AN ASYMPTOTIC ANALYSIS OF A RELIABILITY OF INTERNET TYPE NETWORKS

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Introduction

In this paper a problem of a construction of accuracy and asymptotic formulas for a reliability of internet type networks is solved. Analogously to [1] such network is defined as a tree where each node is connected directly with a circle scheme on a lower level with n>0 nodes. A construction of accuracy and asymptotic formulas for probabilities of an existence of working ways between each pair of nodes of the internet type network is based on a recursive definition of these networks and on asymptotic formulas for a reliability of a random port. This asymptotic formula represents the port reliability as a sum of probabilities of a work for all ways between initial and final nodes of this port. An estimate of a relative error and a complexity of these asymptotic calculations for a radial-circle scheme are shown.

1. An asymptotic formula for a reliability calculation of a port and its accuracy

An asymptotic formula for a reliability of the general type port with low reliable arcs. Consider the no oriented graph Γ with the final nodes set *U*, the arcs set *W*, the fixed initial and final nodes *u*, *v* and the set of the acyclic ways $\{R_1,...,R_n\}$ between *u*, *v*. Suppose that the probability p_w of the arc $w \in W$ work depends on the parameter h > 0: $p_w = p_w(h)$ and $p_w(h) \to 0, \ \ h \to 0.$ Denote $P(U_p)$ - the probability of the event U_p that all arcs $w_1^p,...w_{m_p}^p$ of the way R_p work. Then the reliability of the port Γ is $P_\Gamma = P\left(\bigcup_{p=1}^{n} U_p\right)$, denote $P_\Gamma^* = \sum_{p=1}^{n} P(U_p).$

Remark that for $p \neq q$ the arcs sets $\{w \in R_p\}$, $\{w \in R_q\}$ can not satisfy the inclusion $\{w \in R_p\} \subseteq \{w \in R_q\}$. In an opposite case there is the node u_* in which the ways R_p, R_q diverge by the arcs (u_*, u_p), (u_*, u_q). But as the arc $(u_*, u_p) \in \{w \in R_q\}$ so there is a circle in the way R_q. This statement contradicts with a suggestion that the way R_q. is acyclic. As the inclusion $\{w \in R_p\} \subseteq \{w \in R_q\}$ is not true for $p \neq q$ so the way R_p contains the arc $\overline{w} \notin R_q$ and consequently $P(U_p U_q) = o(P(U_p)), \ \ h \to 0, \ \ p \neq q.$ An induction by *n* gives the inequality

$$P_\Gamma^* - \sum_{1 \le p < q \le n} P(U_p U_q) \le P_\Gamma \le P_\Gamma^*. \tag{1}$$

But

$$\sum_{1 \le p < q \le n} P(U_p U_q) \le n \max_{w \in W} p_w(h) P_\Gamma^*$$

and consequently from the formula (1) we obtain

$$P_\Gamma \sim P_\Gamma^*. \tag{2}$$

Denote by $A = | P_\Gamma^* / P_\Gamma - 1 |$ the relative error of the asymptotic formula (2). It is obvious that

$$A(h) \le n \max_{w \in W} p_w(h) = \Phi(h) \to 0, h \to 0. \tag{3}$$

Assume that $\varphi(h) \to 0,\ h \to 0$ then for the replacement of h by $\varphi(h)$ the upper bound $\Phi(h)$ of the relative error is to be replaced by $\Phi(\varphi(h)) = o(\Phi(h))$.

Radial-circle scheme. Consider the radial-circle scheme represented on the fig. 1. This scheme has the center 0 connected with the nodes $1,\ldots,n$ arranged on the circle.

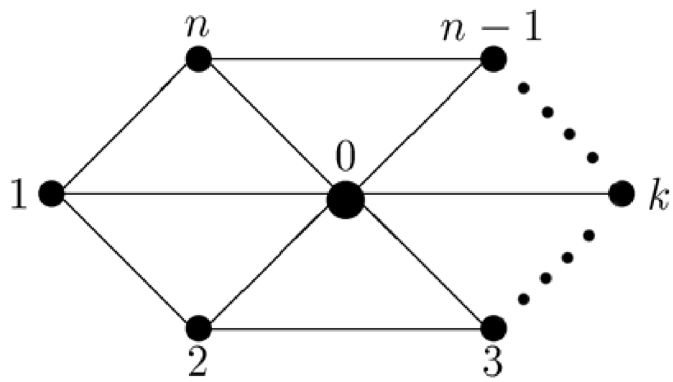

Fig.1 Radial-circle scheme

Each acyclic way from the node i, $1 \le i \le n$, on the circle (the circle node) to the center 0 of this scheme consists of a peace along the circle and a transition to the center 0. A way from the circle node i to the circle node j, $1 \le i \ne j \le n$, has a peace from the node i along the circle, a transition to the center 0, a transition to the circle and a peace along the circle to the node j.

