A STUDY OF ASYMPTOTIC AVAILABILITY MODELING FOR A FAILURE AND A REPAIR RATES FOLLOWING A WEIBULL DISTRIBUTION

Salem Bahri [a], Fethi Ghribi [b], Habib Ben Bacha [a,c]

•

[a] *Electro Mechanical systems laboratory (LASEM), Department of Mechanical Engineering-ENIS*
e-mail: Salem.BenBahri@enis.rnu.tn
[b] *Department of Mathematical and Computer Science National Engineering School of Sfax (ENIS), University of Sfax BP W, Sfax, 3038, Tunisia*
e-mail: fethi.ghribi@enis.rnu.tn
[c] *King Saud University- College of Engineering in Alkharj-P.O Box 655, Elkharj11942, Kingdom of Saudi Arabia*
e-mail: hbacha@ksu.edu.sa

Abstract:

The overall objective of the maintenance process is to increase the profitability of the operation and optimize the availability. However, the availability of a system is described according to lifetime and downtime. It is often assumed that these durations follow the exponential distribution. The work presented in this paper deals with the problem of availability modeling when the failure and repair rates are variable. The lifetime and downtime were both governed by models of Weibull (the exponential model is a particular case). The differential equation of the availability was formulated and solved to determine the availability function. An analytical model of the asymptotic availability was established as a theorem and proved. As results deduced from this study, a new approach of modeling of the asymptotic availability was presented. The developed model allowed an easy evaluation of the asymptotic availability. The existence of three states of availability for a system has been confirmed by this evaluation. Finally, these states can be estimated by comparing the shape parameters of the Weibull model for the failure and repair rates.

Keywords: Availability function, asymptotic availability, failure rate, repair rate, Weibull distribution

1. Introduction

The last two decades witnessed major progress in the development of new maintenance strategies [1]. The primary objectives of these strategies are to reduce equipment downtime, also increase reliability and availability of the equipment which at the same time optimizes the life-cycle costs [2]. The need for high reliability and availability is not just restricted to safety-critical systems [3]. In general, current technology has ensured that the equipments for industrial application, for example, telephone switches, airline reservation systems, process and production control, stock trading system, computerized banking etc. all require very high availability [2]. Reliability is generally described in terms of the failure rate or mean time between failures (MTBF), while availability is normally associated with total downtime [2]. There is some research on increasing system availability [4]. Goel and Soenjoto proposed a generalized model [4]. Markov models are also implemented to analyze the system availability, which combines both software and hardware failures and maintenance processes [4]. Khan and Haddara [1] proposed a methodology for risk-based maintenance to increase availability of a heating, ventilation and air-conditioning (HVAC) system. Garg S. et al. [3] developed a model for a transactions based software system which employs preventive maintenance to maximize availability, minimize probability of loss, minimize response time or optimize a combined measure. The steady state availability can be modelled using standard formulae from Markov regenerative process (MRGP) theory. The Service rate and failure rate are assumed to be functions of real time (Weibull distribution) [3]. The failure and repair rates are supposed constant (λ and μ respectively), so that system availability can be modeled using a Markov chain in Refs. [5,7]. But, Khan and Haddara [1] considered that the Weibull model is more robust than the other models. Dai et al [4] studied the availability of the centralized heterogeneous distributed system (CHDS) and developed a general model for the analysis. The repair time was exponentially distributed. For the failure intensity function (failure rate), the G.O model presented by Goel and Okumoto was used [4]. Some other research considered that the availability depends on both reliability and maintainability and is defined as the ratio of requested service time to practical service time [6, 7]

Nomenclature

A(t)	Availability function
A_∞	Asymptotic availability
λ(t)	Failure rate
μ(t)	Repair rate
β	Shape parameter of Weibull distribution for Failure rate
η	Scale parameter of Weibull distribution for Failure rate
α	Shape parameter of Weibull distribution for repair rate
θ	Scale parameter of Weibull distribution for repair rate

Review of the literature indicates that there is a new trend to use availability and reliability modeling as a criterion to plan maintenance tasks. However, most of the previous studies assumed the failure and/or repair rates are constant. It seems that there is a need for a more generalized methodology that can be applied for variable rates. The present study adopts a new fundamental approach for the asymptotic availability modeling where the failure and repair rates were governed by the Weibull distribution.

This paper is organized as follows. In Section 2, the differential equation of the availability is established. Section 3 is dedicated to the resolution of the differential equation to determine the instantaneous availability. The model of the asymptotic availability is developed in Section 4. Finally, in Section 5, the conclusions along with future research directions are presented.

2. The mathematical formulation of the availability differential equation

According to the standard "Association Française de Normalisation - AFNOR X 06-503" [8, 9] , in order to have a system available at time *t+dt*, there are two possibilities:

- the first is that the system is available at time *t* and does not have breakdown between *t* and *t+dt*
- the second is the system is unavailable at time *t* but it is repaired between *t* and *t+d*.

These expressions are transformed by the following probabilities:

- A(t+dt): The probability that the system is available at time (t+dt),
- *A*(t): the probability that the system is available at time t,
- 1-□(t)dt : The probability that the system does not have breakdown between *t* and *t+dt*, knowing that it had already functioned until the time t,
- 1-*A*(t): The probability that the system is unavailable at time t

- □(t)dt: The probability that the system is repaired between t and (t+dt), knowing that it was already failing until the time t.

