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In reliability, quality control and risk analysis, fuzzy methodologies are more and more involved and inevitably introduced difficulties in seeking fuzzy functional relationship between factors. In this paper, we propose a scalar variable formation of fuzzy regression model based on the credibility measure theoretical foundation. It is expecting our scalar variable treatments on fuzzy regression models will greatly simplify the efforts to seeking fuzzy functional relationship between fuzzy factors. An M-estimator for the regression coefficients is obtained and accordingly the properties and the variance-covariance for the coefficient M-estimators are also investigated in terms of weighted least-squares arguments. Finally, we explore the asymptotic membership function for the coefficient M-estimators.

Applications of multi-state approach to the reliability evaluation of systems composed of independent components are considered. The main emphasis is on multi-state systems with degrading components because of the importance of such an approach in safety analysis, assessment and prediction, and analysing the effectiveness of operation processes of real technical systems. The results concerned with multi-state series systems are applied to the reliability evaluation and risk function determination of a homogeneous bus transportation system. Results on homogeneous multi-state “m out of n” systems are applied to durability evaluation of a steel rope. A non-homogeneous series-parallel pipeline system composed of several lines of multi-state pipe segments is estimated as well. Moreover, the reliability evaluation of the model homogeneous multi-state parallel-series electrical energy distribution system is performed.

A non-stationary approach to reliability analysis of two-state series and consecutive “m out of k: F” systems is presented. Further, the consecutive “m out of k: F”-series system is defined and the recurrent formulae for its reliability function evaluation are proposed.

Generalization of extended family of weakest-link distributions with application to the composite specimen strength analysis is presented. Composite (specifically, monolayer) specimen for tensile strength is modeled as series system but every “link” of this system is modeled as parallel system. Results of successful attempts of using some specific distribution from this family for fitting of experimental dataset of strength of some carbon fiber reinforced specimens are presented.

In this work we make a detailed analysis of the concept of risk, the stress being focused then on various kinds of statistical risks: producer and consumer risks, technical risk, Taguchi`s risk (making a connection with Cpm capability index) and a risk arising in SPC practice.

CRITERION OF THE SUPERVISION ACCURACY OF INDEXES RELIABILITY OF POWER-GENERATING UNITS A STATE DISTRICT POWER STATION.

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Abstract

The automized system is developed, allowing to determine and compare indexes of individual reliability of complex plants in view of a random in character of an initial conditions.

Despite of numerous probing, the quantitative assessment of indexes of reliability of plants EES on former draws notice of technicians. It speaks, first of all, variations in EES: a lifetime of an appreciable unit of plants EES (50÷60)% exceeds rated, that has led to essential body height of working costs. It in turn has inevitably led to variation of the strategy of maintenance. If earlier, in conditions of the regulated scheduled maintenance, indexes of reliability were used mainly for a solution of design problems, and was to evaluate enough some averaged value of indexes of reliability and reparability today the principal direction of probing of reliability of plants EES has a little varied. The strategy of realization of scheduled maintenances more and more is guided by real availability index of product of particular plants. More and more actual the possibility becomes to sample most (least) safe plants. Alongside with indexes of reliability and reparability, the assessment of indexes of longevity since these indexes characterize availability index of product of the equipment is actual. The methodology of problem solving of an assessment and comparison of indexes of reliability is developed insufficiently full, and in practice selection comes true "in the old manner", or at an intuitive level. To number of the fundamental methodical problems concern:

– how to evaluate reliability of particular plant (power-generating unit, a power line and so forth);
– how to calculate accuracy of assessments of indexes of reliability;
– as at matching to consider a random in character of assessments of indexes of reliability.
– In the present paper as plant EES power-generating units (PU) a state district power station are surveyed. Selection of plant not mated. PU a state district power station (SDPS):
– in many respects determine reliability and overall performance EES;
– concern to bunch of complex plants which are characterized by multidimensionality, diversity of types of properties of plant, lack of data on the functional intercoupling of indexes of reliability and the fundamental industrial indexes. The length of pipe ducts PU can be calculated in hundreds kilometers, hundreds units of the various equipment and systems, not speaking already about assemblies of the equipment and systems and their units;
– require an individual approach. The in-service experience displays, that each PU the state district power station has the «weak links», the average duration of working, emergency and standby estates, the periodicity and structure of a between-repairs cycle.

Problems of an assessment of indexes of reliability and reparability PU (named by us – IIR) are surveyed by indexes of individual reliability in [1], and assessments of indexes of longevity which are evaluated according to measurement of diagnostic parameters and inherently, are individual – in [2]. The assessment of accuracy IIR PU is surveyed by a method of simulation

modeling in [3]. Accounts IIR are conducted under static data about estates PU a state district power station, assembled for series of years in the form of the empirical table in which columns match to flock of indications, and strings - to flock of estates. The table allows on purpose-built algorithms and programs [4] practically instantly to evaluate as averaged (on all PU) indexes of reliability and IIR for the preset combination of varieties of indications.

The empirical table represents a final population of static data (further a population) and the separate table lines describing estates particular PU, - sampling of a data set.

The of the same type averaged indexes of reliability PU and IIR as analogue quantities, will differ to some extent. Their accuracy will differ also. The natural problem on a significance of their divergence from here implies. It is known, that:

– sampling of random quantity Y (for example, duration of estates) can be representative and unrepresentable. Unrepresentable sampling in mathematical statistics is fathomed as sampling which $F_m^*(y)$ distribution was nonrandom differs from distribution general or of a final collection $F_M^*(y)$, where m and M – number of random quantities, accordingly samples and collection; * - means an assessment. Unrepresentatively samples are caused with difference of the averaged indexes of reliability and IIR;

– application analytical methods of mathematical statistics oriented on comparison of parameters of distributions two independent representative samples, methods of marshaling of plants on their significances and methods of an automatic group of plants also assume representatively samples from collection;

– comparison of the same type IIR is normal comes true between IIR surveyed PU. The number of such comparisons for one index of reliability is peer $Nc=(n_\delta\text{-}1)!$, where n_δ – number PU. For example, if $n_\delta=8$, Nc=5040. More expediently, in our opinion to compare the of the same type averaged indexes of reliability and IIR PU. Thus Nc=8 and in hundreds times diminishes number of evaluations;

– the representative sampling can be received experimentally by means of random numbers. In actual conditions, in particular, at sampling data of empirical truth table PU, unconditional adoption of the supposition about representatively of sampling is unacceptable, that is quite physically explainable. If now to consider, that by means of the mathematical apparatus of check of statistical hypothesizes it is possible to deny only ours suppositions about representatively of sampling, but never it is impossible to proved justice, and versions of a possible divergence of compared indexes of reliability infinite flock, representatively of sampling appears improvable.

It is possible to simplify a task solution, having received correspondence of watched distribution $F_m^*(y)$ to real regularities of variation of random quantities of sampling. Thus, flock the sample of volume m is divided into three not intersected subsets (we shall designate them through Wo, W1 and W2). Sampling with specified probability is considered representative if (evaluated on experimental data) the statistician describing a divergence $F_M^*(y)$ and $F_m^*(y)$ (we shall designate it through S*), hits in subset Wo, no representatively – if S* hits in subset W_2 and if S* hits in subset W_1 it is considered, that the information has not enough for an adoption of a decision. Therefore, the problem is reduced to presence Wo, W1 and W2 for possible combinations m and M.

– each time when it is necessary to muster a divergence of distribution of random quantities, deal not with one, and with two hypotheses which it is accepted to name initial (H_0) and alternative (H_1) and, accordingly, with errors of two types. First from them - an error of first kind (we shall designate it through α), originates if hypothesis H_0 when actually she is correct is denied. The second type of errors named as an error of second kind (we shall designate it through β), originates if hypothesis H_1 when actually she is correct is denied. What hypothesis to receive for initial and what for alternative soundly is not reserved. Traditionally, for initial H_0 the hypothesis

about a random divergence of compared allocations is received or of their parameters. However, it is not always correct, and at an automatic test can lead to essential errors. It speaks difference of means of simulated implementation of statisticians S.

– in actual conditions the true is unknown. Distribution $F_M^*(y)$ also $F_m^*(y)$ have a random in character. Natural aiming simultaneously to diminish α and β it is impossible, since with decrease α value β increases and on the contrary. The problem consists in making the strategy ensuring minimum risk of an erratic solution. The justified selection of statistical criterions constitutes rather a challenge of modern mathematical statistics [6]. Practical references are reduced to selection on those or to premises are maximal acceptable values α_c (more often 0.1 or 0.05) and further – to selection of the criterion ensuring the least value β. More precise references here are absent, as selection of criterion depends on the big number of the interrelated factors.

In [5] has been offered new non parametric criterion of the supervision of coefficient representativety of sampling (CRS) which short is reduced to following sequence of accounts:

1. The collection $\{y\}_M$ is placed in ascending order and «y» a variation series of a collection the assessment of probability is compared with each value.

$$F_M^*(y) = \frac{i}{M} \quad (i=1,M). \tag{1}$$

2. Analogously p.1 sampling $\{y\}_m$ is placed in ascending order and «y» a variation series of sampling the assessment of probability is compared with each value

$$F_m^*(y) = \frac{i}{m} \quad (i=1,m).$$

3. The divergence is determined

$$\Delta_{m,i}^* = \left|F_m^*(y) - F_M^*(y)\right| = \left|\frac{i}{m} - \frac{L_i}{M}\right| \quad (i=1,m) \tag{2}$$

where a L_i - serial number y_i in variation series of a collection of random quantities $\{y\}_M$

4. The statistician is evaluated

$$\Delta_m^* = \max\left\{\Delta_{m,i_i}\right\}_m. \tag{3}$$

5. If the actual greatest divergence Δ_m^* will appear not less critical value of a statistician $\Delta_m(\alpha)$ the hypothesis about representatively of sampling should be denied.

In [5] was also a task in view of an assessment of preferability of some modifications of criterion CRS in which basis is:

- peak value of a divergence (Δ_m),
- average statistical value of a divergence (Δ_{avr}),
- average quadratic value of a divergence (C_R).

As in the subsequent we should refer repeatedly to statisticians Δ_m, Δ_{AVR}, and C_R, we shall agree to designate them, as well as earlier, through S. Particular labels will be introduced only as required.

The simulation model has been developed, allowing to receive distributions $F^*(S/H_0) = 1-\alpha$ for arbitrary set values m and M.

With reference to the present paper, practical accounts have allowed to establish:

– the discrete character of distributions $F^*(\Delta_m/H_0)$, $F^*(\Delta_{cp}/H_0)$ and $F^*(C_R/H_0)$;

– number of the discrete value of arguments of distributions $F^*(S/H_0)$, the greatest for distributions $F^*(C_R/H_0)$;

– the discrete character of distributions $F^*(S/H_0)$, eliminates a possibility of account of critical value of statisticians (S_K) at the fixed value α_c. Value α_c is in some spacing which width

the is more, than it is less m and M. Events when the lower limit of the spacing which is switching on a design value α_c, appears unfairly small are frequent, and the upper boundary value – is inadmissible greater;

– the time of the automized account of distributions $F^*(S/H_0)$ at arbitrary M, m and at hundreds iterations is calculated by seconds. In conditions when the analytical aspect of laws of distributions $F^*(S/H_0)$ is unknown, simulation modeling is the powerful instrument at statistical probing and is non-comparable to the restrained possibilities of the manual bill. The faultlessness of outcomes of account is easily controlled by a solution technique of "inverse problem".

– on particular instances it is displayed, that outcomes of application of various criteria CRS can differ is essential. Therefore, it is necessary to secrete criterion, which error of second kind the least. For this purpose, first of all, it is necessary to evaluate distribution $F^*(S/H_1)$

Simulation algorithm of distributions $F^*(S/H_1)=\beta$**.** Distributions $F^*(S/H_0)$ in conditions of a solved problem is necessary for:

– comparisons of criteria Δ_m, Δ_{AVR}, and C_R;

– assessments of value of argument S matching critical value of an error of second kind β_c, i.e. minimum from the possible discrete value S, satisfying to a condition $\beta \le \beta_c$;

– assessments of an error value of the second stem β for the greatest value of a divergence of distributions $F_M^*(X)$ and $F_m^*(X)$.

As it has noted been above if the representative sampling from a data set is simulated by means of random numbers with an even distribution in the interval [0,1] it is obvious, that the no representatively sampling can be received, if random numbers mismatch the uniform law in the interval [0,1]. For each law of distribution $F_m^*(y)$ there will be distribution $F(S/H_1)$. In turn, this parity bears that at an assessment $F^*(S/H_1)$ nonparametric criteria S are converted in parametric.

Analysis of statistical data of duration of estates PU displays, that in overwhelming majority of events minimum value of arguments of distributions $F_M^*(y)$ and $F_m^*(y)$ practically coincide. We shall designate a parity of spacing of variation of arguments $F_M^*(y)$ and $F_m^*(y)$ through δ, where $0<\delta\le 1$. This parity, unconditionally, is a particular case. However he is simple enough, obvious, easily controlled on the intermediate evaluations.

The modeling algorithm in this case is similar to algorithm of an assessment of distributions $F^*(S/H_0)$. We shall analogously receive, that distributions $F_m(y)$ and $F_n(y)$ match to the uniform law, accordingly, in spacing $[0;\delta]$ and [0,1].

Modeling algorithm for surveyed statisticians is characterized by following sequence of evaluations:

1. Under the standard program RAND(y) it is simulated n random numbers y with an even distribution in the interval [0,1], mapping value of a distribution function $y=F_n(S)$. We shall designate a block of these numbers through $\{y_{1,i}\}_n$.
2. It is analogously simulated m random numbers «y» in the interval $[0,\delta]$, mapping value of a distribution function $y=F_m(S)$. The block of these numbers will be $\{y_{2,i}\}_m$.
3. The variation series of random numbers $\{y_i\}_M$ and $\{y_{2,i}\}_m$, where M=m+n is constituted.
4. Statistical distribution functions are evaluated $F_M^*(y_j)=j/M$ and $F_m^*(y_{2,i})=i/m$ with j=1,M and i=1,m
5. By formula (2) implementation of a divergence between equal value of arguments of statistical distribution functions are evaluated $F_m^*(y)$ and $F_M^*(y)$

$$\Delta_i=\left|F_m^*(y_{2,i})-F_M^*(y_i)\right| \qquad \text{with i=1, m}$$

6. Implementation are determined

$$\Delta_m = \max\{\Delta_i\}_m, \qquad \Delta_{AVR} = \frac{1}{m}\sum_{i=1}^{m}\Delta_i \quad \text{and} \quad C_R = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\Delta_i^2}$$

7. Having iterated evaluations p.p. 1÷6 N_I time, where N_I – number of iterations, we build a variation series of implementation of each statistician and by that it is calculated assessments $F^*(S/H_1)$.

The graphical case history of sequential account of a statistician Δ_m is reduced on fig.1. For δ=0.7, m=4 and M=10. Some outcomes of accounts of distributions $F^*(\Delta_m/H_1)=\beta^*(\Delta_m)$ of some δ are reduced on fig.2.

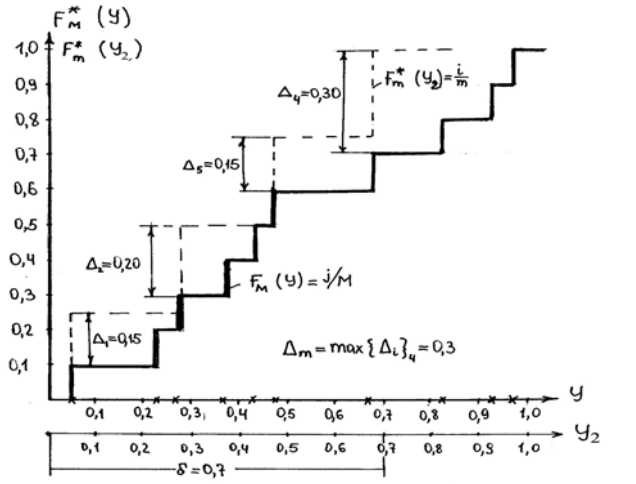

Fig 1. A graphical case history of an assessment of a statistician Δ_m at δ=0.7 m=4 and M=10

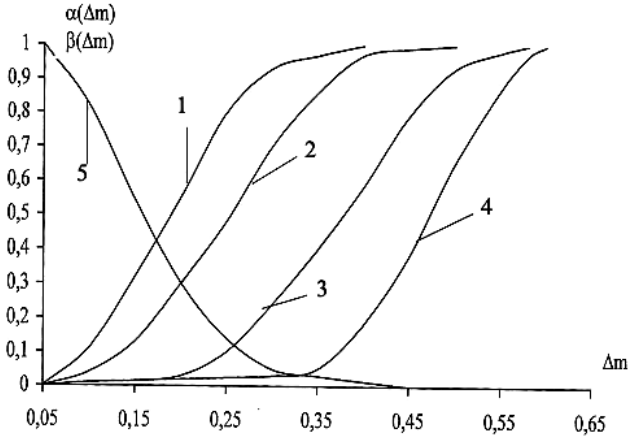

Fig.2, The graphical case history of error distributions of the first and second stem of criterion Δ_m at m=10 and n=40

1- $\delta = 0,3;$ 2- $\delta = 0,5;$ 3- $\delta = 0,7;$ 4- $\delta = 0,85;$ 5- $\delta = 0$

For matching, on fig.2 the assessment of a distribution function $\alpha^*(\Delta_m)=1-F^*(\Delta_m/H_0)$ is reduced at the same m and M, but for an event when sampling is representative $(\delta=0)$. These distributions, first of all, confirm known character of their variation (with decrease α value increases β and on the contrary).

It is established, that: with body height δ the mean of value Δ_m increases; distributions $F^*(S/H_1)$ are discrete; the discrete value of arguments of allocations $F^*(S/H_0)$ and $F^*(S/H_1)$ for conforming statisticians not always coincide; the number of the discrete value of distributions C_R and Δ_{AVR} is essential to statisticians is more, than for a statistician Δ_m. On fig.3 experimental distributions $F^*(\Delta_m/H_1)$ for of some m and n are reduced.

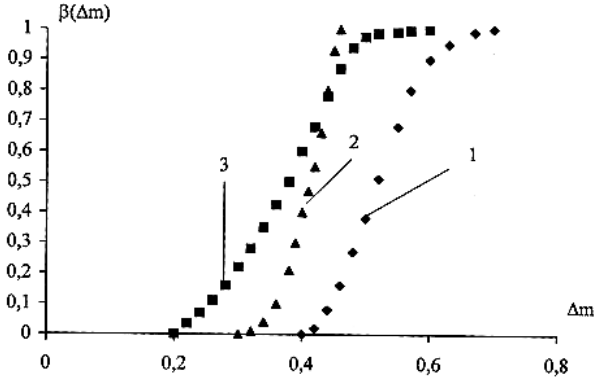

Fig.3. Distribution of the greatest spread of distributions F_m (x) and F_M (x) at $\delta = 0{,}5$
1 –m=10; n=200; 2 -m=10; n =20; 3 -m=50; n=200

As follows from fig.3, with magnifying of number of random quantities Δ_m of sampling (m) the mean $M^*(\Delta_m)$ and an average quadratic deflection $\sigma^*(\Delta_m)$ are diminished (we shall compare distributions 1 and 3 fig.3).

Distributions of random quantities a samples with equal m, taken of collections with a differing number of random quantities (for example $M_1 >> M_2$), will have various $M^*(\Delta_m)$ and $\sigma^*(\Delta_m)$. The M=m+n it is more, and m it is less, the $M^*(\Delta_m)$ it is less, and $\sigma^*(\Delta_m)$ it is more. The magnifying $\sigma^*(\Delta_m)$ at magnifying of M is caused by increasing agency of a random in character of distribution $F_n^*(y)$. Therefore, build-down of spread $F_n^*(y)$ is one of main routes of build - down $\sigma^*(\Delta_m)$. The spread $F_m^*(y)$ depends not only on number of iterations, but also from correspondence of simulated pseudorandom numbers to the uniform law in the interval [0,1]. Analogous outcomes are received and at simulation modeling $F_m(y)$ on $F_m^*(y)$

Method of comparison of criteria. As it has noted been above to compare with criteria, it is necessary to compare with dependences $\beta(S) = f[\alpha(S)]$. Such comparison is most simply realized by a method of simulation modeling by:

- constructions of distribution $F^*(S/H_0)$ and determination $\alpha^*(S) = 1 - F^*(S/H_0)$:
- constructions of distribution $F^*(S/H_1) = \beta^*(S)$:
- constructions of dependence $\beta^*(S) = f^*[\alpha(S)]$.

Some singularities of comparison of criteria of testing of hypothesis about character of a divergence of distribution functions of sampling and a final collection have been surveyed by us in [7]. In the present section we shall try to update methodology of matching of these criteria and modes of build-down of agency of pseudorandom numbers on outcomes of account.

1. Matching of criteria can be carried out by a solution technique of "inverse problem" when character of sampling of a final collection is previously known, namely: sampling is representative

or no representative . It is established, that for representative sampling, than at the fixed arbitrary value of an error of first kind (α), an error of second kind (β) it is more, that reliability of criterion is more. If sampling is no representative, at the fixed arbitrary value α, than β it is less, that reliability of criterion is more.

So that to compare with reliability of surveyed criteria of a test of hypothesis it is necessary to build characteristics $\alpha(s) = f[\beta(s)]$ for representative (R) and no representative (NR) a samples and to compare at the fixed value $\alpha(s) = \alpha_0$ with a design value $\beta(s)$. If to designate criteria as S_1 and S_2 it will be formal condition of preference S_1 above S_2 to look like:

$$\left.\begin{aligned} \beta_R(S_1) = f_1[\alpha_0] < \beta_R(S_2) = f_1[\alpha_0] \\ \beta_{NR}(S_1) = f_1[\alpha_0] < \beta_{NR}(S_2) = f_2[\alpha_0] \end{aligned}\right\} \quad (4)$$

2. Singularities of statistical modeling. One of the fundamental difficulties at simulation analysis of distributions $\alpha^*(S)$ also $\beta^*(S)$ is build-down of agency of pseudo-random values program simulated a samples with an even distribution in the interval [0,1], on outcome of account. Fluctuations of numerical values $\alpha^*(S)$ also $\beta^*(S)$ cause the certain probability of an erratic solution which, in particular, the is more, than it is less number of implementation of sampling (m) and depends on number of iterations a little. Overcoming of this difficulty has been reached as application of a known method of common random numbers [7], and new approaches, in particular:

– applications of criterion of Kolmogorov for the supervision of correspondence of random numbers program simulated a samples to the uniform law. "Classifying" a samples not only reduces fluctuations of implementation $\alpha^*(\Delta_m)$ and $\beta^*(\Delta_m)$, but also fulfills protective functions from imperfection of program implementation a samples at small m and disturbances of a computer;

– removal of agency of distribution of the random quantities adding to a samples up to a data set, i.e. $F_n^*(y)$;

On fig.4. Experimental dependences $\beta(S) = f[\alpha(S)]$ for m=10, M=30 and δ=0.5 are reduced. Analogous dependences are received and for of some other value m, M and δ.

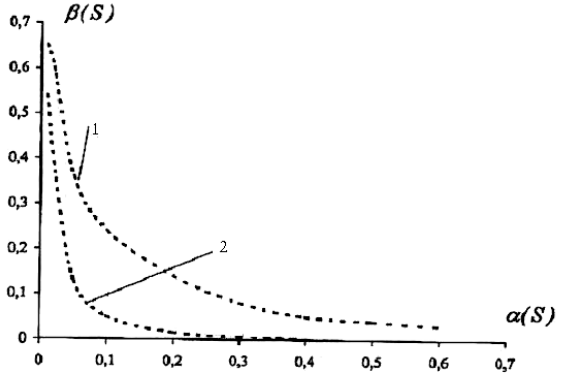

Fig.4. Curve variations of dependence $\beta(S) = f[\alpha(S)]$ at m =10; M=30 and δ=0,5 for criteria (S): 1 – C_R and Δ_{AVR}; 2- Δ_m

Analysis of these data has allowed to conclude:

- the least value β(S) at 0<α(S)<1 occurs for criterion Δ_m;
- value β(S) at 0<α(S)<1 for criteria C_R also Δ_{AVR} are practically peers and is essential above, than for criterion Δ_m. In other words, difference of allocations $F_m(\tau)$ also $F_M(\tau)$ is determined not so much by an average or average quadratic value of their deflections, how much the greatest divergence.

Risk of an erratic solution and some outcomes of accounts. Traditionally, admissible errors at adoption initial or of alternative hypotheses are set, and as a rule, is received, as $\alpha_c \le 0.1$ $\beta_c \le 0.1$. If again to convert to curves fig.2 it is easy to note, that the sum $\alpha^*(\Delta_m)$ and $\beta^*(\Delta_m)$ in process of body height Δ_m varies. First she is diminished, then magnified. Physically the sum $\alpha^*(\Delta_m)$ also $\beta^*(\Delta_m)$ matches to risk of the erratic solution, caused by a random in character Δ_m^*. Generally the risk of an erratic solution is peer

$$\gamma(\Delta_m) = A \cdot \alpha(\Delta_m) + B \cdot \beta(\Delta_m) \qquad (5)$$

where A and B – a relative significance of aftereffects of an erratic solution, where A+B=1. Then it is obvious, that to some value Δ_m there will match minimum value $\gamma(\Delta_m)$ which will be optimum. Having determined $\Delta_{m,opt}$, we receive essentially new effect: α_c also β_c are not set, and pay off proceeding from demands $\gamma(\Delta_m) = \gamma(\Delta_{m,onm})$.

Instance. One of the fundamental indexes of reliability PU is the mean of duration of recovery at emergency cutoff $M_\Sigma^*[\tau_\alpha]$. Under arranged data value $M_\Sigma^*[\tau_\alpha]$=65hrs. At number of implementation M=145 also it is determined as an average arithmetical implementation τ_a. However $M_\Sigma^*[\tau_\alpha]$ insufficiently full mirrors an aspect of cutoff PU and the aftereffects coupled to it. Failures can lead to sudden cutoff (automatically or manually) to be eliminated by cutoff PU under the emergency request (the possibility of cutoff is determined by supervisor EES). Failures can be repeated and at start-up from unloaded reserve or of emergency repair. In each of the enumerated events $M^*[\tau_a]$ and an aftereffect of cutoff PU are various. Is how much essential differ $M^*[\tau_a]$ from the averaged index $M_\Sigma^*[\tau_a]$. In the capacity of $M^*[\tau_a]$ we shall survey average duration of sudden cutoffs $M^*[\tau_{sd}]$ of each PU. The conducted accounts, as one would expect, have displayed, that the distribution function $F_M^*(\tau_a)$ is essential differs from distributions $F_{m.i}^*(\tau_{sd})$ and consequently was nonrandom differ $M_i^*[\tau_a]$ and $M_i^*[\tau_{sd}]$ with i=1,n6. Further the collection of implementation of duration of emergency repair has been surveyed at sudden cutoffs of all PU with M=55 and $M_\Sigma^*[\tau_{sd}]$=37hrs. Outcomes of comparison $M_\Sigma^*[\tau_{sd}]$ and assessments $M_i^*[\tau_{sd}]$ for each PU are reduced in table 1.

Matching of duration of emergency repairs at sudden failures of power-generating units 300 MW.

Table 2

i	m_i	$M^*[\tau_i]$	Δ_m^*	$\gamma^*(\Delta_{onm})$	$\Delta_{m,\alpha}$	$\Delta_{m,\beta}$
1	13	101	0.25	0.01		
2	8	21	0.09	0.18	0,15	0,24
3	8	6	0.27	0.0		
4	6	49	0.31	0.05		
5	4	29	0.09	0.25	0,29	0,27
6	3	7	0.37	0.0		
7	5	6	0.38	0.0		
8	8	3	0.56	0.0		

As follows from this table of an assessment $M_i^*[\tau_{sd}]$ for PU with station numbers 2 and 5, peer, accordingly 21hrs. and 29hrs. (in difference from others PU) is random differ from $M_\Sigma^*[\tau_{sd}]$=31hrs. Given tables 1 together with fig..4. Allow to trace algorithm of an adoption of a decision. As a result of account most (least) safe are secreted PU. In our event it is the eighth (first) PU, and the divergence speaks difference of "weak links» PU and tame duration their emergency repair.

Conclusions

The fundamental outcome of the conducted probing is development of the automized system, allowing to determine and compare indexes of individual reliability PU with a state district power station in view of a random in character of an initial conditions, to submissive staff conforming references and by that, to contribute in junction from the intuitive problem solving, depending from the greatest (least) reliability PU, to a quantitative justification of solutions.

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HOW PROFESSOR B.V. GNEDENKO GOT CAUGHT ON A HOOK IN KHARKOV

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For the first time, I met B.V. Gnedenko in the Fall of 1970 at the USSR conference-school on queueing theory. The conference was organized by the Moscow State University. Boris Vladimirovich Gnedenko was a head of the conference. The conference took place at the fantastic city resort Dilizhan insite the Caucasus mountains in Armenia, in the house of composers. I would have to say that I never attended a more interesting conference in my life. The music was constantly playing from the open windows of the cottages nearby; there was a fascinating harmony of mountains, mathematics and music (additionally, of football- we spent playing football all the free time).

I was inspired by everything I saw, by new ideas, by the people I met, the people who at that time were the face of Soviet school of reliability. There were B.V.Gnedenko, Ya.K. Belyaev, A.D.Soloviev, the authors of a bestseller on the topic of reliability; this book was probably learned word for word by all the attendees of the conference. There were my peers – Igor Ushakov, Vladimir Rykov, Alexander Andronov, Viktor Kashtanov, Illia Gertsbakh, Boyan Dimitrov and many more, who now are the lead professionals in the field of reliability in different countries of the world.

Boris Vladimirovich's hospitality, attention to youth, desire to help solving complicated mathematical problems has been just fascinating.

Later, there were seminars at the Moscow State University that Boris Vladimirovich conducted, the discussions of my articles and my doctoral dissertation followed. The opponents of my dissertation in 1974 were two members of Boris Vladimirovich's team – Alexander Dmitrievich Soloviev and Igor Alekseevich Ushakov.

