Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATION OF COMPLEX TECHNICAL SYSTEMS OPERATION PROCESSES
PART 1
THEORETICAL BACKGROUNDS

K. Kolowrocki, J. Soszynska.

•

Gdynia Maritime University, Gdynia, Poland

e-mail: katmatkk@am.gdynia.pl, joannas@am.gdynia.pl

ABSTRACT

A convenient new tool for solving the problem of reliability and availability evaluation and optimization of complex technical systems is presented. Linking a semi-markov modeling of the system operation processes with a multi-state approach to system reliability and availability analysis is proposed to construct the joint general model of reliability and availability of complex technical systems in variable operation conditions. This joint model and a linear programming is proposed to complex technical systems reliability and availability evaluation and optimization respectively.

1   INTRODUCTION

Most real technical systems are very complex and it is difficult to analyze their reliability and availability. Large numbers of components and subsystems and their operating complexity cause that the evaluation and optimization of their reliability and availability is complicated. The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability characteristics is often very difficult to fix and to analyze. Usually the system environment and infrastructure have either an explicit or an implicit strong influence on the system operation process. As a rule some of the initiating environment events and infrastructure conditions define a set of different operation states of the technical system. A convenient tool for solving this problem is a semi-markov modeling of the system operation processes linked with a multi-state approach for the system reliability and availability analysis and a linear programming for the system reliability and availability optimization.

2   MODELLING SYSTEM OPERATION PROCESS

We assume that the system during its operation process is taking v, $v \in N$, different operation states. Further, we define the system operation process $Z(t)$, $t \in <0,+\infty>$, with discrete operation states from the set of states $Z = \{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the system operation process $Z(t)$ is semi-markov (Grabski 2002) with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b,l = 1,2,\ldots,v$, $b \neq l$. Under these assumptions, the system operation process may be described by [1], [2], [6] the vector of probabilities of the system operation process $Z(t)$ initial operation states $[p_b(0)]_{1 \times v}$, the matrix of probabilities of the system

operation process $Z(t)$ transitions between the operation states $[p_{bl}]_{v \times v}$ and the matrix of conditional distribution functions of the system operation process $Z(t)$ conditional sojourn times θ_{bl} in the operation states $[H_{bl}(t)]_{v \times v}$ or equivalently by the matrix of corresponding conditional density functions $[h_{bl}(t)]_{v \times v}$.

From the formula for total probability it follows that the unconditional distribution functions of the sojourn times θ_b, $b = 1,2,...,v,$ of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,...,v,$ are given by (Kolowrocki, Soszynska 2008)

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \ \ b = 1,2,...,v.$$

(1)

Hence, the mean values $E[\theta_b]$ of the unconditional sojourn times θ_b, $b = 1,2,...,v,$ are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl} , \ b = 1,2,...,v, \tag{2}$$

where M_{bl} are defined by the formula

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t) = \int_0^\infty t h_{bl}(t) dt, \quad b,l = 1,2,...,v, \ b \neq l. \tag{3}$$

The limit values of the transient probabilities $p_b(t)$ at the particular operation states are given by (Grabski 2002, Kolowrocki, Soszynska 2008)

$$p_b = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \ \ b = 1,2,...,v, \tag{4}$$

where M_b, $b = 1,2,...,v,$ are given by (2), while the stationary probabilities π_b of the vector $[\pi_b]_{1 \times v}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \tag{5}$$

3   RELIABILITY, RISK AND AVAILABILITY OF MULTI-STATE SYSTEMS IN VARIABLE OPERAION CONDITIONS

In the multi-state reliability analysis to define systems with degrading (ageing) components we assume that:
– n is the number of system components,
– E_i, $i = 1,2,...,n,$ are components of a system,
– all components and a system under consideration have the state set $\{0,1,...,z\}$, $z \geq 1$,
– the state indexes are ordered, the state 0 is the worst and the state z is the best,

- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
- the system state degrades with time t without repair,
- $e_i(t)$ is a component E_i state at the moment t, $t \in\, <0,\infty)$, given that it was in the state z at the moment $t = 0$,
- $s(t)$ is a system state at the moment t, $t \in\, <0,\infty)$, given that it was in the state z at the moment $t = 0$.