With:

- □(t) : Instantaneous failure rate
- □(t) : Instantaneous repair rate

Fig. 1 shows the state diagram of the system.

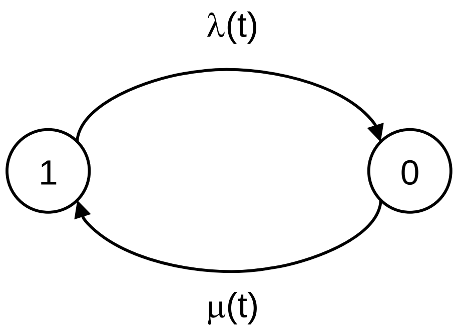

Fig. 1. State transition diagram

A(t+dt)= probabilities (that the system is up at t **and** is no break down between t and (t+dt)**)+ probabilities (**the system to be down at time t and it is repaired between t and (t+dt)**)** [8, 9].

$$A(t+dt) = A(t)[1-\lambda(t)dt] + [1-A(t)]\mu(t)dt \quad (1)$$
$$A(t+dt) - A(t) = [-\lambda(t)A(t) + \mu(t) - A(t)]dt \quad (2)$$
$$\frac{A(t+dt)-A(t)}{dt} = \mu(t) - [\lambda(t)+\mu(t)]A(t) \quad (3)$$

Then:

$$\frac{dA(t)}{dt} = \mu(t) - [\lambda(t)+\mu(t)]A(t) \quad (4)$$

This expression represents the differential equation of first order of the availability. [4,8, 9].

3. The availability function

For t > 0, the failure and repair rates, which are modeled using a Weibull distribution, are given by :

- $\lambda(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$ (5)
- $\mu(t) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}$ (6)
- Eq. (4) can be solved by "Mathematica software", by taking account of the initial conditions A(0)=0 if the system is in the failure state and A(0)=1 if the system is in the functioning state and can be obtained the following solutions:

- If A(0) = 0 then,

$$A(t) = A_0(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\left(\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^{\beta}+\left(\frac{u}{\theta}\right)^{\alpha}} u^{\alpha-1}du\right) \quad (7)$$

- If A(0) = 1 then

$$A(t) = A_1(t) = e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha}\left(1 + \alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^\beta + \left(\frac{u}{\theta}\right)^\alpha} u^{\alpha-1} du\right) \quad (8)$$

It can be deduced that:

$$A_1(t) = e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} + A_0(t) \quad (9)$$

There are four parameters in the availability functions (7), (8), β, η, α, and θ. The sensitivity of different parameters is described in Figures 2, 3, and 4.

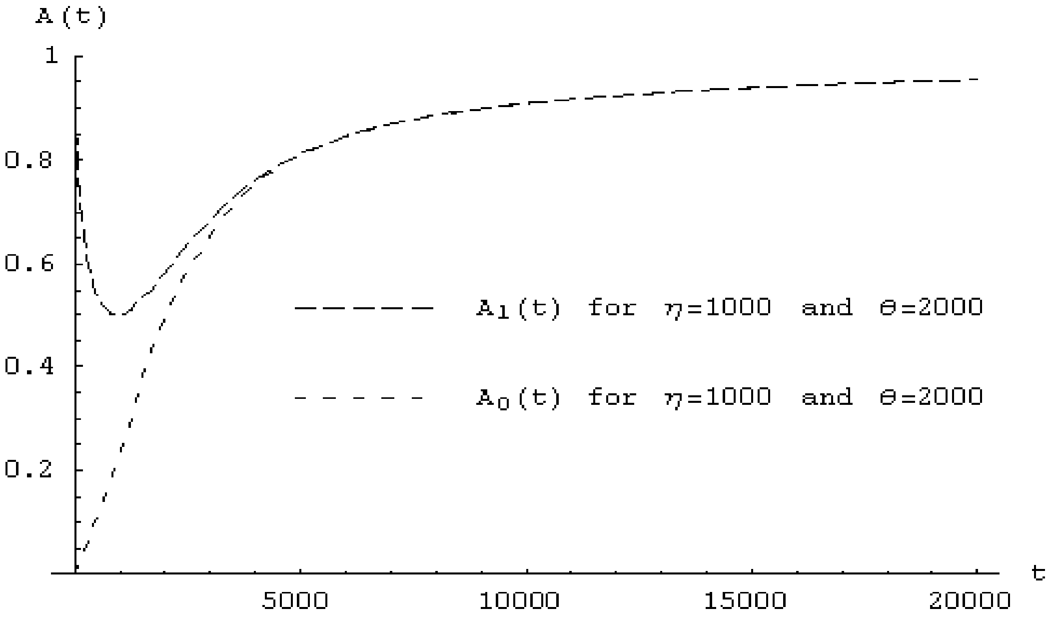

Fig. 2. The availability A(t) for β =0.5, α=1.5 (β < α)

