AN ACCURACY OF ASYMPTOTIC FORMULAS IN CALCULATIONS OF A RANDOM NETWORK RELIABILATY

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INTRODUCTION

In this paper a problem of asymptotic and numerical estimates of relative errors for different asymptotic formulas in the reliability theory are considered. These asymptotic formulas for random networks are similar to calculations of Feynman integrals.

A special interest has analytic and numerical comparison of asymptotic formulas for the most spread Weibull and Gompertz distributions in life time models. In the last case it is shown that an accuracy of asymptotic formulas is much higher.

1. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY LOGARITHM

Consider the nonoriented graph Γ with fixed initial and final nodes and with the arcs set W. Define $\mathcal{R}=\{R_1,...,R_n\}$ as the set of all acyclic ways between the initial and final nodes of the graph Γ. Designate P_R the probability of the way R work. Then in the condition

$$p_w \sim \exp\left(-c_w h^{-d_w}\right),\ h\to 0,\ w\in W,$$

we have:

$$P_R \sim \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right),$$

where $C(R)=\sum\limits_{w:d_w=D(R)} c_w$ and $D'_R < D_R$ is a next by a quantity after $D_R=\max\limits_{w\in R} d_w$ element in the set $\{d_w, w\in R\}$, $C'(R)=\sum\limits_{w:d_w=D'(R)} c_w$. If in the way R this element is absent we put then $D'_R=-\infty$, $C'(R)=0$.

Denote $D_\Gamma=\min\limits_{R\in\mathcal{R}} D_R$ and designate $\mathcal{R}_1=\{R: D_R=D_\Gamma\}$, $\mathcal{R}_2=\mathcal{R}\setminus\mathcal{R}_1$, then the probability P_Γ of the graph Γ work satisfies the formulas

$$P_\Gamma \sim P_\Gamma^1+P_\Gamma^2,\ P_\Gamma^i \sim \sum_{R\in\mathcal{R}_i} P_R = \sum_{R\in\mathcal{R}_i} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right),\ i=1,2.$$

By the definition

$$P_\Gamma^1 \sim \sum_{R\in\mathcal{R}_1} \exp\left(-C(R)h^{-D_\Gamma} - C'(R)h^{-D'_R}(1+o(1))\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\sum_{R\in\mathcal{R}_1} \exp\left(-C'(R)h^{-D'_R}(1+o(1))\right),$$

where $C_\Gamma=\min\limits_{R\in\mathcal{R}_1} C_R$ and $D'_R<D_\Gamma$, $R\in\mathcal{R}_1$, so

$$P_\Gamma^1 \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right),$$

where $D'_\Gamma = \min\limits_{R\in\mathcal{R}_1: C(R)=C_\Gamma} D'_R < D_\Gamma$, $C'_\Gamma = \min\limits_{R\in\mathcal{R}_1: C(R)=C_\Gamma, D'_R=D'_\Gamma} C'(R)$.

And consequently $D_R < D_\Gamma$, $R \in \mathcal{R}_2$, $P_\Gamma^2 = o\left(P_\Gamma^1\right)$:

$$P_\Gamma^2 \sim \sum_{R \in \mathcal{R}_2} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right) \sim \sum_{R \in \mathcal{R}_2} \exp\left(-C(R)h^{-D_R}(1+o(1))\right) =$$
$$= \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \sum_{R \in \mathcal{R}_2} \exp\left(-C_\Gamma h^{-D_\Gamma} - C(R)h^{-D_R}(1+o(1))\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \sum_{R \in \mathcal{R}_2} \exp\left(-C(R)h^{-D_R}(1+o(1))\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \exp\left(-C''_\Gamma h^{-D''_\Gamma}(1+o(1))\right),$$

where

$$D''_\Gamma = \min_{R \in \mathcal{R}_2} D_R > D_\Gamma, \quad C''_\Gamma = \min_{R \in \mathcal{R}_2} C(R).$$

So we have:

$$P_\Gamma \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\left(\exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right) + \exp\left(-C''_\Gamma h^{-D''_\Gamma}(1+o(1))\right)\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right).$$

As a result obtain that

$$\ln P_\Gamma \sim -C_\Gamma h^{-D_\Gamma}\left(1 + Ah^{\Delta_\Gamma}(1+o(1))\right), \ \Delta_\Gamma = D_\Gamma - D'_\Gamma > 0, \ A = C'_\Gamma / C_\Gamma.$$

And consequently

$$\frac{\ln P_\Gamma}{-C(\Gamma)h^{-D_\Gamma}} - 1 \sim Ah^{\Delta_\Gamma}. \tag{1}$$

2. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY

Assume that $P\left(U_p\right)$ is the probability of the event U_p that all arcs $w_1^p, ..., w_{m_p}^p$ of the way R_p work. Then we have

$$P_\Gamma = P\left(\bigcup_{p=1}^{n} U_p\right). \tag{2}$$

Suppose that the probability of the arc $w \in W$ work equals $\exp\left(-c_w h^{-d_w}\right)$, $h > 0$, where c_w, d_w are some positive numbers and for arcs $w' \neq w''$ the constants $d_{w'} \neq d_{w''}$. So we have

$$P\left(U_p\right) = \exp\left(-\sum_{j=1}^{m_p} c_{w_j^p} h^{-d_{w_j^p}}\right).$$

Assume that the enumeration of the arcs in the way R_p satisfies the inequalities

$$d_{w_1^p} > d_{w_2^p} > ... > d_{w_{m_p}^p}.$$

Denote $D^p = \left(d_{w_1^p}, \ldots, d_{w_{m_p}^p} \right)$ and introduce on the vectors set $\left\{ D^p,\ 1 \le p \le n \right\}$ the following order relation. Say that $D^p \succ D^q$, if for some $k \le \min\left(m_p, m_q \right)$ the first $k-1$ components of these vectors coincide and the k component in the vector D^p is larger than in the vector D^q. If there is not such k and in the vectors D^p, D^q all first $\min\left(m_p, m_q \right)$ components coincide then $D^p \succ D^q$ for $m_p < m_q$.

Remark that for some $p \ne q$ the arcs sets $\left\{ w \in R_p \right\}, \left\{ w \in R_q \right\}$ can not satisfy the inclusion $\left\{ w \in R_p \right\} \subseteq \left\{ w \in R_q \right\}$. In the opposite case there is the node u_* in which the ways R_p R_q diverge by the arcs $\left(u_*, u_p \right)$ $\left(u_*, u_q \right)$. But as the arc $\left(u_*, u_p \right) \in \left\{ w \in R_q \right\}$ then the way R_q has a cycle. This conclusion contradicts with the assumption that the way R_q is acyclic.

So as the quantities d_w are different then $D^p \ne D^q$, $p \ne q$. As a result we obtain the order relation on the vectors set $\left\{ D^1, \ldots, D^n \right\}$, and if $D^p \succ D^q$, $h \to 0$, so $P\left(U_q \right) = o\left(P\left(U_p \right) \right)$. It is not difficult to check that this relation is transitive. Consequently the order relation on the set $\left\{ D^1, \ldots, D^n \right\}$ is linear. Assume that the enumeration of the vectors D^p satisfies the formula $D^1 \succ \ldots \succ D^n$. From the formula (2) we have

$$P_\Gamma^* - \sum_{1 \le p < q \le m} P\left(U_p U_q \right) \le P_\Gamma \le P_\Gamma^*, \quad P_\Gamma^* = \sum_{p=1}^{m} P\left(U_p \right). \tag{3}$$

As the inclusion $\left\{ w \in R_p \right\} \subseteq \left\{ w \in R_q \right\}$ is not true for $p \ne q$ so in the way R_p there is an arc which does not belong to the way R_q. Consequently we have

$$P\left(U_q \right) = o\left(P\left(U_p \right) \right),\ 1 \le p < q \le m, \quad \sum_{1 \le i < j \le m} P\left(U_p U_q \right) = o\left(P\left(U_2 \right) \right) \tag{4}$$

The formulas (3), (4) give us the following asymptotic expansion for P_Γ with the first and the second members of the smallness:

$$P_\Gamma \sim P_\Gamma^* \sim P(U_1),\ P_\Gamma - P(U_1) \sim P(U_2),\ P(U_2) = o(P(U_1)),\ h \to 0. \tag{5}$$

3. AN APPLICATION TO LIFE TIME MODELS

Suppose tha τ_w are independent random variables and characterize life times of the arcs $w \in W$. Denote Denote $p_w(h) = P(\tau_w > t)$ and designate the life time of the graph Γ by

$$\tau_\Gamma = \min_{R \in \mathcal{R}} \max_{w \in R} \tau_w.$$

If $h = 1/t$ then we have with $t \to \infty$ the Weibull distributions of the arcs life times and the formula

$$\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma) t^{D_\Gamma}} - 1 = g(t) \sim A t^{-\Delta_\Gamma} = G(t) \tag{6}$$

If $h = \exp(-t)$, $t \to \infty$, then we have the Gompertz distributions of the arcs life times and the formula (1) transforms into

$$\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma)\exp(D_\Gamma t)} - 1 = g_1(t) \sim G_1(t) = G(\exp(t)), \tag{7}$$

so $G_1(t) = o(G(t))$.

