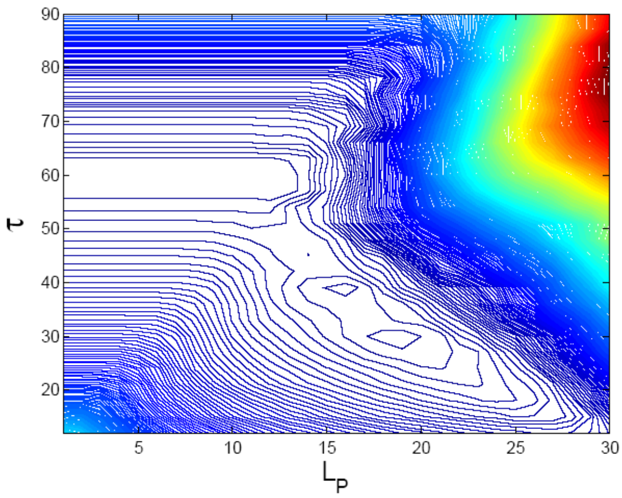


Fig.3 The iso-level curves of EC_∞ for $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for a deteriorating system.

Case 1 (Inexpensive unavailability): $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$;

Case 2 (Expensive unavailability): $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 150$;

Case 3 (Expensive PR): $C_i = 10$, $C_p = 100$, $C_F = 100$ and $C_d = 50$;

Case 4 (Expensive inspection): $C_i = 100$, $C_p = 50$, $C_F = 100$ and $C_d = 50$.

For each case of maintenance cost, we compare the following three values.

(1) Optimal maintenance cost when Z_n come from a general Markov chain;

(2) Optimal maintenance cost when Z_n is fixed to $Z_n = i$ $(i = 1,2,3)$;

(3) Weighted mean of the optimum cost for $Z_n = i$ $(i = 1,2,3)$ with weight given by the steady-state probability:

$$E\overline{C}_\infty = \sum_{k=1}^{3} EC_\infty^*(Z = k)\pi_k$$

Results in Table 2 summarize the results of optimization for a deteriorating system with different maintenance costs. The iso-level curves of expected long-run average cost EC_∞ with $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for such a deteriorating system is depicted in Figure 3,

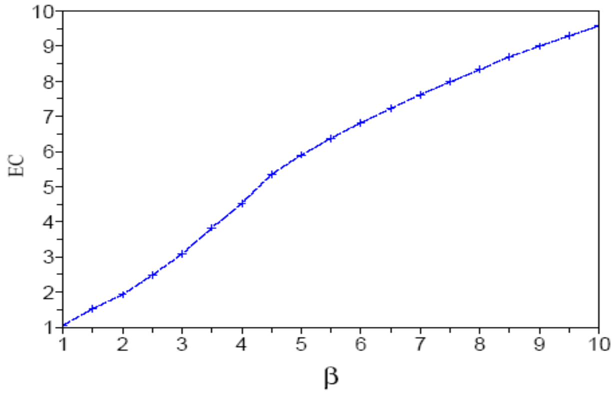


Fig.4 The curve of $EC^*_\infty(\beta)$ for $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for a deteriorating system.

where the optimal parameter values are $L^*_p = 12$, $\tau^* = 60$. These optimal values lead to the optimal expected average cost $EC^* = 1.0607$.

In all cases of the different unit maintenance cost (expensive or inexpensive), the optimal expected average cost under the condition of $Z = 1$ $(\beta = \beta_1)$ are the smallest one. Indeed, for $Z = 1$, the degradation increments are smaller in comparison with other cases. The cost for $Z = 2$ $(\beta = \beta_2)$ is higher than that of $Z = 1$, and cost obtained for $Z = 3$ $(\beta = \beta_3)$ is the highest one. As a consequence, the parameter β can be used to express the influence of the dynamic environment on the deteriorating system.

In order to reveal the way that maintenance cost is influenced by the system parameters β, using the symmetrical property, the optimal expected average cost is computed for various value of β_3 with fixed β_1 and β_2. The result appears in Fig 4. We see that the optimal expected average maintenance cost is an increasing function of the system parameter β_3. In fact, since the regression parameter β expresses the influence of the dynamic environment, the expected average maintenance cost under the worst environment has higher cost than that of better environment.