The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse (Kolowrocki 2004, Kolowrock 2007).

Under these assumptions, a vector

$$R_i(t,\cdot) = [R_i(t,0),R_i(t,1),...,R_i(t,z)], \ t \in (-\infty,\infty), \ i = 1,2,...,n,$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t), \ t \in (-\infty,\infty), \ u = 0,1,...,z,$$

is the probability that the component E_i is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in\, <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Similarly, a vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)], \ t \in (-\infty,\infty),$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t), \ t \in (-\infty,\infty), \ u = 0,1,...,z, \tag{6}$$

is the probability that the system is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in\, <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a system.

A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \leq t), \ t \in (-\infty,\infty),$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the state z at the moment $t = 0$ is called a risk function of the multi-state system or, in short, a risk (Kołowrocki 2004).

Under this definition, from (6), we have

$$\boldsymbol{r}(t) = 1 - P(s(t) \geq r \mid s(0) = z) = 1 - \boldsymbol{R}_n(t,r), \ \ t \in (-\infty,\infty). \tag{7}$$

and if τ is the moment when the risk exceeds a permitted level δ, then

$$\tau = r^{-1}(\delta), \tag{8}$$

where $r^{-1}(t)$, if it exists, is the inverse function of the risk function $r(t)$.

Further, we assume that the changes of the process $Z(t)$ states have an influence on the system multi-state components E_i reliability and the system reliability structure as well. Thus, we denote the conditional reliability function of the system multi-state component E_i while the system is at the operational state z_b, $b = 1,2,...,v$, by (Kołowrocki et all 2008, Limnios, Oprisan 2001, Ross 2007, Soszynska 2006a,b)

$$[R_i(t,\cdot)]^{(b)} = [1, [R_i(t,1)]^{(b)}, ..., [R_i(t,z)]^{(b)}], \ i = 1,2,...,n,$$

where for $t \in <0,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

$$[R_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t \mid Z(t) = z_b)$$

and the conditional reliability function of the system while the system is at the operational state z_b, $b = 1,2,...,v$, by

$$[\boldsymbol{R}_n(t,\cdot)]^{(b)} = [1, [\boldsymbol{R}_n(t,1)]^{(b)}, ..., [\boldsymbol{R}_n(t,z)]^{(b)}],$$

where for $t \in <0,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.

$$[\boldsymbol{R}_n(t,u)]^{(b)} = P(T^{(b)}(u) > t \mid Z(t) = z_b),$$

and $T^{(b)}(u)$ is the system conditional lifetime at the operational state z_b, dependent on the components conditional lifetimes at the operational state z_b.

The reliability function $[R_i(t,u)]^{(b)}$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the state subset $\{u, u+1,...,z\}$ is greater than t, while the process $Z(t)$ is at the operation state z_b. Similarly, the reliability function $[\boldsymbol{R}_n(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the state subset $\{u, u+1,...,z\}$ is greater than t, while the process $Z(t)$ is at the operation state z_b. In the case when the system operation time θ is large enough, the unconditional reliability function of the system

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)], \tag{9}$$

where

$$\boldsymbol{R}_n(t,u) = P(T(u) > t), \ \text{for} \ t \in <0,\infty), \ u = 1,2,...,z,$$

and $T(u)$ is the unconditional lifetime of the system in the system reliability state subsets is given by

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

$$\boldsymbol{R}_n(t,u) \cong \sum_{b=1}^{v} p_b [\boldsymbol{R}_n(t,u)]^{(b)} \text{ for } t \geq 0, \; u = 1,2,...,z, \tag{10}$$

and the mean value of the system unconditional lifetime in the system reliability state subsets is

$$\mu(u) \cong \sum_{b=1}^{v} p_b \mu_b(u), \; u = 1,2,...,z, \tag{11}$$

where

$$\mu_b(u) = \int_0^{\infty} [\boldsymbol{R}_n(t,u)]^{(b)} dt, \; u = 1,2,...,z, \tag{12}$$

and p_b are given by (4) and the variance of the system unconditional lifetime in the system reliability state subsets is

$$\sigma^2(u) = 2\int_0^{\infty} t \, \boldsymbol{R}_n(t,u)dt - [\mu(u)]^2, \; u = 1,2,...,z. \tag{13}$$