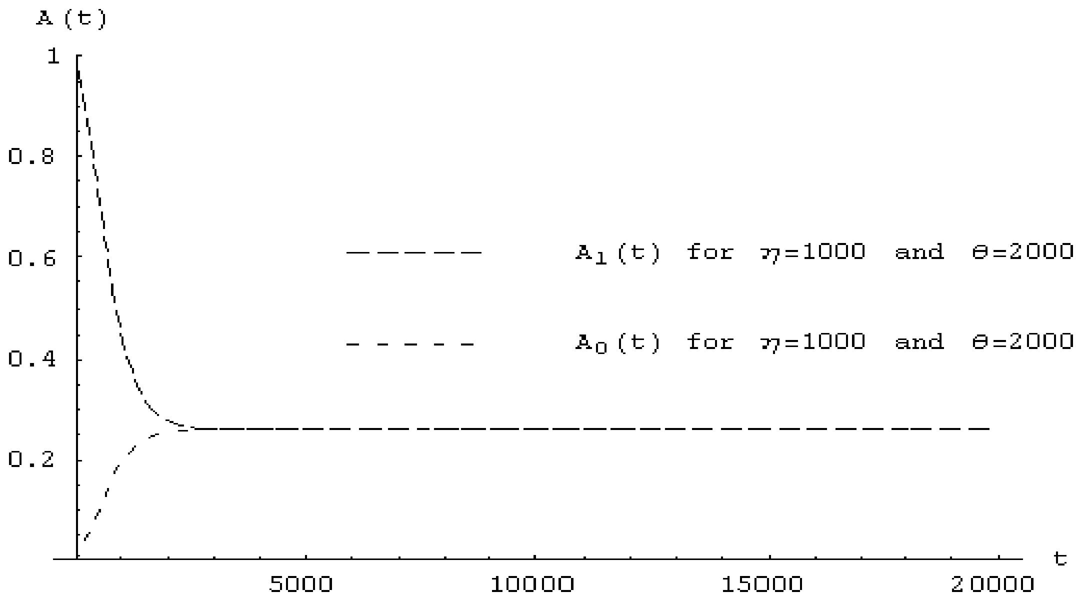

Fig. 3. The availability A(t) for β = α=1.5

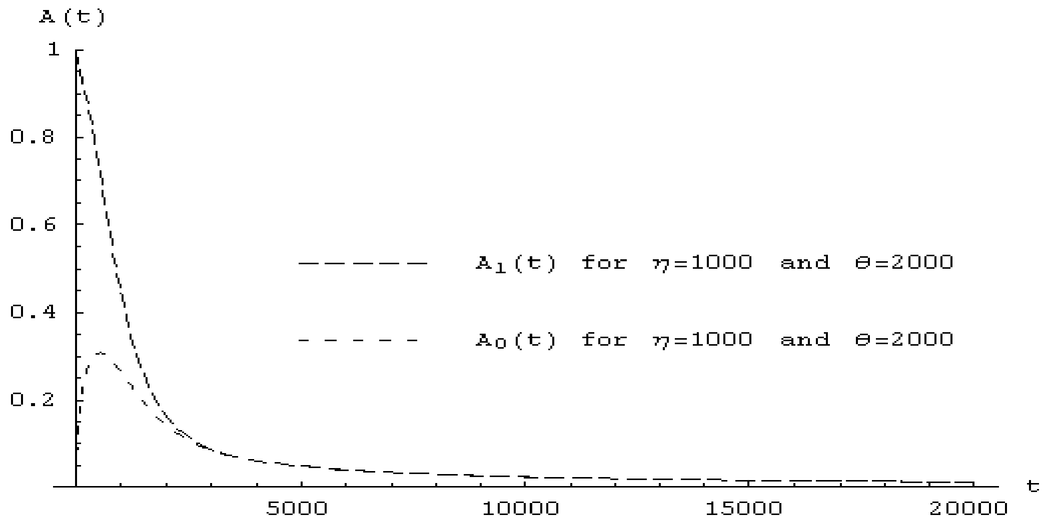

Fig 4. The availability A(t) for β =1.5, α= 0.5 (β > α)

4. The Asymptotic availability

4.1. *Theorem*

$$\lim_{t\to+\infty} A(t) = \lim_{t\to+\infty} \frac{\mu(t)}{\mu(t)+\lambda(t)} \tag{10}$$

Demonstration

It can be assumed that:

$$r(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A(t) \tag{11}$$

Where:

$$r(t) = r_0(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A_0(t), \text{ if } A(t)=A_0(t) \tag{12}$$

And

$$r(t) = r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A_1(t), \text{ if } A(t)=A_1(t) \tag{13}$$

It may be necessary to prove that:

$$\lim_{t\to+\infty} \frac{\mu(t)+\lambda(t)}{\mu(t)} A(t) = 1 \tag{14}$$

There are four intermediate results can be used to explain this.