Now about the hook. During a number of years, I was a head of seminar in Kharkov - Ukraine on the topic of reliability. This seminar took place in the regional Center for Technology, and a number of specalists from the cities of the former USSR presented there. After several years of conducting this event, I got brave enough to invite Boris Vladimirovich to this event. Luckily for me, Boris Vladimirovich happily accepted the invitation: "I am an academican of Ukrainian Academia of Science but in Kharkov, the first capital of Ukraine, was not for quite some time."

Boris Vladimirovich came to Kharkov. Besides the usual participants of the seminar, a lot of mathematicians and engineers came to listen to Boris Vladimirovich. The grand room of the Center of Technology was filled with people. Boris Vladimirovich's presentation was on a topic of how mathematics and the theory of reliability is connected to the use of technical systems and equipment that is being developed and produced in Kharkov. Boris Vladimirovich was extremely well informed about large plants based in Kharkov that produced turbines, tractors, electrical engines and electrical equipment. He even knew that there was one of the largest in USSR tank factory; and this fact at that time was considered top secret. Boris Vladimirovich had almost a unique ability to talk in simple terms about complicated matters, and his presentation that day was a direct evidence of it.

The seminar went for a long time, Boris Vladimirovich was faced with a lot of questions, and he tried to give a full explanation to each. After the seminar, we started to walk slowly to my house, where among all delicious foods, a big pike awaited us.

"I am from Volga river," said Boris Vladimirovich proudly and chose a pike's head. You can only imagine how horrified my wife and my parents were when they realized that there was a hook in pike's head, but by this time Boris Vladimirovich already had this piece of fish with hook

in his mouth. Happily enough, the hook did not get too far. “That is it, now I am hooked by you,” said Boris Vladimirovich. “This hook I am going to take with me as a memory and a proof that you caught me with a hook.”

The next day was dedicated to paintings. “In every city that I visit, I always go to the local museum, and every museum has something interesting to look at,” after Boris Vladimirovich said this, we went to the local museum, but unfortunately, the museum was closed on that day. I have no idea how Boris Vladimirovich has done it, but he managed to persuade museum employees to let us in; we were the only ones in the museum. Afterwards we went to city of Chuguev, a small city near Kharkov, where a famous Russian painter Illia Repin was born, and where a museum in his names is located. Later in the evening I took Boris Vladimirovich to the railway station.

Today, at my house I have a number of books by Boris Vladimirivoch signed by him. I look at his books and my book on the a subject of control systems reliability that was published in Moscow in 1982 with the foreword by professor Gnedenko. I think that I got very luck in my life to work with Boris Vladimirovich, and with his followers.

DIRECT CALCULATIONS OF A REACHING MOMENT DISTRIBUTION FOR AN AUTOREGRESSIVE RANDOM SEQUENCE BY RECCURENT INTEGRAL EQUALITIES

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In this paper we consider an autoregressive random sequence

$$X_k = RX_{k-1} + \eta_{k-1},\ X_0 = 0,$$

with $0 < R < 1$ and assume that η_k has exponentials mixture distribution with

$$P(\eta_k > t) = \sum_{r=1}^{l} p_r \exp(-\lambda_r t).$$

Our problem is to calculate a distribution of a reaching moment $\tau = \inf(k : X_k \ge X)$. This problem origins in the risk theory, in the financial mathematics, in the statistics of random processes and in the reliability theory. Interest to mixtures of exponentials as approximations of distributions with heavy tails is initiated by papers [1], [2]. At first look this problem may be solved by martingale technique. But in different applications when $R > 1$ or R, X depend on k or we need a distribution of a jump over X it is too complicated. In this paper we apply some recurrent integral equalities to get over these difficulties. We solve the considered problem in symbols and illustrate obtained solutions by numerical calculations.

Denote the deriving moment for this sequence reaching some boundary and designate

$$X_k^j = XR^j,\ k \ge 1,\ 0 \le j \le k-1,\ X_k^k = 0.$$

Theorem 1. *The following formula are true for* $k \ge 1$:

$$T_k(x) = P(X_k > x, \tau \ge k) = \sum_{r=1}^{l} \sum_{j=0}^{k-s} a_{k\,k-s\,j\,r} \exp\left(-\frac{\lambda_r x}{R^j}\right), \tag{1}$$

$X_k^{k-s+1} \le x \le X_k^{k-s}$, $s = 1,...,k$, *and*

$$P(\tau = k) = \sum_{r=1}^{l} a_{k\,0\,0\,r} \exp(-\lambda_r X). \tag{2}$$

Here for $1 \le r, q \le l$

$$a_{100r} = p_r, \tag{3}$$

$$a_{k+1\,k+1-s\,0\,q} = I(s \ne 1) \sum_{t=1}^{s-1} \sum_{j=0}^{k-t} \sum_{r=1}^{l} \frac{p_q \lambda_r a_{k\,k-t\,j\,r}}{\lambda_r - R^{j+1}\lambda_q} A(k,t,q,r,j) +$$

$$+ I(s \ne k+1) \sum_{j=1}^{k-s} \sum_{r=1}^{l} \frac{p_q \lambda_r a_{k\,k-s\,j\,r}}{\lambda_r - R^{j+1}\lambda_q} B(k,s,q,r,j),\ 1 \le s \le k+1, \tag{4}$$

$$a_{k+1\,k+1-s\,j\,r} = -\sum_{q=1}^{l} \frac{p_q \lambda_r a_{k\,k-s\,j-1\,r}}{\lambda_r - R^j \lambda_q},\ 0 < j \le k+1-s,\ 1 \le s \le k, \tag{5}$$

with

$$B(k,s,q,r,j) = \exp\left(-X_{k+1}^{k+2-s}\left(\frac{\lambda_r}{R^{j+1}} - \lambda_q\right)\right),$$

$$A(k,t,q,r,j) = B(k,t,q,r,j) - \exp\left(-X_{k+1}^{k+1-t}\left(\frac{\lambda_r}{R^{j+1}} - \lambda_q\right)\right).$$

Proof. As $T_1(x)=\sum_{r=1}^{l} p_r \exp(-\lambda_r x)$, $0\le x\le X$, so we have an equality $P(\tau=1)=\sum_{r=1}^{l} p_r \exp(-\lambda_r X)$ and (1)-(3) are true.

Denote $Q_k(x)=P(RX_k>x,\ \tau\ge k+1)$ and calculate

$$Q_1(x)=\sum_{r=1}^{l} p_r \exp\left(-\frac{\lambda_r x}{R}\right),\ 0\le x\le XR.$$

Then we have

$$T_2(x)=-\int_0^{\min(x,XR)}\sum_{q=1}^{l} p_q \exp\left(-\lambda_q (x-u)\right)dQ_1(u)=$$

$$=\sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r \exp\left(-\lambda_q x\right)}{\lambda_r-R\lambda_q}\left(1-\exp\left(-\min(x,XR)\left(\frac{\lambda_r}{R}-\lambda_q\right)\right)\right)=$$

$$=\begin{cases}\sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q}\left(\exp\left(-\lambda_q x\right)-\exp\left(-\frac{\lambda_r x}{R}\right)\right), & 0\le x\le XR,\\ \sum_{q=1}^{l}\sum_{r=1}^{l} p_q p_r \frac{\lambda_r \exp\left(-\lambda_q x\right)}{\lambda_r-R\lambda_q}\left(1-\exp\left(-X\left(\lambda_r-R\lambda_q\right)\right)\right), & XR\le x\le X.\end{cases}$$

So

$$T_2(x)=\sum_{r=1}^{l}\sum_{j=0}^{2-s} a_{2\,2-s\,j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right),\ X_2^{3-s}\le x\le X_2^{2-s},\ s=1,2,$$

with

$$a_{210q}=\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q},\ a_{211q}=-\sum_{q=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q},$$

$$a_{200q}=\sum_{r=1}^{l} p_q p_r \frac{\lambda_r}{\lambda_r-R\lambda_q}A(1,1,q,r,0),\ A(1,1,q,r,0)=\left(1-\exp\left(-X\left(\lambda_r-R\lambda_q\right)\right)\right).,$$

Then we have $P(\tau=2)=\sum_{r=1}^{l} a_{200r}\exp(-\lambda_r X)$. So for $k=1$ the formulas (4), (5) are true also.

Assume that the formulas (1), (2) are true for fixed k then

$$-dQ_k(u)=\sum_{r=1}^{l}\sum_{j=0}^{k-s} a_{k\ k-s\ j\ r}\frac{\lambda_r}{R^{j+1}}\exp\left(-\frac{\lambda_r u}{R^{j+1}}\right)du,\ X_{k+1}^{k+2-s}\le u\le X_{k+1}^{k+1-s},\ s=1,...,k.$$

So

$$T_{k+1}(x)=-\int_0^{\min\left(x,X_{k+1}^{1}\right)}\sum_{q=1}^{l} p_q \exp\left(-\lambda_q (x-u)\right)dQ_k(u)=$$

$$=\sum_{s=1}^{k}\int_{\min\left(x,X_{k+1}^{k+2-s}\right)}^{\min\left(x,X_{k+1}^{k+1-s}\right)}\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{a_{k\,k-s\,j\,r}p_q\lambda_r\exp\left(-\lambda_q x\right)}{R^{j+1}}\exp\left(-\frac{u\lambda_r}{R^{j+1}}+u\lambda_q\right)du=$$

$$=\sum_{s=1}^{k}\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp\left(-\lambda_q x\right)a_{k\,k-s\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}A_1(k,s,q,r,j),\ 0\le x\le X,$$

with

$$A_1(k,s,q,r,j)=\exp\left(-\min\left(x,X_{k+1}^{k+2-s}\right)\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right)-\exp\left(-\min\left(x,X_{k+1}^{k+1-s}\right)\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right).$$

From another side we search $T_{k+1}(x)$ as follows:

$$T_{k+1}(x)=\sum_{r=1}^{l}\sum_{j=1}^{k+1-s} a_{k+1\,k+1-s\ j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right),\ X_{k+1}^{k+2-s}\le x\le X_{k+1}^{k+1-s},\ s=1,...,k+1.$$

So for $s=1$ we have

$$\sum_{r=1}^{l}\sum_{j=0}^{k} a_{k+1\,k\,j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right)=\sum_{j=0}^{k-1}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r a_{k\,k-1\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}\left(\exp\left(-\lambda_q x\right)-\exp\left(-\frac{\lambda_r x}{R^{j+1}}\right)\right)$$

and obtain (4), (5). For $2\le s\le k+1$ we have

$$\sum_{r=1}^{l}\sum_{j=0}^{k+1-s} a_{k+1\,k+1-s\,j\,r}\exp\left(-\frac{\lambda_r x}{R^j}\right)=\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp\left(-\lambda_q x\right)a_{k\,k-t\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}A(k,s,q,r,j)+$$

$$+I\left(s\ne k+1\right)\sum_{j=0}^{k-s}\sum_{q=1}^{l}\sum_{r=1}^{l}\frac{p_q\lambda_r\exp\left(-\lambda_q x\right)a_{k\,k-s\,j\,r}}{\lambda_r-R^{j+1}\lambda_q}\left(B(k,s,q,r,j)-\exp\left(-x\left(\frac{\lambda_r}{R^{j+1}}-\lambda_q\right)\right)\right).$$

As a result obtain (2), (4), (5).

Denominators in (4), (5) may be small or even zero. This circumstance creates difficulties in calculations. These difficulties may be got over by a following statement.

Lemma 1. *Suppose that* $\lambda_1,...,\lambda_l$ *are different positive numbers and* $R=m/n$ *where* m,n *are integers and mutually simple. Then for any* $\varepsilon>0$ *satisfying inequalities* $\left|\lambda_i-\lambda_j\right|>2\varepsilon$, $1\le i\ne j\le l$, *there are rational numbers* $\tilde\lambda_1,...,\tilde\lambda_l$ *so that*

$$\left|\lambda_i-\tilde\lambda_i\right|<\varepsilon,\ \tilde\lambda_i\ne R^k\tilde\lambda_j,\ 1\le i\ne j\le l,\ k\ge 0. \qquad (6)$$

Proof. Denote $L=mn$ for any $\varepsilon>0$ there are integers $N,k_1,...,k_l$ so that

$$\frac{1}{NL}<\frac{\varepsilon}{2},\ \left|\lambda_i-\frac{k_i}{N}\right|<\frac{\varepsilon}{2},\ 1\le i\le l,$$

then

$$\left|\lambda_i-\tilde\lambda_i\right|<\varepsilon,\ \text{with}\ \tilde\lambda_i=\frac{k_iL+1}{NL},\ 1\le i\le l.$$

As $\left|\lambda_i-\lambda_j\right|>2\varepsilon$ so $\tilde\lambda_i\ne\tilde\lambda_j$, $1\le i\ne j\le l$. Each pair of integers (k_iL+1,L), $1\le i\le l$, has not joint divisors larger 1 so $(k_iL+1)n^k\ne\left(k_jL+1\right)m^k$, $1\le i\ne j\le l$, $k\ge 0$.

Remark 1. *Suppose that* $X=1,R=0.9,l=13$ *and* λ_i, $p_i,1\le i\le l,$ *are described by Table 1 then in an accordance with Theorem 1 we obtain Table 2.*

i	λ_i	p_i
1	4.491	0.193963
2	1.422	0.651199
3	0.371	0.147817
4	0.076	0.006832
5	0.014	1.88×10^{-4}
6	0.03	4.61×10^{-6}
7	5×10^{-4}	1.11×10^{-7}
8	8.8×10^{-5}	2.65×10^{-9}
9	1.6×10^{-5}	6.35×10^{-11}
10	2.9×10^{-6}	1.52×10^{-12}
11	5.4×10^{-7}	3.63×10^{-14}
12	9.7×10^{-8}	8.61×10^{-16}
13	1.5×10^{-8}	1.72×10^{-17}

Table 1.

k	$P(\tau=k)$
3	0.267786
6	0.214032
9	0.001387
12	0.000051
15	1.621×10^{-6}
18	4.747×10^{-8}
21	1.345×10^{-9}
24	3.755×10^{-11}
27	1.042×10^{-12}
30	2.9×10^{-14}

Table 2.

Remark 2. *The results of Theorem 1 remain valid for variable boundary*

$$\tau = \inf\left(k : X_k \geq X\left(k\right)\right)$$

with the replacement of the designations

$$X_k^j = \min\left(X_{k-1}^{j-1}R, X\left(k\right)\right),\ j = 1,...,k-1,\ X_k^k = 0,\ X_k^0 = X\left(k\right),\ k \geq 1.$$

Remark 3. *Obtained formulas are true for variable* R*:*

$$X_k = R_{k-1}X_{k-1} + \eta_{k-1},\ 0 \leq R_{k-1} \leq 1,$$

with the replacement of the designations

$$X_k^j = XR_k^j,\ R_k^j = R_{k-1}^{j-1}R_{k-1},\ 0 \leq j \leq k-1,\ R_k^0 = 1,\ X_k^k = 0.$$

Remark 4. *In an accordance with (2) we have that a jump of* X_k*,* $k \geq 1$*, over a level* X *may be characterized by the following formula:*

$$P(X_k > X + y / \tau = k) = \frac{\sum_{r=1}^{l} a_{k00r} \exp(-\lambda_r (X + y))}{\sum_{r=1}^{l} a_{k00r} \exp(-\lambda_r X)}, y > 0\ .$$

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RELIABILITY ASSESSMENT DUE TO WEAR

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Evaluation of structural reliability under processes of deterioration presents very important problem in design. The structure's wear shows a reduction of bearing capacity in time that for one's turn leads to increasing the probability of failure. The reasons for long duration and irreversible change of structural features can be corrosion in steel structures, decomposition in wood structures, ageing in polymer structures, and processes of abrasion or erosion also. The problem of defects accumulation should be mentioned too, when reduction of the bearing capacity connects with load's value and its duration.

The models and peculiarities of corrosion wear and its influence on bearing capacity are discussed in this paper.

1. MODELS OF CORROSION WEAR

Corrosion is an important factor in reducing of reliability and durability due to different kinds of structures or equipments. From 10% to 12% of fabricated and used steel is lost annually due to destructive effects of corrosion. In spite of widely used protection methods, the quantity of steel destroyed is growing almost proportionally to the accumulated stores of steel. Losses from corrosion average are between 2% to 4% of GDP in almost every country. About 30% of structural steel is subjected to atmospheric corrosion, and 75% is subjected to atmospheric and aggressive corrosion simultaneously [1]. Under corrosion's influence the initial cross-section of a structural element is decreased, and consequently so its bearing capacity. Fig.1 presents the types of corrosion for structural steel.

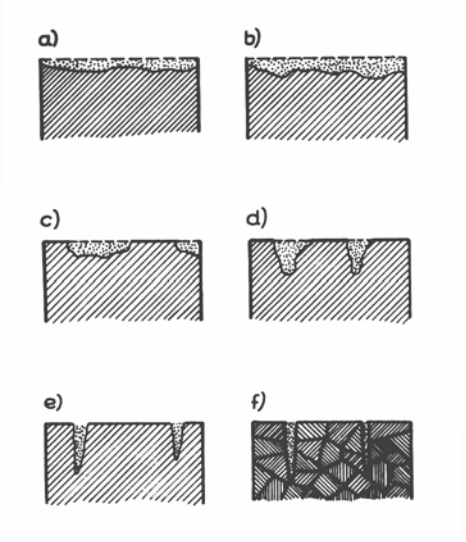

Fig.1 Types of corrosion of a structural steel.
a) Uniformly distributed wear. b) Irregular distributed wear.
c) Corrosion with spots. d) Corrosion with ulcers.
e) Corrosion with points. f) Corrosion with cracks.

The speed of a corrosion process depends upon degree of aggressive environment and is changing with 0.05mm/year to 1.6mm/year. The damage of structural steel in soil depends on the duration of an exposure, as shown in Fig.2. Data are based on 16 types of soil. Similarly, the damage of the steel from atmospheric corrosion is shown in Fig.3. Distribution of corrosion speed (measured at the inner reservoir surface along its height) for different products is presented in Fig.4.

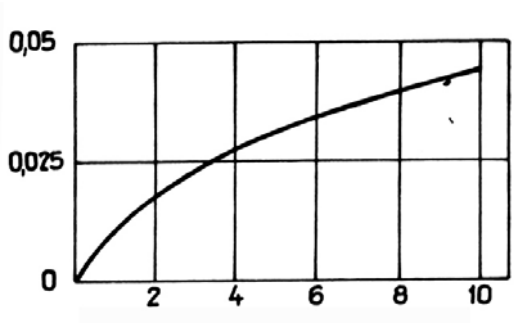

Fig.2 Corrosion of structural steel in soil over years.
Y-*axis shows the mean depth of corrosion in mm.;* X-axis shows the years of duration.

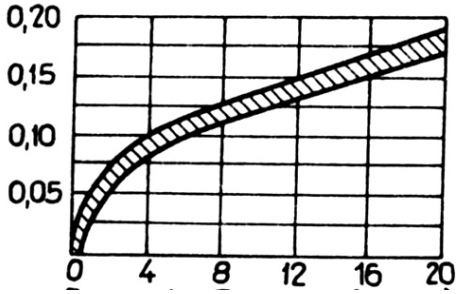

Fig. 3 Corrosion of structural steel in open air.
Y-axis shows the average depth of corrosion (mm).
X-axis shows the years of duration.

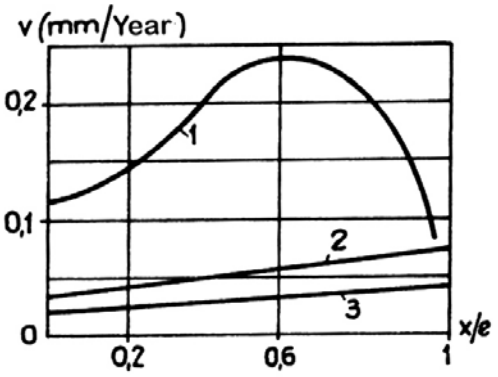

Fig. 4 Variation of corrosion's speed
1. Gasoline. 2. Kerosene. 3. Diesel.

The evaluation of structural durability depends essentially on the choice of the model that is capable to reflect the influence of an aggressive environment. When modeling corrosion processes, there are important damage characteristics to consider, such as depth of defect (δ) and corrosion speed (v=dδ/dt). Classification of mathematical models of corrosion (based on empirical approach)

presents in Table 1 [1,2,3]. The kinetics of the corrosion process in different metals for different aggressive environments looks very similar, and this fact presents the opportunity to use these models in design.

In general, processes of wear can be presented as time-dependent random functions of time. Type of processes depends on maintenance conditions, methods of structure's fabrications, steel's composition and others.

Table 1

#	Models of corrosion	Functional relationship
1	$\delta = v_0 t$	Linear
2	$v_t = kt^{-n}$	Power
3	$\delta = a + b\lg t$	Logarithmic
4	$\delta = \ln(kt)$	Logarithmic
5	$v_t = v_0 \exp(-\alpha t)$	Exponential
6	$v_t = mt^2 \exp(-t/\tau)$	Exponential
7	$\delta = \delta_0[1-\exp(-t/\tau)]$	Exponential
8	$\delta = \frac{a}{1+b\exp(-ct)}$	Exponential
9	$v_t = \frac{t}{at^2+bt+c}$	Fractionally linear
10	$\delta = \frac{\delta_0 t}{1+at}$	Fractionally linear

Models of long-term processes presents as random time processes, but its uncertainty defines, due to random, independent from time parameters. Such kind of random processes were called "deterministic random processes "[4].

In the case that all loads F_i presents independent random values, probability of no failure during working life can be expressed as:

$$\text{P(n)=P}\,[R_1 > F_1, R_2 > F_2, ..., R_n > F_n], \quad (1)$$

where $R_1, R_2, ..., R_n$ - values of bearing capacity in considered time intervals. If designate $R_n = R_0\varphi(n)$, then n=t –term of maintenance in years; R_0-initial (random) value of bearing capacity; $\varphi(n)$ - monotonically decreasing nonnegative function (i=1,2,3,n.), satisfying to the conditions: $\varphi(0) = 1;\ \varphi(\infty) = 0; d\varphi/dt < 0$. It should be mentioned also the additive property of φ (t) function, independence of wear's process in the subsequent time interval t_i from previous process's value in time t_{i-1}, i.e. $\varphi(t_1)\varphi(t_2) = \varphi(t_1 + t_2)$.

$F_1, F_2, ..., F_n$ - Loads, corresponding to considered time intervals.

2. UNIFORMLY DISTRIBUTED CORROSION WEAR

This problem is illustrating in considering a steel pipeline's section (cylindrical tube), subjected to inner pressure, changes of the temperature and corrosion. The inner pressure F and steel yield stress R_y are random values with given distributions. The corrosion process considers deterministic. The limit state condition is taken in the form: $S_i \le Ry$. Here S_i- intensiveness of stresses in considered cylindrical shell. In accordance with Guber-Mises condition [5], general case looks as:

$$S_i = \frac{1}{\sqrt{2}}\sqrt{(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_1 - S_3)^2} \tag{2}$$

In discussed situation S_2=0, and the radial and the tangential stresses reads: $S_1 = \frac{FD_i}{2h}, S_3 = \frac{FD_i}{4h} - \alpha E \Delta\theta.$

Here F is the inner pressure, and its maximum value is random for some time intervals; D_i- inner diameter of the pipe; α-parameter of linear extension; E-modulus of elasticity; $\Delta\theta$- temperature drop (difference between temperature of the pipeline during use and assembly).

The reliability condition expresses as follows:

$$\frac{3F^2 D_i^2}{16h^2} + \alpha^2 E^2 \Delta\theta^2 \le R_y^2 \tag{3}$$

As temperature's drop presents an uncertain value with unknown distribution, then temperature's stresses are given as some part of the yield stress.

$$\alpha E \Delta\theta = R_y \sin\chi \tag{4}$$

χ is a value of angle in the given interval [0,π/2]. The condition (3) presents now in the form:

$$F \le \frac{4h}{\sqrt{3}D_i} R_y \cos\chi \tag{5}$$

Corrosion wear causes a reduction of tube thickness as $h = h_0\varphi(t)$, where h_0 is the initial thickness. In accordance with the Table 1 one can takes:

$$\varphi(t) = \exp(-t/\tau) \tag{6}$$

From (6) comes:

$$h = h_0 - \delta[1 - \exp(-t/\tau)] \tag{7}$$

where δ is the depth of corrosion bubble. It is assumed that the corrosion process in interval t_2 is independent of the preceding values in interval t_1, so that $\varphi(0, t+t_1) = \varphi(o,t) + \varphi(t, t+t_1)$. It is assumed also that time t takes only discrete values: t=n, where n is number of years or months. An assumption is made for pressure F supposing that statistic data belong to some period of time, a month, for example. From all observations, maximum values selects only. If the time interval is large in comparison with correlation zone, then Fisher-Tippet distribution (second type) of maximum values can be used [6].

$$\mathrm{P}(x) = \exp[-(x/\xi)^{-\eta}] \tag{8}$$

If $v_F = s_F / \overline{F}, \overline{F}$ are correspondingly the coefficient of variation and the mean value, then parameters ξ and η are determinated from the solution of two equations, which includes gamma functions.

$$\begin{aligned} 1 + v_F{}^2 &= \Gamma(a)\Gamma(b) \\ \xi &= \overline{F} / \Gamma(a) \end{aligned} \tag{9}$$

Gamma functions are:

$$\Gamma(a) = \int_0^\infty e^{-z} z^{a-1} dz,$$
$$\Gamma(b) = \int_0^\infty e^{-z} z^{b-1} dz. \quad (10)$$

The case when b=0 and η=2 is excluded.

For yield stress Weibull distribution is applied.

$$P(x) = 1 - \exp[-(x/\omega)^{\mu}] \quad (11)$$

Form's parameter μ is expressed through coefficient of variation $v_R = s_R / \overline{R}$:

$$v_R = \frac{\sqrt{\Gamma(1+2\mu) - [\Gamma(1+1/\mu)]^2}}{\Gamma(1+1/\mu)} \quad (12)$$

Values v_R and μ define scale parameter ω.

Taking into account (8) and (11) the reliability function is written in the form:

$$P(n) = -\int_0^\infty \exp[-(\frac{4h_0 x \cos\chi}{\sqrt{3} D_i \xi})^{-\eta}] \sum_{i=0}^{n-1} \varphi^{-\eta}(t) d[\exp\{-(x/\omega)^{\mu}\}] \quad (13)$$

Example. After statistic data processing of pressure in pipelines and yield stress the following values of the distribution parameters were defined: ξ = 73.5; η = 65; ω = 42.5; μ = 23.5.Coefficients of variations are: $v_F = 0.0201; v_R = 0.0522$. Temperature stresses (9.4) show essential influence on pipeline's reliability. When χ = π / 3, P (n) is close to zero. P (n) values for different n are presented in the Table (2).

Table 2

τ	χ	Values of function P (n)						
		Time in years						
		1	5	10	15	20	25	30
100	0	0.9989	0.9989	0.9989	0.9989	0.9987	0.9962	0.9860
100	6	0.9989	0.0087	0.9968	0.9880	0.9590	0.8600	0.6000
100	4	0.9560	0.8500	0.5800	0.1800	-	-	-
120	0	0.9989	0.9989	0.9989	0.9989	0.9989	0.9975	0.9872
120	6	0.9989	0.9941.	0.9941	0.9750	0.9600	0.8990	0.8060
120	4	0.9560	0.8790	0.6870	0.3790	-	-	-
150	0	0.9989	0.9989	0.9989	0.9989	0.9988	0.9985	0.9900
150	6	0.9989	0.9988	0.9980	0.9900	0.9760	0.9570	0.3200
150	4	0.9989	0.8820	0.7500	0.5200	0.3800	-	-

From (13) the member responsible for corrosion process's influence is picked out:

$$\lambda = \left[\sum_{i=0}^{n-1} \varphi^{-\eta}(i)/n\right]^{\frac{\mu}{\eta+\mu}} \quad (14)$$

where λ characterizes decreasing of reliability in regard of corrosion's development.

Parameter τ in (6) and in Table 2 defines intensiveness of uniform corrosion. Physical sense of this value consists in decreasing of initial tube's thickness. This essential decreasing is possible under large values of τ = 100…150.

Results of many experiments and real observations demonstrated [1,.3] the influence of stresses in structures to the speed of corrosion. Especially large is this influence in places of concentrations of stresses. Dependence between corrosion's speed and increasing level of stresses

can be as linear as nonlinear. If to take dependence between the intensiveness of stresses and the depth of the corrosion's penetration such as $\delta = \alpha t^{\beta} \exp(kS_i)$, and substituting it in the formula for the circular stresses in cylindrical shell $S_1 = \frac{FD_i}{2h}$, then the condition of the failure reads:

$$\frac{FD_i}{2\left[h_o - \alpha t^{\beta} \exp(kS_i)\right]} > R_y \tag{15}$$

After decomposition into the row $\exp(kS_i) \cong 1 + kS_i$, expression (15) performs to:

$$\mathrm{F} < 2\left[h_o - \alpha t^{\beta}\left(1 + \sqrt{3}kRy/2\right)\right]/D_i \tag{16}$$

Here the expression in brackets takes into account influence of stress state at speed of corrosion. If to take the same distribution for inner pressure (8) and for yield stress (11), and to consider process of corrosion as a function of discrete argument then the expression for reliability function can be written in the form:

$$\mathrm{P}(\mathrm{n}) = -\int_0^{\infty} \prod_{i=1}^{n} \exp\left\langle -\frac{2\left[x\left(h_o - \alpha i^{\beta}\left(1 + \sqrt{3}xk/2\right)\right)\right]}{D_i \xi} \right\rangle^{-\eta} d \exp\left[-\left(\frac{x}{\omega}\right)\right]^{\mu} \tag{17}$$

Expression (17) allows to evaluate the reliability of pipelines, subjected to continuous corrosion and to take into account influence of stress state to the corrosion's depth penetration or corrosion's speed.