Additionally, according to (3.19) (Blokus et all 2008), we get the following formulae for mean values of the unconditional lifetime of the system in particular reliability states

$$\overline{\mu}(u) = \mu(u) - \mu(u+1), \; u = 0,1,...,z-1, \; \overline{\mu}(z) = \mu(z). \tag{14}$$

The main characteristics of multi-state renewal system with ignored time of renovation related to their operation process can be approximately determined by using results of the research report (Blokus et all 2008) formulated in the form of the following theorem.

Theorem 3.1

If components of the multi-state renewal system with ignored time of renovation at the operational states have exponential reliability functions and the system reliability critical state is r, $r \in \{1,2,...,z\}$, then:

i) the distribution of the time $S_N(r)$ until the *Nth* exceeding of reliability critical state r of this system, for sufficiently large *N*, has approximately normal distribution $N(N\mu(r), \sqrt{N}\sigma(r))$, i.e.,

$$F^{(N)}(t,r) = P(S_N(r) < t) \cong F_{N(0,1)}(\frac{t - N\mu(r)}{\sqrt{N}\sigma(r)}), t \in (-\infty, \infty), \; r \in \{1,2,...,z\},$$

ii) the expected value and the variance of the time $S_N(r)$ until the *Nth* exceeding the reliability critical state *r* of this system take respectively forms

$$E[S_N(r)] = N\mu(r), \; D[S_N(r)] = N[\sigma(r)]^2, r \in \{1,2,...,z\},$$

iii) the distribution of the number $N(t,r)$ of exceeding the reliability critical state *r* of this system up to the moment $t, t \geq 0$, for sufficiently large *t*, is approximately of the form

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

$$P(N(t,r) = N) \cong F_{N(0,1)}(\frac{N\mu(r)-t}{\sigma(r)\sqrt{\dfrac{t}{\mu(r)}}}) - F_{N(0,1)}(\frac{(N+1)\mu(r)-t}{\sigma(r)\sqrt{\dfrac{t}{\mu(r)}}}), \ N = 0,1,2,..., \ r \in \{1,2,...,z\},$$

iv) the expected value and the variance of the number $N(t,r)$ of exceeding the reliability critical state r of this system at the moment $t, t \geq 0,$ for sufficiently large t, approximately take respectively forms

$$H(t,r) = \frac{t}{\mu(r)}, \ \ D(t,r) = \frac{t}{[\mu(r)]^3}[\sigma(r)]^2, \ r \in \{1,2,...,z\},$$

where and $\mu(r)$ and $\sigma(r)$ are given by (11)-(13) for $u = r$.
The main characteristics of multi-state renewal system with non-ignored time of renovation related to their operation process can be approximately determined by using results of the research report [1] formulated in the form of the following theorem.
Theorem 3.2
If components of the multi-state renewal system with non-ignored time of renovation at the operational states have exponential reliability functions, the system reliability critical state is r, $r \in \{1,2,...,z\},$ and the successive times of system's renovations are independent and have an identical distribution function with the expected value $\mu_o(r)$ and the variance $\sigma_o^2(r)$, then:

i) the distribution function of the time $\overline{\overline{S}}_N(r)$ until the *Nth* system's renovation, for sufficiently large *N*, has approximately normal distribution

$$N(N(\mu(r) + \mu_o(r)), \sqrt{N(\sigma^2(r) + \sigma_o^2(r))}) , \text{ i.e.,}$$

$$\overline{\overline{F}}^{(N)}(t,r) = P(\overline{\overline{S}}_N(r) < t) \cong F_{N(0,1)}(\frac{t - N(\mu(r) + \mu_o(r))}{\sqrt{N(\sigma^2(r) + \sigma_o^2(r))}}), \ t \in (-\infty, \infty), \ N = 1,2,..., \ r \in \{1,2,...,z\},$$