- **1st result:**

$$\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} r_0(t) = \lim_{t\to+\infty} r_1(t) \tag{15}$$

Proof :

From Eq. (12), this can be obtained by substituting $A_1(t)$ by Eq.(9)

$$r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} \left(e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} + A_0(t) \right) = \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} + \frac{\mu(t)+\lambda(t)}{\mu(t)} A_0(t) \tag{16}$$

An analogy with Eq. (9) can be deduced:

$$r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} + r_0(t) \tag{17}$$

Therefore, from Eq. (17), it is necessary, to verify the 1st result, to prove.

$$\lim_{t\to+\infty} \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} = 0 \tag{18}$$

$$\frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} = \left(1+\frac{\lambda(t)}{\mu(t)}\right) e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} \tag{19}$$

Referring to Eqs. (5) and (6) :

$$\frac{\lambda(t)}{\mu(t)} = \frac{\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}}{\frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}} = \frac{\frac{\beta\eta}{\eta t}\left(\frac{t}{\eta}\right)^{\beta}}{\frac{\alpha\theta}{\theta t}\left(\frac{t}{\theta}\right)^{\alpha}} = \frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)} \quad (20)$$

Then, Eq. (19) will become:

$$\left(1+\frac{\lambda(t)}{\mu(t)}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} + \frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}(t)^{\beta-\alpha}e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} \quad (21)$$

The limit study of Eq. 18 gives:

Figure 1: $\lim_{t\to+\infty} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} = 0$

(22)

And

$$\lim_{t\to+\infty}\frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}(t)^{\beta-\alpha}e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} = 0 \quad (23)$$

So, $\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} r_0(t) = \lim_{t\to+\infty} r_1(t)$ and the 1st result is verified.

- **2nd result :**

$$r_0(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv \quad (24)$$

Proof:

From Eqs. 16 and 17, the $r_0(t)$ function is written as:

$$r_0(t) = \left(1+\frac{\lambda(t)}{\mu(t)}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} \quad (25)$$

According to Eqs. (7) and (20), The Eq. (25) will become as follow:

$$r_0(t) = \left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^{\beta}+\left(\frac{u}{\theta}\right)^{\alpha}}u^{\alpha-1}du \quad (26)$$

A change of variables is applied in Eq. (26): $u = t\cdot v \Rightarrow du = tdv$

$$r_0(t) = \left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{tv}{\eta}\right)^{\beta}+\left(\frac{tv}{\theta}\right)^{\alpha}}(tv)^{\alpha-1}tdv \quad (27)$$

$$r_0(t) = \left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\left(\frac{t}{\theta}\right)^{\alpha}\int_0^t e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv \quad (28)$$

Therefore, $r_0(t) = \left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\int_0^t e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv$ and the 2nd Result is verified.

- **3rd result :**

According to the shape parameters β and α, the $r_0(t)$ function should satisfy the two following inequalities:

a) If **β≤ α**, then

$$\frac{\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \le r_0(t) \le \left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \quad (29)$$

Or

b) If β≥ α then

$$\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \le r_0(t) \le \frac{\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \quad (30)$$

Proo*f:*

a) For **β≤ α**

$$\forall v\in\,]0,1],\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha-1} \le \beta\left(\frac{t}{\eta}\right)^{\beta}v^{\beta-1}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha-1} \quad (31)$$

And $\left(\left(\frac{t}{\eta}\right)^{\beta} + \left(\frac{t}{\theta}\right)^{\alpha}\right) v^{\alpha} \le \left(\frac{t}{\eta}\right)^{\beta} v^{\beta} + \left(\frac{t}{\theta}\right)^{\alpha} v^{\alpha}$ (32)

By referring to the second result (24) and the two above mentioned inequalities (31) and (32), so, the $r_0(t)$ function can be put under the form of the following inequality:

$$\left(\beta\left(\frac{t}{\eta}\right)^{\beta} + \alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}} v^{\alpha-1}dv \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le$$
$$\int_0^1 \left(\beta\left(\frac{t}{\eta}\right)^{\beta} v^{\beta-1} + \alpha\left(\frac{t}{\theta}\right)^{\alpha} v^{\alpha-1}\right) e^{\left(\left(\frac{t}{\eta}\right)^{\beta}(v)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)} dv \quad (33)$$

The calculations of exponential integral allow to express the inequality (33) as follow:

$$\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}\right]_0^1 \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le \left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}\right]_0^1 \quad (34)$$

$$\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} - 1\right] \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le \left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} - 1\right] \quad (35)$$

So, $\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1 - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \le r_0(t) \le \left[1 - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]_0^1$, the first inequality (29) is satisfied.

b) For **$\beta \ge \alpha$**

A similar development and demonstration is used for this case also

$$\forall v \in]0,1],\ \beta\left(\frac{t}{\eta}\right)^{\beta} v^{\beta-1} + \alpha\left(\frac{t}{\theta}\right)^{\alpha} v^{\alpha-1} \le \left(\beta\left(\frac{t}{\eta}\right)^{\beta} + \alpha\left(\frac{t}{\theta}\right)^{\alpha}\right) v^{\alpha-1} \quad (36)$$

And

$$\left(\frac{t}{\eta}\right)^{\beta} v^{\beta} + \left(\frac{t}{\theta}\right)^{\alpha} v^{\alpha} \le \frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta} + \left(\frac{t}{\theta}\right)^{\alpha}\right) v^{\alpha} \quad (37)$$