Consequently for the Gompertz distributions the convergence rate in the asymptotic (7) is much faster than for the Weibull distributions in (6).

If $h = 1/t$, $t \to \infty$, then for the Weibull distributions of the arcs life times the formula (5) transforms into

$$P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right), \tag{8}$$

$$\frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right)} - 1 = f(t) \sim F(t) = o(1), \; F(t) = \exp\left(-\sum_{j=1}^{m_2} c_{w_j^2} t^{d_{w_j^2}} + \sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right).$$

If $h = \exp(-t)$, $t \to \infty$, then for the Gompertz distributions of the arcs life times the formula (5) transforms into

$$P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} \exp\left(d_{w_j^1} t\right)\right), \; \frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} \exp\left(d_{w_j^1} t\right)\right)} - 1 = f_1(t) \sim F_1(t) = F(\exp(t)), \tag{9}$$

so $F_1(t) = o(F(t))$.

Consequently for the Gompertz distributions the convergence rate in the asymptotic (9) is much faster than for the Weibull distributions in (8).

For $h = 1/t$ denote $\left|P_\Gamma^* - P_\Gamma\right| / P_\Gamma = A(t)$, and for $h = \exp(-t)$ designate $\left|P_\Gamma^* - P_\Gamma\right| / P_\Gamma = A_1(t)$. It is clear that $A_1(t) = A(\exp(t))$ tends to zero for $t \to \infty$ much faster than $A(t)$.

From this section we see that the Gompertz distributions of the arcs life times (these distributions are preferable in life time models of alive [1] and of complex information [2] systems), give much more accuracy asymptotic formulas than the Weibull distributions. These both distributions are limit for a scheme of a minimum of independent and identically distributed random variables.

4. RESULTS OF NUMERICAL EXPERIMENTS FOR BRIDGE SCHEMES

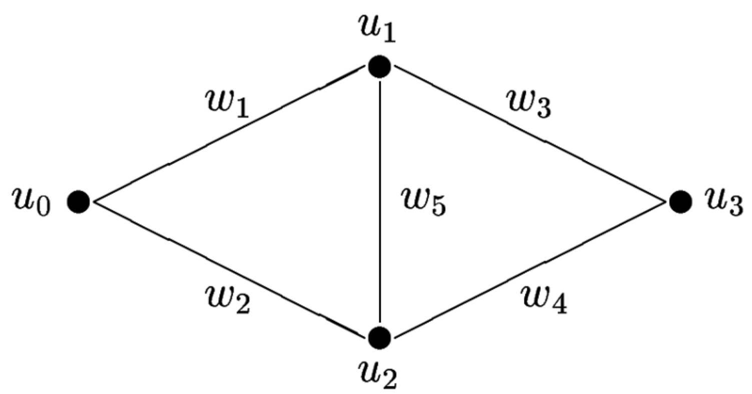

Fig.1 The bridge scheme.

Consider the bridge scheme Γ represented on the Fig. 1 with the parameters $d_1 = 0.02$, $d_2 = 0.09$, $d_3 = 0.5$, $d_4 = 0.72$, $d_5 = 0.2$. Calculate the functions $f(t), f_1(t), A(t), A_1(t), g(t), g_1(t)$.