3. IRREGULAR DISTRIBUTED CORROSION WEAR

A problem of structural durability and the protection from a local corrosion turns out to be very important as well. Local corrosion leads to some local destruction seen on the surface of the structure in the form of spots, ulcers, points or cracks (Fig.1). Appearance of this destruction in time is random too.

Corrosion cavities' ensemble is based on the following assumptions:

- Events, which have to do with the appearance of various numbers of cavities at disjoint time intervals are independent.
- Probability of corrosion's cavity appearance in the arbitrary time interval t is proportional to the length of this interval with the factor of proportionality equal to μ.
- Probability of the two or more events appearance through an extremely small time interval presents an infinitely small value of more high order.

The simultaneous realization of all these assumptions should be present and have an existence of the primary flow of events – a uniform Poisson process. Such process can be described by the system of differential equations:

$$\begin{aligned} \frac{dP_0}{dt} &= \mu P_0 \\ &\ldots\ldots\ldots\ldots \\ \frac{dP_n}{dt} &= \mu(P_{n-1} - P_n) \end{aligned} \tag{18}$$

Initial conditions for this system of equations are:

$$\begin{aligned} &\mathrm{P_n}(\mathrm{t}) = 1, \text{ when } \mathrm{n}=0 \\ &\mathrm{P_n}(\mathrm{t}) = 0, \text{ when } \mathrm{n}=1,2, \end{aligned} \tag{19}$$

There will be only one solution for the system (9.18) and together with the conditions (9.19) it can be presented as the Poisson distribution:

$$P_n(t) = \frac{[\mu(t-t_0)^n}{n!\exp[-\mu(t-t_0)]} \tag{20}$$

From (20) probability of the fact follows that in the moment $t \geq t_0$ the system is in the state n (n = 1,2,3,). If the number of cavities appearing in some time interval submits to Poisson distribution, then the amount time before appearance of the next cavity possesses exponential distribution [7].

$$P(t) = \exp(-\mu t). \tag{21}$$

The number of experimental data that connects with investigations of kinetic due to cavity growth or an increase of cavities number is very small. Experimental dependences were received in [8]:

$$\mu = \mu_{gr}(1 - e^{-\beta t}) \tag{22}$$

Here μ_{gr}, β are empiric coefficients. Value μ_{gr} varies in wide limits and measures as number of defects to unit of structural surface.

Important parameters for considered type of irregular corrosion are - maximum depth of a cavity, its diameter and square of a cavity.

The random value of a cavities depth δ_k (k-random point on structural surface) is distributing in the final interval $[0,h_0]$, where h_0 is the thickness of structural element. It is considered that this value had uniform distribution, i.e.

$$P_\delta(x) = \begin{cases} 0 & x < 0 \\ x / h_0 & 0 \leq x \leq h_0 \\ 1 & x > h_0 \end{cases} \tag{23}$$

Distribution of the maximum depth for n cavities, i.e. $\delta_n = \max \{x_1, x_2, x_3,\ldots,x_n\}$ is well known from theory of extreme values [9] and can be taken as exponential.

$$\begin{aligned} P_{\delta n} &= \exp\ [-n\,(h_0 - x)] \qquad 0 \leq x \leq h_0 \\ P_{\delta n} &= 1 \qquad\qquad\qquad\qquad x > h_0 \end{aligned} \tag{24}$$

The next important parameter is the diameter of considered cavity, due to an assumption that this cavity has cylindrical form (Fig.5). Let the depth of the cavity is equal to x. Then the possible region for variation of diameter is the chord AB with the length $2\sqrt{2rx - x^2}$, and r is the external radius. An assumption is taken that the random value of diameter y_i has uniform distribution in the interval $[0,\ 2\sqrt{2rx - x^2}\ ]$.

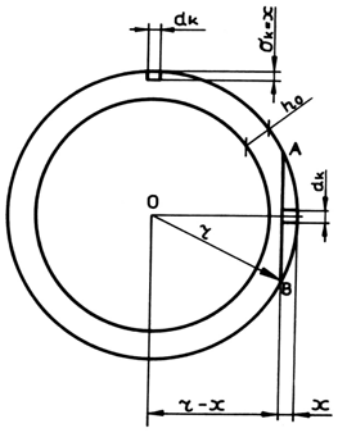

Fig.5. Element of ring's cross-section

$P_d(y) = 0$ if $y<0$

$$P_d(y) = \frac{y}{2\sqrt{2rx - x^2}} \qquad \text{if } 0 < y < 2\sqrt{2rx - x^2} \tag{25}$$

$$P_d(y) = 1 \qquad \text{if } y > 2\sqrt{2rx - x^2}$$

Distribution of the maximum diameter for n cavities $d_n = \max(y_1, y_2, y_3, \ldots y_n)$ is:

$$P_{dn} = \exp[-n(2\sqrt{2rx - x^2}, \qquad 0 \le y \le 2\sqrt{2rx - x^2}$$

$$P_{dn} = 1, \qquad y > 2\sqrt{2rx - x^2} \tag{26}$$

Third parameter of this cavity is its square A_k. The knowledge of the maximum square value is important in solution of the considered problem. There are some difficulties, however, unclear even in the theory of order statistics. The point is that the maximum δ_n value doesn't always correspond to the maximum value of d_n. If to agree with this position then the solution will be received in safety margin. Two kinds of versions can be offered, distribution of maximum depth's value δ_n and distribution of diameter's value d_k for k-cavity in the first case, and otherwise: distribution of maximum diameter's value d_n, and distribution of depth's value δ_k in the second case.

Types of $P_A(x)$ distributions are written for three cases:

Case 1:

$$P_{\delta n}(x) = \exp[-n(h_0 - x)], x \in [0, h_0],$$

$$P_{dn}(y) = \frac{y}{2\sqrt{2rx - x^2}}, 0 \le y \le 2\sqrt{2rx - x^2}. \tag{27}$$

The square of the cavity A_k is equal to the square of the segment at Fig.5:

$$A_k = r^2 \arcsin\frac{y}{2r} - \frac{y}{2}\sqrt{r^2 - \frac{y^2}{4}} + [x - r + \sqrt{r^2 - \frac{y^2}{4}}]y \tag{28}$$

The maximum possible value of the cavity square A_k will be when $x = h_0$ and $y = 2\sqrt{2rh_0 - h_0^2}$.

In pipes of large diameter x / r, y / 2r values are highly small numbers and possible reasonable approximation will be $A_k = xy$, and it follows:

$$P_{Ak}(A) = \frac{An}{2\sqrt{2r}} \int_0^{h_0} \exp[-n(h_0 - x)] x^{-\frac{3}{2}} dx \tag{29}$$

A_k is here uniformly distributed at interval [0,A*] random value.

Case2:

$$P_{\delta n}(x) = \frac{x}{h_0}, x \in [0, h_0]$$

$$P_{dn}(y) = \exp\left[-n\left(2\sqrt{2rx - x^2} - y\right)\right], y \in \left[0, 2\sqrt{2rx - x^2}\right] \tag{30}$$

Distribution of A_k is:

$$P_{Ak}(A) = \frac{1}{h_0} \int_0^{h_0} \exp\left[-n\left(2\sqrt{2rx - x^2} - \frac{A}{x}\right)\right] dx \tag{31}$$

Case 3:

$$P_{\delta n}(x) = \exp[-n(h_0 - x)], x \in [0, h_0]$$

$$P_{dn}(y) = \exp\left[-n\left(2\sqrt{2rx - x^2} - y\right)\right], y \in \left[0.2\sqrt{2rx - x^2}\right] \tag{32}$$

It follows:

$$P_{Ak}(A) = \int_0^{h_0} \exp\left(-2\sqrt{2rx - x^2} - \frac{A}{x}\right) d\left[\exp\left(-n\left(h_0 - x\right)\right)\right] \quad (33)$$

The last case, as it was written before, leads to safety margin.

Example 1. Reliability of pipeline subjected to one-sided irregular corrosion.

Dimensions of the resulting cavity-depth and diameter are increasing in time in such degree that the failure of pipe will occur i.e. formation of a reach-through hole will take place. Time, t_n before this hole will appear calculates from the expression:

$$\int_0^{t_n} v(t)dt = h_0 - \delta_n \quad (34)$$

Here δ_n – maximum depth from an ensemble of n cavities; $v(t) = v_0\exp(-\alpha t)$ – corrosion's speed (Table 1,5). From (34) we get:

$$t_n = \frac{1}{\alpha}\ln\frac{v_0}{h_0 - \delta_n} \quad (35)$$

Time distribution $P(t_n < t)$ to reach –through hole can be written as:

$$P_n(t) = P\left\{\delta_n \geq \left[h_0 - \frac{v_0}{\alpha}\left(1 - \exp\left(-\alpha t\right)\right)\right]\right\} = 1 - \exp\left[-n\frac{v_0}{\alpha}\left(1 - \exp\left(-\alpha t\right)\right)\right] \quad (36)$$

After averaging on "n" it follows:

$$P(t) = \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} \exp\left\{1 - \exp\left[1 - \exp\left(-\alpha t\right)\right]\right\} \quad (37)$$

Example 2. Design of structural members under central tension.

Cylindrical element having a ring cross-section is considered. This element is subjected to irregular corrosion under deterministic load F. If A_0 is initial value of cross-section (t = 0), A_k is square of cavity with given distribution $PA_k(A,)$ then the condition of no failure will be:

$$F / (A_o - A_k) < R_y \quad \text{or} \quad A_k < A_0 - F / R_y \quad (38)$$

Substituting the last expression into distribution function as an argument and carrying out an average on n and R_y probability of no failure in t moment is:

$$P(t) = \exp(-\mu t)\sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} \int_0^{\infty} P_{Ak}\left(A_0 - \frac{F}{R_y}\right) p\left(R_y\right) dR_y \quad (39)$$

Here $p(R_y)$ – is density of yield stress distribution. In numerical example the following data are taken. External diameter D = 6.26in; initial thickness h_0 = 0.24in; F = 127929ft; $\mu = \mu_{gr}[1 - \exp(-\beta t)]$ and β = 0.05; $\overline{R}_y$ = 290Mpa; s_{Ry} = 25Mpa. Parameters of the cavity are $\overline{d}_k = \overline{\delta}_k = 0.008$in. Results of numerical realization are shown at Fig.6

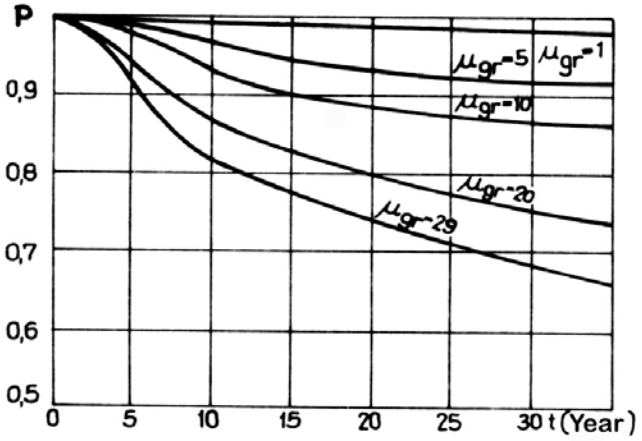

Fig.6 The reliability function

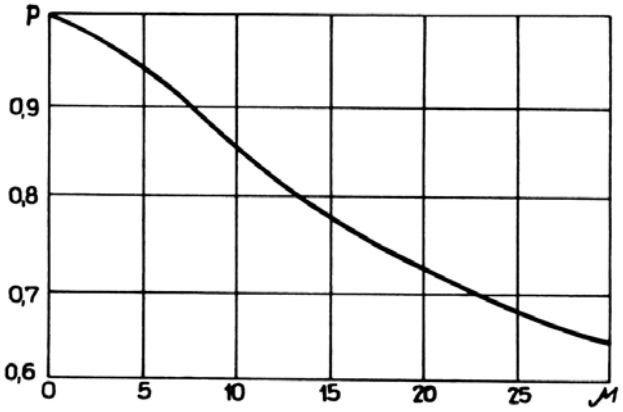

Fig.7 Variation of the function of reliability
due to number of cavities

4 CALIBRATION OF MODEL PARTIAL FACTOR

Partial factor for model uncertainties can be determined from comparison with identical structures operating in normal or in aggressive environment. Let us consider the structure under load F and with resistance equal to R. In case when random value of the load maximum for the definite period of time (one year, for example) has distribution $P_F(x)$ and year's load maximums are independent random values, the reliability function can be written as follows

$$P_1(n) = \int_0^\infty P_F^n(x) dP_R(x) \tag{40}$$

It is assumed that there is a structure operating in aggressive environment in the terms of uniformly corrosion. To guarantee the sufficient reliability level in the design it is necessary to go on additional expense of structural material such as increasing cross-section square, for example. The condition of no failure is

$$\widetilde{F} \le \gamma_D \widetilde{R} \tag{41}$$

Though the corrosion process is continuous in time it is proposed to consider the function $\varphi(t)$ which influences to geometric characteristics of cross-section as a function of discrete argument $\varphi(n)$. (41) to nth year could be rewritten as

$$F \le \gamma_D R\varphi(n) \tag{42}$$

Reliability function will be

$$P_2(n) = \int_0^{\infty} \prod_{i=1}^{n} P_F\left[\gamma_D x\varphi(i)\right] dP_R(x) \tag{43}$$

The equation for definition of γ_D arrives from the equality (40) to (43).

$$\int_0^{\infty} \prod_{i=1}^{n} P_F\left[\gamma_D x\varphi(i)\right] dP_R(x) = \int_0^{\infty} P_F^n(x) dP_R(x) \tag{44}$$

If we consider tensioned non-corrosive structural element then (41) can be presented as

$$F \le RA_0 \tag{45}$$

Where A_0 -initial cross-section square.

Function of reliability will be

$$P_1(n) = \int_0^{\infty} P_F(xA_0) dP_R(x) \tag{46}$$

For corroding structural element cross-section square is - $A_0\gamma_D$, and γ_D>1. The failure condition can be expressed as

$$F \le \gamma_D A_0 \varphi(n) R \tag{47}$$

Reliability function (43) will be

$$P_2(n) = \int_0^{\infty} \prod_{i=1}^{n} P_F\left[\gamma_D A_0 x\varphi(i)\right] dP_R(x) \tag{48}$$

Equality (44) for the fast n value allows to determinate γ_D. Fisher-Tippet distribution (8) was chosen for $P_F(x)$. Equality (44) will be performed to

$$\int_0^{\infty} \exp\left[-n\left(\frac{xA_0}{\xi}\right)^{-\eta}\right] dP_R(x) = \int_0^{\infty} \prod_{i=1}^{n} \exp\left[-\left(\frac{\gamma_D A_0 x\varphi(i)}{\xi}\right)^{-\eta}\right] dP_R(x) \tag{49}$$

From here, it follows

$$\gamma_D = \frac{1}{[n\sum_{i=1}^{n} \varphi^{-\eta}(i)]^{\frac{1}{\eta}}} \tag{50}$$

Introducing corrosion model in the form of (6) and presenting the sum in (50) in the row we will get

$$\gamma_D = \frac{1}{\left\{n\left[\exp\left(\frac{\eta}{\tau}\right) + \exp\left(\frac{2\eta}{\tau}\right) + \ldots + \exp\left(\frac{n\eta}{\tau}\right)\right]\right\}^{\frac{1}{\eta}}} \tag{51}$$

After transformation we get

$$\gamma_D = \frac{\exp\left(\frac{n+1}{\tau}\right)-1}{n\left[\exp\left(\frac{\eta}{\tau}\right)-1\right]} \tag{52}$$

Table 3 contains modal factor's values in accordance with (52)

Table 3

n	η	γ_D $\tau = 100$	γ_D $\tau = 150$	γ_D $\tau = 200$
10	10	1.0666	1.0461	1.0364
	20	1.1304	1.0828	1.0612
	30	1.2098	1.1280	1.0920
15	10	1.0657	1.0439	1.0337
	20	1.1374	1.0851	1.0618
	30	1.2263	1.1354	1.0958
20	10	1.0665	1.0433	1.0327
	20	1.1443	1.0881	1.0632
	30	1.2401	1.1424	1.0999

Aggressiveness of environment can be classified depending on parameter τ: heavily aggressive $\tau = 100$, middle aggressive $\tau = 150$, weakly aggressive

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ACCURACY SOLUTION OF A.A. NOVIKOV PROBLEM

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Introduction

In this paper we consider the Laplas model described by the following autoregressive random sequence

$$X_k = RX_{k-1} + \eta_k,\ X_0 = 0,\ \tau = \inf(k : X_k \ge X), \tag{1}$$

$$P(\eta_k > t) = \frac{\exp(-\lambda t)}{2},\ t > 0,\ P(\eta_k \le t) = \frac{\exp(\lambda t)}{2},\ t \le 0,.$$

Our problem is to calculate a distribution of a reaching moment τ. This problem was put before the authors of this paper by A.A. Novikov. The problem origins in the risk theory and in the reliability theory. M. Jacobson found approximate solution of this problem by martingale technique. V.V. Mazalov suggested to solve the problem for $R < 1$ approximately by some recurrent procedure which includes a compressing operator. In this paper we apply some recurrent integral equalities to find accuracy solution represented by mixtures of exponentials. This solution is illustrated by numerical calculations. Our solution may be used for an arbitrary R and when as R so X depend on k.

1. Main results

Denote $X_k^k = 0$, $X_k^{k-s} = XR^{k-s}$, $s = 1,...,k,$ and designate for $k \ge 1$

$$T_k(x) = P(X_k > x, \tau \ge k),\ x \ge 0,\ S_k(x) = P(X_k \le x, \tau \ge k),\ x < 0,$$
$$P(\tau = k) = T_k(X).$$

Denote

$$B_1(k,s,j) = \exp\left(-\lambda X_{k+1}^{k+2-s}\left(\frac{1}{R^{j+1}} - 1\right)\right),\ B_2(k,s,j) = \exp\left(\lambda X_{k+1}^{k+2-s}\left(\frac{1}{R^{j+1}} + 1\right)\right),$$

$$B_3(k,s,j) = \exp\left(-\lambda X_{k+1}^{k+1-s}\left(\frac{1}{R^{j+1}} + 1\right)\right),\ B_4(k,s,j) = \exp\left(\lambda X_{k+1}^{k+1-s}\left(\frac{1}{R^{j+1}} - 1\right)\right),$$

$$A_1(k,s,j) = B_1(k,s,j) - B_4^{-1}(k,s,j),\ \ A_2(k,s,j) = B_2(k,s,j) - B_3^{-1}(k,s,j),$$

$$A_3(k,s,j) = B_2^{-1}(k,s,j) - B_3(k,s,j),\ \ A_4(k,s,j) = B_1^{-1}(k,s,j) - B_4(k,s,j).$$

Theorem 1. *The following formulas are true for* $k \ge 0$, *and for* $x \ge 0$

$$T_k(x) = \sum_{j=0}^{k-s} a_{k\,k-s\,j} \exp\left(-\frac{\lambda x}{R^j}\right) + \sum_{j=0}^{k-s} b_{k\,k-s\,j} \exp\left(\frac{\lambda x}{R^j}\right) + c_{k\,k-s} \tag{2}$$

with $X_k^{k-s+1} \le x \le X_k^{k-s}$ *for some* $s \in \{1,...,k\}$ *and for* $x < 0$

$$S_k(x) = \sum_{j=0}^{k-1} d_{k\,j} \exp\left(\frac{\lambda x}{R^j}\right) \tag{3}$$

and

$$P(\tau = k) = a_{k\,0\,0} \exp(-\lambda X). \tag{4}$$

Here

$$a_{1\,0\,0}=\frac{1}{2},\ b_{1\,0\,0}=0,\ c_{1\,0}=0,\ d_{1\,0}=\frac{1}{2},\ c_{1\,-1}=0 \tag{5}$$

and

$$a_{k+1\,k+1-s\,j}=-a_{k\,k-s\,j-1}\frac{R^{2j}}{1-R^{2j}},\ 0<j\le k-s+1,\ 1\le s\le k, \tag{6}$$

$$b_{k+1\,k+1-s\,j}=-b_{k\,k-s\,j-1}\frac{R^{2j}}{1-R^{2j}},\ 0<j\le k-s+1,\ 1\le s\le k, \tag{7}$$

$$a_{k+1\,k+1-s\,0}=\frac{1}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\,k-t\,j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\,k-t\,j}}{1+R^{j+1}}\right]+\right.$$
$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\,k-s\,j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\,k-s\,j}}{1+R^{j+1}}\right]\right),\ 1\le s\le k+1, \tag{8}$$

$$b_{k+1\,k+1-s\,0}=-\frac{1}{2}\left(\sum_{t=s+1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,t,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{A_4(k,t,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]+\right.$$
$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_3(k,s,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{B_4(k,s,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]\right),\ 1\le s\le k,\ b_{k+1\,0\,0}=0, \tag{9}$$

$$c_{k+1\,k+1-s}=c_{k\,k-s}-a_{k\,0\,0}\exp(-\lambda X),\ 1\le s\le k,\ \ c_{k+1\,0}=c_{k+1\,-1}=0, \tag{10}$$

$$d_{k+1\,0}=\frac{1}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1-R^{j+1}}+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\,k-s\,j}}{1+R^{j+1}}A_3(k,s,j)+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\,k-s\,j}}{1-R^{j+1}}A_4(k,s,j)\right), \tag{11}$$

$$d_{k+1\,j+1}=-d_{k\,j}\frac{R^{2(j+1)}}{1-R^{2(j+1)}},\ 0\le j\le k-1. \tag{12}$$

2. Theorem 1 proof

It is obvious that

$$T_1(x)=\frac{\exp(-\lambda x)}{2},\ \ x>0,\ \ S_1(x)=\frac{\exp(\lambda x)}{2},\ \ x\le 0, \tag{13}$$

and

$$P(\tau=1)=\frac{\exp(-\lambda X)}{2}.$$

Calculate now for $x\ge 0$

$$T_2(x)=\left(\int_{-\infty}^{0}dS_1\left(\frac{u}{R}\right)-\int_{0}^{\min(x,XR)}dT_1\left(\frac{u}{R}\right)\right)\frac{\exp(-\lambda(x-u))}{2}\ -\ \int_{\min(x,XR)}^{XR}dT_1\left(\frac{u}{R}\right)\left(1-\frac{\exp(\lambda(x-u))}{2}\right).$$

As a result obtain

$$T_2(x)=\frac{1}{2(1-R^2)}\left(\exp(-\lambda x)-R^2\exp\left(-\frac{\lambda x}{R}\right)\right)-\frac{1}{2}\exp(-\lambda X)+$$
$$+\frac{\exp(\lambda x)\exp(-\lambda X(1+R))}{4(1+R)},\ 0\le x\le XR, \tag{14}$$

$$T_2(x)=\frac{\exp(-\lambda x)}{2}\left(\frac{1}{1-R^2}-\frac{\exp(\lambda(R-1)X)}{2(1-R)}\right),\ XR\le x\le X. \tag{15}$$

Calculate now for $x<0$

$$S_2(x)=\int_{-\infty}^{x}dS_1\left(\frac{u}{R}\right)\left(1-\frac{\exp(-\lambda(x-u))}{2}\right)+\left(\int_{x}^{0}dS_1\left(\frac{u}{R}\right)-\int_{0}^{XR}dT_1\left(\frac{u}{R}\right)\right)\frac{\exp(\lambda(x-u))}{2}.$$

As a result obtain

$$S_2(x)=\exp\left(\frac{\lambda x}{R}\right)\left(\frac{1}{2}-\frac{1}{4(1+R)}-\frac{1}{4(1-R)}\right)+$$

$$+\frac{\exp(\lambda x)}{2}\left(\frac{1}{2(1-R)}+\frac{1}{2(1+R)}(1-\exp(-\lambda X(1+R)))\right),\ x<0. \tag{16}$$

In an accordance with (14) - (16) assume by an induction that for $x\ge 0$ the formula (2) is true and for $x<0$ the formula (3) takes place. Then for $x\ge 0$

$$T_{k+1}(x)=J_1(x)+J_2(x)+J_3(x)+J_4(x). \tag{17}$$

Here

$$J_1(x)=\int\limits_{-\infty}^{0} dS_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}. \tag{18}$$

Calculate now

$$J_2(x)=-\int\limits_{0}^{\min\left(x,X_{k+1}^1\right)} dT_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=$$

$$=\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+ \tag{19}$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\,k-t\,j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)},\ x\ge 0.$$

Then we have for $X_{k+1}^{k+2-s}\le x\le X_{k+1}^{k+1-s}$

$$J_3(x)=-\int\limits_{\min\left(x,X_{k+1}^1\right)}^{X_{k+1}^1} dT_k\left(\frac{u}{R}\right)=-a_{k\,0\,0}\exp(-\lambda X)+$$

$$+\sum_{j=0}^{k-s}a_{k\,k-s\,j}\exp\left(-\frac{\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\,k-s\,j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\,k-s} \tag{20}$$

and

$$J_4(x)=\int\limits_{\min\left(x,X_{k+1}^1\right)}^{X_{k+1}^1} dT_k\left(\frac{u}{R}\right)\frac{\exp(-\lambda(x-u))}{2}=$$

$$=\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\,k-s\,j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{\min\left(\max\left(x,X_{k+1}^{k+2-s}\right),X_{k+1}^1\right)}^{\min\left(\max\left(x,X_{k+1}^{k+1-s}\right),X_{k+1}^1\right)}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\,k-s\,j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(\max\left(x,X_{k+1}^{k+2-s}\right),X_{k+1}^1\right)}^{\min\left(\max\left(x,X_{k+1}^{k+1-s}\right),X_{k+1}^1\right)}. \tag{21}$$

for $s\in\{1,...,k\}$,

$$J_3(x)=J_4(x)=0,\ s=k+1. \tag{22}$$

From (17) - (22) we have for $x\ge 0,\ s\in\{1,...,k\}$

$$T_{k+1}(x)=-a_{k\,0\,0}\exp(-\lambda X)+\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+ \tag{23}$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\,k-t\,j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\sum_{j=0}^{k-s}a_{k\,k-s\,j}\exp\left(\frac{-\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\,k-s\,j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\,k-s}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{\min\left(\max(x,X_{k+1}^{k+1-s}),X_{k+1}^{1}\right)}^{\min\left(\max(x,X_{k+1}^{k+1-s}),X_{k+1}^{1}\right)}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(\max(x,X_{k+1}^{k+1-s}),X_{k+1}^{1}\right)}^{\min\left(\max(x,X_{k+1}^{k+2-s}),X_{k+1}^{1}\right)}=$$

and for $s=k+1$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+ \tag{24}$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{a_{k\,k-t\,j}}{1-R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}+$$

$$+\frac{\exp(-\lambda x)}{2}\sum_{t=1}^{k}\sum_{j=0}^{k-t}\frac{b_{k\,k-t\,j}}{1+R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{\min\left(x,X_{k+1}^{k+1-t}\right)}^{\min\left(x,X_{k+1}^{k+2-t}\right)}.$$

Using the formulas (2) rewrite the formulas (23), (24) as follows: for $s\in\{1,...,k\}$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\,k-t\,j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\,k-t\,j}}{1+R^{j+1}}\right]+\right. \tag{25}$$

$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\,k-s\,j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\,k-s\,j}}{1+R^{j+1}}\right]\right)-$$

$$-\frac{1}{2}\sum_{j=0}^{k-s}\left[\frac{\exp\left(-\frac{\lambda x}{R^{j+1}}\right)a_{k\,k-s\,j}}{1-R^{j+1}}+\frac{\exp\left(\frac{\lambda x}{R^{j+1}}\right)b_{k\,k-s\,j}}{1+R^{j+1}}\right]+$$

$$+\sum_{j=0}^{k-s}a_{k\,k-s\,j}\exp\left(-\frac{\lambda x}{R^{j+1}}\right)+\sum_{j=0}^{k-s}b_{k\,k-s\,j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)+c_{k\,k-s}-a_{k\,0\,0}\exp(-\lambda X)-$$

$$-\frac{\exp(\lambda x)}{2}\left(\sum_{t=s+1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,t,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{A_4(k,t,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]-\right.$$

$$\left.-\sum_{j=0}^{k-s}\left[\frac{B_3(k,s,j)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{B_4(k,s,j)b_{k\,k-s\,j}}{1-R^{j+1}}\right]\right)-\frac{1}{2}\sum_{j=0}^{k-s}\left[\frac{\exp\left(-\frac{\lambda x}{R^{j+1}}\right)a_{k\,k-s\,j}}{1+R^{j+1}}+\frac{\exp\left(\frac{\lambda x}{R^{j+1}}\right)b_{k\,k-s\,j}}{1-R^{j+1}}\right]=$$

$$=\sum_{j=0}^{k+1-s}a_{k+1\;k+1-s\,j}\exp\left(-\frac{\lambda x}{R^{j}}\right)+\sum_{j=0}^{k+1-s}b_{k+1\;k+1-s\,j}\exp\left(\frac{\lambda x}{R^{j}}\right)+c_{k+1\;k+1-s},$$

for $s=k+1$

$$T_{k+1}(x)=\frac{\exp(-\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\,j}}{1+R^{j+1}}+\sum_{t=1}^{s-1}\sum_{j=0}^{k-t}\left[\frac{A_1(k,t,j)a_{k\,k-t\,j}}{1-R^{j+1}}+\frac{A_2(k,t,j)b_{k\,k-t\,j}}{1+R^{j+1}}\right]+\right.$$

$$\left.+\sum_{j=0}^{k-s}\left[\frac{B_1(k,s,j)a_{k\,k-s\,j}}{1-R^{j+1}}+\frac{B_2(k,s,j)b_{k\,k-s\,j}}{1+R^{j+1}}\right]\right)=a_{k+1\,0\,0}\exp(-\lambda x). \tag{26}$$

Calculate now for $x<0$

$$S_{k+1}(x)=\int\limits_{-\infty}^{x} dS_k\left(\frac{u}{R}\right)\left(1-\frac{\exp(-\lambda(x-u))}{2}\right)+\left(\int\limits_{x}^{0} dS_k\left(\frac{u}{R}\right)-\int\limits_{0}^{X_{k+1}^1} dT_k\left(\frac{u}{R}\right)\right)\frac{\exp(\lambda(x-u))}{2}= \qquad (27)$$

$$=\sum_{j=0}^{k-1} d_{k\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)\left(1-\frac{1}{2\left(1+R^{j+1}\right)}-\frac{1}{2\left(1-R^{j+1}\right)}\right)+$$

$$+\frac{1}{2}\sum_{j=0}^{k-1}\frac{d_{k\ j}\exp(\lambda x)}{1-R^{j+1}}+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{a_{k\ k-s\ j}}{1+R^{j+1}}\exp\left(-\lambda u\left(\frac{1}{R^{j+1}}+1\right)\right)\Bigg|_{X_{k+1}^{k+1-s}}^{X_{k+1}^{k+2-s}}+$$

$$+\frac{\exp(\lambda x)}{2}\sum_{s=1}^{k}\sum_{j=0}^{k-s}\frac{b_{k\ k-s\ j}}{1-R^{j+1}}\exp\left(\lambda u\left(\frac{1}{R^{j+1}}-1\right)\right)\Bigg|_{X_{k+1}^{k+1-s}}^{X_{k+1}^{k+2-s}}=$$

$$=\sum_{j=0}^{k-1} d_{k\ j}\exp\left(\frac{\lambda x}{R^{j+1}}\right)\left(1-\frac{1}{2\left(1+R^{j+1}\right)}-\frac{1}{2\left(1-R^{j+1}\right)}\right)+$$

$$+\frac{\exp(\lambda x)}{2}\left(\sum_{j=0}^{k-1}\frac{d_{k\ j}}{1-R^{j+1}}+\sum_{s=1}^{k}\sum_{j=0}^{k-s}\left[\frac{A_3(k,s,j)a_{k\ k-s\ j}}{1+R^{j+1}}+\frac{A_4(k,s,j)b_{k\ k-s\ j}}{1-R^{j+1}}\right]\right)=\sum_{j=0}^{k} d_{k+1\ j}\exp\left(\frac{\lambda x}{R^{j}}\right).$$

From the equalities (13) - (16), (25) - (27) we have the formulas (4) - (12). Theorem 1 is proved.