ii) the expected value and the variance of the time $\overline{\overline{S}}_N(r)$ until the *Nth* system's renovation take respectively forms

$$E[\overline{\overline{S}}_N(r)] \cong N(\mu(r) + \mu_o(r)), \ \ D[\overline{\overline{S}}_N(r)] \cong N(\sigma^2(r) + \sigma_o^2(r)), \ r \in \{1,2,...,z\},$$

iii) the distribution function of the time $\overline{S}_N(r)$ until the *Nth* exceeding the reliability critical state r of this system takes form

$$\overline{F}^{(N)}(t,r) = P(\overline{S}_N(r) < t) \cong F_{N(0,1)}(\frac{t - N(\mu(r) + \mu_o(r)) + \mu_o(r)}{\sqrt{N(\sigma^2(r) + \sigma_o^2(r)) - \sigma_o^2(r)}}), \ t \in (-\infty, \infty), \ N = 1,2,..., $$

$$r \in \{1,2,...,z\},$$

iv) the expected value and the variance of the time $\overline{S}_N(r)$ until the *Nth* exceeding the reliability critical state r of this system take respectively forms

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

$$E[\bar{S}_N(r)] \cong N\mu(r) + (N-1)\mu_o(r), \quad D[\bar{S}_N(r)] \cong N\sigma^2(r) + (N-1)\sigma_o^2(r), \quad r \in \{1,2,...,z\},$$

v) the distribution of the number $\bar{\bar{N}}(t,r)$ of system's renovations up to the moment $t, t \geq 0,$ is of the form

$$P(\bar{\bar{N}}(t,r) = N) \cong F_{N(0,1)}\left(\frac{N(\mu(r) + \mu_o(r)) - t}{\sqrt{\dfrac{t}{\mu(r) + \mu_0(r)}(\sigma^2(r) + \sigma_o^2(r))}}\right) - F_{N(0,1)}\left(\frac{(N+1)(\mu(r) + \mu_o(r)) - t}{\sqrt{\dfrac{t}{\mu(r) + \mu_0(r)}(\sigma^2(r) + \sigma_o^2(r))}}\right),$$

$N = 1,2,...,$ $r \in \{1,2,...,z\},$

vi) the expected value and the variance of the number $\bar{\bar{N}}(t,r)$ of system's renovations up to the moment $t, t \geq 0,$ take respectively forms

$$\bar{\bar{H}}(t,r) \cong \frac{t}{\mu(r) + \mu_o(r)}, \quad \bar{\bar{D}}(t,r) \cong \frac{t}{(\mu(r) + \mu_o(r))^3}(\sigma^2(r) + \sigma_o^{\ 2}(r)), \quad r \in \{1,2,...,z\},$$

vii) the distribution of the number $\bar{N}(t,r)$ of exceeding the reliability critical state r of this system up to the moment $t, t \geq 0,$ is of the form

$$P(\bar{N}(t,r) = N) \cong F_{N(0,1)}\left(\frac{N(\mu(r) + \mu_o(r)) - t - \mu_0(r)}{\sqrt{\dfrac{t + \mu_0(r)}{\mu(r) + \mu_0(r)}(\sigma^2(r) + \sigma_o^2(r))}}\right) - F_{N(0,1)}\left(\frac{(N+1)(\mu(r) + \mu_o(r)) - t - \mu_0(r)}{\sqrt{\dfrac{t + \mu_0(r)}{\mu(r) + \mu_0(r)}(\sigma^2(r) + \sigma_o^2(r))}}\right),$$

