PHASE TRANSITION IN RENEWAL SYSTEMS WITH COMMON RESERVE

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Introduction

Mathematical models of renewal systems with a common reserve have been introduced and analyzed detailed in the monograph [1]. In [2] a phenomenon of a phase transition in the aggregated renewal system with the unload reserve is analyzed as analytically so numerically. But the mathematical method applied in this paper is too specific to analyze the phase transition phenomenon in general renewal systems with the common reserve. This phenomenon is connected with a reform of municipal engineering systems.

In this paper a method based on a definition of a state in which a birth and death process describing this system has a maximal limit probability is suggested. This method allows to construct convenient upper bounds of the limit probability for other states and to analyze phase transition phenomenon. The obtained bounds depend on transition intensities of the birth and death processes which describe aggregations of renewal systems with unload, under load and load reserves. The suggested method allows an analyzing of a renewal system with a competition between the repair places also.

Preliminaries

At first consider a renewal system with an unload reserve. This system has a single work place with failure intensity a, a single repair place with repair intensity b, $c = b/a$ and two elements. Suppose that work and repair times are independent random variables and have exponential distributions with the parameters a,b accordingly. Take n independent copies of this system and aggregate them so that we have a renewal system with an unload reserve and with n work places, n repair places and $2n$ elements. Our initial problem is to analyze a limit by $n \to \infty$ of the stationary probability P_n that there are elements in all n work places. This probability may be interpreted as a probability that all n aggregated renewal subsystems work. It was shown in [2] that the formulas

$$\lim_{n\to\infty} P_n = 1,\ c > 1,\ \lim_{n\to\infty} P_n = 0,\ c < 1, \tag{1}$$

describing the phase transition in the aggregated system are true.

Consider now a renewal system with an unload reserve which has p work places, q repair places and r elements, $p \le r, q \le r$. Aggregate n independent copies of this system and obtain a renewal system with an unload reserve and with np work places, nq repair places and nr elements. Our problem is to analyze the limit by $n \to \infty$ of the stationary probability Π_n that there are elements in all np work places of the aggregated system. Remark that for a family of independent renewal systems this probability equals $Q^n \to 0$, $n \to \infty$, where $Q < 1$ is the stationary probability that the renewal system with p work places, q repair places and r elements work.

Introduce the birth and death process $x_n(t)$ which characterizes a number of elements in a work phase of the aggregated system that is in the work places and in a queue to these places. This process has the states set $U_n = \{0,...,r\}$ and the birth and death intensities

$$\lambda_n(k) = b\min(nr - k, nq), \ \mu_n(k) = a\min(k, np), \ c_n(k) = \frac{\lambda_n(k-1)}{\mu_n(k)}. \quad (2)$$

The discrete Markov process $x_n(t)$ with the finite states set U_n is ergodic and its limit distribution $\pi_n(k) = \lim_{t\to\infty} P(x_n(t) = k)$ satisfies the equalities

$$\pi_n(k-1) = \frac{\pi_n(k)}{c_n(k)}, \quad k \in U_n, \quad k > 0, \quad (3)$$

and the limit probability, that all or practically all subsystems in the aggregated system work, satisfy the equalities

$$\Pi_n = \lim_{t\to\infty} P(x_n(t) \ge np) = \sum_{k \ge np} \pi_n(k),$$
$$\Pi_{n,\varepsilon} = \lim_{t\to\infty} P(x_n(t) \ge n(p - 2\varepsilon)) = \sum_{k \ge n(p-2\varepsilon)} \pi_n(k).$$

Auxiliary Statements

Introduce the monotonically no increasing and continuous functions

$$C_n(v) = \frac{c\min\left(r - v + \frac{1}{n}, q\right)}{\min(v, p)}, \ \frac{1}{n} \le v \le r; \ C_n(v) = \frac{c\min(r - v, q)}{\min(v, p)}, \ 0 \le v \le r, \quad (4)$$

with $C_n(1/n) = C(0) = +\infty$. Remark that the equality

$$C_n(k/n) = c_n(k), \ 0 \le k \le r, \quad (5)$$

and the inequalities

$$C(v) \le C_n(v) \le C\left(v - \frac{1}{n}\right), \ \frac{1}{n} \le v \le r, \quad (6)$$

are true. Define the sets

$$V_n = \left[\frac{1}{n}, r - q\right] \cap \left[p + \frac{1}{n}, r\right], \ V = [0, r - q] \cap [p, r]$$

and note that these sets contain more than a single point if and only if $r - q > p$.