Remark 1. *Obtained formulas remain true in a case of variable boundary and interest rate:*

$$X_k=R_{k-1}X_{k-1}+\eta_{k-1},\ \tau=\inf\left(k: X_k\ge X(k)\right)$$

then we rewrite

$$X_k^k=0,\ X_k^0=X(k),\ X_k^j=\min\left(X_{k-1}^{j-1}R_{k-1}, X(k)\right),\ R_k^0=1,\ R_k^j=R_{k-1}^{j-1}R_{k-1},$$

and replace R^{j+1} *by* R_{k+1}^{j+1} *and* R^j *by* R_{k+1}^j *in previous formulas,* $1\le j\le k-1$, $k\ge 1$, *without assumption* $R_{k-1}<1$.

Remark 2. *The proof of Theorem 1 contains sufficiently complicated and long symbol transformations. The transformations create manifold mistakes. To avoid these mistakes we examined the transformations by numerical calculations.*

Remark 3. *Suppose that X=1, R=0.5, λ=0.4491 then in an accordance with Theorem 1 we obtain Table 1.*

k	$P(\tau=k)$
10	0.03052
20	0.00512968
30	0.000861841
40	0.000144798
50	0.0000243276
60	4.08729×10^{-6}
70	6.86708×10^{-7}
80	1.15374×10^{-7}
90	1.93841×10^{-8}
100	3.25672×10^{-9}

Table 1.

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DISTRIBUTED COMPUTING ENVIRONMENT FOR RELIABILITY-ORIENTED DESIGN[1]

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ABSTRACT

A theoretical approach and applied techniques for designing analogous electronic devices and systems with due account of random variations in system parameters and reliability specifications are considered. The paper discusses the problem of choosing nominal values of parameters of electronic devices and systems for which the system survival probability or the performance assurance probability for the predetermined time period is maximized. Several algorithms for region of acceptability location, modelling and discrete optimization using parallel and distributed processing are discussed. For seeking a numerical solution of the parametric design problem a distributed computer-aided reliability-oriented design system is proposed.

1 INTRODUCTION

One of basic problems of Computer-Aided Design (CAD) systems design and usage is high computation cost of simulation, multivariate analysis and optimization. Solutions of these tasks constitute the basis of system design.

System design with account of stochastic regularity of parameter deviations and reliability requirements is one of the most computational-intensive tasks. In this task, the simulation of stochastic processes of parameter deviations, statistical simulation and optimization are added to necessity of dynamic and often nonlinear systems simulation. The optimization, in addition, is performed using stochastic criteria.

Despite the fact of continuous development of CAD tools for electronic circuit design, the examples of their successful use and particularly when the optimal design with the account of reliability criteria is used are virtually non-existent. However, in recent years a radical way to improve the efficiency of solving problems of high computational cost is successfully developed. It is based on the technology of parallel and distributed computing. The creation of CAD systems using the technology of parallel computing is very interesting and promising.

This work is an attempt to outline the tasks which arise during development of parallel (distributed) CAD systems for electronic circuits and the ways to solve them.

As a subject area the optimal parametric synthesis of analog electronic circuits with respect to random processes of parameters variations and the requirements of reliability is considered.

2 PARAMETRIC SYNTHESIS PROBLEM

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Suppose that we have a system which depends on a set of n parameters $\mathbf{x}=(x_1,\ldots, x_n)$. We will say that system is acceptable if $\mathbf{Y}(\mathbf{x})$ satisfy the conditions (1):

$$\mathbf{a} \le \mathbf{Y} \le \mathbf{b}, \tag{1}$$

where $\mathbf{Y}$, $\mathbf{a}$ and $\mathbf{b}$ are m-vectors of system responses (output parameters) and their specifications, e.g. $Y_1(\mathbf{x})$ is average power, $Y_2(\mathbf{x})$ - delay, $Y_3(\mathbf{x})$ - gain.

The inequalities (1) define a region D_x in the space of input parameters

$$D_x = \{\mathbf{x} \in R^n \mid \mathbf{a} \le \mathbf{Y} \le \mathbf{b}\}. \tag{2}$$

D_x is called tolerance margin domain (region of acceptability) for the system. It is a region inside the input parameters space.

The engineering system parameters are subject to random variations (aging, wear, temperature variations) and the variations may be considered as stochastic processes:

$$\mathbf{X}(t) = \{X_1(t),\ldots, X_n(t)\}.$$

In general the parametric optimization (optimal parametric synthesis) problem can be stated as follows (Abramov 1992).

Let the characteristics of random processes $\mathbf{X}(t)$ of system parameters variations, a region of admissible deviation and a service time T are given, find such a deterministic vector of parameter ratings (nominals) $\mathbf{x}_r=(x_{1r},\ldots,x_{nr})$ that the reliability

$$P_r(\mathbf{x}_r, T) = P_r\left\{\begin{array}{l}[X_1(t)-x_{1r},\ldots,X_n(t)-x_{nr}] \in D_x, \\ \forall t \in [0,T]\end{array}\right\} = \max. \tag{3}$$

Any optimization technique requires, first, a method of objective function calculation and, secondly, an extremum searching method which allows to find a solution with a minimal cost.

3 OBJECTIVE FUNCTION ESTIMATION

The practical algorithm of the stochastic criterion calculation is based on the conventional Monte Carlo method and on the method of "critical sections" (Abramov 1992, Abramov 2006).

At the beginning, the random vector of parameters is generated (this vector means random manufacturing device realization), and then the internal parameters degradation is simulated using degradation model. For example parameters variations can be approximated as follows

$$X(t) = \sum_{k=0}^{m} x_k u_k(t),$$

where x_k is a random variable; $\{u_k(t)\}$, $k=0,\ldots,m$ are continuous deterministic functions of time.

The Monte Carlo method approximates $P_r(\mathbf{x}_r, T)$ by the ratio of number of acceptable realizations (falling into region D_x)-N_a to the total number of trials - N.

$$P_r = N_a / N.$$

Unfortunately often the region D_x is unknown. It is given only implicitly through system's equations and the systems response functions. If we do not know the region D_x, the Monte Carlo evaluation of probability $P_r(\mathbf{x}_r, T)$ at particular nominal value $\mathbf{x}_r$ requires N system analyses for each trial set of parameter $\mathbf{x}_r$. Typically, hundreds of trials are required to obtain a reasonable estimate for $P_r(\mathbf{x}_r, T)$.

Optimization requires the evaluation of the probability $P_r(\mathbf{x}_r, T)$ for many different values of the nominal values of parameters $\mathbf{x}_r$. Since objective function calculation is based on the numerical technique we can only use the non-gradient-based optimization methods. These optimization methods require top computing powers. Particularly effective way to decrease total design time on the phase of simulation and statistical optimization is to use modern supercomputing technologies and distributed parallel processing techniques. The easiest implementation of this idea would be the

use of distributed processing technologies. In this case computational tasks can be distributed over the set of networked workstations. Realization means reflection of all computing scheme to parallel architecture of the computer, taking into account topology of interprocessor communications and providing correctness of interaction of set of process in parallel carried out separately from each other (Foster 1995).

Using of parallel calculations within the Monte Carlo method is the easiest way of reduction of computational cost input of process of parametrical synthesis as the idea of parallelism - recurrences of some typical process with the various data - is incorporated in the structure of a method.

It is intuitively clear that use of s separate processors, by distribution of independent tests between them, will reduce of computational cost input of statistical modelling in s times as expenses for final summation and averaging of results are practically insignificant. The final rating can be received under the formula:

$$\hat{P} = \sum_{i=1}^{s} n_g^i / N ,$$

where n^i_g - the number of "good" realizations for each of processors, N – the required number of tests.

4 SEQUENTIAL ALGORITHM OF OBJECTIVE FUNCTION CALCULATION.

The yield estimation, based on a Monte Carlo and “critical sections” method, is made as follows.

Algorithm 1.

Let an initial vector of nominal values of parameters is $\boldsymbol{x}^{(1)}_{nom}$.

1. Proceeding from the defined distribution laws of parameters x_1..., x_n, we generate realization of a random vector of parameters $\boldsymbol{x}^{(k)}$.
2. For the realization of values of parameters we calculate output parameters:

$$y_j = F(\mathbf{x}^{(k)}), j = 1,...,m .$$

This stage is the most cumbersome since calculation of output parameters is quite often associated with the solution of systems of differential (and not always linear) equations.

3. Conditions of serviceability are checked

$$y \in D_y ,$$

where $D_y=\{\boldsymbol{y} \mid \boldsymbol{a} \le \boldsymbol{y}(\boldsymbol{x}) \le \boldsymbol{b}\}$ is the known area of allowable values of output parameters $\boldsymbol{y}$.

Satisfaction of step 3 allows to refer the given realization $\boldsymbol{x}^{(k)}$ to the number of "good" (providing an efficient status of system) or "bad".

The first internal cycle is concluded with this operation and there is a return to item 1.

We generate the next realization $\boldsymbol{x}^{(k+1)}$ and pass steps 2 and 3 again.

The total number of iterations N is determined by the necessary accuracy of probability estimation:

$$\hat{P} = n_g / N ,$$

where n_g - the number of "good" realizations from the total number N of tests.

For calculation of probability of non-failure operation during the certain time $P(t)$ the mentioned above procedure 1-3 is carried out several times, determined by the number of t sections, and looks as follows.

Algorithm 2.

Let the following data is known:

- The distribution laws of parameters x_1..., x_n.

- The model of change of parameters in time, specifying number of time sections *l*.

1. *k=1*, we set an initial vector of nominal values of parameters $\boldsymbol{x}^{(1)}_{nom}$.
2. Proceeding from the set distribution laws of parameters $x_1...,x_n$, we generate realization of a random vector of parameters $\boldsymbol{x}_0^{(k)}$ at the moment of time *t=0.*
3. For the set realization of values of parameters calculate output parameters

$$y_j = F(\mathbf{x}^{(k)}), j = 1,...,m .$$

4. Conditions of serviceability are checked

$$\mathbf{y} \in D_y ,$$

where D_y is the known area of allowable values of output parameters $\boldsymbol{y}$.

5. In the case of satisfaction of conditions (step 4) on the given time section proceeding from models of change of parameters we form realization of a random vector of parameters $\boldsymbol{x}_0^{(k)}$ for the following time section $\boldsymbol{x}_i^{(k)}$, *i=1,...,l*. We carry out steps 3 and 4 of the given algorithms. Satisfaction of conditions of acceptability on all time sections allows to relate the given realization to the number of "good", we increase the counter $n_g = n_g + 1$.

If on the next time section the conditions (4) are not carried out, all realization concerns to the number of "bad".

6. If *k <N, k=k+1,* we pass on to step 2 - generation of the following realization of a random vector of parameters.
7. We receive the final rating

$$\hat{P} = n_g / N .$$

5 DISTRIBUTED PARALLEL MONTE-CARLO ALGORITHM.

The main processor (master):

1. Makes an exchange of seeds with the subordinated processes (initializes random numbers generator).
2. Appoints the amount of Monte Carlo calculations n_i to each processor. n_i is the volume of sample for the processor number *i*. Thus

$$\sum_{i=1}^{s} n_i = N .$$

3. Carries out statistical tests using algorithm 2. As the result a number of "good" tests n^i_g is received.
4. Receives from the subordinated processors results of Monte Carlo calculations n^i_g, *i = 1,...,s.*
5. Forms a final rating

$$\hat{P} = \sum_{i=1}^{s} n_g^i / N .$$

The subordinated processors (slaves):

1. Receive from the main processor a seed for random numbers generator.
2. Receive the amount of statistical tests which are necessary to carry out by each of them.
3. Carry out statistical tests using algorithm 2.
4. Send to the main process a number of "good" tests n^i_g.

At the distributed parallel Monte Carlo method both message passing time and sleep time are reduced to a minimum.

6 DISCRETE OPTIMIZATION DISTRIBUTED ALGORITHMS.

Evaluation of *extr* $P_r(\mathbf{x}_r,T)$ requires a global optimization. The simplest method of global optimization is scanning (full enumeration) method. However, such method is considered computationally inefficient. The effective way to decrease optimization time is data decomposition.

The region of extremum seeking is divided into non-overlapping subregions. These subregions are distributed between separate computation processes which perform extremum seeking. After calculations, the results are passed to main process which composes final result.

The nominal values of the schematic components $\boldsymbol{x}_r$ commonly used for engineering systems should lie inside the predefined set of values as it is required by various standards and technical recommendations, it is sometimes more preferable to search the optimal vector inside the discrete set of values that conforms to the standards and lies in the acceptable region D_x.

The information on a variation of values of internal parameters can be presented as limits of their values, i.e.

$$x_{i\min} \le x_i \le x_{i\max}, i=\overline{1,n}.$$

The area inside the space of internal parameters assigned by these relations represents n-dimensional orthogonal parallelepiped called box of tolerances (tolerance region) B_d:

$$B_d = \{\mathbf{x} \in R^n \mid x_{i\min} \le x_i \le x_{i\max}, i=\overline{1,n}\}$$

Using the algorithm described in (Abramov, Katueva, Nazarov 2006) the circumscribed box $B_o \subseteq B_d$ is determined with the following equations:

$$B_o = \{\mathbf{x} \in R^n \mid a_i^0 \le x_i \le b_i^0 \ \forall i = \overline{1,n}\}$$

where

$$a_i^0 = \min_{\mathbf{x}\in D_x} x_i, b_i^0 = \max_{\mathbf{x}\in D_x} x_i$$

This algorithm is based on Monte-Carlo method and can be performed in parallel mode with linear speedup.

Circumscribed box makes it possible to narrow the region of extremum searching (Abramov, Katueva, Nazarov 2006). Circumscribed box constraints do not exceed tolerance region's ones.

Figure 1 schematically illustrates tolerance region B_d , circumscribed parallelepiped B_o and acceptable region D_x in the case of 2-dimensional space of internal parameters.

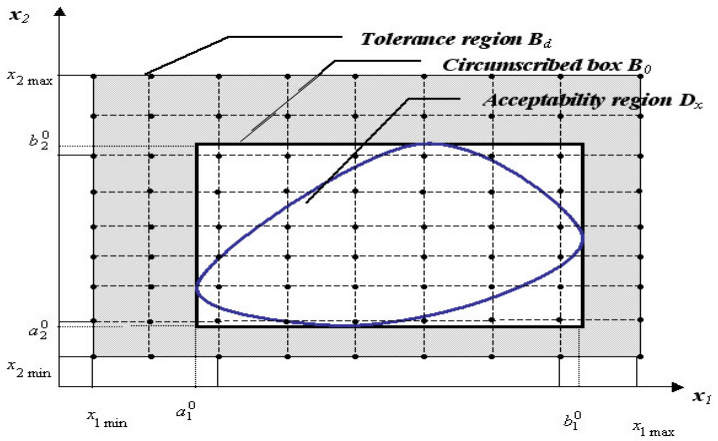

Figure 1. The approach to tolerance region discretization

Standard nominal values of input parameters form a grid inside the circumscribed box. D_o is a discrete set of grid nodes.

Let the vector of internal parameters values $\mathbf{x}_r \in D_o$ is known. Therefore at the each point of discrete set

$$D_r^{in} = \{x_r^{in} \mid x_r \in D_x \cap D_o\}$$

we need to find the $P_r(\mathbf{x}_r^{in})$ estimation (3). The optimum of nominal vector $\mathbf{x}_r$ we are looking for can be found as a solution of the following task

$$\mathbf{x}_r^{opt} = \max_{\mathbf{x}_r} P_r(\mathbf{x}_r^{in}) \tag{4}$$

In the simplest case the solution can be found by complete verification of each element of the set D_r^{in} with the probability estimation for each of them. The construction of set D_r^{in} can be implemented as a preliminary procedure that puts the element values to the database.

The optimum search process can be performed in parallel mode.

The set D_r^{in} is distributed between separate processes. Each of the processes searches solution of task (4) on the subset given (local optimization). Then each of the processes passes the result of solution to main process. Main (Master) process composes final result (global optimization). The average speedup for distributed discrete optimization is close to linear.

7 COMPUTER-AIDED RELIABILITY-ORIENTED DISTRIBUTED DESIGN SYSTEM

All algorithms described above were included in the computer-aided reliability-directed distributed deign system (CARD). The CARD system was developed for parametric synthesis of analogous electronic devices with respect to reliability requirements.

The CARD system includes:

1. the simulation module (it facilitates the use of a variety of simulation programs for electronic circuits design);

2. the module for multivariate (deterministic and statistical) analysis;

3. the module for objective function (reliability and/or manufacturing yield) calculation;

4. the optimization module.

The system is organized from group of computers connected to a network. Such system allows using all advantages of client-server technology. It is necessary, however, to notice that tasks of the server and client stations in such system differ from usual client-server architecture. Let's consider it in detail.

The first task is to connect clients to the server. Thus we do not only increase computing resources but also obtain the amount of prospective tasks for the analysis and decomposition of electronic circuits. In this case connections scheme at LAN looks as shown in Figure 2.

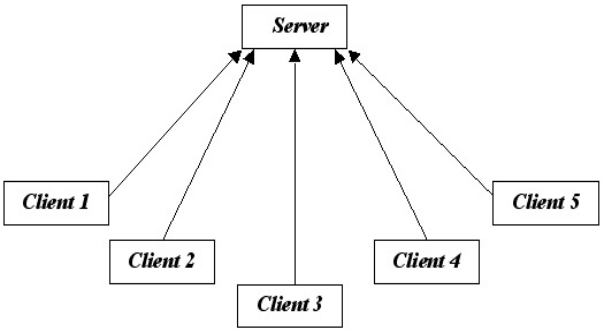

Figure 2. Connection to the server

Any circuit represents a set of elements and a set of connections between these elements (node points). The decomposition of a circuit can be implemented by nodes or by elements. In both

cases the circuit is easier for representing as undirected graph where nodes represent circuit elements and edges represent connections between the elements. The other way of circuit decomposition is splitting an element. Necessity of implementation of transient element algorithms considerably complicates the task. In the first case need only to transfer volt-ampere characteristic of one circuit part to another place of circuit and obtain this characteristic. Then these data is analysed on the both parts.

But taking into account digitization aspect of data transmitted there is a small computation inaccuracy and insignificant delays of signal stabilization and, as probably, unexpected signal attenuation in cyclic circuits. It is expedient to examine the extended model of network interactions.

LAN represents set of the clients connected by switching device. The connections for data transfer can be established between any computers of the network. The network is represented by the complete graph. The server's duty of switching data blocks is not a purpose for great amount of clients. In this case, to increase the rate of data transmission at the network it is necessary to establish connections between the clients; it reduces volume of the data transmitted twice. Thus, the network graph construction is required and then the rules of connection to be organized:

- installation of uniform connection between two separate computers if necessary;
- classification clients onto groups "connected" and "expecting" (distribution of client-server role between the clients).

The example of splitting and connection for 5 clients is shown on Figure 3.

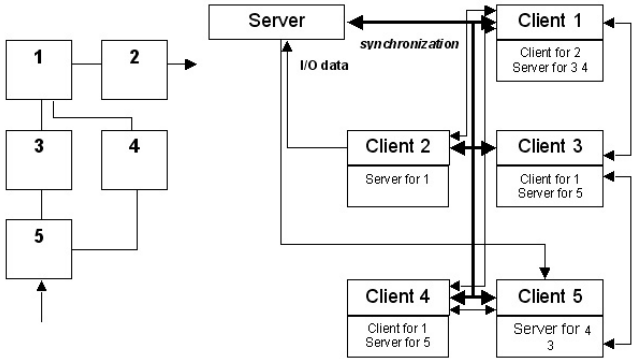

Figure 3. Fragment of network CAD system architecture

The following tasks are executed by server:

1. generation of initial circuit according to the certain requirements, or granting of convenient manual input;
2. gathering information about clients, IP address and information of clients performance;
3. splitting the circuit with special algorithms by pieces for modelling realization;
4. transfer the data to client stations and start designing;
5. receiving of the circuit parts finally optimized and associate it to the uniform circuit.

The following tasks are executing by clients:

1. reception of the information from the server and sending signal about readiness for begin designing;
2. simulation on basis of the algorithm chosen;
3. transfer results to the server.

For network communications sockets are used. Socket is a final point of the network communications. Every socket used has a type and process associated with it. Sockets exist inside communication domains. The domains are abstraction which means concrete addressing structure and set of protocols and defines various sockets types inside the domain. As the protocol of sockets exchange TCP/IP protocol is used. It concerns to the transfer protocols with guaranteed delivery. It

is most convenient in this case to use stream sockets. With stream sockets the pipe created between two applications in stream form. The streams can be input or output, normal or formatted, with or without buffering. Note that stream sockets allow transferring the data only between two applications, as they assume a channel between these applications. However sometimes it is necessary to provide interaction of several client applications with one server or several client applications with the several server applications. The separate tasks and separate channels for each client application are created in this case in the server application.

The CARD system uses a modification of widely distributed PSPICE circuit simulation program that allows simulating a large class of analogous devices in direct current, frequency and time domains. CARD also consists of features for nominal design, design centering, tolerances assignment, etc. Mathematical models of semi-conductor devices are used in many similar programs, and the lists of connections of the circuit in a format SPICE are made by the majority of applications (Micro-Cap, Dr. Spice, OrCAD, P-CAD, ACCEL EDA, Viewlogic, COMPASS, Design Architect etc.). These ones and the subsequent versions use the same algorithms as SPICE, the same format of the input data. PSPICE allows simulation and support of circuit development containing as analogue, and digital components without manufacturing real circuits. The circuits on input influences, circuit behaviour on various frequencies, noises and other characteristics of the circuit can be designed by the user. PSPICE allow user to create "a computer model of circuit" for testing and debugging of the developed circuit before the beginning of its manufacturing. Using the circuit tests it is possible to be convinced that in all cases PSPICE works in 1.3-30 times faster, than other similar programs. The CARD system has been tested on a number of complex designs involving filters, amplifiers and control systems.

8 CONCLUSION

We have attempted to describe some our work in progress on the problem of facilitating the phase of reliability analysis and optimization based on distributed CAD system. On the negative side, reliability optimization requires many stochastic function evaluations which can be expensive in terms of circuit simulation and optimization cost. The expenses of optimization can be reduced by the implementation of efficient parallel algorithms and distributed processing technologies. New computer-aided reliability-oriented distributed design (CARD) system was described. This CARD system had some initial success towards making reliability optimization applicable.

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DETERMINING A PRIORI DISTRIBUTION OF ERROR-FREE RUNNING TIME FOR HIGH-RELIABILITY COMPONENTS BY DELPHI METHOD

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ABSTRACT

We have considered the approach to determining a priori distribution of error-free running time for high-reliability components by the method of paired comparisons useful for the increase of their reliability indicators. We have introduced the distinct variables, whose grades of membership are interpreted as subjective probabilities of finding the error-free running time and its characteristics at various time intervals. The method of recording the expert evaluation accuracy has been suggested.

1 INTRODUCTION

To control the meeting of the requirements by such high-reliability components as components of nuclear reactors [1], aircraft and space-and-missile engineering [2], gas equipment [3], etc., it is necessary to evaluate small (below 0.01) failure probabilities for the preset error-free running time. When there is practically no statistics on the failures of these components during their operation, the error-free running time distribution law is required for evaluating the reliability indicators with acceptable accuracy. This permits, in particular, the subsequent use of information pooling techniques [4], for instance, Bayesian methods of pooling the a priori information and observational data [5].

Determination of error-free running time distribution belongs to intricately formalized problems (there are no sufficiently accurate mathematical models for its solution in most cases [1, 6]). There is also no sufficiently representative statistics on the failures of high-reliability components [2]. Therefore, to obtain the a priori distribution of error-free running time for a component, it is expedient to employ expert evaluation [7]. The necessity of using non-formal experience and appreciating the physical nature of failures is also caused by the fact that, as is shown by simulation modeling of various distribution laws, small samples with the same mean values may result in considerable differences in description of distribution tail areas, which substantially influences on the accuracy of determining the reliability indicators of high-reliability components.

The aim of the article consists in employing expert evaluation for finding out the type and parameters of distribution of error-free running time for high-reliability components.

2 PROBLEM FORMULATION

Let $F(t) = P(T < t)$ be the law of distribution of error-free running time, T be the random value of error-free running time for a component. Select on the time axis n periods, within which the failures of the component under consideration are expected. During the operation a random component from a certain main entity may fail in the z-th (z = 1, 2, …, n) period, namely, its possible state. The error-free running time is associated with the discrete state of the component by relationship

$$t = t_0 + \Delta t \cdot z \tag{1}$$

where t_0 is the maximum error-free running time, until which the component failures have not been observed yet; Δt is the duration of the time period corresponding to the discrete state of the component. Then, discrete random value Z described by means of bar chart $\widetilde{f}_Z(z)$ corresponds to continuous variate T of the component with probability density function $f_T(t)$. In its turn, continuous variate T can be made to correspond to discrete random value Z set by any method.

It is necessary to obtain the expected distribution of the component's error-free running time.

3 SOLUTION METHOD

The problem of obtaining bar chart $\widetilde{f}_Z(z)$ can be solved by the method of paired comparisons (analysis of hierarchies) [8] developed by T. Saaty [9]. All pairs of the component's states are presented to the expert and the latter each time determines which of them is preferable with respect to a possibility of finding the component's error-free running time. In the course of assessment the experts take into account the following: available data on all kinds of reliability tests of the component and its failures during the operation; own experience in evaluating the reliability indicators of similar components by various methods and other factors. The evaluation process results in paired comparisons matrix $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 2 \text{ if } i \succ j; \\ 1 \text{ if } i \approx j; \\ 0 \text{ if } i \prec j; \end{cases} \quad (i,j = 1,\ldots,n),$$

where i, j are the component's compared states of n possible ones.

The constituents of normalized maximum characteristic vector q of paired comparisons matrix B are taken to be relative weights characterizing a possibility of the component's staying in each state.

The method modification, namely, the method of paired comparisons based on a qualitative attribute with quantitative preference judgment. In the course of paired comparisons and filling-in of matrix B the expert not only selects the preferable state in each pair, but also indicates how many times this state is preferable with respect to a possibility of finding the error-free running time than in another state of the pair. The method does not require compulsory transitivity of the expert's preferences, while the processing of the paired comparisons matrices is easily realizable on computers. However, the method has no clear physical interpretation and is unable to treat obtained evaluations q as subjective probabilities [10]. This hampers a possibility of employing the conceptual and mathematical apparatuses well developed in the theory of probability and mathematical statistics for further operations with the obtained results. Therefore, let us complement the method with a fuzzy model [8].

Let us introduce the following indistinct variables:

1) "Possible error-free running time" (basic) – for evaluating the possibility of finding the error-free running time, i.e., probability density function $f_T(t)$.

Auxiliary indistinct variables can be also introduced for solving such subproblems as calibration - clarification of the parameters obtained with the aid of the first indistinct distribution variable, evaluation of the expert's assessment accuracy, etc.;

2) "Expected error-free running time" – for evaluating average error-free running time t_{cp};

3) "Most probable error-free running time" – for evaluating the distribution mode.

Let the considered indistinct variable be determined on discrete multitude $Z = \{z\}$ from n possible component states. Indistinct multitude $\widetilde{Z}$ on multitude Z appears to be an aggregate of pairs $\widetilde{Z} = \{\langle \mu_Z(z)/z \rangle\}$ where $\mu_Z(z)$ is the function of the error-free running time's membership in the fuzzy set, whose sense is formalized by the chosen indistinct variable. The function of membership is made up of degrees of membership (relative weights q_z) of states $z \in Z$ in multitude $\widetilde{Z}$. We shall treat them as subjective probabilities of finding error-free running time z. Meant by the subjective probability is the estimate of probability (relative weight) of finding the error-free running time within a certain time period obtained as a result of processing the experts' opinions rather than mathematically on the basis of the statistic data on frequency of failures getting into this time period as it happens in case of objective probability.