$N = 1,2,...,$ $r \in \{1,2,...,z\},$

viii) the expected value and the variance of the number $\bar{N}(t,r)$ of exceeding the reliability critical state r of this system up to the moment $t, t \geq 0,$ for sufficiently large $t,$ are approximately respectively given by

$$\bar{H}(t,r) \cong \frac{t + \mu_0(r)}{\mu(r) + \mu_o(r)}, \quad \bar{D}(t,r) \cong \frac{t + \mu_0(r)}{(\mu(r) + \mu_o(r))^3}(\sigma^2(r) + \sigma_o^{\ 2}(r)), \quad r \in \{1,2,...,z\},$$

ix) the availability coefficient of the system at the moment t is given by the formula

$$A(t,r) \cong \frac{\mu(r)}{\mu(r) + \mu_o(r)}, \quad t \geq 0, \ r \in \{1,2,...,z\},$$

x) the availability coefficient of the system in the time interval $<t,t+\tau), \tau > 0,$ is given by the formula

$$A(t,\tau,r) \cong \frac{1}{\mu(r) + \mu_o(r)} \int_\tau^\infty \boldsymbol{R}_n(t,r)dt, \quad t \geq 0, \ \tau > 0, \ r \in \{1,2,...,z\},$$

where $\boldsymbol{R}_n(t,r)$ is given by the formula (10) and $\mu(r)$ and $\sigma(r)$ are given by (11)-(13) for $u = r.$

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

4 OPTIMIZATION OF A SYSTEM OPERATIONPROCESS

4.1 Optimal transient probabilities maximizing system lifetime

Considering the equation (10), it is natural to assume that the system operation process has a significant influence on the system reliability. This influence is also clearly expressed in the equation (11) for the mean values of the system unconditional lifetimes in the reliability state subsets.

From linear equation (11), we can see that the mean value of the system unconditional lifetime $\mu(u)$, $u = 1,2,..., z$, is determined by the limit transient probabilities p_b, $b = 1,2,..., \nu$, of the system operation states given by (4) and the mean values $\mu_b(u)$, $b = 1,2,..., \nu$, $u = 1,2,..., z$, of the system conditional lifetimes in the reliability state subsets $\{u, u+1,..., z\}$, $u = 1,2,..., z$, given by (3.6). Therefore, the system lifetime optimization approach based on the linear programming can be proposed. Namely, we may look for the corresponding optimal values $\dot{p}_b$, $b = 1,2,..., \nu$, of the transient probabilities p_b, $b = 1,2,..., \nu$, in the system operation states to maximize the mean value $\mu(u)$ of the unconditional system lifetimes in the reliability state subsets $\{u, u+1,..., z\}$, $u = 1,2,..., z$, under the assumption that the mean values $\mu_b(u)$, $b = 1,2,..., \nu$, $u = 1,2,..., z$, of the system conditional lifetimes in the reliability state subsets are fixed. As a special case of the above formulate system lifetime optimization problem, if r, $r = 1,2,..., z$, is a system critical reliability state, then we want to find the optimal values $\dot{p}_b$, $b = 1,2,..., \nu$, of the transient probabilities p_b, $b = 1,2,..., \nu$, in the system operation states to maximize the mean value $\mu(r)$ of the unconditional system lifetimes in the reliability state subset $\{r, r+1,,..., z\}$, $r = 1,2,..., z$, under the assumption that the mean values $\mu_b(r)$, $b = 1,2,..., \nu$, $r = 1,2,..., z$, of the system conditional lifetimes in this reliability state subset are fixed. More exactly, we formulate the optimization problem as a linear programming model with the objective function of the following linear form

$$\mu(r) = \sum_{b=1}^{\nu} p_b \mu_b(r) \tag{15}$$

for a fixed $r \in \{1,2,..., z\}$ and with the following bound constraints

$$\sum_{b=1}^{\nu} p_b = 1, \tag{16}$$

$$\breve{p}_b \leq p_b \leq \widehat{p}_b, \; b = 1,2,..., \nu, \tag{17}$$

where

$$\mu_b(r), \; \mu_b(r) \geq 0, \; b = 1,2,..., \nu,$$

are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ and

$$\breve{p}_b, \qquad 0 \leq \breve{p}_b \leq 1 \qquad \text{and} \qquad \widehat{p}_b, \qquad 0 \leq \widehat{p}_b \leq 1, \qquad \breve{p}_b \leq \widehat{p}_b, \; b = 1,2,..., \nu,$$