According to the second result (24) and the two above mentioned inequalities (36) and (37), so, the $r_0(t)$ function can be put under the form of the following inequality:

$$\int_0^1 \left(\beta\left(\frac{t}{\eta}\right)^{\beta} v^{\beta-1} + \alpha\left(\frac{t}{\theta}\right)^{\alpha} v^{\alpha-1}\right) e^{\left(\left(\frac{t}{\eta}\right)^{\beta}(v)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)} dv \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le$$
$$\left(\beta\left(\frac{t}{\eta}\right)^{\beta} + \alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}} v^{\alpha-1}dv \quad (38)$$

The calculations of exponential integral allow to express the inequality (38) as follow:

$$\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}\right]_0^1 \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}\right]_0^1 \quad (39)$$

$$\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} - 1\right] \le e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} r_0(t) \le \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} - 1\right] \quad (40)$$

$$\left[1 - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \le r_0(t) \le \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\frac{\beta}{\alpha}-1\right)\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)} - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \quad (41)$$

So, $\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \leq r_o(t) \leq \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]$, the second inequality (30) is satisfied.

- **4th result :**

$$\lim_{t\to+\infty} r_o(t) = 1 \tag{42}$$

Proof:

a) β≤ α

By referring to the third result "inequality (29)", to prove the fourth result, it can be sufficient to show that the limits:

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=1 \tag{43}$$

And

$$\lim_{t\to+\infty}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=1 \tag{44}$$

Then

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(\left(\frac{t}{\eta}\right)^{\beta}\ \left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=\lim_{t\to+\infty}\frac{\alpha\left(\frac{t}{\theta}\right)^{\alpha}\left(\frac{\beta\theta^{\alpha}}{\alpha\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}{\alpha\left(\frac{t}{\theta}\right)^{\alpha}\left(\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}\left[1-e^{\left(\left(\frac{t}{\eta}\right)^{\beta}\ \left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \tag{45}$$

$$\lim_{t\to+\infty}\frac{\left(\frac{\beta\theta^{\alpha}}{\alpha\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}{\left(\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]= 1[1-0]=1 \tag{46}$$ And

$$\lim_{t\to+\infty}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=[1-0]=1 \tag{47}$$

So, $\lim_{t\to+\infty} r_o(t) = 1$ if β≤ α

b) β ≥ α

In the same way as explained in the previous case, according to inequality (30), to prove the fourth result, it can be sufficient to show that the limits:

$$\lim_{t\to+\infty}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=1 \tag{48}$$

And

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=1 \tag{49}$$

Then

$$\lim_{t\to+\infty}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=[1-0]=1 \tag{50}$$

And

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=\lim_{t\to+\infty}\frac{\beta\left(\frac{t}{\eta}\right)^{\beta}\left(1+\frac{\alpha\eta^{\beta}}{\beta\theta^{\alpha}}t^{(\alpha-\beta)}\right)}{\beta\left(\frac{t}{\eta}\right)^{\beta}\left(1+\frac{\eta^{\beta}}{\theta^{\alpha}}t^{(\alpha-\beta)}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \tag{51}$$

$$\lim_{t\to+\infty}\frac{\left(1+\frac{\alpha\eta^{\beta}}{\beta\theta^{\alpha}}t^{(\alpha-\beta)}\right)}{\left(1+\frac{\eta^{\beta}}{\theta^{\alpha}}t^{(\alpha-\beta)}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]=1[1-0]=1 \tag{52}$$

So, $\lim_{t\to+\infty} r_o(t) = 1$ and the 4th result (42) is proven also for α≤ β

Finally, the theorem (10) ensues therefore of results 1 and 4

The availability A(t) is plotted together with The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function in figure 5 for β<α, figure 6 for β=α and figure 7 for β>α.

The three figures show that the availability A(t) " with its two solution $A_0(t)$ and $A_1(t)$ " and the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function have tendency to converge towards the same limit when the time t is more important.

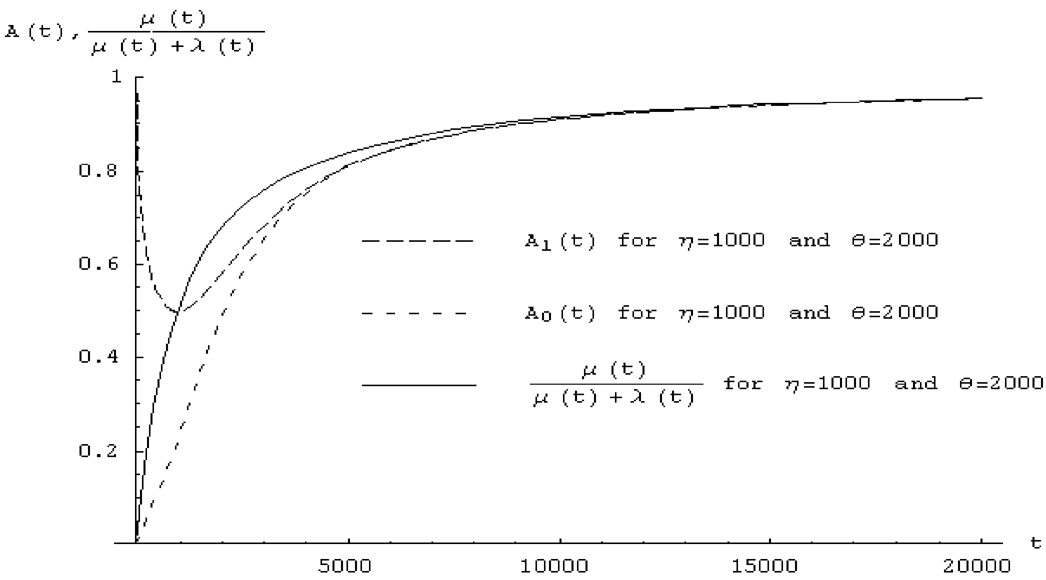

Fig.5. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for β =0.5, α=1.5 (β < α)