Define the connection matrix $\mathbf{P} = \| P_{ij} \|_{i,j=0}^{n}$ of the radial-circle scheme in which P_{ij} is the probability that there is a working way between the nodes i, j of this scheme. Represent the results of the matrix **P** calculation with n=6 and

$p_{01} = 0.0471595$	$p_{02} = 0.0469944$
$p_{03} = 0.0287418$	$p_{04} = 0.0499121$
$p_{05} = 0.0135117$	$p_{06} = 0.00822811$
$p_{12} = 0.0490761$	$p_{23} = 0.0340865$
$p_{34} = 0.0442866$	$p_{45} = 0.0004677$
$p_{56} = 0.00818179$	$p_{16} = 0.0173955$

Here the matrix **P** is calculated by the Monte-Carlo method with 1000000 realizations during 14 hours. Denote by $\mathbf{P^*} = \| P_{ij}^* \|_{i,j=0}^{n}$ the connection matrix with elements calculated by the asymptotic formula (2). The matrix **P*** have been calculated during one minute that is approximately 1000 times faster. As $P_{ij} = P_{ji}$ and $P_{ii} = 1$ we show only the elements P_{ij}, P^*_{ij} with $1 \le i < j \le n$.

P*=

–	0.0496627	0.050371	0.0326335	0.0512658	0.0136101	0.00920024
–	–	0.0523323	0.00549431	0.00262581	0.000817883	0.0178084
–	–	–	0.0378183	0.00408933	0.000753723	0.00131303
–	–	–	–	0.0458115	0.000532299	0.00112028
–	–	–	–	–	0.00123165	0.00129662
–	–	–	–	–	–	0.00912706
–	–	–	–	–	–	–

P=

–	0,049758	0,049859	0,03268	0,051263	0,013703	0,009279
–	–	0,052157	0,005359	0,002637	0,000844	0,017839
–	–	–	0,037725	0,004073	0,000743	0,001327
–	–	–	–	0,045997	0,000555	0,001108
–	–	–	–	–	0,001253,	0,001301
–	–	–	–	–	–	0,009229
–	–	–	–	–	–	–

The matrix of the relative errors $\mathbf{A} = \| A_{ij} \|_{0 \le i < j \le 6}$ satisfies the equality:

A=

–	0.00191948	0.0101643	0.00142467	0.0000537835	0.00682727	0.00856111
–	–	0.00335063	0.0246274	0.0042632	0.0319321	0.00171756
–	–	–	0.00246654	0.00399357	0.0142263	0.0106371
–	–	–	–	0.00404871	0.0426474	0.0109635
–	–	–	–	–	0.0173307	0.00337921
–	–	–	–	–	–	0.0111695
–	–	–	–	–	–	–

Remark. *Analogously it is possible to obtain asymptotic formulas for a general type network or a radial circle scheme with high reliable arcs. But in this case it is necessary to replace a work probability by a failure probability and a way by a cross section.*

2. Recursively defined networks

A calculation of the connection matrix in recursively defined networks. Suppose that D_* is the set of networks Γ with no intersected sets of arcs. Define recursively the networks class D, $D_* \subset D$ by the condition

$$\Gamma_1 = \{U_1, W_1\} \in D\ ,\ \Gamma_2 = \{U_2, W_2\} \in D_*,\ W_1 \cap W_2 = Ш, \quad (4)$$
$$U_1 \cap U_2 = \{z\}, \text{(z is a single node)} \to \Gamma_1 \cup \Gamma_2 \in D.$$

Analogously to [2] in this paper we calculate $\{P_\Gamma,\ u, v \in U,\ u \ne v\}$, not its single element. These calculations are based on the recursive formulas: if $\Gamma' \in D,\ \Gamma'' \in D_*,\ U' \cap U'' = \{z\}$, then

$$P_{\Gamma' \cup \Gamma''} = \begin{cases} P_{\Gamma'},\ u, v \in U', \\ P_{\Gamma''},\ u, v \in U'', \\ P_{\Gamma'} P_{\Gamma''},\ u \in U', v \in U''. \end{cases} \quad (5)$$

In the last equality the quantity $P_{\Gamma'}$, characterizes the connection between the nodes *u, z* and the quantity $P_{\Gamma''}$ – the connection between the nodes *z, v.* The number of arithmetical operations $n(P_{\Gamma})$ necessary to calculate $\{P_{\Gamma}, u,v \in U, u \neq v\}$, by the recursive formulas (5) is characterized by the following statement.