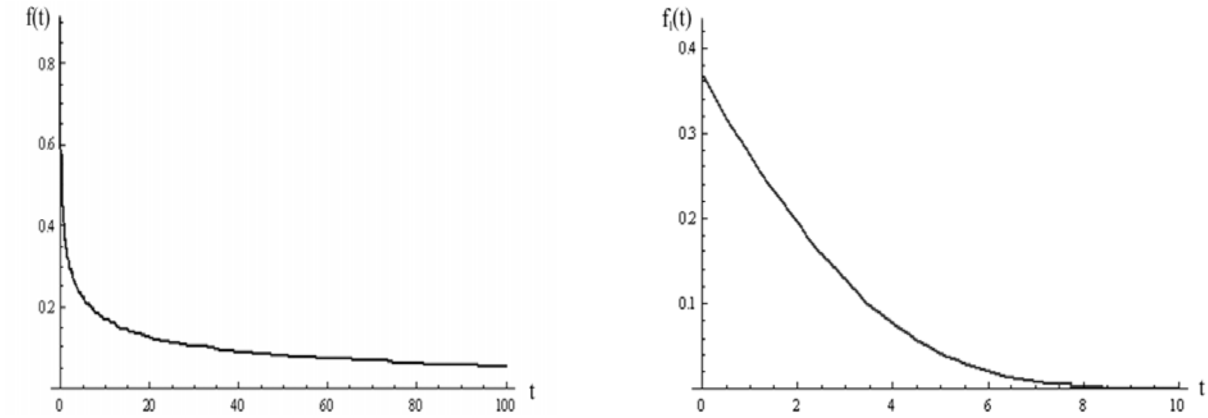

Fig.2 The relative errors $f(t)$ and $f_1(t)$ in the reliability P_Γ calculations

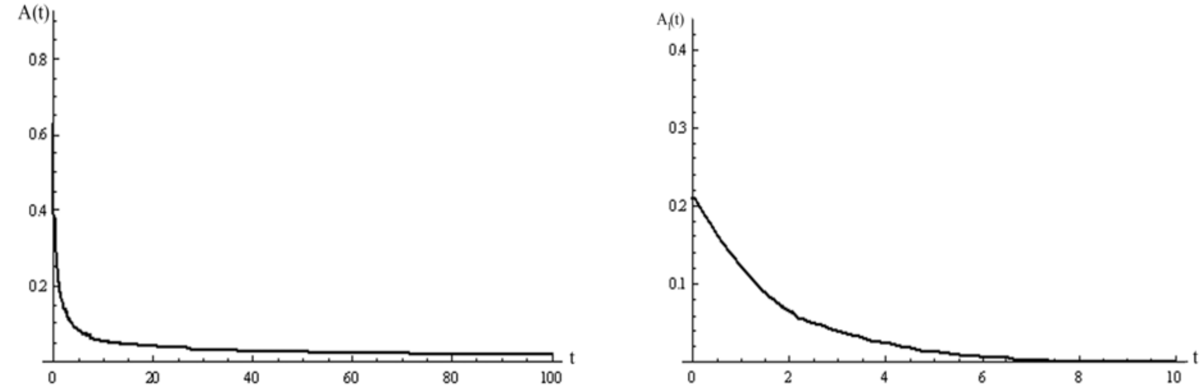

Fig.3 The relative errors $A(t)$ and $A_1(t)$ in the reliability P_Γ calculations

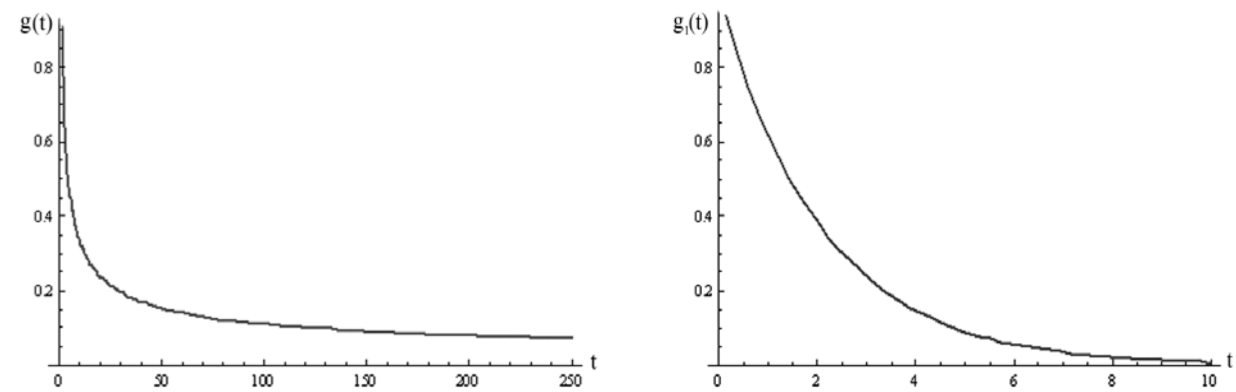

Fig.4 The relative errors $g(t)$ and $g_1(t)$ in $\ln P_\Gamma$ calculations.

The results of the numerical experiments represented above show that a transition from the Weibull to the Gompertz distribution decreases significantly relative errors in calculations of the reliability and its logarithm. The asymptotic estimate P_{Γ}^{*} of the reliability P_{Γ} is better than $P(U_1)$. The relative error of the $\ln P_{\Gamma}$ calculation is larger than the relative error of the P_{Γ} calculation. But a complexity of the $\ln P_{\Gamma}$ calculation is smaller.

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