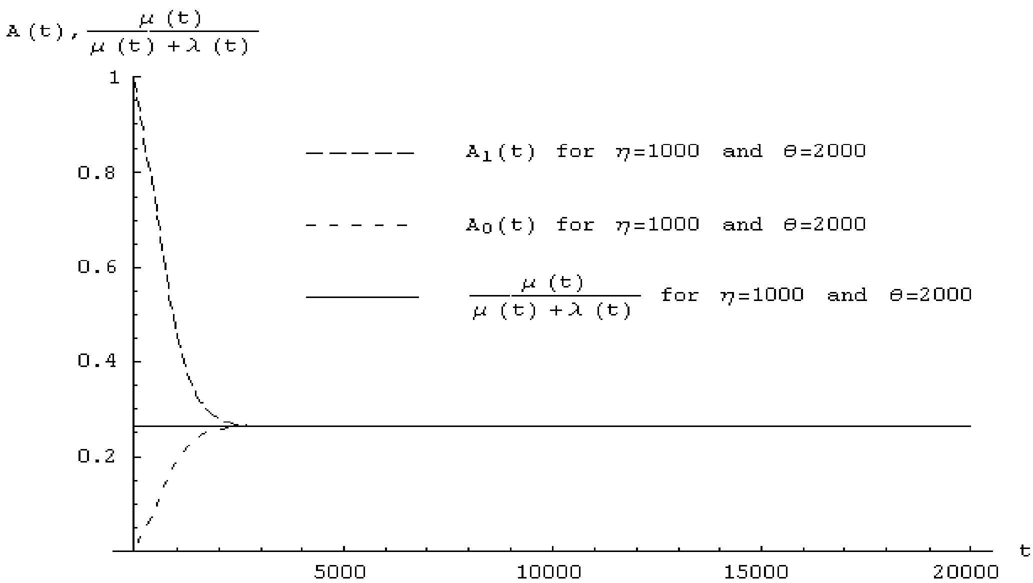

Fig.6. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for A(t) for β = α=1.5

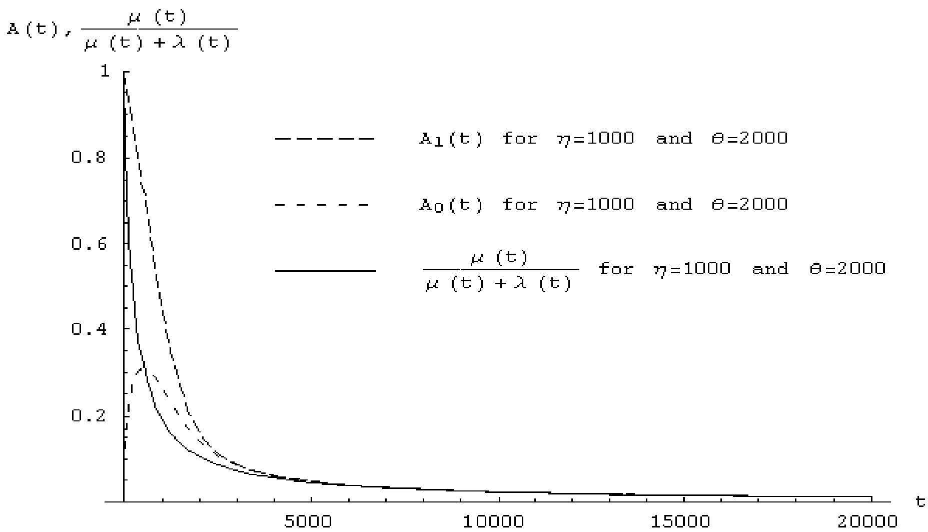

Fig.7. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for A(t) for β =1.5, α=0.5 (β > α)

4.2. *Asymptotic availability evaluation*

According to (10), the asymptotic availability is defined by:

$$A_\infty = \lim_{t\to+\infty} A(t) = \lim_{t\to+\infty} \frac{\mu(t)}{\mu(t)+\lambda(t)} \quad (53)$$

$$A_\infty = \lim_{t\to+\infty} \frac{1}{1+\frac{\lambda(t)}{\mu(t)}} \quad (54)$$

The study of the limit of the function will be done according to three following cases:

1st case: β<α

$$\lim_{t\to+\infty} \frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty} \frac{\beta}{\alpha}\frac{\theta^\alpha}{\eta^\beta} t^{\beta-\alpha} = 0 \quad (55)$$

Then,

$$A_\infty = \lim_{t\to+\infty} \frac{1}{1+\frac{\lambda(t)}{\mu(t)}} = \frac{1}{1+0} = 1 \quad (56)$$

The converge of the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function, when the time t is more important, to the A_∞= **1** with the sensitivity of the scale parameters (η<θ, η=θ or η>θ) is shown in Fig.8.