Greater values of $\mu_Z(z)$ correspond to the states conforming, to greater extent, to the meaning of the chosen indistinct variable (i.e., with a greater possibility of finding the component's error-free running time in these state and time period).

As usually $n \geq 4$, the approximated method [11] is recommended for finding the normalized maximum characteristic vector. To do this, introduced is normalized characteristic vector $q^{(r)} = Bq^{(r-1)}$ of the paired comparisons matrix where r is the No. of the approximate computation algorithm's step. Then, let us assume that the relative weights are represented by the constituents of the normalized vector at the r-th iteration step determined from the formula

$$\hat{q}^{(r)} = B\hat{q}^{(r-1)} \Big/ \lambda^{(r)},$$

where $\lambda^{(r)}$ is the sum of the constituents of vector $B\hat{q}^{(r-1)}$ or

$$\hat{q}_i^{(r)} = \frac{1}{\lambda^{(r)}} \sum_{j=1}^{m} b_{ij} \hat{q}_i^{(r-1)}; \qquad \lambda^{(r)} = \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} \hat{q}_j^{(r-1)}$$

till reaching preset accuracy ε. The required accuracy of calculation of the characteristic vector constituents is preset beforehand (henceforth we accept $\varepsilon = 0.0001$ for further computations) and the calculation is stopped at step r if n conditions $\left| \hat{q}_i^{r-1} - \hat{q}_i^{r} \right| \leq \varepsilon \quad \forall i$ are satisfied.

4 PROCEDURE OF EXPERT EVALUATION AND INTERPRETATION OF ITS RESULTS

It is expedient to begin expert evaluation by the selected method from plotting the function of the possible error-free running time's membership in the fuzzy set, whose meaning is formalized by the first indistinct variable. To do this, first one should indicate the range of the component's possible error-free running time: monad $\langle t_0, \Delta t, n \rangle$ expected in the initial approximation. Paired comparisons matrix B is obtained by way of interrogating the experts on the extent, to which, in

their opinion, state *i* corresponds to the meaning of the "Possible error-free running time" indistinct variable more than state *j*. When matrix *B* is filled in, the expert compares, with the aid of the scale developed by T. Saaty [9] (Table 1), all pairs of discrete states with respect to a possibility of the component failure in them. So, to allot marks b_{ij}, the expert actually compares the expected densities of probability of finding the error-free running time (frequency of failures) during various time periods.

Table 1

Modified paired comparisons scale developed by T. Saaty

Preference degree b_{ij}	Definition	Explanation
1	States are equally likely	Possibility of finding the (average or most probable)* component's error-free running time in both time periods of the compared pair is similar
2	Intermediate meaning	
3	Poor superiority	The expert's experience makes it possible to consider the possibility of finding the (average or most probable)* component's error-free running time in the first time period of the pair somewhat higher than in the second one
4	Intermediate meaning	
5	Strong superiority	The expert considers that the possibility of finding the (average or most probable)* component's error-free running time in the first time period of the pair is definitely higher than in the second one
6	Intermediate meaning	
7	Apparent superiority	The expert considers that the possibility of finding the (average or most probable)* component's error-free running time in the first time period of the pair is apparently higher than in the second one, while the available statistics of failures of the analyzed components under the similar conditions, as well as the model calculations conform this fact
8	Intermediate meaning	
9	Absolute superiority	The expert has no doubts with respect to the fact that the possibility of finding the (average or most probable)* component's error-free running time in the first time period of the pair is absolutely higher than in the second one

* Here and hereinafter in Table 1 the text in brackets pertains to either second or third indistinct variable.

As a result of processing matrix *B* we shall obtain function $\mu_Z(z)$ of the error-free running time's membership in fuzzy set $\widetilde{Z}$, the meaning of which is formalized by indistinct variable "Possible error-free running time". The membership function is formed by the membership degrees, which can be represented by the components of the normalized maximum characteristic vector of matrix *B*. Let us interpret this function of this function as the bar chart of the observed random value of the component's error-free running time, including the error of its expert evaluation. This bar chart can help determining the kind and parameters of the observed distribution of error-free running time and, in particular, giving an approximated estimate of observed average error-free

running time $\widetilde{m}'_{cp} = \sum_{z=1}^{n} z \cdot \widetilde{f}(z)$ and observed error-free running time dispersion $(s')^2 = \sum_{z=1}^{n} \left(z - \widetilde{m}'_{cp}\right)^2 \cdot \widetilde{f}(z)$.

By changing over to t according to (1), we shall respectively obtain

$$\widetilde{t}'_{cp} = \sum_{z=1}^{n} t_z \cdot \widetilde{f}(z) \tag{2}$$

and

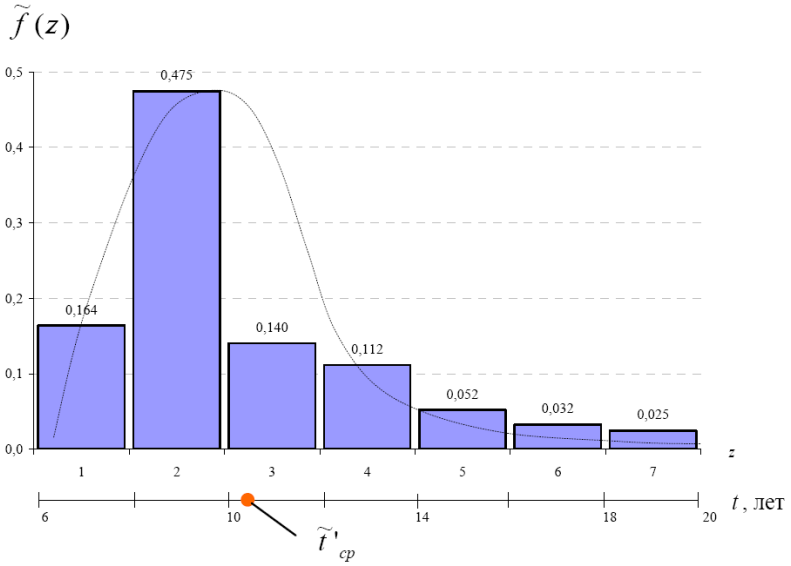
$$\left(s'_t\right)^2 = \sum_{z=1}^{n_1} \left(t - \widetilde{t}'_{cp}\right)^2 \cdot \widetilde{f}(z) \tag{3}$$

where t_z is the error-free running time meaning corresponding to the middle of the z-th time period.

E x a m p l e 1 . The expert is asked to evaluate the component's error-free running time in seven time periods, each 2 years long, beginning from the 6th year of its operation. Paired comparisons matrix B from the expert's judgments pertinent to distribution of the possible component's error-free running time is shown in Fig. 1 a.

	1	2	3	4	5	6	7
1	1	1/7	1/3	1/3	9	9	9
2	7	1	9	9	8	8	8
3	3	1/9	1	3	3	4	5
4	3	1/9	1/3	1	3	4	5
5	1/9	1/8	1/3	1/3	1	3	3
6	1/9	1/8	1/4	1/4	1/3	1	2
7	1/9	1/8	1/5	1/5	1/3	1/2	1

(a)



(b)

Fig. 1. Paired comparisons matrix of possible component's error-free running time (a) and respective bar chart of observed error-free running time (b)

Determine the type and parameters of the form of the expected error-free running time.

Solution. As a result of processing matrix B_1 we use the approximative method to obtain the components of normalized characteristic vector q_z, having the meaning of relative weights (probabilities) of finding the component within certain time periods z on time axis t, i.e. bar chart $\widetilde{f}(z)$ of the component's error-free running time in Fig. 1 (b).

The average error-free running time is found from formula (2) and amounts to $\widetilde{t}'_{cp}$ = 7·0.164 + 9·0.475 + 11·0.140 + 13·0.112 + 15·0.052 + 17·0.032 + 19·0.025 = 10.22 years, while the root-mean-square deviation with allowance for (3) amounts to s_t' = 3.99 years.

The type of a priori distribution of error-free running time corresponding to the bar chart obtained by the expert method can be determined by various methods: method of moments, with the aid of statistical criteria, etc. However, the use of known goodness-of-fit tests (Pearson's, Kolmogorov's, Shapiro's, Wilk's, Bartlett's test, Mann's, etc.) requires getting the answer to the question: to which number of statistic observations do the results of expert evaluation of the error-free running time by the group of experts correspond? Therefore, let us use the method of moments. When balancing the "statistical" rows by this method, use is often made of the system of Pearson curves [12]. The values of the coordinates obtained in the form of distribution bar chart $f_T(t)$ in the diagram (Fig. 2) make up $\beta_1 = 1.641$ and $\beta_2 = 4.250$. So, the distribution of error-free running time by the expert method obtained can be adjusted by distribution from the family of *J*-shaped beta distributions (Fig. 2).

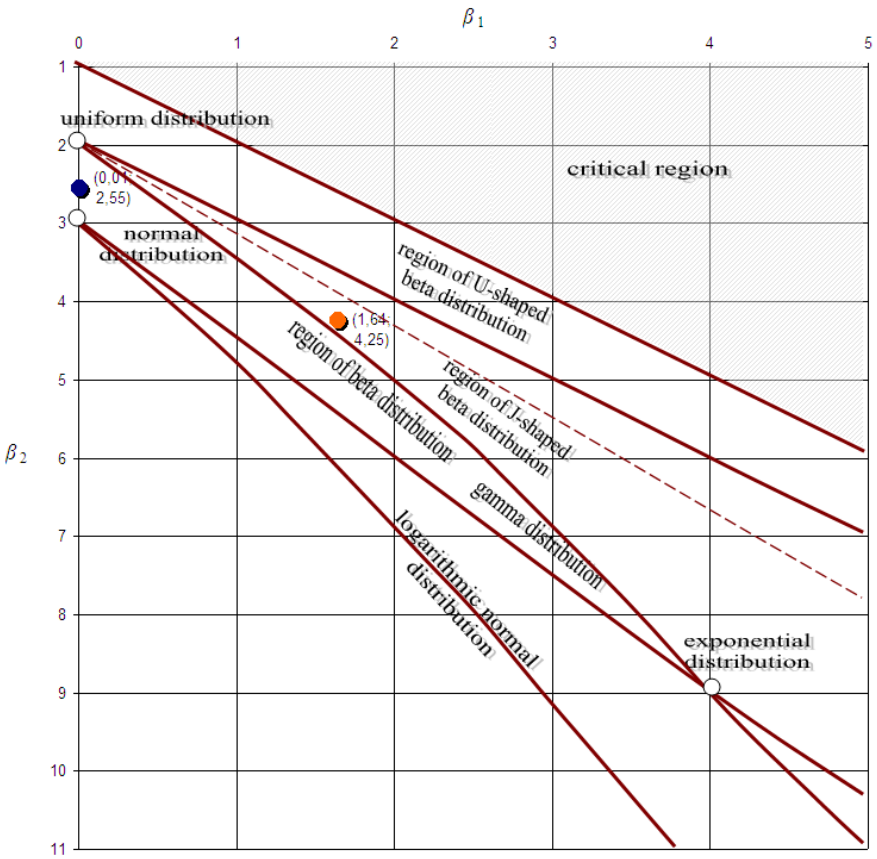


Fig. 2. Location of obtained distribution on diagram of distributions of family of Pearson curves

Now let us consider the "Expected error-free running time" indistinct variable and indicate to the experts the range of its possible values $\langle t_2, \Delta t_2, n_2 \rangle$ with the aid of information on $\widetilde{t}'_{cp}$ and bar chart $\widetilde{f}_T(t)$. During the repeated evaluation the experts compare the pairs of possible time periods with respect to the possibility of finding the average error-free running time within them. As a result of processing paired comparisons matrix B_2 formed in such a manner, we shall obtain bar chart $\widetilde{f}_2(z)$ with discrete random value Z_2 of the average error-free running time (Fig. 3 b). This bar chart is helpful in specifying the estimate of the average error-free running time:

$$\widetilde{t}_{cp} = \sum_{z=1}^{n_2} t_z \cdot \widetilde{f}_2(z) \qquad (4)$$

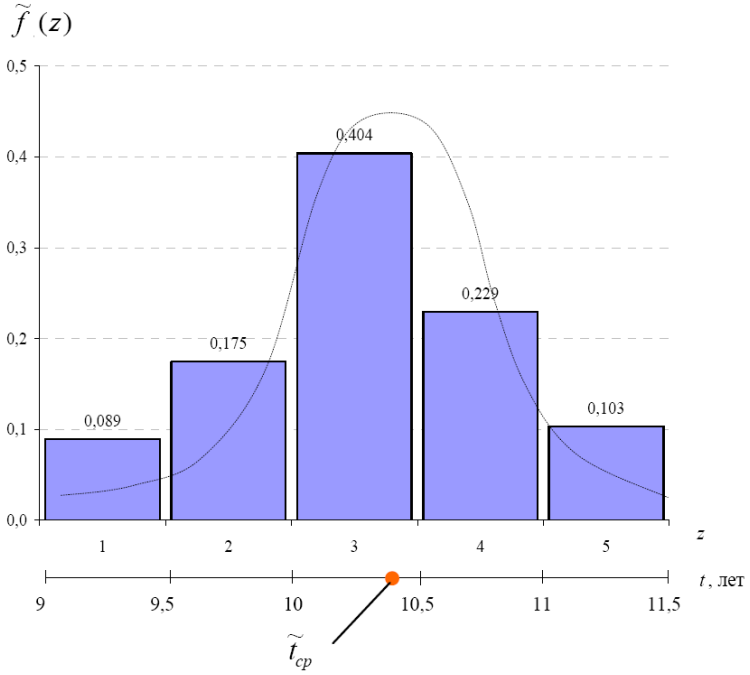
and obtain the dispersion of its evaluation

$$s_{cp}^2 = \sum_{z=1}^{n_2} \left(t_z - \tilde{t}_{cp}\right)^2 \cdot \tilde{f}_2(z). \qquad (5)$$

E x a m p l e 2. Basing on the data of Example 1 for assessing the average error-free running time, let us choose $n_2 = 5$ time periods, beginning from the 9th year of the component's operation, half a year long each. Obtained as a result of expert evaluation is paired comparisons matrix B_2 for a possibility of finding the average error-free running time within these time periods (Fig. 3 b).

	1	2	3	4	5
1	1	1/2	1/4	1/3	1
2	2	1	1/2	1	1
3	4	2	1	2	5
4	3	1	1/2	1	3
5	1	1	1/5	1/3	1

(a)



(b)

Fig. 3. Paired comparisons matrix (a) and bar chart of average error-free running time (b) obtained as a result of expert evaluation

It is necessary to determine the component's average error-free running time and accuracy of its evaluation by the expert.

Solution. Let us obtain the bar chart of the average error-free running time (Fig. 3 b) by the approximative method. Take estimation of expectation $m_{cp} = M[Z_2]$ of discrete random variable Z_2 as the average error-free running time. While changing over to variable t, we obtain the specified estimate of the average error-free running time

$$\tilde{t}_{cp} = \sum_{z=1}^{n_2} t_z \cdot \tilde{f}_2(z) = 9.25 \cdot 0.089 + 9.75 \cdot 0.175 + 10.25 \cdot 0.404 + 10.75 \cdot 0.229 + 11.25 \cdot 0.103 = 10.29 \text{ years.}$$

The root-mean-square deviation of this estimate with allowance for (5) will amount to 0.15 year.

The values of the coordinates for the system of Pearson curves determined from bar chart $f_2(t)$ make up $\beta_1 = 0.010$ and $\beta_2 = 2.553$ (see Fig. 2). Their position in the diagram shows that the distribution of the average error-free running time estimate obtained by the expert method is predictably close to the normal one.

So, as a result of expert evaluation with the aid of the first indistinct variable, when the experts compare the selected time periods with respect to the possibility of the component's failure during it, bar chart $\widetilde{f}_T(t)$ of the component's error-free running time was obtained and, with its aid of the information on dispersion of the component's error-free running time. However, the measurement errors are imposed on the results of expert evaluation of the error-free running time. The expert (by convention, a highly knowledgeable specialist) acts as a "measuring instrument" for expert evaluation. Therefore, bar chart $\widetilde{f}_T(t)$ describes observed component's error-free running time T' obtained with the aid of expert evaluation. Let us assume that the dispersion of the error-free running time with respect to the average value determined by the expert method includes two additive constituents:

$$\Delta' = \Delta + \Delta^e \qquad (5)$$

where Δ is the actual dispersion of the error-free running time, which should be taken into account in determining the component's reliability indicators; Δ^e is the expert evaluation error.

The problems of accuracy of the expert methods are discussed in [13]. However, to assess accuracy, use is made, as a rule, of indirect indicators not relying on the characteristics employed in the theory of accuracy. It is obvious that accuracy of expert evaluation of the component's error-free running time depends on the quality of experts, namely, their competence, objectivity, and information awareness. The a priori and a posteriori assessment of the expert [13] can be done with the aid of usually interrelated indicators:

– "weight" of the expert normalized with respect to other experts under a certain rule (the expert's "weight" depends on his education, academic degree, knowledge of physics of the component's failures, practical experience in determination of the reliability indicators) and set by the decision of the "absolutely competent" person – a priori estimate;

– accuracy of the estimates made by the experts – a posteriori estimate.

If we consider the expert to be the measuring instrument, to analyze accuracy of expert evaluation, generally accepted metrological performances serve turn. Of them the most universal one is root-mean-square deviation σ^e of the "measurement" result relative to the true (or average – in the absence of systematic errors) meaning. Accuracy of the estimates expressed in terms of value σ^e can be determined by the following methods:

– by deviations of the expert estimates from the true meaning. This method is implemented by testing the experts on the problems with the a priori known result or with the result instrumentally (statistically) determined after expert evaluation. The method advantage consists in exclusion of systematic errors, while the disadvantage, in considerable expenses;

– by means of the dispersion characteristics ("concentration") of the obtained expert estimates relative to the true (average) meaning. This method is also applicable in the situations, when the true state of the object being evaluated is unknown, but it does not take into account possible systematic errors.

The disadvantage of both methods consists in the necessity of a certain sample for finding the dispersion.

Let us assume that the characteristic of accuracy of expert evaluation of the error-free running time under the second method is represented by the root-mean-square deviation in determining a certain fixed state [14]. We shall consider the average error-free running time as such a state in the problem being solved. The method advantage consists in the fact that it is obtained in case of determining a single considered component. The more contradictory and inconsistent the expert's judgment on the possible component's state, the higher the value of σ^e. As the dispersion meaning depends on the expert's quality, therefore it can serve as the measure of this quality.

Under the conditions of the considered example $\sigma^e = s_{av} = 0.15$ year.

If the expert evaluation error is known, in some cases it can be taken into account. On the condition of satisfying supposition (5)

$$\Delta = \Delta' - \Delta^e .$$

Allowance for the errors introduced by the expert is possible by way of correcting the parameters of the shape of distribution often functionally bound with the value of σ. Table 2 contains certain distributions of the error-free running time with indication of correlations required to eliminate the measurement errors.

Table 2

Correlations for elimination of expert evaluation error for certain distribution laws of error-free running time

Distribution law	Kind of F(t)	Correlation for correcting the form parameter
Uniform	$\frac{t-t_a}{t_b-t_a}$	$t_a = t_{cp} - \sqrt{3\left[(\sigma')^2 - (\sigma^e)^2\right]}$ $t_b = t_{cp} + \sqrt{3\left[(\sigma')^2 - (\sigma^e)^2\right]}$
Unimodal		
Normal	$\Phi\left\{\frac{t-m}{\sigma}\right\}$	$\sigma = \sqrt{(\sigma')^2 - (\sigma^e)^2}$
Unimodal with positive skew		
α- distribution	$\Phi\left\{\alpha_0\left(1-\frac{1}{at}\right)\right\}$	$\alpha_0 = m \Big/ \sqrt{(\sigma')^2 - (\sigma^e)^2}$
Weibullized ($b \geq 1$)	$1-\exp\left\{-\left(\frac{t-c}{a}\right)^b\right\}$	$b = \left[\frac{1}{m}\sqrt{\left[(\sigma')^2 - (\sigma^e)^2\right]}\right]$
Logarithmic-normal	$\Phi\left\{\frac{y-m_y}{\sigma_y}\right\}$	$\sigma_y = \sqrt{\left[(\sigma_y')^2 - (\sigma_y{}^e)^2\right]}$

* The following designations are taken in Table 2: $\Phi(\cdot)$ is the Laplace's function; $m = M[T]$ is the expectation; *a, b,* and *c* are the scale, form, and shift of the Weibull distribution, respectively; *v* is the coefficient of variation, $\alpha_0 = 1/v$; t_a, t_b are the uniform distribution parameters; $y = \lg t$; $m_y = M[\lg T]$; $\sigma_y = D[\lg T]^{1/2}$; $\sigma'_y = D[\lg T']^{1/2}$; $\sigma_y^э = D[\lg \Delta^э]^{1/2}$.

5 CONCLUSION

The suggested approach makes it possible to improve the extent of justification of setting the a priori distribution of the components' error-free running time for provision of acceptable accuracy of determination of reliability indicators for high-reliability components.

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RISK ANALYSIS AND MANAGEMENT. BASIC CONCEPTS AND PRINCIPLES

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ABSTRACT

This paper reviews and discusses some key concept and principles of risk analysis and risk management, based on a set of statements, formulated as myths about risk. Examples of such myths are: risk is equal to the expected value, risk equals uncertainty, risk can be expressed by probabilities, risk is equal to an event, risk acceptance criteria contribute to obtaining a high safety standard, and ALARP can be verified by cost-benefit analyses. It is concluded that risk needs to address both the consequences and the uncertainties about the consequences, and that it is necessary to see beyond expected values and probabilities.

1 INTRODUCTION

There is an enormous drive and enthusiasm in various industries, services and society as a whole nowadays to implement risk management in the organizations. There are high expectations, that risk management is the proper framework for obtaining high levels of performance. We see a lot of initiatives to establish adequate concepts and tools. However, the risk management discipline is young, and there are many difficult issues and challenges. These relate in particular to the foundation and use of risk analyses; how to express risk, how to handle uncertainties, and how to use risk analysis in a decision-making context. These issues are addressed in this paper. The purpose of the paper is to review and discuss some key concept and principles of risk analysis and risk management. We do this by formulating a set of statements, which can be seen as *myths* about risk. These myths are presented and discussed in the following section. Some conclusions are provided in Section 3. Risk management is defined as all measures and activities carried out to manage risk. Risk management deals with balancing the conflicts inherent in exploring opportunities on the one hand, and avoiding losses, accidents, and disasters, on the other (Aven & Vinnem 2007).

To support decision-making on design and operation, risk analyses are conducted. The analyses include identification of hazards and threats, cause analyses, consequence analyses and risk description. The results of the analyses are then evaluated. The totality of the analyses and the evaluations are referred to as risk assessment. Risk assessment is followed by risk treatment, which is a process involving the development and implementation of measures to modify risk, including measures designed to avoid, reduce ("optimize"), transfer or retain risk. Risk transfer means sharing with another party the benefit or loss associated with a risk. It is typically affected through insurance. The terminology is in line with the ISO standard on risk management terminology (ISO 2002).

By carrying out a risk analysis one can:

- Establish a risk picture
- Compare different alternatives and solutions in terms of risk
- Identify factors, conditions, activities, systems, components, etc. that are important (critical) with respect to risk
- Demonstrate the effect of various measures on risk.

This provides a basis for:

- Choosing between various alternative solutions and activities while in the planning phase of a system
- Choosing between alternative designs of a solution or a measure.
- Drawing conclusions on whether specific solutions and measures meet stated requirements
- Setting requirements for various solutions and measures, for example related to the performance of the preparedness systems
- Documenting an acceptable safety and risk level.

2 MYTHS ABOUT RISK

We will discuss the following myths about risk:

1. Risk is equal to the expected value
2. Risk equals uncertainty
3. Risk is equal to an event
4. Risk is a probability or a probability distribution
5. Risk equals expected disutility
6. Risk is restricted to the case of known probabilities
7. Risk based on subjective probabilities is the same as risk perception
8. Objective risk exists
9. Risk is determined by the historical data
10. Risk relates to negative consequences
11. Risk and probability cannot be determined in case of large uncertainties
12. There are large inherent uncertainties in risk analyses
13. Risk acceptance criteria contribute to obtaining a high safety standard
14. ALARP can be verified by cost-benefit analyses
15. The cautionary/pre-cautionary principles and risk management cannot be meaningfully integrated.

2.1 Risk is equal to the expected value

It is common to refer to risk as probability multiplied by consequences (losses), i.e. what is called the expected value in probability calculus. If C is the quantity of interest, for example the number of future attacks, the number of fatalities, the costs etc., the expected value would be a good representation of risk if this value is approximately equal to C, i.e. EC ≈ C. But since C is unknown at the time of the assessment, how can we be sure that this approximation would be accurate? Can the law of large numbers be applied, expressing that the empirical mean of independent identically distributed random variables converges to the expected value when the number of variables increases to infinity? Or the portfolio theory (Levy & Sarnat 1990) saying that the value of a portfolio of projects is approximately equal to the expected value, plus the systematic risk (uncertainties) caused by events affecting the whole market?

Yes, it is likely that if C is the sum of a number of projects, or some average number, our expected value could be a good prediction of C. Take for example the number of fatalities in traffic in a specific country. From previous years we have data that can be used to accurately predict the number of fatalities next year (C). In Norway about 250 people were killed last year, and using this number as EC and predictor for the coming year, we would be quite sure that this number is close to the actual C.

However, in many cases the uncertainties are much larger. Looking at the number of fatalities in Norway caused by terrorist attacks the next year, the historical data would give a poor basis. We may assign an EC but obviously EC could be far away from C. The accuracy increases when we extend the population of interest. If we look at one unit (e.g. country) in isolation the C number is in general more uncertain than if we consider many units (e.g. countries). Yet, there will always be uncertainties, and in a world where the speed of change is increasing, relevant historical data are scarce and will not be sufficient to obtain accurate predictions.

Even so, some researchers define risk by the expected value. Consider the terrorism case discussed in Willis (2007). Willis (2007) defines risk as follows:

Terrorism risk: The expected consequences of an existent threat, which for a given target, attack mode, target vulnerability, and damage type, can be expressed as

$$\text{Risk} = P(\text{attack occurs}) \cdot P(\text{attacks results in damage} \mid \text{attacks occurs}) \cdot E[\text{damage} \mid \text{attacks occurs and results in damage}]$$

Willis (2007) refers to Haimes (2004) who highlights that expected value decision-making is misleading for rare and extreme events. The expected value (the mean or the central tendency) does not adequately capture events with low probabilities and high consequences. Nonetheless, Willis represents risk by the expected value as the basis for his analysis. The motivation seems to be that the expected value provides a suitable practical approach for comparing and aggregating terrorism risk, as it is based on just one number.

For terrorism risk, where the possible consequences could be extreme and the uncertainties in underlying phenomena and processes are so large, it is however obvious that the expected value may hide important aspects of concern for risk management. The expected value can be small, say 0.01 fatalities, but extreme events with millions of fatalities may occur, and this needs special attention.

Hence we need to see beyond the expected values. We have to take into account uncertainties and risks. Risk management is concerned about how to assess these uncertainties and risk, and how to handle them.

2.2 Risk equals uncertainty

Risk is sometimes associated with uncertainty, for example, in Cabinet Office (2002), risk refers to uncertainty of outcome, of actions and events. Often the uncertainty is seen in relation to the expected value, and the variance is used as a measure of risk.

As an example, consider the problem of investing money in a stock market. Suppose the investor considers two alternatives, both with expectation 1, and variances 0.16 and 0.08, respectively. As alternative 2 has the lowest risk (uncertainty), expressed by the variance, this alternative would normally be chosen.

As another example, consider the number of fatalities in traffic next year in a specific country. Then the variance is rather small, as the number of fatalities shows rather small variations from year to year. Hence according to this definition of risk, we must conclude that the risk is small, even

though the number of fatalities are many thousands each year. Clearly, this definition of risk fails to capture an essential aspect, the consequence dimension. Uncertainty cannot be isolated from the intensity, size, extension etc. of the consequences. Take an extreme case where only two outcomes are possible, 0 and 1, corresponding to 0 and 1 fatality, and the decision alternatives are A and B, having uncertainty (probability) distributions (0.5,0.5), and (0.0001, 0.9999), respectively. Hence for alternative A there is a higher degree of uncertainty than for alternative B, meaning that risk according to this definition is higher for alternative A than for B. However, considering both dimensions, both uncertainty and the consequences, we would of course judge alternative B to have the highest risk as the negative outcome 1 is nearly certain to occur.

2.3 Risk is equal to an event

Risk is also being defined as an event or as a consequence:

1. Risk is a situation or event where something of human value (including humans themselves) is at stake and where the outcome is uncertain (Rosa 1998, 2003)
2. Risk is an uncertain consequence of an event or an activity with respect to something that human value (Renn 2005).

We do not distinguish between the definitions 1) and 2). They are basically expressing the same: Risk is an event or a consequence of an event. See Figure 1. The activity considered could produce events and consequences and these are subject to uncertainties. Something of human value is at stake.

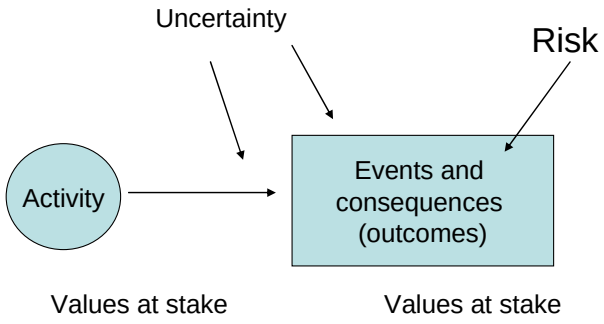

Figure 1. Risk defined as an event or a consequence (Aven & Renn 2008)

According to these definitions, risk expresses a state of the world independent of our knowledge and perceptions. Referring to risk as an event or a consequence, we cannot conclude on risk being high or low, or compare options with respect to risk. Compared to standard terminology in risk research and risk management, they lead to conceptual difficulties that are incompatible with the everyday use of risk in most applications, as discussed in the following.