The expected average maintenance cost for system with a Markov chain is always greater than the weighted mean of the optimal costs for the three static environments, since we have less information for the deteriorating system under a Markov chain than under static environment. The weighted mean of the optimal costs gives the lower bound for the cost of a deteriorating system.

5. Conclusion

This paper deals with the periodic inspection/replacement policy for a monotonic deteriorating system with covariates, where the covariates form temporally homogenous finite states Markov chain. We use a method similar to the proportional hazards model to induce the influence of dynamic covariates on the degradation of the system. Expected average cost is estimated and optimum periodic inspection/replacement policies are derived for different maintenance cost per unit. The numerical results show that the optimal average cost is an increasing function of the regression parameters β. Therefore the parameters β can be used to express the effect of the environment. The relationship between the optimal cost in the case of a covariates Markov chain and a combination of fixed covariates (with stead-state distribution) shows that the first is greater than the later. It will be interesting to apply the methods exposed in this paper on non-monotonic systems.

Appendix: The distribution of the increments of increasing degradation system

We prove the conclusion by mathematical induction. For $n=2$, we have

$$\begin{aligned} f_2(y) &= f_{X_1+X_2}(y) \\ &= \int_0^y \lambda_1 \exp(-\lambda_1 x)\lambda_2 \exp(-\lambda_2(y-x))dx \\ &= \lambda_1\lambda_2 \frac{\exp(-\lambda_1 y)-\exp(-\lambda_2 y)}{\lambda_2-\lambda_1} \end{aligned}$$

Suppose that for $y>0$, $f_n(y)=\left(\prod_{i=1}^{n}\lambda_i\right)\sum_{i=1}^{n-1}\frac{\exp(-\lambda_i y)-\exp(-\lambda_n y)}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}$

then

$$f_{n+1}(y)=\int_0^y f_n(x) f_{X_{n+1}}(y-x)dx$$

$$=\left(\prod_{i=1}^{n}\lambda_i\right)e^{-\lambda_{n+1}y}\int_0^y\left(\sum_{i=1}^{n-1}\frac{e^{-(\lambda_i-\lambda_{n+1})x}-e^{-(\lambda_n-\lambda_{n+1})x}}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}\right)dx$$

$$=\left(\prod_{i=1}^{n+1}\lambda_i\right)\left[\sum_{i=1}^{n-1}\frac{\exp(-\lambda_i y)-\exp(-\lambda_{n+1}y)}{\prod_{1\le j\le n+1; j\ne i}(\lambda_j-\lambda_i)}+\frac{\exp(-\lambda_n y)-\exp(-\lambda_{n+1}y)}{\lambda_{n+1}-\lambda_n}\sum_{i=1}^{n-1}\frac{1}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}\right].$$

Setting $A=\sum_{i=1}^{n-1}\frac{1}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}$, we have

$$A=\frac{\sum_{i=1}^{n-1}(-1)^{i+1}\prod_{k>j\ne i}(\lambda_k-\lambda_j)}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)}=\frac{1}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)}\left(\begin{vmatrix} 1 & & 0 & 0 \\ 1 & & 1 & 1 \\ \lambda_1 & \ddots & \lambda_{n-1} & \lambda_n \\ \lambda_1^{n-2} & & & \lambda_n^{n-2} \end{vmatrix}+\cdots+\begin{vmatrix} 0 & & 1 & 0 \\ 1 & & 1 & 1 \\ \lambda_1 & \ddots & \lambda_{n-1} & \lambda_n \\ \lambda_1^{n-2} & & & \lambda_n^{n-2} \end{vmatrix}\right)$$

$$= \frac{(-1)^{n+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ \lambda_1 & \lambda_{n-1} & \lambda_n \\ & \ddots & \\ \lambda_1^{n-2} & & \lambda_n^{n-2}\end{vmatrix}}{\prod\limits_{1\le i<j\le n}(\lambda_j-\lambda_i)} = \frac{1}{(\lambda_1-\lambda_n)(\lambda_2-\lambda_n)\cdots(\lambda_{n-1}-\lambda_n)},$$

so we obtain the conclusion:

$$f_{n+1}(y) = \left(\prod_{i=1}^{n+1}\lambda_i\right)\sum_{i=1}^{n}\frac{\exp(-\lambda_i y)-\exp(-\lambda_n y)}{\prod\limits_{1\le j\le n+1;\, j\ne i}(\lambda_j-\lambda_i)}.$$

□

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