Lemma 1. *1)Suppose that* $cq \neq p$ *or* $r-q \leq p$ *then there are the real numbers* w_n, w *which are the single roots of the equations*

$$C_n(v)=1\ ,\ \frac{1}{n} \leq v \leq r\ , \tag{7}$$

$$C_n(v)=1,\ 0 \leq v \leq r\ , \tag{8}$$

respectively and satisfy the inequalities

$$w \leq w_n \leq w+\frac{1}{n}\ . \tag{9}$$

There is the integer k_n *which satisfies the formulas*

$$c_n(k)>1,\ k<k_n\ ;\ c_n(k_n)\geq 1;\ c_n(k)<1,\ k>k_n\ , \tag{10}$$

$$nw-1 \leq k_n \leq nw+1 \tag{11}$$

2) *If the conditions* $cq=p,\ r-q>p$ *are true then the equations (7), (8) have the roots sets* V_n, V *respectively and for* $k_n^- = np+1$, $k_n^+ = n(r-q)$ *we have*

$$c_n(k)>1,\ k<k_n^-\ ;\ c_n(k)=1,\ k_n^- \leq k \leq k_n^+\ ; c_n(k)<1,\ k>k_n^+\ . \tag{12}$$

Proof. The functions $C_n(v), C(v)$ are monotonically no increasing. If the condition $r-q \leq p$ takes place then the sets V_n, V contain no more than a single point. So the functions $C_n(v), C(v)$ are strictly monotonically decreasing. If $cp \neq q, r-q>p$ then the sets V_n, V contain more than a single point but on these sets the functions $C_n(v)=C(v)=cp/q \neq 1$. Consequently in the condition of the statement 1) there are single roots w_n, w of the equations (7), (8) respectively.

The inequality (9) is a corollary of the formula (6). Define k_n as a minimal integer which is not larger than w_n .Then from the formula (5) we obtain (10) and from the formulas (10) and from the formulas (9), (10) - the inequalities (11).

If the condition of the statement 2) is true then the sets V_n, V are segments which contain more than a single point and $C_n(v)=C(v)=cp/q=1$ on these sets. Consequently the segments V_n, V are the sets of the equations (7), (8) roots respectively. The formula (12) is a corollary of the formula (5).

Lemma 2. *1)Suppose that* $cp \neq q$ *or* $r-q \leq p$ *then there is a positive number* ε_1 *so that for any* $\varepsilon,\ 0<\varepsilon<\varepsilon_1$, *we have*

$$A_n = \sum_{0 \leq k \leq n(w-2\varepsilon)-1} \pi_n(k) \to 0\ ,\ n \to \infty\ . \tag{13}$$

$$B_n = \sum_{n(w+2\varepsilon) \leq k \leq nr} \pi_n(k) \to 0\ ,\ n \to \infty\ . \tag{14}$$

2) If the condition $cp=q,\ \ r-q>p$ *are true then there is a positive number* ε_1 *so that for any* $\varepsilon,\ \ 0<\varepsilon<\varepsilon_1$*, the following formula is true*

$$C_n=\sum_{0\le k\le n(p-2\varepsilon)-1}\pi_n(k)\to 0\ \ ,\ \ n\to\infty\,. \tag{15}$$

Proof. 1) Denote $S_n=\pi_n(k_n)<1$, from the lemma 1 we have that the limit distribution of the process $x_n(t)$ satisfies the equalities

$$\pi_n(k)=S_n\prod_{k<i\le k_n}C_n^{-1}\left(\frac{i}{n}\right),\ k<k_n\,; \tag{16}$$

$$\pi_n(k)=S_n\prod_{k_n<i\le k}C_n\left(\frac{i}{n}\right),\ k>k_n\,. \tag{17}$$