An analogous definition to 1-2) is found in reliability theory. Here the term unavailability is normally used as the expected fraction of time the system being considered is unavailable, i.e. is not functioning (Aven & Jensen 1999), but we also see unavailability defined in the sense 1 as a state of the world, expressed by the actual fraction of time the system is unavailable (ISO 2005). Then we may consider failures in the system as the "events" according to the definition 1 and the fractional downtime as the consequences. The events and consequences are subject to uncertainties.

The definitions 1-2) mean that risk and assessment of risk are separate domains of the world. The occurrence of a leakage in a process plant is a risk (according to 1). This event is subject to

uncertainties, but the risk concept is restricted to the event "leakage" – the uncertainties and how people judge the uncertainties is a different domain. Hence a risk assessment according to 1-2) cannot conclude for example that the risk is high or low, or that option A has a lower or higher risk than option B, as it makes no sense to speak about a high or higher leakage. Instead the assessment needs to conclude on the uncertainty or the probability of the risk being high or higher.

A similar comment can be made on risk perception.

We conclude that any judgement about risk need to take into account uncertainties/ likelihoods, so why not include this dimension into the risk concept?

We refer to Aven and Renn (2008) for further discussion of this risk perspective.

2.4 Risk is a probability or a probability distribution

We often see risk defined by probabilities. Here are some examples:

- Risk is the probability of an undesirable event (Campbell 2005)
- Risk is the probability of an adverse outcome (Graham & Weiner 1995)
- Risk is a measure of the probability and severity of adverse effects (Lowrance 1976)
- Risk is the combination of probability of an event and its consequences (ISO 2002)
- Risk is defined as a set of scenarios s_i, each of which has a probability p_i and a consequence c_i (Kaplan & Garrick 1981, Kaplan 1991).

Clearly, the first definition is inadequate as a description of risk, as the consequences and outcomes are not taken into account. If we consider the undesirable event "machine failure", the consequences can range from negligible to disaster depending on the availability and performance of a set of barriers, as well as the extent of exposure of human lives, and other objects that human values.

However, also the other probability based definitions can be challenged. A probability is not capturing all aspects of concern. To explain this we need to first introduce the two common ways of interpreting a probability: the classical relative frequency interpretation and the subjective Bayesian interpretation.

According to the classical relative frequency paradigm, a probability is interpreted as the relative fraction of times the events occur if the situation analyzed were hypothetically "repeated" an infinite number of times. The underlying probability is unknown, and is estimated in the risk analysis. Hence if this interpretation is adopted in the above definitions of risk, we have to take into account that the risk estimates could be more or less accurate relative to the underlying true risk. The uncertainties in the estimates could be very large, and difficult to express.

The alternative (the Bayesian perspective) considers probability as a measure of uncertainty about events and outcomes (consequences), seen through the eyes of the assessor and based on the available background information and knowledge. Probability is a subjective measure of uncertainty, conditional on the background information. The reference is a certain standard such as drawing a ball from an urn. If we assign a probability of 0.4 for an event A, we compare our uncertainty of A to occur with drawing a red ball from an urn having 10 balls where 4 are red. Objective probabilities do not exist.

Following this perspective, we assign a probability by performing uncertainty assessments, and there is no reference to a correct or true probability. A probability is always conditional on the background knowledge, and given this background knowledge there is no uncertainty related to the assigned probability, as it is an expression of uncertainty.

However, a probability is not a perfect tool for this purpose. The assigned probabilities are conditional on a specific background knowledge, and they could produce poor predictions.

Surprises relative to the assigned probabilities may occur, and by just addressing probabilities such surprises may be overlooked (Aven 2007a, 2008).

Let us look at two examples:

Maintenance

We consider an offshore petroleum installation where the operations management is concerned about the deterioration of some critical equipment. The maintenance discipline ensures that the deterioration will not cause safety problems. It refers to a special maintenance program that will be implemented, which will cope with the problem. So what is the risk associated with hydrocarbon leakages caused by operational problems. Given the background information of the maintenance discipline, a leakage probability (for a defined size) of 10% is assigned. This number is based on relevant historical data, and do not in any respect reflect the concern of the operations management. The assignment assumes that the maintenance program will be effective. But surprises could occur. Production of oil over time lead to changes in operating conditions, such as increased production of water, H_2S and CO_2 content, scaling, bacteria growth, emulsions, etc.; problems that to large extent need to be solved by the addition of chemicals. These are all factors causing increased likelihood of corrosion, material brittleness and other conditions that may cause leakages.

By the assignment of 10% we hide an important element of uncertainty. In a risk analysis a number of such probability assignments are performed, and the hidden uncertainties could create surprising outcomes someplace. You do not know where it will come, but it certainly could happen.

Offshore diving activities

Consider the risk, seen through the eyes of a risk analyst in the 1970s, related to future health problems for divers working on offshore petroleum projects. An assignment is to be made for the probability that a diver would experience health problems (properly defined) during the coming 30 years due to the diving activities. Let us assume that an assignment of 1% is made. This number is based on the available knowledge at that time. There are not strong indications that the divers will experience health problems. However, we know today, that these probabilities led to poor predictions. Many divers have experienced severe health problems (Aven & Vinnem 2007, p. 7). By restricting risk to the probability assignments alone, we see that aspects of uncertainty and risk are hidden. There is a lack of understanding about the underlying phenomena, but the probability assignments alone are not able to fully describe this status.

2.5 Risk equals expected disutility

If X is the outcomes (consequences) and u(X) the utility function, risk defined by the expected disutility is given by -Eu(X) (Campbell 2005). Hence the preferences of the decision maker is a part of the risk concept. The result is a mixture of scientific assessments of uncertainties about X and the decision makers preferences concerning different values of X. We consider this to be an unfortunate mixture. There will be a strong degree of arbitrariness in the choice of the utility function, and some decision makers would also be reluctant to specify the utility function as it reduces their flexibility to weight different concerns in specific cases. Risk should be possible to describe also in case that the decision maker is not able or willing to define his/her utility function.

2.6 Risk is restricted to the case of known probabilities

In economic applications a distinction has traditionally been made between risk and uncertainty, based on the availability of information. Under risk the probability distribution of the

performance measures can be assigned objectively, whereas under uncertainty these probabilities must be assigned or estimated on a subjective basis (Douglas 1983). This definition goes back to Knight (1921).

Although this definition is often referred to, it is not so often used in practice. The problem is of course that we seldom have known distributions, and then we cannot refer to the risk concept. The Knightian definition violates the intuitive interpretation of risk, which is related to situations of uncertainty and lack of predictability.

2.7 Risk based on subjective probabilities is the same as risk perception

In cultural theory and constructivism, *risk is the same as risk perception* (Rosa 1998). Risk coincides with the perceptions of it (Douglas & Wildavsky 1982, Freudenburg 1989). Beck (1992) concludes that "because risks are risks in knowledge, perceptions of risks and risk are not different things, but one and the same". Beck argues that the distinction between risk and risk perception is central to a scientific myth of expertise, according to which the population "perceives risks" but science determines (i.e., identifies and quantifies) risk (Campbell & Currie 2006, p. 152).

This viewpoint of risk being the same as risk perception is, however, not confined to these paradigms and scientists (Rosa 1998). Rosa (1998) refers for example to the leading risk psychometrician Paul Slovic who has written: 'Human beings have invented the concept of "risk" . . . there is no such thing as "real risk" or "objective risk" (Slovic 1992: 119).

But rejecting the idea that there exists a "real risk" or an "objective risk", does not mean that risk is the same as perceived risk. If probability is a way of expressing uncertainties, seen through the eyes of the assigner (a subjective probability), there is no "real risk" or "objective risk". However, subjective probabilities and related risk assignments are not the same as risk perception. You may assign a probability equal to 0.000000001 for an event to occur, but still find the risk to be intolerable. Our judgments about risk are as we know from many risk perception studies influenced by a number of factors outside the realm of the probabilistic world. The assigned probability and the risk perceptions are different dimensions, or separate domains of the world using Rosa's words.

In the case that the risk perspective is based on the idea that a true risk exists, it is obvious that the risk = risk perception thesis is wrong, refer Campbell (2005, p. 230). The above analysis shows that this thesis is invalid also for other risk perspectives.

We refer to Aven and Renn (2008).

2.8 Objective risk exists

The classical relative frequency approach to risk is based on the idea of an underlying true, objective probability. We understand the meaning of this perspective in gambling situations, but what does this idea of a true probability mean in a more complex context. Consider for example the probability of a terrorist attack, i.e. P(attack occurs). How can this probability be understood by reference to a thought-constructed repeated experiment?

It does not work at all. It makes no sense to define a large set of "identical", independent attack situations, where some aspects (for example related to the potential attackers and the political context) are fixed and others (for example the attackers motivation) are subject to variation. Say that the attack probability is 10%. Then in 1000 situations, with the attackers and the political context specified, the attackers will attack in about 100 cases. In these situations the attackers are motivated, but not in the remaining ones. Motivation for an attack in one situation does not affect the motivation in another. For independent random situations such "experiments" are meaningful, but not for deliberate actions as in the terrorism case.

As another example, consider the probability of at least one fatality during one year in a production facility. According to the relative frequency view, this probability is interpreted as the proportion of facilities with at least one fatality when considering an infinite number of similar facilities. This is of course a thought experiment - in real life we just have one such facility. Therefore, the probability is not a property of the unit itself, but the population it belongs to. How should we then understand the meaning of similar facilities? Does it mean the same type of buildings and equipment, the same operational procedures, the same type of personnel positions, the same type of training programmes, the same organizational philosophy, the same influence of exogenous factors, etc. As long as we speak about similarities on a macro level, the answer is yes. But something must be different, because otherwise we would get exactly the same output result for each facility, either the occurrence of at least one fatality or no such occurrence. There must be some variation on a micro level to produce the variation of the output result. So we should allow for variations in the equipment quality, human behaviour, etc. But the question is to what extent we should allow for such variation. For example, in human behaviour, do we specify the safety culture or the standard of the private lives of the personnel, or are these factors to be regarded as factors creating the variations from one facility to another, i.e. the stochastic (aleatory) uncertainty? We see that we will have a hard time specifying what should be the framework conditions of the experiment and what should be stochastic uncertainty. In practice we seldom see such a specification carried out, because the framework conditions of the experiment are tacitly understood. As seen from the above example, it is not obvious how to make a proper definition of the population.

2.9 Risk is determined by the historical data

To many people, risk is closely related to accident statistics. Numerous reports and tables are produced showing the number of fatalities and injuries as a result of accidents. The statistics may cover the total number of accidents associated with an activity within different consequence categories (loss of lives, personal injuries, material losses, etc.) and they could be related to different types of accidents, such as industrial accidents and transport accidents.

Often the statistics are related to time periods, and then time trends can be identified. More detailed information is also available in some cases, related to, for example, occupation, sex, age, operations, type of injury, etc.

Do these data provide information about the future, about risk? Yes, although the data are historical data, they would usually provide a good picture of what to expect in the future. If the numbers of accidental deaths in traffic during the previous five years are 1000, 800, 700, 800, 750, we know a lot about risk, even though we have not explicitly expressed it by formulating predictions and uncertainties. This is risk related to the total activity, not to individuals. Depending on your driving habits, these records could be more or less representative for you.

However, historical data may exclude extreme observations, but this does not preclude such observations to occur in the future. The historical figures can obviously lead to poor predictions.

By attempting to understand the data, by looking for trends, we may be able to improve the predictions. But we may also end up "over-interpreting" the data in the sense that we look for all sorts of explanations for why the historical figures are as they are. The data may indicate that the quantity of interest (for example the injury rate) is increasing, but perhaps the trend arrow will be reversed next month. We can analyse possible underlying conditions that can affect the quantity, but it is not easy to reflect what the important factors are, and what is "noise" or arbitrariness.

An analysis based on the historical numbers could easily become too narrow and imply that extreme outcomes are ignored. Surprises occur from time to time, and suddenly an event could occur that dramatically changes the development, with the consequence that the quantity of interest

jump up or down. In a risk analysis such events should be identified. However, the problem is that we do not always have the knowledge and insights to be able to identify such events, because they are extremely unexpected.

2.10 Risk relates to negative consequences

Most people associate the word risk with something undesirable and negative. There are, however, good reasons for not restricting the risk concept to negative consequences, and many definitions of risk relate risk to both negative and positive consequences. What is a negative consequence or outcome? To some, an outcome can be negative, and for others a positive outcome. We wish to avoid a discussion on whether a consequence is classified in the correct category. In a risk assessment, the aim is to uncover all relevant consequences, then assess uncertainties and assign probabilities.

2.11 Risk and probability cannot be determined in case of large uncertainties

It is common to hear statements saying that risk and probability cannot be determined in case of large uncertainties. It is however a myth. Risk analyses can always be carried out. Risk can always be expressed, regardless of access to input data. Through the risk analysis the knowledge and lack of knowledge one has concerning various quantities are expressed. Of course, in a case of large uncertainties, it will be difficult to establish good predictions, but the purpose of the analysis is to describe the uncertainties. See the following section.

2.12 There are large inherent uncertainties in risk analyses

The risk analysis systemizes available knowledge and uncertainties about phenomena, systems, and activities that are being studied. What can go wrong, why, and what are the consequences? This knowledge and this uncertainty are described and discussed, and thereby we obtain a basis on which we can evaluate what is important and compare different solutions and measures.

If one has a large and relevant data base, the probabilities derived from it could be precise in the sense that they may be able to provide accurate predictions of future events. For example, assume that one has observed 200 failures in a population of 10 000 units of type T over a one year period. The derived probability of failure for one arbitrary chosen unit is then 2%, and we will predict for example 20 failures per thousand units. We can express the uncertainty, for example, using a 95% prediction interval: [13, 31]. The number of failures will lie within this interval with a 95% probability. To establish this interval, let X denote the number of failures among 1000 units. Then X has a binomial distribution, which can be approximated by a normal distribution with mean 20 and standard deviation 4.4, and this gives $P(11 \leq X \leq 29) = 0.95$.

In a risk analysis context, we often focus on rare events, for example, the occurrence of a fatal accident, an accident that causes impairment of a main safety function, etc. We have only one unit or activity, and we are able to give a good prediction about the future: no fatal accidents will occur the next year. Fortunately, such a prediction will normally provide correct results. The risk analysis however, should also express the likelihood associated with whether the event will occur. This raises the question about precision in the probability assignment.

Many risk analyses today are characterized either by silence on the subject, or by general statements such as:

> The analyses are based on the "best estimates" obtained by using the company's standards for models and data. It is acknowledged that there are uncertainties associated with all elements in the analysis, from the hazard identification to the models and probability calculations. It is concluded that the precision of the analysis is limited, and that one must take this into considerations when comparing the results with the risk acceptance criteria and tolerability limits.

The above statement is not very convincing, and it is obvious that there is no clarity regarding what the analyses express, and what uncertainty means in a risk analysis context.

In any event, does this acknowledgment -- that a considerable amount of uncertainty exists -- affect the analyses and the conclusions? Only very rarely! Our impression is that one writes such statements just to meet a requirement, and then they are put aside. This says a lot about the quality of the analyses.

If the goal of the risk analysis is to obtain reliable, i.e. accurate, estimates of some true risk, we can quickly conclude that risk analysis fails as a scientific method. Referring to Section 2.4, we can conclude that the classical approach to risk analysis does not work in situations of large uncertainties. The uncertainties of the risk estimates are too large.

Alternatively, we may consider risk analysis as a tool for assessing uncertainties about risk and risk estimates. Since the assessment's aim then is to express uncertainties about the true risk, reliability is not related to the accuracy in the results but rather the "accuracy" of the transformation of uncertainties to probabilities. Risk analysis is then not about bounding and reducing uncertainties, but to describe uncertainties. Two prevailing approaches for describing the uncertainties are:

Traditional statistical methods such as confidence intervals

The probability of frequency approach, i.e. assessing epistemic uncertainties about the risk by means of subjective probabilities. In this approach there are two levels of probability introduced; i) the relative frequency interpreted probabilities reflecting variation within populations and ii) the subjective probabilities reflecting the analyst's uncertainty what the correct relative frequency probabilities are (see e.g. Kaplan & Garrick (1981) and Aven (2003)). In Garrick et al. (2004) the probability of frequency approach is suggested for risk analysis of attacks. Garrick et al. (2004) refer to a probability distribution saying for example that there is a probability of 20% that the attackers would succeed in 10% of their attacks.

However, confidence intervals would not work in this setting as we do not have sufficient amount of relevant data.

The ambition of the probability of frequency approach is to express the epistemic uncertainties of the probability p of an attack, and take into account all relevant factors causing uncertainties. The analysis may produce a 90% credibility interval for p, [a, b], saying that the analyst is 90% confident that p lies in the interval [a, b]. In practice it is difficult to perform a complete uncertainty analysis following this approach. In theory an uncertainty distribution on the total model and parameter space should be established, which is impossible to do. So in applications only a few marginal distributions on some selected parameters are normally specified, and therefore the uncertainty distributions on the output probabilities are just reflecting some aspects of the uncertainty. This makes it difficult to interpret the produced uncertainties.

If the risk perspective is based on probability being a measure of uncertainty seen through the eyes of the assessor, and based on a background knowledge, we can argue along the same lines. As for the probability of frequency approach, we conclude that this approach in general meets the reliability requirement, if reliability is associated with subjective probability assignments and these follow the standards established for such assignments.

2.13 Risk acceptance criteria contribute to obtaining a high standard of safety

To manage safety and security, it is common to use a hierarchy of goals, criteria and requirements, such as

- overall ideal goals, for example "our goal is to have no accidents"
- risk acceptance criteria (defined as upper limits of acceptable risk) or tolerability limits, controlling the accident risk, for example "the individual probability of being killed in an accident shall not exceed 0.1 %"
- requirements related to the performance of safety systems and barriers, such as a reliability requirement for a safety system
- requirements related to the specific design and operation of a component or subsystem, for example the gas detection system.

According to the standard procedures for using such goals, criteria and requirements, they are to be specified before alternatives are generated and subsequently analysed. The point is to look for what to obtain before looking for possible ways of implementation. For example, the Norwegian offshore petroleum regulations state that risk acceptance criteria (expressed as upper limits of acceptable risk) should be developed, and before the risk analyses are carried out (PSA 2001, Aven 2007b). Note that we in the following, when using the term "risk acceptance criteria", always have in mind such upper limits.

Are such criteria appropriate for managing investments in safety and security? With large uncertainties, it is not meaningful to use such limits as the precision level is poor. However, it is also questionable to use such criteria where there is no discussion about the risk picture as such.

Consider the following criterion for an offshore installation:

The probability of getting an oil spill during one year of operation causing an environmental damage having a restitution period of more than z years, should not exceed $1 \cdot 10^{-x}$.

At the political level it is obvious that it would not be possible to establish consensus about such a limit. Different parties, would have different preferences. But for the Government it should be possible to establish such a number? Say that it would make an attempt to do this. And suppose that it considers two options, a weak limit, say $1 \cdot 10^{-3}$ and a strong limit say $1 \cdot 10^{-4}$. What limit should it choose? The answer would be the weak limit, as the strong limit could mean lack of flexibility in choosing the overall best solution. If the benefits are sufficient large, the level $1 \cdot 10^{-3}$ could be acceptable. Following this line of arguments, the use of such limits leads to the formulation of weak limits, which are met in most situations. Risk assessments are then used to test whether risks are acceptable in relation to these weak limits. It is to large extent waste of money, the conclusions are obvious.

At the operational level, the same type of arguments will apply. The oil company is to determine an acceptance criterion, and it faces the same type of dilemmas as above. Why should it specify strong limits? It would restrict the company from obtaining the overall best solutions. The result is that weak limits are specified and risk assessments play the role of verification, a role that adds not much value.

If a high level of safety or security is to be obtained, other mechanisms need to be implemented than risk acceptance criteria. If such criteria are established, they give a focus on obtaining a minimum safety standard, instead of continuous improvement and risk reduction.

The ALARP principle represents such a mechanism. The ALARP principle expresses that the risk should be reduced to a level that is as low as reasonably practicable. A risk reducing measure

should be implemented provided it cannot be demonstrated that the costs are grossly disproportionate relative to the gains obtained (HSE 2001).

2.14 ALARP can be verified by cost-benefit analyses

The ALARP principle implies what could be referred to as the principle of 'reversed onus of proof'. This implies that the base case is that all identified risk reduction measures should be implemented, unless it can be demonstrated that there is gross disproportion between costs and benefits. To verify ALARP, procedures mainly based on engineering judgments and codes are used, but also traditional cost-benefit analyses and cost effectiveness analyses. When using such analyses, guidance values are often used, to specify what values that define 'gross disproportion'.

The practice of using traditional cost-benefit analyses and cost effectiveness analyses to verify ALARP has been questioned (Aven & Abrahamsen 2007). The ALARP principle is an example of application of the cautionary principle (see Section 2.15). Uncertainty should be given strong weight, and the grossly disproportionate criterion is a way of making the principle operational. However, cost-benefit analyses calculating expected net present values to large extent ignore the risks (uncertainties) and the use of this approach to weight the risk and uncertainties is therefore problematic. The same applies to the cost-effectiveness indices such as the expected cost per expected number of saved lives (referred to as the implied cost of averting a statistical fatality, ICAF) which are often used instead of full cost-benefit analyses. If a measure costs 2 million euros and the risk analysis shows that the measure will bring about a reduction in the number of expected fatalities by 0.1, then the ICAF is equal to 2/0.1 = 20 million euros. By comparing this number with reference values, we can express the effectiveness of the measure.

Modifications of the traditional cost-benefit analysis are suggested to cope for this problem, see e.g. Aven & Flage (2008). In these methods, adjustments are made on either the discount rate or the contribution from the cash flows. This latter case could be based on the use of certainty equivalents for the uncertain cash flows. Although arguments are provided to support these methods, their rationale can be questioned. There is a significant element of arbitrariness associated with the methods, in particular when seen in relation to the standard given by the expected utility theory.

2.15 The cautionary/pre-cautionary principles and risk management cannot be meaningfully integrated

It is common among engineers, economist and others to consider a rational decision in face of uncertainties to be based on risk and decision analysis, including cost-benefit analysis. It is irrational to give weight to the cautionary and pre-cautionary principles. However, a further look into this view shows that is cannot be justified.

The cautionary principle is a basic principle in risk management, expressing that in the face of uncertainty, *caution* should be a ruling principle, for example by not starting an activity, or by implementing measures to reduce risks and uncertainties (HSE 2001, Aven & Vinnem 2007). The level of caution adopted will of course have to be balanced against other concerns such as costs. However, all industries would introduce some minimum requirements to protect people and the environment, and these requirements can be considered justified by the reference to the cautionary principle.

For example, in the Norwegian petroleum industry it is a regulatory requirement that the living quarters on an installation should be protected by fireproof panels of a certain quality, for walls facing process and drilling areas. This is a standard adopted to obtain a minimum safety level. It is based on established practice of many years of operation of process plants. A fire may occur

which represents a hazard for the personnel, and in the case of such an event, the personnel in the living quarters should be protected. The assigned probability for the living quarter on a specific installation being exposed to fire may be judged as low, but we know that fires occur from time to time in such plants. It does not matter whether we calculate a fire probability of x or y, as long as we consider the risks to be significant; and this type of risk has been judged to be significant by the authorities. The justification is experience from similar plants and sound judgments. A fire may occur, since it is not an unlikely event, and we should then be prepared. We need no references to cost-benefit analysis. The requirement is based on a cautionary thinking.

Risk analyses, cost-benefit analyses and similar types of analyses are tools providing insights into risks and the trade-offs involved. But they are just tools - with strong limitations. Their results are conditioned on a number of assumptions and suppositions. The analyses do not express objective results. Being cautious also means reflecting this fact. We should not put more emphasis on the predictions and assessments of the analyses than what can be justified by the methods being used.

In the face of uncertainties related to the possible occurrences of failures, hazardous situations and accidents, we are cautious and adopt principles of risk management, such as

- robust design solutions, such that deviations from normal conditions are not leading to hazardous situations and accidents,
- design for flexibility, meaning that it is possible to utilise a new situation and adapt to changes in the frame conditions,
- implementation of safety barriers, to reduce the negative consequences of hazardous situations if they should occur, for example a fire,
- improvement of the performance of barriers by using redundancy, maintenance/ testing, etc.
- quality control/ quality assurance,
- the precautionary principle, saying that in the case of lack of scientific certainty on the possible consequences of an activity, we should not carry out the activity.
- the ALARP-principle, saying that the risk should be reduced to a level which is as low as reasonably practicable.

Thus the precautionary principle may be considered a special case of the cautionary principle, as it is applicable in cases of scientific uncertainties (Sandin 1999, Löfstedt 2003, Aven 2006). There are however many definitions of the precautionary principle. The well-known 1992 Rio Declaration use the following definition:

In order to protect the environment, the precautionary approach shall be widely applied by States according to their capabilities. Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation.

Seeing beyond environmental protection, a definition such as the following reflects what is a typical way of understanding this principle:

The precautionary principle is the ethical principle that if the consequences of an action, especially the use of technology, are subject to scientific uncertainty, then it is better not to carry out the action rather than risk the uncertain, but possibly very negative, consequences.

In the following we will refer to the cautionary and precautionary principles, and in this way avoid a discussion about whether we refer to the cautionary principle or the precautionary principle. The distinction is not essential for the purpose of this paper.

We have to acknowledge that there exists no simple and mechanistic method or procedure for dealing with uncertainties and balancing different concerns. This is also recognised by many others

analysts, see e.g. the approach adopted by the risk governance framework (Renn 2005) and the risk framework used by the UK Cabinet Office (2002).

Uncertainty is an important aspect of risk, and hence the cautionary and precautionary principles constitute important aspect of risk management.

3. CONCLUSIONS

We suggest the following definition of risk (Avenn 2007a):

> *By risk we understand the two-dimensional combination of i) events A and the consequences of these events C, and ii) the associated uncertainties U (wil A occurs and what value will C take)* (I)
>
> We refer to this definition as the (C,U) risk definition. For simplicity, we write only C, instead of A and C.

We may rephrase this definition by saying that risk associated with an activity is to be understood as (Aven & Renn 2008):

Uncertainty about and severity of the consequences of an activity (I'),

where severity refers to intensity, size, extension, and so on, and is with respect to something that humans value (lives, the environment, money, etc). Losses and gains, for example expressed by money or the number of fatalities, are ways of defining the severity of the consequences.

The main features of the definition are illustrated in Figure 2.

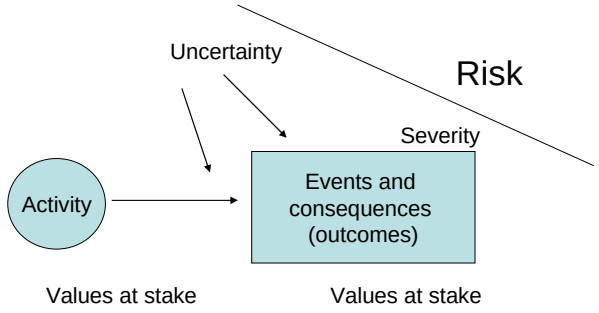

Figure 2. Illustration of the risk definition (I, I')

The uncertainty relates to both the event and the consequences given that this event occurs.

We see that the definition is based on the combined dimensions Consequences (outcome stakes) and Uncertainties. It is essential that the second dimension is uncertainties and not probabilities. The concept of risk should cover uncertainties beyond probabilities. Probabilities and expected values are not perfect tools for expressing uncertainties, as discussed in Section 2.4.

Note that following our definition of risk (I), a low degree of uncertainty does not necessarily mean a low risk, or a high degree of uncertainty does not necessarily mean a high level of risk. As risk is defined as the two-dimensional combination of consequences and uncertainties, any judgment about the level of risk, needs to consider both dimensions. See example in Section 2.2.

The risk concept is supporting a broad perspective on risk, as for example shown by the way risk assessment are conducted. In stead of a traditional quantitative risk assessment approach, we

recommend a more qualitative approach or semi-quantitative approach (Aven 2008). The basic features of the approach can be summarised as follows:

A broad qualitative risk picture is established highlighting

- Potential hazards/threats and accident scenarios
- Barriers and the effectiveness of these barriers
- Risk influencing factors and possible risk reducing measures
- Uncertainties in phenomena and processes
- Vulnerabilities
- Special features of the consequences
- Manageability factors

Crude risk categorisations are defined based on this risk picture, reflecting

- Probabilities/frequencies of hazards/threats
- Expected losses given the occurrence of such a hazard/threat
- Factors that could create large deviations between expected outcomes and the actual outcomes (uncertainties, vulnerabilities)

Evaluations of the risk picture and categorisations to compare alternatives and make judgments about risk acceptance.

Quantifying risk using risk indices such as expected number of fatalities gives an impression that risk can be expressed in a very precise way. However, in most cases, the arbitrariness is large, and the semi-quantitative approach acknowledges this by providing crude risk numbers, including assessments of the factors that can cause "surprises" relative to the probabilities and expected values. We are not opposed to detailed risk quantification as such, but quantification often requires strong simplifications and assumptions and as result, important factors could be ignored or given too little (or much) weight. In a qualitative or semi-quantitative analysis a more comprehensive risk picture can be established, taking into account underlying factors influencing risks. In contrast to the prevailing use of quantitative risk assessments, the precision level of the risk description is in line with the accuracy of the risk assessment tool. In addition, risk quantification is very resource demanding. We need to ask whether the resources are used in the best way. We conclude that in many cases more is gained by pursuing a broader more qualitative approach, which allows for considerations beyond the probabilities and expected values.

For problems with large uncertainties, risk assessments could support decision making, but other principles, measures and instruments are required, such as the cautionary principle.

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CREDIBILISTIC FUZZY REGRESSION

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ABSTRACT

In reliability, quality control and risk analysis, fuzzy methodologies are more and more involved and inevitably introduced difficulties in seeking fuzzy functional relationship between factors. In this paper, we propose a scalar variable formation of fuzzy regression model based on the credibility measure theoretical foundation. It is expecting our scalar variable treatments on fuzzy regression models will greatly simplify the efforts to seeking fuzzy functional relationship between fuzzy factors. An *M*-estimator for the regression coefficients is obtained and accordingly the properties and the variance-covariance for the coefficient *M*-estimators are also investigated in terms of weighted least-squares arguments. Finally, we explore the asymptotic membership function for the coefficient *M*-estimators.