(18)

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

are lower and upper bounds of the unknown transient probabilities p_b, $b = 1,2,...,\nu$, respectively. Now, we can obtain the optimal solution of the formulated by (15)-(18) the linear programming problem, i.e. we can find the optimal values $\dot{p}_b$ of the limit transient probabilities p_b, $b = 1,2,...,\nu$, that maximize the objective function (15). First, we arrange the system conditional lifetime mean values $m^{(b)}(r)$, $b = 1,2,...,\nu$, in non-increasing order

$$\mu_{b_1}(r) \geq \mu_{b_2}(r) \geq \ldots \geq \mu_{b_\nu}(r), \text{ where } b_i \in \{1,2,...,\nu\} \text{ for } i = 1,2,...,\nu.$$

Next, we substitute

$$x_i = p_{b_i}, \; \breve{x}_i = \breve{p}_{b_i}, \; \widehat{x}_i = \widehat{p}_{b_i} \text{ for } i = 1,2,...,\nu \tag{19}$$

and we maximize with respect to x_i, $i = 1,2,...,\nu$, the linear form (15) that after this transformation takes the form

$$\mu(r) = \sum_{i=1}^{\nu} x_i \mu_{b_i}(r) \tag{20}$$

for a fixed $r \in \{1,2,...,z\}$ with the following bound constraints

$$\sum_{i=1}^{\nu} x_i = 1, \tag{21}$$

$$\breve{x}_i \leq x_i \leq \widehat{x}_i, \; i = 1,2,...,\nu, \tag{22}$$

where

$$\mu_{b_i}(r), \; \mu_{b_i}(r) \geq 0, \; i = 1,2,...,\nu,$$

are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ arranged in non-increasing order and

$$\breve{x}_i, \; 0 \leq \breve{x}_i \leq 1 \text{ and } \widehat{x}_i, \; 0 \leq \widehat{x}_i \leq 1, \; \breve{x}_i \leq \widehat{x}_i, \; i = 1,2,...,\nu, \tag{23}$$

are lower and upper bounds of the unknown limit transient probabilities x_i, $i = 1,2,...,\nu$, respectively.
We define

$$\breve{x} = \sum_{i=1}^{\nu} \breve{x}_i, \; \widehat{y} = 1 - \breve{x} \tag{24}$$

and

$$\breve{x}^0 = 0, \; \widehat{x}^0 = 0 \text{ and } \breve{x}^I = \sum_{i=1}^{I} \breve{x}_i, \; \widehat{x}^I = \sum_{i=1}^{I} \widehat{x}_i \quad \text{for } I = 1,2,...,\nu. \tag{25}$$

Next, we find the largest value $I \in \{0,1,...,\nu\}$ such that

Kolowrocki Krzysztof, Joanna Soszynska – RELIABILITY, RISK AND AVAILABILITY BASED OPTIMIZATIO OF COMPLEX TECHNICAL SYSTEMS
OPERATION PROCESS, PAT 1, THEORETICAL BACKGROUNDS

R&RATA # 4
(Vol.2) 2009, December

$$\widehat{x}^I - \breve{x}^I < \widehat{y} \tag{26}$$

and we fix the optimal solution that maximize (20) in the following way:

i) if $I = 0$, the optimal solution is

$$\dot{x}_1 = \widehat{y} + \breve{x}_1 \ \text{ and } \ \dot{x}_i = \breve{x}_i \ \text{ for } \ i = 2,3,..., \nu; \tag{27}$$

ii) if $0 < I < \nu$, the optimal solution is

$$\dot{x}_i = \widehat{x}_i \,\text{for}\ i = 1,2,..., I, \ \dot{x}_{I+1} = \widehat{y} - \widehat{x}^I + \breve{x}^I + \breve{x}_{I+1}$$

and

$$\dot{x}_i = \breve{x}_i \ \text{ for } \ i = I + 2, I + 3,..., \nu; \tag{28}$$

iii) if $I = \nu$, the optimal solution is

$$\dot{x}_i = \widehat{x}_i \ \text{ for } \ i = 1,2,..., \nu. \tag{29}$$

Finally, after making the inverse to (19) substitution, we get the optimal limit transient probabilities

$$\dot{p}_{b_i} = \dot{x}_i \ \text{ for } \ i = 1,2,..., \nu, \tag{30}$$

that maximize the system mean lifetime given by the linear form (15) giving its optimal value in the following form

$$\dot{\mu}(r) = \sum_{b=1}^{\nu} \dot{p}_b \mu_b(r) \tag{31}$$

for a fixed $r \in \{1,2,..., z\}$.