As $w\notin V$ so there is $\varepsilon_1>0$ so that for any $\varepsilon,\ \ 0<\varepsilon<\varepsilon_1$, the following inequalities

$$\psi^{-1}=C(w-\varepsilon)>C(w)=1>C(w+\varepsilon)=\varphi \tag{18}$$

are true. Then for $k<n(w-2\varepsilon)<k_n$ from the formulas (4)-(6), (16), (18) we obtain

$$\pi_n(k)\le S_n\prod_{k<i\le n(w-\varepsilon)}C^{-1}\left(\frac{i}{n}\right)\le\prod_{k<i\le n(w-\varepsilon)}C^{-1}(w-\varepsilon)$$

and consequently $\pi_n(k)\le\psi^{n(w-\varepsilon)-k-1}$, $0\le k\le n(w-\varepsilon)-1$. Summarize the last inequality by $k, 0\le k\le n(w-2\varepsilon)-1$, and obtain the formula

$$A_n\le\frac{\psi^{n\varepsilon}}{1-\psi}\to 0\,,\ n\to\infty$$

so the formula (13) is proved.

For $k>n(w+\varepsilon)+1>k_n$ from the formulas (4)-(6), (17), (18) we obtain

$$\pi_n(k)\le S_n\prod_{n(w+\varepsilon)+1\le i\le k}C\left(\frac{i-1}{n}\right)\le\prod_{n(w+\varepsilon)\le i\le k-1}C(w+\varepsilon)$$

and consequently $\pi_n(k)\le\varphi^{k-n(w+\varepsilon)-1}$, $n(1+\varepsilon)\le k\le nr$. Summarize the last inequality by k, $n(1+2\varepsilon)\le k\le nr$, and obtain the formula

$$B_n\le\frac{\varphi^{n\varepsilon-2}}{1-\varphi}\to 0\,,\ n\to\infty$$

so the formula (14) is proved.

2) Denote $R_n=\pi_n\left(k_n^-\right)<1$, from the lemma 1 we have that the limit distribution of the process $x_n(t)$ satisfies the equalities

$$\pi_n(k) = R_n \prod_{k<i\le k_n^-} C_n^-\left(\frac{i}{n}\right),\ k < k_n^- \ .$$

As $V = [p, r-q]$ so for any ε, $0 < \varepsilon < p$, we have

$$\delta^{-1} = C(p-\varepsilon) > C(p) = 1.$$

Then for $k < n(p-2\varepsilon) < k_n^-$ the formulas (4)-(6) we obtain

$$\pi_n(k) \le S_n \prod_{k<i\le n(w-\varepsilon)} C^{-1}\left(\frac{i}{n}\right) \le \prod_{k<i\le n(p-\varepsilon)} C^{-1}(p-\varepsilon).$$

And so $\pi_n(k) \le \delta^{n(p-\varepsilon)-k-1}$, $0 \le k \le n(p-\varepsilon)-1$. Summarize the last inequality by k, $0 \le k \le n(p-2\varepsilon)-1$, and obtain the formula

$$C_n \le \frac{\delta^{n\varepsilon}}{1-\delta} \to 0,\ n \to \infty.$$

So the formula (15) is proved.

Main results

For the aggregation of n renewal systems with the unload reserve and with np work places, nq repair places and nr elements the following statement takes place.

Theorem 1. *1) Suppose that* $cp \ne q$ or $r-q \le p$ *then we have*

$$\lim_{n\to\infty} \Pi_n = 1,\ w > p;\ \lim_{n\to\infty} \Pi_n = 0,\ w < p. \tag{19}$$

2) If the conditions $cp = q$, $r-q > p$ are *true then there is a positive number* ε_1 *so that for any* ε, $0 < \varepsilon < \varepsilon_1$ *the following formula is true*

$$\Pi_{n,\varepsilon} = 1 - C_n \to 1,\ n \to \infty. \tag{20}$$

Proof. 1) Suppose that $w > p$ and choose ε from the conditions $0 < \varepsilon < \varepsilon_1$, $w - 2\varepsilon > p$ then we have

$$1 - \Pi_n = \sum_{0\le k<np} \pi_n(k) \le \sum_{0\le k\le n(w-2\varepsilon)-1} \pi_n(k) = A_n \to 0,\ n \to \infty.$$

So the first formula from (19) is true.

Suppose that $w < p$ and choose ε from the conditions $0 < \varepsilon < \varepsilon_1$, $w + 2\varepsilon < p$ then we obtain

$$\Pi_n = \sum_{k\ge np} \pi_n(k) \le \sum_{n(w+2\varepsilon)\le k\le nr} \pi_n(k) = B_n \to 0\ ,\ n \to \infty\ .$$

So the second formula from (19) is true also.