1 INTRODUCTION

In statistical theory, regression is an important topic for modeling the functional relationship between response variable(s) and exploratory variable(s) under random uncertainty assumptions. When data is fuzzy, fuzzy regression models were also developed although mostly on the ground of Zadeh's fuzzy mathematics (1965, 1978). We noticed that more and more system dynamics researchers in reliability, quality and risk analysis engage into fuzzy approach. However, whenever we use fuzzy set theory for practical modeling, we will face a sequence of fundamental issues:

The first one is the self-duality in its theoretical foundation. Fuzzy mathematics initiated by Zadeh (1965) facilitated a foundation dealing with vague phenomena in fuzzy modeling. Nevertheless, the fuzzy mathematical foundation initiated by Zadeh (1965, 1978) is membership function and possibility measure based and widely used. The possibility measure was originally expected to play the role of probability measure in probability theory, but could not because it does not possess self-duality property as that in probability theory.

The second issue is the variable-orientation issue. In standard probability theory, random variable and the distribution function play important roles for converting set-based arguments into variable-based arguments, which result in great conveniences in applications. Kaufmann (1975) first proposed the concept of fuzzy variable with the intention of creating its counterpart in probability theory. Unfortunately, Kaufmann's fuzzy variable is in fact another name for a fuzzy subset, and the mathematical operations are difficult to handle.

The third one is the membership specification issue. During almost four decades, fuzzy researchers have to specify membership function and set up the parameter values in terms of their own working experiences. Compared to the probabilistic counterpart, for random variable and its

distribution, very rich (data-oriented) statistical estimation and hypothesis testing theory have been developed. The fuzzy statistical theory developed very slowly, and its applications are difficult due to the set-oriented foundation.

To resolve the three dilemmas, Liu (2004, 2007) proposed an axiomatic foundation for modelling fuzzy phenomena, credibility measure theory. The credibility measure possesses self-duality property and is able to play the role of that in probability theory. Furthermore, fuzzy variable concept and its (credibility) distribution, which are parallel to these in probability theory, are developed.

Many set variable oriented fuzzy regression models were proposed: Tanaka et al. (1980, 1982) initiated an approach of fuzzy regression which minimizes the fuzziness as an optimal criterion; Diamond (1987, 1988) used least-squared errors as a decision criterion; Interval regression oriented fuzzy regression model was also presented (Dubois and Prade, 1980),(Kacprzyk and Fedrizzi, 1992). It is also noticed that there is variable treatment in fuzzy regression in terms of numerical valued approaches in terms of the usage of representative values, say, the fuzzy mode, the fuzzy average, the fuzzy median, or the mid-range of α -cut set of fuzzy membership function to specify the fuzzy subset. However the fundamental weakness of these numerical valued treatments on fuzzy subsets lies on the utilization of the partial information of the fuzzy subsets under study.

In this paper, based on Liu's (2004, 2007) classical credibility measure theory, i.e., (Ъ,Ш)-credibility measure theory, we develop a scalar variable oriented treatment in terms of an M-estimation for the fuzzy regression coefficients, which leads to weighted least-squares formation.

The structure of this paper is as follows. Section two is used for reviewing Liu's credibility measure theory, defining the scalar fuzzy variable and further comparing Liu's and Zadeh's fuzzy theories. In Section 3, the *M*-function utilizing maximum membership grades is introduced and therefore, the *M*-estimators for regression coefficients are derived. In Section 4, the properties of regression coefficient *M*-estimators are investigated and in Section five, we propose asymptotic membership function under the assumptions of fuzzy errors being taken normal membership function, which will lead to asymptotic credibility distribution. In section 6, we explore the fuzzy regression formation when the sample data are taken from different memberships. Section 7 offers an extension to multiple regression treatment. Finally a few concluding remarks are offered in Section 8.

2 A REVIEW ON CREDIBILITY MEASURE THEORY

Let Θ be a nonempty set, and 2^{Θ} the power set on Θ. A power set is the set class containing all the possible subsets of nonempty set Θ, i.e., $2^{\Theta} = \{A : A \subset \Theta\}$. It is obvious that a power set 2^{Θ} is the largest σ-algebra on Θ. Each element of a power set, say, $A \subset \Theta$, $A \in 2^{\Theta}$ is called an event. A number denoted as $\mathrm{Cr}\{A\}$, $0 \le \mathrm{Cr}\{A\} \le 1$, is assigned to an arbitrary event $A \in 2^{\Theta}$, which indicates the credibility grade with which event $A \in 2^{\Theta}$ occurs. For any $A \in 2^{\Theta}$, set function $\mathrm{Cr}\{A\}$ satisfies following axioms (Liu, 2004, 2007):

Axiom 1: $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2: $\mathrm{Cr}\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$.

Axiom 3: $\mathrm{Cr}\{\cdot\}$ is self-dual, i.e., for any $A \in 2^{\Theta}$, $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$.

Axiom 4: $\mathrm{Cr}\{\bigcup_i A_i\} \wedge 0.5 = \sup_i [\mathrm{Cr}\{A_i\}]$ for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\} \le 0.5$.

Axiom 5: Let set functions $\mathrm{Cr}_k\{\cdot\} : 2^{Q_k} \to [0,1]$ satisfy **Axioms 1-4**, and $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_p$, then:

$$\begin{aligned} &\mathrm{Cr}\{q_1, q_2, \ldots, q_p\} \\ &= \mathrm{Cr}_1\{q_1\} \amalg \mathrm{Cr}_2\{q_2\} \amalg \cdots \amalg \mathrm{Cr}_p\{q_p\} \end{aligned} \tag{1}$$

where $\{q_1, q_2, \mathrm{L}, q_p\} \mathrm{O} 2^{\mathrm{Q}}$.

Definition 2.1: (Liu, 2004, 2007) Any set function $\mathrm{Cr}: 2^{\mathrm{Q}} \mathrm{R} [0,1]$ satisfies **Axioms 1-4** is called a **(Ъ,Ш)**-credibility measure (or simply a credibility measure). The triple $(\mathrm{Q}, 2^{\mathrm{Q}}, \mathrm{Cr})$ is called the credibility measure space.

A credibility measure satisfies all the properties of uncertainty measure and also many of its own. For space limitation reason, we can only review minimal materials, but more technical details can be found in Liu (2007).

Definition 2.2: (Liu, 2004, 2007) A fuzzy variable ξ is a mapping from credibility space $(\mathrm{Q}, 2^{\mathrm{Q}}, \mathrm{Cr})$ to the set of real numbers, i.e., $\xi: (\Theta, 2^{\Theta}, \mathrm{Cr}) \to \mathbf{R}$.

We should be fully aware that on the credibility measure platform, a fuzzy variable is recorded as a real-valued number similar to that of a random variable. Definitely, similar to random variable, a real number as a realized value of a fuzzy variable has a distributional grade associated with it.

Definition 2.3: (Liu, 2004, 2007) The credibility distribution $\Lambda: \mathrm{R} \to [0,1]$ of a fuzzy variable ξ on $(\mathrm{Q}, 2^{\mathrm{Q}}, \mathrm{Cr})$ is:

$$\Lambda(x) = \mathrm{Cr}\{\theta \in \Theta | \xi(\theta) \le x\} \tag{2}$$

The credibility distribution $\Lambda(x)$ is the accumulated *credibility grade* that the fuzzy variable ξ takes a value less than or equal to a real-number $x \in \mathrm{R}$. Generally speaking, the credibility distribution $\Lambda(\cdot)$ is neither left-continuous nor right-continuous. What we will deal with is absolutely continuous fuzzy variables with continuous credibility density functions and thus poses no further restrictions on our developments.

Definition 2.4: (Liu, 2004, 2007) Let $\Lambda(\cdot)$ be the credibility distribution of the fuzzy variable ξ. Then function $\lambda: \mathrm{R} \to [0, +\infty)$ of a fuzzy variable ξ is called a credibility density function such that,

$$\Lambda(x) = \int_{-\infty}^{x} \lambda(y)\mathrm{d}y, \quad \forall x \in \mathrm{R} \tag{3}$$

The axiomatic credibility measure foundation is the starting point of Liu's fuzzy theory, while the definition of a membership function is the fundamental starting point of Zadeh's fuzzy set theory. Zadeh (1978) further proposed possibility measure based theoretical framework and expected the possibility measure could be a counterpart of probability measure, nevertheless, Zadeh failed his own mission. Table 1 offers comparisons between the two fuzzy theories:

Table 1. Comparison between Zadeh's and Liu's Fuzzy Theories

Item	Zadeh's	Liu's
Cornerstone concept	Possibility measure $\mathrm{Poss}\{\cdot\}$	Credibility measure $\mathrm{Cr}\{\cdot\}$
Axiomatic Foundation	No	Yes, four axioms
Membership	Initial concept $\mu_A: \Theta \to [0,1]$	Induced $\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1$
Measure space	$(\mathrm{Q}, \%, \mathrm{Poss})$	$(\mathrm{Q}, 2^{\mathrm{Q}}, \mathrm{Cr})$
Self-duality	No, $\mathrm{Poss}\{A\} + \mathrm{Poss}\{A^c\} \ne 1$	Yes, $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$
Identical transmogrification	No. For any fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C}$, $\tilde{A} = \tilde{B} + \tilde{C}$; Does not imply $\tilde{B} = \tilde{A} - \tilde{C}$	Yes. For any fuzzy variables $\tilde{\eta}, \tilde{\gamma}, \tilde{\zeta}$, $\tilde{\eta} = \tilde{\gamma} + \tilde{\zeta}$; Does imply $\tilde{\gamma} = \tilde{\eta} - \tilde{\zeta}$

Item	Zadeh's	Liu's
Link between two fuzzy theories	$\mu(x)=\left(2\mathrm{Cr}\{\xi=x\}\right)\wedge 1$	$\mathrm{Cr}\{B\}=\frac{1}{2}\left(\mathrm{Poss}\{B\}-\mathrm{Nec}\{B\}\right)$
Membership	Initially defined concept $\mu_A:\Theta\to[0,1]$	Secondarily defined concept $\mu(x)=\left(2\mathrm{Cr}\{\xi=x\}\right)\wedge 1$

The next definition describes membership in terms of Liu's credibility measure, together with related theorems, a link between Zadeh's membership-initiated fuzzy mathematics and Liu's credibility measure-oriented fuzzy mathematics has been established in nature. The linkage definitely gives an intuitive understanding of Liu's credibility measure concept and also paves the way of applying credibility measure in practices, particularly, for those who are familiar with membership function concept.

Definition 2.5: (Liu, 2004, 2007) The (induced) membership function of a fuzzy variable ξ on $\left(\mathrm{Q},2^{\mathrm{Q}},\mathrm{Cr}\right)$ is:

$$\mu(x)=\left(2\mathrm{Cr}\{\xi=x\}\right)\wedge 1,\quad x\in\mathbf{R} \tag{4}$$

Conversely, for given membership function the credibility measure is determined by the credibility inversion theorem.

Theorem 2.6: (Liu, 2004, 2007) Let ξ be a fuzzy variable with membership function m, then for $"B\,\mathbf{M}\,\mathbf{R}$,

$$\mathrm{Cr}\{\xi\in B\}=\frac{1}{2}\left(\sup_{x\in B}\mu(x)+1-\sup_{x\in B^c}\mu(x)\right),\ B\subset\mathbf{R} \tag{5}$$

As an example, if the set B is degenerated into a point x, then:

$$\mathrm{Cr}\{\xi=x\}=\frac{1}{2}\left(\mu(x)+1-\sup_{y\neq x}\mu(y)\right),\ \forall x\in\mathbf{R} \tag{6}$$

Theorem 2.7: (Liu, 2004, 2007) Let ξ be a fuzzy variable on $\left(\mathrm{Q},2^{\mathrm{Q}},\mathrm{Cr}\right)$ with membership function μ. Then its credibility distribution,

$$\Lambda(x)=\frac{1}{2}\left(\sup_{y\le x}\mu(y)+1-\sup_{y>x}\mu(y)\right),\ \forall x\in\mathbf{R} \tag{7}$$

It is necessary to emphasize here that with or without membership function fuzzy phenomena in real world can be accurately described by the credibility measure models. Linking between credibility measure and membership plays role of bridging Zadeh's fuzzy mathematics and the new axiomatic fuzzy theory and thus provides a conversion channel.

It is critical to emphasize again at the end of this section that different from Zehad's fuzzy set theory, the fuzzy variable on the Liu's credibility measure foundation is scalar real-valued function characterized by its credibility distribution. Therefore, the mathematical treatment of fuzzy variables on the platform is easier than that based on fuzzy sets as variable in Zadeh's fuzzy set theory.

3 AN *M*-ESTIMATOR FOR REGRESSION COEFFICIENTS

A fuzzy linear model describes a functional relationship containing fuzzy uncertainty. For simplicity, let us start with the simple fuzzy regression model:

$$Y=\alpha+\beta x+\varepsilon \tag{8}$$

where x is exploratory (or independent or controllable) variable, Y is the fuzzy response (or dependent variable), ε is a fuzzy error term with $\mathrm{E}[\varepsilon]=0$ and $V[\varepsilon]=\sigma^2$. Note that the expectation is taken with respect to the credibility distribution. Denote the empirical (or fitted) fuzzy linear regression of Y with respect to x by

$$\hat{Y} = a + bx \tag{9}$$

where vector (a,b) is the estimate of regression coefficient vector (α,β).

Let $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$ be a simple random sample. $Y_1,Y_2,\cdots,Y_n$ denote the *n* observed response corresponding to $x_1,x_2,\cdots,x_n$. In (probabilistic) linear model theory, $\varepsilon \sim N(0,\sigma^2)$ and ε_i and ε_j are uncorrelated. Thus, $Y_i \sim N(\alpha+\beta x_i,\sigma_i^2)$ and Y_i and Y_j are uncorrelated too.

From the credibilistic point of view, the universe should facilitate the error events. Mathematically

$$\Theta = \{\varepsilon_i \mid \varepsilon_i = Y_i - (\alpha+\beta x_i), i = 1,2,\cdots,n\} \tag{10}$$

Let $\tilde{R}$ be a fuzzy event defined on Θ, which connects to a fuzzy concept, {error is close to zero}, denoted by k. A membership, denoted by $\mu_{\tilde{R}}(\varepsilon_i)$, represents the degree of belongingness to the fuzzy concept k. Then the regression model fitting problem now becomes one of finding an empirical linear regression equation $\hat{Y} = a + bx$ based on the observations $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$ such that the membership $\mu_{\tilde{R}}(\varepsilon_i)$ is maximized.

Now, it is ready to investigate the forms of *M*-functions for regression coefficients estimation. Let the residual take a membership function of form

$$\mu_\varepsilon\left(Y_i - (a+bx_i)\right) = g\left(-\left(Y_i - (a+bx_i)\right)^2\right) \tag{11}$$

where g(0) is a differentiable function with g(0)= 1, and $\lim_{x\to-\infty} g(x) = 0$.

The basic idea underlying the searching the coefficients is to maximize the membership grade for any individual observation pair (x_i,Y_i). Then for all *n* pairs of observations $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$, The sum of the membership grades should be maximized. In mathematical language, the object function

$$J \equiv \sum_{i=1}^{n} \mu_\varepsilon\left(Y_i - (a+bx_i)\right) \tag{12}$$

It is obvious that for any individual observation Y_i the contribution to model goodness-of-fit is measured by the membership grade $\mu_\varepsilon(\cdot)$. The closer the observed value Y_i to the fitted value $\hat{Y}_i = (a+bx_i)$, the nearer to membership grade of the difference, i.e., the error, $\tilde{\varepsilon}_i = Y_i - (a+bx_i)$ to 1, which implies that the degree of fuzzy event $\tilde{\varepsilon}_i$ belonging to concept k is high. Therefore, it is reasonable to use the sum of all the membership grades of the *n* observations $Y_1,Y_2,\cdots,Y_n$, for measuring the overall degree of belongingness of observations $Y_1,Y_2,\cdots,Y_n$. A typical membership function satisfying Equation (11) takes a normal form:

$$g(e_i) = \exp\left(-w\left(Y_i - (a+bx_i)\right)^2\right) \tag{13}$$

Now, we can define the *M*-functional equation system for fuzzy regression coefficients (a,b).

Definition 3.1: Given the differentiable membership function $g\left(-\left(Y_i-(a+bx_i)\right)^2\right)$, which measures the degree of belongingness to empirical linear regression line $\hat{Y} = a + bx$ at observation pair (Y_i,x_i), then the normal formed *M*-functional system based on the *n* observations $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$ takes the form

$$\begin{cases} \sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(Y_i-(a+bx_i)\right) = 0 \\ \sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(Y_i-(a+bx_i)\right)x_i = 0 \end{cases} \tag{14}$$

where $h(x) = g'(x) = dg/dx$.

Theorem 3.2: Let a simple regression model $Y=\alpha+\beta x+\varepsilon$ assumes a fuzzy error ε $\mathrm{E}[\varepsilon]=0$ & $V[\varepsilon]=\sigma^2$ and membership function $g\left(-\left(Y_i-(a+bx_i)\right)^2\right)$. For given *n*-pair independent observations $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$, a general *M*-estimator of the coefficients for fitted regression $\hat{Y}=a+bx$, (a,b) is the solution to the general *M*-function equation system Equation (14). Furthermore, the *M*-estimator (a,b) takes a weighted least-square estimator form as

$$\begin{cases} b=\dfrac{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)(x_i-\bar{x}_h)(Y_i-\bar{Y}_h)}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)(x_i-\bar{x}_h)^2} \\ a=\bar{Y}_h-\bar{b}x_h \end{cases} \tag{15}$$

where

$$\begin{aligned} \bar{x}_h &= \sum_{i=1}^{n}\frac{h\left(-\left(Y_i-(a+bx_i)\right)^2\right)}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)}x_i \\ \bar{Y}_h &= \sum_{i=1}^{n}\frac{h\left(-\left(Y_i-(a+bx_i)\right)^2\right)}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)}Y_i \end{aligned} \tag{16}$$

Theorem 3.2 is easy to prove by expanding the left side terms of Equation (14), re-arrange them, utilizing Equation (16) for obtaining Equation (15).

Let

$$\underline{Y}=\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix},\ X=\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix},\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{17}$$

where $d_i=h\left(-\left(Y_i-(a+bx_i)\right)^2\right)$, $i=1,\mathrm{L}\ ,n$. Further, let

$$\underline{\Gamma}=\begin{bmatrix} a \\ b \end{bmatrix},\ W=D^{-1}=\begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \tag{18}$$

Then, the *M*-functional equation system Equation (14) can be re-written in a matrix form:

$$X^TW^{-1}X\underline{\Gamma}=X^TW^{-1}\underline{Y} \tag{19}$$

Equation (19) takes the weighted least-squares normal equation form in statistical linear model theory. However, the weighted least-squares formation of the *M*-functional equation system of Equation (14) will help further mathematical treatments. For example, Equation (15) can be re-expressed in matrix form:

$$\underline{\Gamma}=\left(X^TW^{-1}X\right)^{-1}X^TW^{-1}\underline{Y} \tag{20}$$

as long as the inverse matrix exists.

Example 3.3: Let then membership function g take a normal form:

$$\begin{aligned} &g\left(-\left(Y-(a+bx)\right)^2\right) \\ &=\exp\left(-w\left(Y-(a+bx)\right)^2\right) \end{aligned} \tag{21}$$

Then the derivative of membership function

$$\begin{aligned} & h\left(-w\left(Y-\left(a+bx\right)\right)^2\right) \\ & =\exp\left(-w\left(Y-\left(a+bx\right)\right)^2\right) \end{aligned} \tag{22}$$

The factor w is sample-dependent and can be defined by

$$w=\frac{2}{d_{\max}-d_{\min}} \tag{23}$$

where

$$\begin{aligned} d_{\max} &= \max_{i\in\{1,2,\cdots,n\}}\left\{\left(Y_i-\left(a+bx_i\right)\right)^2\right\} \\ d_{\min} &= \min_{i\in\{1,2,\cdots,n\}}\left\{\left(Y_i-\left(a+bx_i\right)\right)^2\right\} \end{aligned} \tag{24}$$

Then the *M*-estimators for regression coefficients are

$$\begin{cases} b=\dfrac{\sum_{i=1}^{n}\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)\left(x_i-\bar{x}_\infty\right)\left(Y_i-\bar{Y}_\infty\right)}{\sum_{i=1}^{n}\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)\left(x_i-\bar{x}_\infty\right)^2} \\ a=\bar{Y}_\infty-\bar{b}\bar{x}_\infty \end{cases} \tag{25}$$

where

$$\begin{aligned} \bar{x}_\infty &= \sum_{i=1}^{n}\frac{\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)}{\sum_{i=1}^{n}\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)}x_i \\ \bar{Y}_\infty &= \sum_{i=1}^{n}\frac{\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)}{\sum_{i=1}^{n}\exp\left(-w\left(Y_i-\left(a+bx_i\right)\right)^2\right)}Y_i \end{aligned} \tag{26}$$

4 PROPERTIES OF THE M-ESTIMATORS

It should be emphasized at the beginning of this section that the issue of properties of estimator of fuzzy regression coefficients was not really deeply explored. Furthermore, it should be also fully aware that the issue of variance-covariance structure of regression coefficient estimators in statistical linear model is a standard exercise, however, in fuzzy regression developed so far, most of the modeling exercises stopped at obtaining the estimators of regression coefficients. These difficulties rooted in the set-level variable treatments in Zadeh's fuzzy set theory.

Lemma 4.1: *M*-Estimator b is a linear function of observations $\{Y_1,Y_2,\cdots,Y_n\}$. In other words,

$$b=\sum_{i=1}^{n}\kappa_i Y_i \tag{27}$$

where

$$\kappa_i=\frac{h\left(-\left(Y_i-\left(a+bx_i\right)\right)^2\right)\left(x_i-\bar{x}_h\right)}{\sum_{i=1}^{n}h\left(-\left(Y_i-\left(a+bx_i\right)\right)^2\right)\left(x_i-\bar{x}_h\right)^2},\quad i=1,2,\cdots,n \tag{28}$$

The proof of Lemma 4.1 is a straightforward manipulation of Equation (18).

Theorem 4.2: *M*-Estimator for the regression coefficients (α,β), denoted as (a,b), are (conditionally) unbiased. In other words,

$$\mathrm{E}[a]=\alpha,\ \ \mathrm{E}[b]=\beta \tag{29}$$

The proof is a straightforward task by noticing that $\sum_{i=1}^{n} k_i = 0$ and $\sum_{i=1}^{n} k_i = 1$. Furthermore, we should notice that k_i is conditionally on the roots of *M*-function equation system.

Theorem 4.3: The estimated variance-covariance matrix of the regression coefficient *M*-estimators is given by

$$\hat{\mathrm{Cov}}_0[\underline{\Gamma}] = s^2 \left(X^T W^{-1} X\right)^{-1} X^T V[\underline{Y}] W^{-1} X \left[\left(X^T W^{-1} X\right)^{-1}\right]^T \tag{30}$$

where

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n} h\left(-\left(Y_i - (a+bx_i)\right)^2\right)\left(Y_i - (a+bx_i)\right)^2 \tag{31}$$

The proof is a matrix manipulation.

Let

$$\hat{\mathrm{Cov}}_0[\underline{\Gamma}] = \begin{bmatrix} \hat{\sigma}^2(a) & \hat{\sigma}(a,b) \\ \hat{\sigma}(a,b) & \hat{\sigma}^2(b) \end{bmatrix} \tag{32}$$

Corollary 4.4: The estimated variances for the regression coefficient *M*-estimators a and b respectively are

$$\begin{aligned} \hat{\sigma}^2(a) &= s^2\left[\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\right]^2 \\ &\times \frac{\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(\frac{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i^2}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i} - x_i\right)^2}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)^2} \end{aligned} \tag{33}$$

and

$$\hat{\sigma}^2(b) = s^2 \frac{\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)^2}{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)^2} \tag{34}$$

Furthermore, the correlation between the regression coefficient *M*-estimators a and b is

$$\begin{aligned} \hat{\sigma}(a,b) &= \frac{s^2}{\left[\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_g\right)^2\right]^2} \times \\ &\left[-\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i^2\right)\times\left(\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)\right)\times\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i\right)\right. \\ &+\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i\right)^2\times\left(\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i\right) \\ &+\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i^2\right)\times\left(\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i\right)\times\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\right) \\ &\left.-\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i\right)\times\left(\sum_{i=1}^{n} h^2\left(-\left(Y_i-(a+bx_i)\right)^2\right)x_i^2\right)\times\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\right)\right] \end{aligned} \tag{35}$$

Theorem 4.5: The estimated correlation coefficient for regression model $Y = \alpha + \beta x + \varepsilon$ is

$$r = \frac{\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)\left(Y_i-\bar{Y}_h\right)}{\sqrt{\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)^2\right)\times\left(\sum_{i=1}^{n} h\left(-\left(Y_i-(a+bx_i)\right)^2\right)\left(Y_i-\bar{Y}_h\right)^2\right)}} \tag{36}$$

Note here that accurately, we should say the correlation coefficient between X and Y. However, in the fuzzy regression model assumptions x_i are clearly assumed to be certain real-

valued number. Therefore, in Theorem 4.5, which gives the formula for r, r reveals the association between random variate Y and exploratory variable x and thus is regarded as an inherent index to the regression.

5 ASYMPTOTIC MEMBERSHIP FOR *M*-ESTIMATOR

It would be difficult task to discuss the asymptotic membership for coefficient estimators if we assume the membership function $g\left(-(Y-(a+bx))^2\right)$ takes very general form. Nevertheless, if the normal membership function is assumed, then the discussions will be slightly simplified.

Theorem 5.1: Let membership function for residual error takes normal form

$$m(e)=\exp\left(-w(Y-(a+bx))^2\right) \tag{37}$$

Then the asymptotic membership function for $(a-a)$ is

$$m_{(a-a)}(v)=\exp\left(-w\frac{v^2}{\sqrt{s^2(a)}}\right) \tag{38}$$

and the asymptotic membership function for $(b-b)$ is

$$m_{(b-b)}(u)=\exp\left(-w\frac{u^2}{\sqrt{s^2(b)}}\right) \tag{39}$$

where $\sqrt{s^2(a)}$ and $\sqrt{s^2(b)}$ are given in Theorem 4.4 respectively.

Theorem 5.2: The asymptotic joint membership function for vector

$$\begin{pmatrix} a-a \\ b-b \end{pmatrix} \tag{40}$$

is

$$m_{\begin{pmatrix} a-a \\ b-b \end{pmatrix}}(u,v)=\exp\left(-w\begin{pmatrix} u \\ v \end{pmatrix}^T C^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right) \tag{41}$$

Due to the feature that a normal form membership function only requires first two moments (mean and variance), therefore, utilizing the mean-variance information from Section 4 it is reasonable to establish the asymptotic membership function for $a-a$, $b-b$, and their bivariate joint asymptotic membership function. Once an asymptotic membership function is found the asymptotic credibility distribution can be easily derived in terms of Equation (7).

6 AN EXTENSION TO MULTI-MEMBERSHIP ERRORS

The *M*-estimation to the simple fuzzy regression lies on assuming that the errors have the same credibility distribution, or equivalently, the same membership function. However, we can extend our treatments into the case that error terms comes from multiple credibility distributions.
Let a sample be $\{(x_1,Y_1),(x_2,Y_2),\cdots,(x_n,Y_n)\}$, which is assumed from multiple membership functions, i.e.,

$$m_i(e_i)=g\left(-w_i(Y_i-(a+bx_i))^2\right) \tag{42}$$

Then, the optimization criterion is still to maximize the total membership grades:

$$\mathrm{J}=\max_{a,b}\sum_{i=1}^{n} g\left(-w_i(Y_i-(a+bx_i))^2\right) \tag{43}$$

and the *M*-functional equation system

$$\begin{cases} \sum_{i=1}^{n} w_i h\left(-w_i\left(Y-(a+bx_i)\right)^2\right)\left(Y-(a+bx_i)\right)=0 \\ \sum_{i=1}^{n} w_i h\left(-w_i\left(Y-(a+bx_i)\right)^2\right)\left(Y-(a+bx_i)\right)x_i=0 \end{cases} \tag{44}$$

Theorem 6.1: A general *M*-estimator of the coefficients for fitted regression $\hat{Y}=a+bx$, (a,b) is the solution to the general *M*-function equation system Equation (43). Furthermore, the *M*-estimator (a,b) takes a weighted least-square estimator form as

$$\begin{cases} b=\dfrac{\sum_{i=1}^{n} h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)\left(Y_i-\bar{Y}_h\right)}{\sum_{i=1}^{n} h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)\left(x_i-\bar{x}_h\right)^2} \\ a=\bar{Y}_h-\bar{b}x_h \end{cases} \tag{45}$$

where

$$\begin{aligned} \bar{x}_h &= \sum_{i=1}^{n} \frac{h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)}{\sum_{i=1}^{n} h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)} x_i \\ \bar{Y}_h &= \sum_{i=1}^{n} \frac{h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)}{\sum_{i=1}^{n} h\left(-w_i\left(Y_i-(a+bx_i)\right)^2\right)} Y_i \end{aligned} \tag{46}$$

However, we should notice that the matrix form of *M*-functional equation system, i.e., the weighted least-squares formed normal equation will be similar but we need take care of the variance-covariance matrix since the model assumption is changed to,

$$\mathrm{E}(e_i)=0 \;\&\; V(e_i)=s_i^2,\ \mathrm{E}(e_i e_j)=0 \tag{47}$$

The remaining investigations can be carried on in a similar way but the error variance-covariance matrix needs care.