From the above, replacing r by u, $u = 1,2,..., z$, we obtain the corresponding optimal solutions for the mean values of the system unconditional lifetimes in the reliability state subsets $\{u, u + 1,..., z\}$ of the form

$$\dot{\mu}(u) = \sum_{b=1}^{\nu} \dot{p}_b \mu_b(u) \ \text{ for } \ u = 1,2,..., z,$$

(32)

and by (13) the corresponding values of the variances of the system unconditional lifetimes in the system reliability state subsets is

$$\dot{\sigma}^2(u) = 2\int_0^\infty t \, \dot{R}_n(t,u) dt - [\dot{\mu}(u)]^2, \ \ u = 1,2,..., z, \tag{33}$$

where $\dot{\mu}(u)$ is given by (32) and $\dot{R}_n(t,u)$, according to (9)-(10), is the coordinate of the corresponding optimal unconditional multistate reliability function of the system

$$\dot{\boldsymbol{R}}_n(t,\cdot) = [1,\ \dot{\boldsymbol{R}}_n(t,1),\ ...,\ \dot{\boldsymbol{R}}_n(t,z)\,], \tag{34}$$

given by

$$\dot{\boldsymbol{R}}_n(t,u) \cong \sum_{b=1}^{v} \dot{p}_b [\boldsymbol{R}_n(t,u)]^{(b)} \text{ for } t \geq 0,\ \ u = 1,2,...,z, \tag{35}$$

and by (14) the optimal solutions for the mean values of the system unconditional lifetimes in the particular reliability states are of the form

$$\dot{\bar{\mu}}(u) = \dot{\mu}(u) - \dot{\mu}(u+1),\ \ u = 0,1,...,z-1,\ \ \dot{\bar{\mu}}(z) = \dot{\mu}(z). \tag{36}$$

Moreover, considering (7) and (8), the corresponding optimal system risk function and the moment when the risk exceeds a permitted level δ, respectively are given by

$$\dot{\boldsymbol{r}}(t) = 1 - \dot{\boldsymbol{R}}_n(t,r),\ \ t \in (-\infty,\infty), \tag{37}$$

and

$$\dot{\tau} = \dot{\boldsymbol{r}}^{-1}(\delta),$$

(38)

where $\dot{\boldsymbol{r}}^{-1}(t)$, if it exists, is the inverse function of the risk function $\dot{\boldsymbol{r}}(t)$.

Finally, replacing $\mu(r)$ by $\dot{\mu}(r)$ and $\sigma(r)$ by $\dot{\sigma}(r)$ in the expressions for the renewal systems characteristics pointed in Theorem 1 and Theorem 2, we get their corresponding optimal values.

4.2. Optimal sojourn times in operation states maximizing system lifetime

Replacing in (4) limit transient probabilities p_b in operational states by their optimal values $\dot{p}_b$ found in the previous section and the mean values M_b of the unconditional sojourn times in operational states by their corresponding unknown optimal values $\dot{M}_b$ we get the system of equations

$$\dot{p}_b = \frac{\pi_b \dot{M}_b}{\sum_{l=1}^{v} \pi_l \dot{M}_l},\ \ b = 1,2,...,v. \tag{39}$$