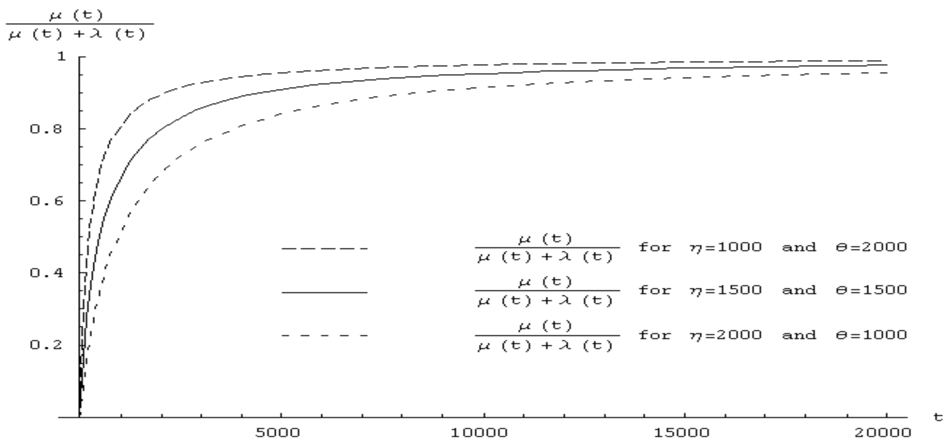

Fig.8. The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function limit studies for β=0.5, α=1.5 (β < α).

2nd case: β=α

$$\lim_{t\to+\infty}\frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty}\frac{\beta\,\theta^{\alpha}}{\alpha\,\eta^{\beta}}t^{\beta-\alpha} = \left(\frac{\theta}{\eta}\right)^{\beta} \qquad (57)$$

In this case, the asymptotic availability is defined to be equal to

$$A_{\infty} = \frac{1}{1+\left(\frac{\theta}{\eta}\right)^{\beta}} = \frac{1}{1+\left(\frac{\eta}{\theta}\right)^{-\beta}} = \frac{\theta^{-\beta}}{\theta^{-\beta}+\eta^{-\beta}} \qquad (58)$$

Particular cases:

β=α=1: the exponential models

- $A_{\infty} = \frac{\mu}{\lambda+\mu}$ (59)

  With $\mu = \frac{1}{\theta}$ and $\lambda = \frac{1}{\eta}$

  if η=θ, then

- $A_{\infty} = \frac{1}{2}$ (60)

Fig. 9 shows the asymptotic availability plotted with □=□ with the sensitivity of the scale parameters (η<θ, η=θ or η>θ).

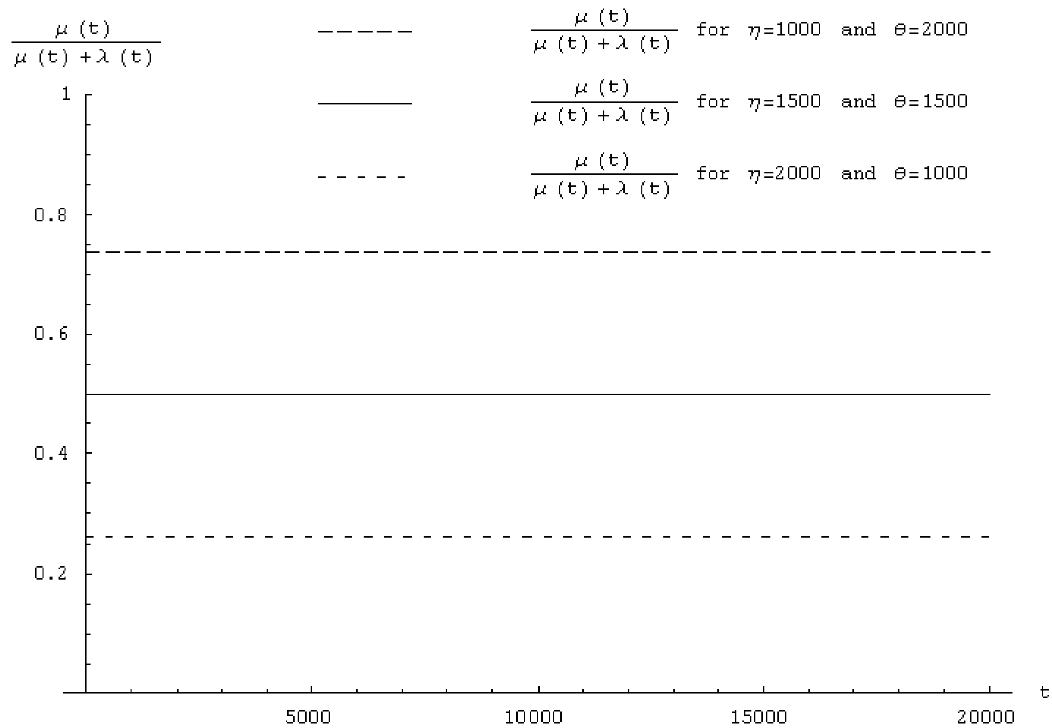

Fig.9. The function $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ if β= α=1

3rd case: β>α

$$\lim_{t\to+\infty}\frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty}\frac{\beta\,\theta^{\alpha}}{\alpha\,\eta^{\beta}}t^{\beta-\alpha} = +\infty \qquad (61)$$

$$\lim_{t\to+\infty}\frac{1}{1+\frac{\lambda(t)}{\mu(t)}} = \frac{1}{1+\infty} = 0 \qquad (62)$$

The converge of the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function, when the time t is more important, to the A_{∞}= **0** with the sensitivity of the scale parameters (η<θ, η=θ or η>θ) is shown in Fig.10.