2) The formula (20) is a direct corollary of the formulas (15).

Consider an aggregation of n renewal systems with a load reserve which have np work places, nq repair places and nr elements. In the renewal system with the load reserve elements which are in a queue to the work places fail with the same intensity as working elements. The birth and death process $x_n(t)$ which characterizes a number of elements in the work phase is ergodic. It has the states set U_n and its birth and death intensities satisfy the equalities $\lambda_n(k) = b\min(nr-k, nq)$, $\mu_n(k) = ak$ instead of (2). For the aggregation of n renewal systems with the load reserve the following statement is true.

Theorem 2. *The formulas (19) are true with* w *which is the single root of the equation* $\min(r-v,q) = cv$ *by* v.

Proof. The theorem 2 proof practically repeats the proof of the section 1) from the theorem 1. A main difference is in a replacement of (4) by the equalities

$$C_n(v) = \frac{\min(r-v,q)}{c\left(v-\frac{1}{n}\right)},\ \frac{1}{n} \le v \le r;\ C_n(v) = \frac{\min(r-v,q)}{cv},\ 0 \le v \le r, \tag{21}$$

and in a fact that the equation $C(v) = 1$ has only the single root.

Consider now an aggregation of n renewal systems with an under load reserve. It has np work places, nq repair places and nr elements. In this system elements in a queue to work places may fail with the positive intensity $d < a$, $c' = d/b$. The birth and death process $x_n(t)$ which characterizes a number of elements in the work phase is ergodic. It has the states set U_n and its birth and death intensities satisfy the equalities $\lambda_n(k) = b\min(nr-k, nq)$, $\mu_n(k) = dk + (a-d)\min(k, np)$ instead of (2). For the aggregation of n renewal systems with the under load reserve the following statement is true.

Theorem 3. *The formulas (19) are true with* w *which is the single root of the equation* $c'v + (1-c')\min(v,p) = c\min(r-v,q)$ *by* v.

Proof. The theorem 3 proof practically repeats the theorem 2 proof. A main difference is in a replacement of (21) by the equalities

$$C_n(v) = \frac{c\min(r-v,q)}{c'\left(v-\frac{1}{n}\right) + (1-c')\min\left(v-\frac{1}{n}, p\right)},\ \frac{1}{n} \le v \le r,$$

$$C_n(v) = \frac{c\min(r-v,q)}{c'v + (1-c')\min(v,p)},\ 0 \le v \le r.$$

Remark 1. *As in the theorems 1- 3 conditions the inequality* $w > p$ *(the inequality* $w < p$*) is equivalent to the inequality* $C(p) > 1$ *(to the inequality* $C(p) > 1$*) so the formula (19) may be rewritten as follows*

$$\lim_{n\to\infty} \Pi_n = 1,\ c\min(r-p,q) > p;\ \lim_{n\to\infty} \Pi_n = 0,\ c\min(r-p,q) < p. \tag{22}$$

Consider the same aggregated systems but with a competition between the repair places. An element arrives in the repair phase and receives information about times of its repair at all places. Then the place with a minimal time of a repair is selected. During this time all other repair places do not work.

These systems are described by similar birth and death processes with the following redefinition of the birth intensity $\lambda_n(k) = nbq\min(nr - k, 1)$.

Theorem 4. *For the aggregated renewal systems with the common unload (load or under load) reserve the following formulas are true*

$$\lim_{n\to\infty}\Pi_n = 1,\ cq > p;\ \lim_{n\to\infty}\Pi_n = 0,\ cq < p. \tag{23}$$

Proof. Consider the aggregated renewal system with the common reserve. The theorem 4 proof practically repeats the theorem 1 proof. A main difference is in a replacement of (4) by the following equalities: for the unload reserve

$$C_n(v) = C(v) = \frac{cq}{\min(v, p)}$$

and for the load reserve

$$C_n(v) = C(v) = \frac{cq}{v}$$

and for the under load reserve

$$C_n(v) = C(v) = \frac{cq}{c'v + (1-c')\min(v, p)}.$$

Remark 2. *A comparison of the theorems 1-3 with the theorem 4 allows to derive that the common reserve with the competition of the repair places gives better results in terms of the work probability* Π_n.

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