After simple transformations the above system takes the form

$$(\dot{p}_1 - 1)\pi_1 \dot{M}_1 + \dot{p}_1 \pi_2 \dot{M}_2 + ... + \dot{p}_1 \pi_v \dot{M}_v = 0$$

$$\dot{p}_2 \pi_1 \dot{M}_1 + (\dot{p}_2 - 1)\pi_2 \dot{M}_2 + ... + \dot{p}_2 \pi_v \dot{M}_v = 0$$

$$.$$
$$.$$
$$.$$

$$\dot{p}_v \pi_1 \dot{M}_1 + \dot{p}_v \pi_2 \dot{M}_2 + ... + (\dot{p}_v - 1)\pi_v \dot{M}_v = 0, \tag{40}$$

where $\dot{M}_b$ are unknown variables we want to find, $\dot{p}_b$ are optimal limit transient probabilities determined by (30) and π_b are probabilities determined by (2).

Since the above system is homogeneous then it has nonzero solutions when the determinant of the system equations main matrix is equal to zero, i.e. if its rank is less than ν. Moreover, in this case the solutions are ambiguous. Anyway, if we fix the optimal values $\dot{M}_b$ of the mean values M_b of the unconditional sojourn times in operational states, for instance by arbitrary fixing one or a few of them, then it is also possible to look for the optimal values $\dot{M}_{bl}$ of the mean values M_{bl} of the conditional sojourn times in operational states using the following system of equations

$$\sum_{l=1}^{v} p_{bl}\dot{M}_{bl} = \dot{M}_b, \quad b = 1,2,...,v, \tag{41}$$

obtained from (2) by replacing M_b by $\dot{M}_b$ and M_{bl} by $\dot{M}_{bl}$, were p_{bl} are known probabilities of the system operation process transitions between the operation states.

5   REFERENCES

Blokus-Roszkowska, A., Guze, S., Kołowrocki K., Kwiatuszewska-Sarnecka, B., Soszy ska J. 2008. *Models of safety, reliability and availability evaluation of complex technical systems related to their operation processes*. WP 4 – Task 4.1 – English – 31.05.2008. Poland-Singapore Joint Project. MSHE Decision No. 63/N-Singapore/2007/0.

Grabski, F. 2002. *Semi-Markov Models of Systems Reliability and Operations*. Warsaw: Systems Research Institute, Polish Academy of Sciences.

Kołowrocki, K. 2004. *Reliability of Large Systems*, Elsevier.

Kołowrocki, K. 2007a. Reliability modelling of complex systems – Part 1. *International Journal of Gnedenko e-Forum "Reliability: Theory & Application"*, Vol. 2, No 3-4, 116-127.

Kołowrocki, K. 2007b. Reliability modelling of complex systems – Part 2. *International Journal of Gnedenko e-Forum "Reliability: Theory & Application"*, Vol. 2, No 3-4, 128-139.

Kołowrocki, K. & Soszy ska, J. 2008. *A general model of technical systems operation processes related to their environment and infrastructure*. WP 2 – Task 2.1 – English – 31.05.2008. Poland-Singapore Joint Project. MSHE Decision No. 63/N-Singapore/2007/0.

Kołowrocki, K., Soszy ska, J., Baranowski, Z., Golik, P. 2008. *Preliminary modeling, statistical identification and evaluation of reliability, risk, availability and safety of port, shipyard and ship technical systems in constant and variable operation conditions*. WP4 - Task 4.2. Preliminary reliability, risk and availability analysis and evaluation of a port oil transportation system in constant and variable operation conditions. WP4 - Sub-Task 4.2.1 – English – 30.11.2008. Poland-Singapore Joint Project. MSHE Decision No. 63/N-Singapore/2007/0.

Limnios, N., Oprisan, G. 2001. *Semi-Markov Processes and Reliability*. Birkhauser, Boston.

Ross, S. M. 2007. *Introduction to Probability Models*. Elsevier, San Diego, 2007.

Soszy ska, J. 2006a. Reliability of large series-parallel system in variable operation conditions. *International Journal of Automation and Computing*. Vol. 3, No 2: 199-206.

Soszy ska, J. 2006b. Reliability evaluation of a port oil transportation system in variable operation conditions. *International Journal of Pressure Vessels and Piping*. Vol. 83, Issue 4: 304-310.