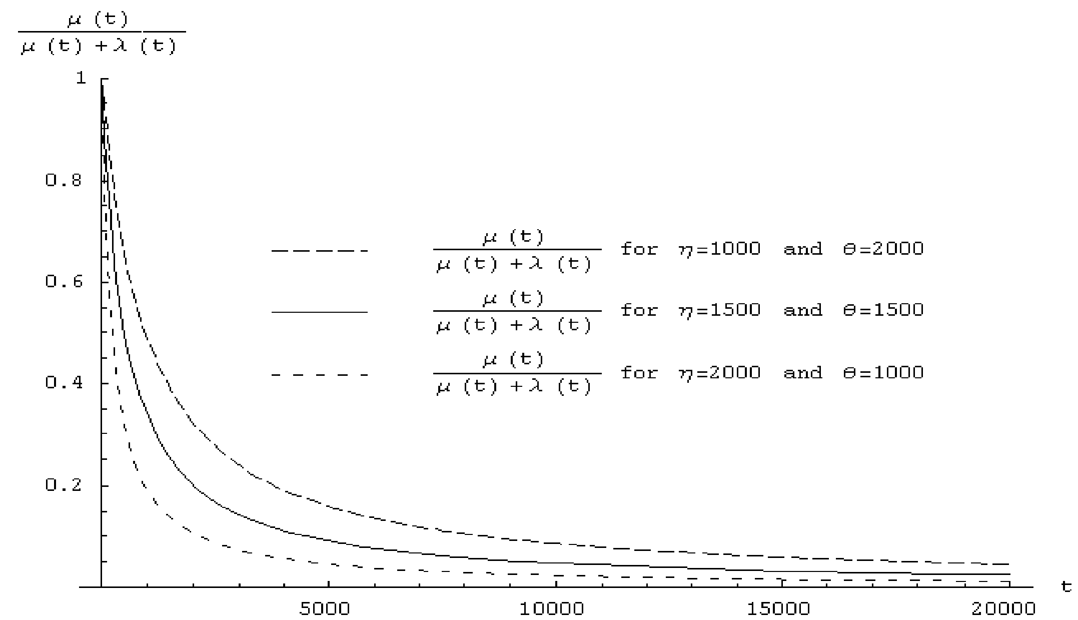

Fig.10. The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function limit studies for β=1.5, α=0.5 (β>α).

5. Conclusion

In this paper, the presented work extended the classic availability model to a new asymptotic availability model when the failure and repair rates are distributed according to the Weibull model. The analysis of asymptotic behavior of the system according to the developed model allowed to extract the following result:

The asymptotic availability depends only on the shape parameters of the Weibull models β and α. The scale parameters η and θ do not have an influence in the limit of the availability.

- If $b < a$ then, the system is fully available
- If $b > a$ the system resides in the down state, then, it is unavailable
- If $b = a$, in this case, the asymptotic behavior of the system is analogous to a system governed by the exponential model.

Thus, the future plan includes the research on a novel approach, which will be the combination of two different models (Weibull, Gamma,) or (Weibull, lognormal).

REFERENCES

1. Khan F.I., Haddara M. M. Risk-based maintenance (RBM): a quantitative approach for maintenance/inspection scheduling and planning. Journal of Loss Prevention in the Process Industries 2003;16: 561–573
2. Ogaji S.O.T., Singh R. Advanced engine diagnostics using artificial neural networks. Applied Soft Computing 2003;3: 259–271
3. Garg S., Puliafito A., Telek M., Trivedi K. S. Analysis of Preventive Maintenance in Transactions Based Software Systems. IEEE Trans. Comput. 1998; 47/1: 96–107 (special issue on dependability of computing systems).

4. Dai Y.S., Xie M., Poh K.L., Liu G.Q. A study of service reliability and availability for distributed systems. Reliab Engng Syst Safety 2003; 79: 103–112.
5. Volovoi V. Modeling of system reliability Petri nets with aging tokens. Reliab Engng Syst Safety 2004;8:4149–161.
6. Tsai Y.T., Wang K.S., Tsai L. C. A study of availability-centred preventive maintenance for multi-component systems. Reliab Engng Syst Safety 2004; 84: 261–270
7. Ji1 M., Yu1 S.H. Availability Modeling for Reliable Routing Software. Proceedings of the 2005 Ninth IEEE International Symposium on Distributed Simulation and Real-Time Applications (DS-RT'05) IEEE Computer Society; 2005.
8. AFNOR, Recueil des normes française: maintenance industrielle, AFNOR Paris 1988. p 436-573
9. Monchy F, Maintenance méthodes et organisations, Paris, édition Dunod, 2000 p 137-233.