Theorem. *Suppose that* $\Gamma_1,...,\Gamma_l$ *is the sequence of networks with the no intersected sets of arcs. If* D_* *consists of sequences of independent probability copies of* $\Gamma_1,...,\Gamma_l$, *then for each* $\Gamma \in D$ *the inequalities*

$$\frac{l(\Gamma)(l(\Gamma)-1)}{2} \leq \sum_{u,v\in U, u\neq v} n(P_{\Gamma}) \leq \frac{l(\Gamma)(l(\Gamma)-1)}{2} + \sum_{i=1}^{l} \sum_{u,v\in U_i,\, u\neq v} n(P_{\Gamma_i}) \qquad (6)$$

are true with $l(\Gamma)$ *the number of nodes in the graph* Γ.

From the inequalities (6) obtain that

$$\lim_{l(\Gamma)\to\infty} \frac{2 \sum\limits_{u,v\in U, u\neq v} n(P_{\Gamma})}{l(\Gamma)(l(\Gamma)-1)} = 1.$$

So asymptotically when $l(\Gamma) \to \infty$ to calculate a connection probability for a single pair of nodes it is necessary a single arithmetical operation.

Proof. Suppose that the inequality (6) is true for Γ' then from the recursive formulas (5) and the equality $l(\Gamma' \cup \Gamma'') = l(\Gamma') + l(\Gamma'') - 1$ we have

$$\sum_{u,v\in U'\cup U'', u\neq v} n(P_{\Gamma'\cup\Gamma''}) \leq \sum_{i=1}^{l} \sum_{u,v\in U_i,\, u\neq v} n(P_{\Gamma_i}) + \frac{l(\Gamma_1)(l(\Gamma_1)-1)}{2} + \frac{l(\Gamma_2)(l(\Gamma_2)-1)}{2} +$$
$$+ (l(\Gamma_1)-1)(l(\Gamma_2)-1) = \sum_{i=1}^{l} \sum_{u,v\in U_i,\, u\neq v} n(P_{\Gamma_i}) + \frac{l(\Gamma_1\cup\Gamma_2)(l(\Gamma_1\cup\Gamma_2)-1)}{2}.$$

A calculation of the transition matrices in the internet type networks. Analogously to [1] define the class of the internet type networks as the recursively defined class of networks D with the set of originating schemes D_* which consists of radial-circle schemes and in the formula (4) the node *z* is the center of the radial-circle scheme Γ_2.

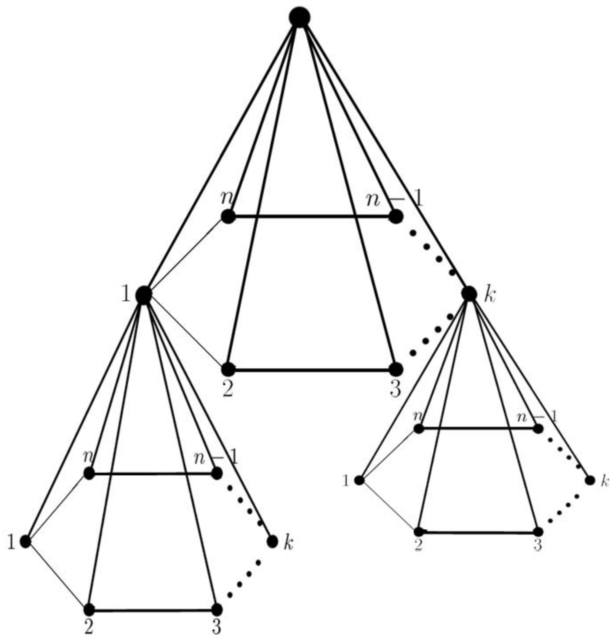

Fig.2. The internet type network

So if we have the transition matrix for the radial-circle schemes it is possible to calculate the transition matrix of the internet type network by the formula (5). This algorithm is significantly faster than general type algorithm from [1]. It contains fast algorithm to calculate the transition matrix in the radial-circle scheme and practically optimal algorithm to calculate the transition matrix for the internet type networks.

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