7 AN EXTENSION TO MULTIPLE FUZZY REGRESSION

Let us assume that the response variable Y functionally related to *p* exploratory variables, $x_1, x_2, \ldots, x_p$. Let $\{(Y_i, x_{1i}, \ldots, x_{pi}),\ i=1,\ldots,n\}$ be the sample observations have model assumptions:

$$\begin{aligned} &\mathrm{E}\left(Y_i-\sum_{k=0}^{p} b x_{ik}\right)=0 \\ &V\left(Y_i-\sum_{k=0}^{p} b x_{ik}\right)=s_i^2 \\ &\mathrm{E}\left(Y_i-\sum_{k=0}^{p} b x_{ik}\right)\left(Y_j-\sum_{k=0}^{p} b x_{jk}\right)=0 \end{aligned} \tag{48}$$

Then the optimization criterion is

$$J=\sum_{i=1}^{n} g\left(w_i\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)\right)^2 \tag{49}$$

Let $\underline{\beta}=(b_0,b_1,\ldots,b_p)^T$, then $\underline{\hat{\beta}}=(\hat{b}_0,\hat{b}_1,\ldots,\hat{b}_p)^T$ denotes the *M*-estimator of regression coefficients. The *M*-functional equation system is thus

$$\begin{cases} \sum_{i=1}^{n} w_i h\left(-w_i\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)^2\right)\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)=0 \\ \sum_{i=1}^{n} w_i h\left(-w_i\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)^2\right)\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)x_{1i}=0 \\ \quad\vdots \\ \sum_{i=1}^{n} w_i h\left(-w_i\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)^2\right)\left(Y_i-\sum_{k=1}^{p} b_k x_{ki}\right)x_{pi}=0 \end{cases} \tag{50}$$

Notice that the error variance-covariance matrix

$$V=\begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} \tag{51}$$

The matrix form – weighted least-squares normal equation is

$$X^T W^{-1} X\underline{\beta}=X^T W^{-1}\underline{Y} \tag{52}$$

where

$$\underline{Y}=\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix},\ X=\begin{bmatrix} 1 & x_{11} & \cdots & x_{p1} \\ 1 & x_{12} & \cdots & x_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & \cdots & x_{pn} \end{bmatrix},\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{53}$$

and

$$W^{-1}=\operatorname{diag}\left(\sigma_i^2 h\left(-w_i\left(Y_i-\sum_{k=1}^{p}\beta_k x_{ki}\right)^2\right)\right) \tag{54}$$

Then the *M*-estimator for the multiple regression coefficients are

$$\underline{\hat{\beta}}=\left(X^T W^{-1} X\right)^{-1} X^T W^{-1}\underline{Y} \tag{55}$$

as long as the inverse matrix exists.

The properties and the variance-covariance structure of $\underline{\hat{\beta}}$ could be investigated as that of weighted least-squares regression formality.

8 CONCLUSION

The major advantage of this paper is its scalar variable treatments of fuzzy observations because fuzzy variable concept is established on the credibility measure foundation. Therefore, we are able to propose an *M*-estimation approach for simple fuzzy regression model. The optimization criterion is minimizing the fuzzy uncertainty by seeking the fitted errors with the membership grades as large as possible. In this sense, our simple fuzzy regression has the similar optimization criterion as Tanaka et al. (1980, 1982). We notice that the *M*-functional equation system can be re-written in weighted least-squares normal equation formation, which enables heavily to borrow arguments similar to statistical linear model theory. Finally, it is necessary to point out that in similar manner, the *M*-functional equation system could be defined for fuzzy multivariate

regression model.

9 ACKNOWLEDGEMENT

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RELIABILITY AND RISK ANALYSIS OF MULTI-STATE SYSTEMS WITH DEGRADING COPONENTS

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ABSTRACT

Applications of multi-state approach to the reliability evaluation of systems composed of independent components are considered. The main emphasis is on multi-state systems with degrading components because of the importance of such an approach in safety analysis, assessment and prediction, and analysing the effectiveness of operation processes of real technical systems. The results concerned with multi-state series systems are applied to the reliability evaluation and risk function determination of a homogeneous bus transportation system. Results on homogeneous multi-state "*m* out of *n*" systems are applied to durability evaluation of a steel rope. A non-homogeneous series-parallel pipeline system composed of several lines of multi-state pipe segments is estimated as well. Moreover, the reliability evaluation of the model homogeneous multi-state parallel-series electrical energy distribution system is performed.

1 INTRODUCTION

Many technical systems belong to the class of complex systems as a result of the progressive ageing of components they are built of and their complicated operating processes. Taking into account the importance of the safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis. These more general and practically important complex systems composed of multi-state components are considered among others in (Abouammoh & Al-Kadi 1991; Amari & Misra 1997; Aven 1985, 1993; Barlow & Wu 1978; Bausch 1987; Boedigheimer & Kapur 1994; Brunelle & Kapur 1999; Butler 1982; El-Neweihi, Proschan & Setchuraman 1978; Fardis & Cornel 1981; Griffith 1980; Huang, Zuo & Wu 2000; Hudson & Kapur 1982, 1983a, b, 1985, Kolowrocki 2004; Levitin Lisnianski, Haim & Elmakis 1998; Levitin & Lisnianski 1998, 1999, 2000a, b, 2001, 2003; Meng 1993; Natvig 1982, 1984; Ohio & Nishida 1984; Piasecki 1995; Polish Norm; Pourret, Collet & Bon 1999; Xue 1985; Xue & Yang 1995a, b; Yu, Koren & Guo 1994). An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is defined in the paper for systems with and degrading (ageing) in time components (Barlow & Wu 1978; Kolowrocki 2004; Xue 1985; Xue & Yang 1995a, b; Yu). The assumption that the systems are composed of multi-state components with reliability states degrading in time without repair gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its

distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

2 MULTI-STATE RELIABILITY ANALYSIS

In the multi-state reliability analysis to define systems with degrading components we assume that:

– E_i, $i = 1,2,...,n$, are components of a system,
– all components and a system under consideration have the state set $\{0,1,...,z\}$, $z \geq 1$,
– the state indexes are ordered, the state 0 is the worst and the state z is the best,
– $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$,
– $T(u)$ is a random variable representing the lifetime of a system in the state subset $\{u,u+1,...,z\}$ while it was in the state z at the moment $t = 0$,
– the system state degrades with time t without repair,
– $e_i(t)$ is a component E_i state at the moment t, $t \in <0,\infty)$,
– $s(t)$ is a system state at the moment t, $t \in <0,\infty)$.

The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse. The way in which the components and the system states change is illustrated in **Figure 1**.

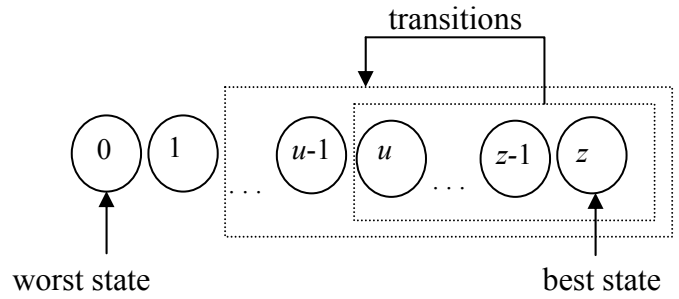

Figure 1. Illustration of states changing in system with ageing components.

Definition 1. A vector

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1),..., R_i(t,z)], \ t \in <0,\infty),$$

where

$$R_i(t,u) = P(e_i(t) \geq u \mid e_i(0) = z) = P(T_i(u) > t)$$

for $t \in <0,,\infty)$, $u = 0,1,...,z$, $i = 1,2,...,n$, is the probability that the component E_i is in the state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Definition 2. A vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)], \ t \in <0,\infty),$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \tag{1}$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, is the probability that the system is in the state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a system.

Under this definition we have

$$\boldsymbol{R}_n(t,0) \geq \boldsymbol{R}_n(t,1) \geq \ldots \geq \boldsymbol{R}_n(t,z),\ t \in <0,\infty),$$

and if

$$p(t,u) = P(s(t) = u \mid s(0) = z),\ t \in <0,\infty),$$

for $u = 0,1,...,z$, is the probability that the system is in the state u at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, then

$$\boldsymbol{R}_n(t,0) = 1,\ \boldsymbol{R}_n(t,z) = p(t,z),\ t \in <0,\infty), \tag{2}$$

and

$$p(t,u) = \boldsymbol{R}_n(t,u) - \boldsymbol{R}_n(t,u+1),\ t \in <0,\infty),\ \text{for } u = 0,1,...,z. \tag{3}$$

Moreover, if

$$\boldsymbol{R}_n(t,u) = 1 \text{ for } t \leq 0,\ u = 1,2,...,z,$$

then

$$M(u) = \int_0^\infty \boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \tag{4}$$

is the mean lifetime of the system in the state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{N(u) - [M(u)]^2}\ ,\ u = 1,2,...,z, \tag{5}$$

where

$$N(u) = 2\int_0^\infty t\, \boldsymbol{R}_n(t,u)dt,\ u = 1,2,...,z, \tag{6}$$

is the standard deviation of the system sojourn time in the state subset $\{u, u+1,..., z\}$ and moreover

$$\overline{M}(u) = \int_0^\infty p(t,u)dt,\ u = 1,2,...,z, \tag{7}$$

is the mean lifetime of the system in the state u while the integrals (4), (6) and (7) are convergent. Additionally, according to (3), (4) and (7), we get the following relationship

$$\overline{M}(u) = M(u) - M(u+1), \quad u = 0,1,...,z-1, \quad \overline{M}(z) = M(z). \tag{8}$$

Definition 3. A probability

$$r(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t), \ t \in <0,\infty),$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the state z at the moment $t = 0$ is called a risk function of the multi-state system or, in short, a risk.

Under this definition, from (1), for $t \in <0,\infty),$ we have

$$r(t) = 1 - P(s(t) \ge r \mid s(0) = z) = 1 - \boldsymbol{R}_n(t,r), \tag{9}$$

and if τ is the moment when the risk exceeds a permitted level δ, then

$$\tau = r^{-1}(\delta), \tag{10}$$

where $r^{-1}(t)$, if it exists, is the inverse function of the risk function $r(t)$.

3 BASIC MULTI-STATE RELIABILITY STRUCTURES

3.1 Multi-state series system

Definition 4. A multi-state system is called series if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \le i \le n} \{T_i(u)\}, \ u = 1,2,...,z.$$

The above definition means that a multi-state series system is in the state subset $\{u, u+1,..., z\}$ if and only if all its components are in this subset of states.
It is easy to work out the following results.

Corollary 1. The reliability function of the multi-state series system is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1), ..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = \prod_{i=1}^{n} R_i(t,u), \ t \in <0,\infty), \ u = 1,2,...,z.$$

Corollary 2. If the multi-state series system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \ \text{ for } \ t \in <0,\infty), \ u = 1,2,...,z, \ i = 1,2,...,n,$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_n(t,\cdot) = [1, \overline{\boldsymbol{R}}_n(t,1), ..., \overline{\boldsymbol{R}}_n(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n(t,u) = [R(t,u)]^n \text{ for } t \in <0,\infty), \ u = 1,2,...,z.$$

Example 1 (*a bus transportation system*). The city transportation system is composed of n, $n \geq 1$, buses necessary to perform its communication tasks. We assume that the bus lifetimes are independent random variables and that the system is operating in successive cycles (days) $c = 1,2,...$. In each of the cycles the following three operating phases of all components are distinguished:

f_1 – components waiting for inclusion in the operation process, lasting from the moment t_0 up to the moment t_1,

f_2 – components' activation for the operation process, lasting from t_1 up to t_2,

f_3 – components operating, lasting from t_2 up to $t_3 = t_0$.

Each of the system components during the waiting phase may be damaged because of the circumstances at the stoppage place. We assume that the probability that at the end moment t_1 of the first phase the ith component is not failed is equal to $p_i^{(1)}$, where $0 \leq p_i^{(1)} \leq 1$, $i = 1,2,...,n$. Since component lifetimes are independent then the system availability at the end moment t_1 of phase f_1 is given by

$$p^{(1)} = \prod_{i=1}^{n} p_i^{(1)}. \tag{11}$$

In the activation phase f_2 system components are prepared for the operation process by the service. They are checked and small flaws are removed. Sometimes the flaws cannot be removed and particular components are not prepared to fulfill their tasks. We assume that the probability that at the end moment t_2 of the first phase the ith component is not failed is equal to $p_i^{(2)}$, where $0 \leq p_i^{(2)} \leq 1$, $i = 1,2,...,n.$ Since component lifetimes are independent then the system availability at the end moment t_2 of the phase f_2 is given by

$$p^{(2)} = \prod_{i=1}^{n} p_i^{(2)}. \tag{12}$$

Thus, finally, the system availability after two phases is given by

$$p^{(1,2)} = p^{(1)} \cdot p^{(2)}, \tag{13}$$

where $p^{(1)}$ and $p^{(2)}$ are defined respectively by (11) and (12).

In the operating phase f_3, during the time $t_4 = t_3 - t_2$, each of the system components is performing one of two tasks:

z_1 – a first task (working at normal communication conditions),

z_2 – a second task (working at a communication peak),

with probabilities respectively equal to r_1 and r_2, where $0 \leq r_1 \leq 1$, $r_2 = 1 - r_1$.

Let

$$R^{(1)}(t,\cdot) = [1, R^{(1)}(t,1), R^{(1)}(t,2)],$$

where

$$R^{(1)}(t,u)=1 \text{ for } t<0,$$

$$R^{(1)}(t,u)=\exp[-\frac{1}{15-5u}t] \text{ for } t\geq 0,\ u=1,2,$$

be the reliability function of the ith component during performance of task z_1 and

$$R^{(2)}(t,\cdot)=[1,R^{(2)}(t,1),R^{(2)}(t,2)],$$

where

$$R^{(2)}(t,u)=1 \text{ for } t<0,$$

$$R^{(2)}(t,u)=\exp[-\frac{1}{10-2u}t] \text{ for } t\geq 0,\ u=1,2,$$

be the reliability function of the ith component during performance of task z_2.
Thus, by *Definition 4*, the considered transportation system is a homogeneous three-state series system and according to the formula for total probability, after applying *Corollary 2*, we conclude that

$$\overline{\boldsymbol{R}}_n(t,\cdot)=[1,\overline{\boldsymbol{R}}_n(t,1),\overline{\boldsymbol{R}}_n(t,2)],$$

where

$$\overline{\boldsymbol{R}}_n(t,1)=1 \text{ for } t<0,$$

$$\overline{\boldsymbol{R}}_n(t,1)=r_1\exp[-\frac{n}{15-5u}t]+r_2\exp[-\frac{n}{10-2u}t] \text{ for } t\geq 0,\ u=1,2, \qquad (14)$$

is the reliability function of the system performing two tasks.
The mean values of the system lifetimes $T(u)$ in the state subsets, according to (4), are:

$$M(u)=E[T(u)]=\frac{r_1(15-5u)+r_2(10-2u)}{n} \text{ for } u=1,2.$$

If we assume that

$$n=30,\ r_1=0.8,\ r_2=0.2,$$

then from (14), we get

$$\overline{\boldsymbol{R}}_{30}(t,\cdot)=[1,\ 0.8\exp[-3t]+0.2\exp[-3.75t]+0.8\exp[-6t]+0.2\exp[-5t]] \text{ for } t\geq 0 \quad (15)$$

and

$$M(1) \cong 0.32,\ M(2) \cong 0.17.$$

Thus, considering (8), the expected values of the sojourn times in the particular states are:

$$\overline{M}(1) \cong 0.15,\ \overline{M}(2) \cong 0.17.$$

If a critical state is $r = 1$, then according to (9), the system risk function is given by

$$\mathbf{r}(t) = 1 - 0.8\exp[-3t]+0.2\exp[-3.75t] \text{ for } t \geq 0.$$

The moment when the system risk exceeds a permitted level $\delta = 0.05$, according to (10), is

$$\tau = \mathbf{r}^{-1}(\delta) \cong 0.016 \text{ years} \cong 6 \text{ days}.$$

At the end moment of the system activation phase, which is simultaneously the starting moment of the system operating phase t_2 the system is able to perform its tasks with the probability $p^{(1,2)}$ defined by (13). Therefore, after applying the formula (15), we conclude that the system reliability in c cycles, $c = 1,2,\ldots$, is given by the following formula

$$G(c,\cdot) = [1,\ p^{(1,2)}\,0.8\exp[-3ct_4]+0.2\exp[-3.75\ ct_4],\ p^{(1,2)}\,0.8\exp[-6\ ct_4]+0.2\exp[-5\ ct_4]],$$

where $t_4 = t_3 - t_2$ is the time duration of the system operating phase f_3. Further, assuming for instance

$$p^{(1,2)} = p^{(1)}p^{(2)} = 0.99 \cdot 0.99 = 0.98,$$

$$t_4 = 18 \text{ hours} = 0.002055 \text{ years}$$

for the number of cycles $c = 7$ days = 1 week, we get

$$G(7,\cdot) \cong [1,\ 0.966,\ 0.902].$$

This result means that during 7 days the considered transportation system will be able to perform its tasks in state not worse than the first state with probability 0.966, whereas it will be able to perform its tasks in the second state with probability 0.902.

3.2. Multi-state parallel system

Definition 5. A multi-state system is called parallel if its lifetime $T(u)$ in the state subset $\{u, u+1,\ldots, z\}$ is given by

$$T(u) = \max_{1 \leq i \leq n}\{T_i(u)\},\ u = 1,2,\ldots,z.$$

The above definition means that the multi-state parallel system is in the state subset $\{u, u+1,\ldots, z\}$ if and only if at least one of its components is in this subset of states.

Corollary 3. The reliability function of the multi-state parallel system is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1 - \prod_{i=1}^{n} F_i(t,u), \ t \in <0,\infty), \ u = 1,2,...,z.$$

Corollary 4. If the multi-state parallel system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \ \text{ for } \ t \in <0,\infty), \ u = 1,2,...,z, \ i = 1,2,...,n,$$

then its reliability function is given by

$$\boldsymbol{R}_n(t,\cdot) = [1, \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],$$

where

$$\boldsymbol{R}_n(t,u) = 1 - [F(t,u)]^n \ \text{ for } \ t \in <0,\infty), \ u = 1,2,...,z.$$

3.3. Multi-state "*m* out of *n*" system

Definition 6. A multi-state system is called an "*m* out of *n*" system if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = T_{(n-m+1)}(u), m = 1,2,...,n, u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m*th maximal order statistic in the sequence of the component lifetimes

$$T_1(u), T_2(u),..., T_n(u).$$

The above definition means that the multi-state „*m* out of *n*" system is in the state subset $\{u, u+1,..., z\}$ if and only if at least m out of its n components are in this state subset; and it is a multi-state parallel system if $m = 1$ and it is a multi-state series system if $m = n$.

Corollary 5. The reliability function of the multi-state "*m* out of *n*" system is given either by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1, \boldsymbol{R}_n^{(m)}(t,1),..., \boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le m-1}}^{1} [R_i(t,u)]^{r_i} [F_i(t,u)]^{1-r_i} \ \text{ for } \ t \in <0,\infty), \ u = 1,2,...,z, \text{ or by}$$

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,u) = \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le \overline{m}}}^{1} [F_i(t,u)]^{r_i} [R_i(t,u)]^{1-r_i} \text{ for } t \in <0,\infty),\ \overline{m} = n-m,\ u = 1,2,...,z.$$

Corollary 6. If the multi-state "*m* out of *n*" system is homogeneous, i.e. if

$$R_i(t,u) = R(t,u) \text{ for } t \in <0,\infty),\ u = 1,2,...,z,\ i = 1,2,...,n,$$

then its reliability function is given by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1, \boldsymbol{R}_n^{(m)}(t,1),..., \boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{k=0}^{m-1} [R(t,u]^k [F(t,u)]^{n-k} \text{ for } t \in <0,\infty),\ u = 1,2,...,z, \text{ or by}$$

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1), ..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,u) = \sum_{k=0}^{\overline{m}} [F(t,u)]^k [R(t,u)]^{n-k} \text{ for } t \in <0,\infty),\ \overline{m} = n-m,\ u = 1,2,...,z.$$

Example 2 (*a three-stratum rope, durability*). Let us consider the steel rope of type *M*-80-200-10 described in [36]. It is a three-stratum rope composed of 36 strands: 18 outer strands, 12 inner strands and 6 more inner strands. All strands consist of seven still wires. The rope cross-section is presented in **Figure 2**.

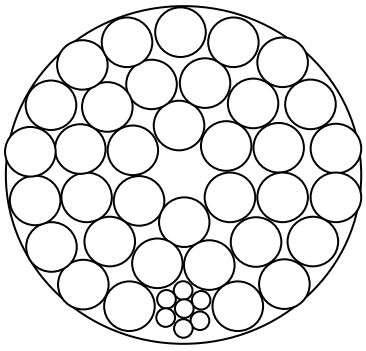

Figure 2. The steel rope *M*-80-200-10 cross-section

Considering the strands as basic components we conclude that the rope is a system composed of $n = 36$ components (strands). Due to [38] concerned with the evaluation of wear level, the following reliability states of the strands are distinguished:
state 3 – a strand is new, without any defects,
state 2 – the number of broken wires in the strand is greater than 0% and less than 25% of all its wires, or corrosion of wires is greater than 0% and less than 25%, abrasion is up to 25% and strain is up to 50%,

state 1 – the number of broken wires in the strand is greater than or equal to 25% and less than 50% of all its wires, or corrosion of wires is greater than or equal to 25% and less than 50%, abrasion is up to 50% and strain is up to 50%,
state 0 – otherwise (a strand is failed).
Thus, the considered steel rope composed of $n = 36$ four-state, i.e. $z = 3$. Let us assume that the rope strands have identical exponential reliability functions with transitions rates between the state subsets

$$\lambda(u) = 0.2u/\text{year}, \; u = 1,2,3.$$

Assuming that the rope is in the state subset $\{u, u+1,..., z\}$ if at least $m = 10$ of its wires are in this state subset, according to *Definition 6*, we conclude the rope is a homogeneous four-state "10 out of 36" system. Thus, by *Corollary 6*, its reliability function is given by

$$\boldsymbol{R}_{36}^{(10)}(t,\cdot) = [1, \boldsymbol{R}_{36}^{(10)}(t,1), \boldsymbol{R}_{36}^{(10)}(t,2), \boldsymbol{R}_{36}^{(10)}(t,3)], \tag{16}$$

where

$$\boldsymbol{R}_{36}^{(10)}(t,1) = 1 \text{ for } t < 0,$$

$$\boldsymbol{R}_{36}^{(10)}(t,1) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.2t][1 - \exp[-0.2t]]^{36-i} \text{ for } t \geq 0,$$

$$\boldsymbol{R}_{36}^{(10)}(t,2) = 1 \text{ for } t < 0,$$

$$\boldsymbol{R}_{36}^{(10)}(t,2) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.4t][1 - \exp[-0.4t]]^{36-i} \text{ for } t \geq 0,$$

$$\boldsymbol{R}_{36}^{(10)}(t,3) = 1 \text{ for } t < 0,$$

$$\boldsymbol{R}_{36}^{(10)}(t,3) = 1 - \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.6t][1 - \exp[-0.6t]]^{36-i} \text{ for } t \geq 0.$$

By (16), the approximate mean values of the rope lifetimes $T(u)$ in the state subsets and their standard deviations in years are:

$$M(1) \cong 6.66, \; M(2) \cong 3.33, \; M(3) \cong 2.22,$$

$$\sigma(1) \cong 1.62, \; \sigma(2) \cong 0.81, \; \sigma(3) \cong 0.54,$$

whereas, the approximate mean values of the rope lifetimes in the particular reliability states are:

$$\overline{M}(1) \cong 3.33, \; \overline{M}(2) \cong 1.11, \; \overline{M}(3) \cong 2.22.$$

If the critical state is $r = 2$, then the rope risk function is approximately given by

$$\boldsymbol{r}(t) = \sum_{i=0}^{9} \binom{36}{i} \exp[-i0.4t][1 - \exp[-0.4t]]^{36-i} \text{ for } t \geq 0.$$

The moment when the risk exceeds an admissible level $\delta = 0.05$, after applying (10), is

$$\tau \cong 2.074 \text{ years.}$$

The behaviour of the rope system reliability function and its risk function are illustrated in Table 1.

Table 1. The values of the still rope multi-state reliability function and risk function

t	$\boldsymbol{R}_{36}^{(10)}(t,1)$	$\boldsymbol{R}_{36}^{(10)}(t,2)$	$\boldsymbol{R}_{36}^{(10)}(t,3)$	$\boldsymbol{r}(t)$
0.2	1.00000	1.00000	1.00000	0.00000
0.6	1.00000	0.99998	0.99979	0.00002
1.0	0.99999	0.99961	0.99425	0.00039
1.4	0.99995	0.99641	0.94590	0.00359
1.8	0.99979	0.98014	0.77675	0.01986
2.2	0.99928	0.92792	0.49332	0.07208
2.6	0.99783	0.81520	0.23107	0.18480
3.0	0.99425	0.64221	0.08058	0.35779
3.4	0.98649	0.44415	0.02168	0.55585
3.8	0.97157	0.26782	0.00469	0.73218
4.2	0.94590	0.14130	0.00085	0.85870
4.6	0.90602	0.06584	0.00013	0.93416
5.0	0.84969	0.02742	0.00002	0.97258
5.4	0.77675	0.01034	0.00000	0.98966
5.8	0.68965	0.00357	0.00000	0.99643
6.2	0.59314	0.00114	0.00000	0.99886
6.6	0.49332	0.00034	0.00000	0.99966
7.0	0.39645	0.00010	0.00000	0.99990
7.4	0.30784	0.00003	0.00000	0.99997
7.8	0.23107	0.00001	0.00000	0.99999

3.4. Multi-state series-parallel system

Other basic multi-state reliability structures with components degrading in time are series-parallel and parallel-series systems. To define them, we assume that:

– E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are components of a system,
– all components E_{ij} have the same state set as before $\{0,1,...,z\}$,
– $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are independent random variables representing the lifetimes of components E_{ij} in the state subset $\{u, u+1,..., z\}$, while they were in the state z at the moment $t = 0$,
– $e_{ij}(t)$ is a component E_{ij} state at the moment t, $t \in < 0, \infty)$, while they were in the state z at the moment $t = 0$.

Definition 7. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1),..., R_{ij}(t,z)] \text{ for } t \in < 0, \infty),\ i = 1,2,...,k, j = 1,2,...,l_i,,$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t)$$

for $t \in <0,\infty)$, $u = 0,1,...,z$, is the probability that the component E_{ij} is in the state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

Definition 8. A multi-state system is called series-parallel if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \max_{1\leq i\leq k}\{\min_{1\leq j\leq l_i}\{T_{ij}(u)\}\}, \; u = 1,2,...,z.$$

Corollary 7. The reliability function of the multi-state series-parallel system is given by

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,1),..., \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,u) = 1 - \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} R_{ij}(t,u)] \text{ for } t \in <0,\infty), \; u = 1,2,...,z,$$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Corollary 8. If the multi-state series-parallel system is homogeneous, i.e.

$$R_{ij}(t,u) = R(t,u) \text{ for } t \in <0,\infty), \; u = 1,2,...,z, \; i = 1,2,...,k, j = 1,2,...,l_i,,$$

then its reliability function is given by

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,1),..., \boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\boldsymbol{R}_{k,l_1,l_2,...,l_k}(t,u) = 1 - \prod_{i=1}^{k}[1 - [R(t,u)]^{l_i}] \; for \; t \in <0,\infty), u = 1,2,...,z,$$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Corollary 9. If the multi-state series-parallel system is homogeneous, i.e.

$$R_{ij}(t,u) = R(t,u) \text{ for } t \in <0,\infty), \; u = 1,2,...,z, \; i = 1,2,...,k, j = 1,2,...,l_i,$$

and regular, i.e.

$$l_1 = l_2 = \ldots = l_k = l, \; l \in N.$$

then its reliability function is given by

$$\boldsymbol{R}_{k,l}(t,\cdot) = [1,\boldsymbol{R}_{k,l}(t,1),...,\boldsymbol{R}_{k,l}(t,z)],$$

and

$$\boldsymbol{R}_{k,l}(t,u) = 1-[1-[R(t,u)]^{l}]^{k} \text{ for } t\in<0,\infty), \ u = 1,2,...,z,$$

where k is the number of series subsystems linked in parallel and l is the number of components in the series subsystems.

Example 3 (*a pipeline system*). Let us consider the pipeline system composed of $k = 3$ lines of pipe segments linked in parallel, each of them composed of $l = 100$ five-state identical segments linked in series. The scheme of the considered system is shown in **Figure 3**.

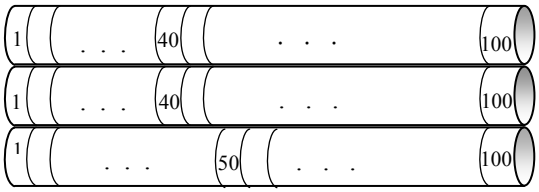

Figure 3. The model of a regular series-parallel pipeline system

Considering pipe segments as basic components of the pipeline system, according to *Definition 8*, we conclude that it is a homogeneous regular five-state series-parallel system. Therefore, by *Corollary 9*, the pipeline system reliability function is given by

$$\boldsymbol{R}_{3,100}(t,\cdot) = [1,\boldsymbol{R}_{3.1000}(t,1),\boldsymbol{R}_{3,100}(t,2),\boldsymbol{R}_{3,100}(t,3),\boldsymbol{R}_{3,100}(t,4)],$$

where

$$\boldsymbol{R}_{3,100}(t,u) = 1-[1-[R(t,u)]^{100}]^{3} \text{ for } t\in(-\infty,\infty), \quad u = 1,2,3,4.$$

Taking into account pipe segment reliability data given in their technical certificates and expert opinions we assume that they have Weibull reliability functions

$$R(t,\cdot) = [1, R(t,1), R(t,2), R(t,3), R(t,4)],$$

where

$$R(t,u) = 1 \text{ for } t < 0,$$

$$R(t,u) = \exp[-\beta(u)t^{\alpha(u)}] \text{ for } t \geq 0, \ u = 1,2,3,4,$$

with the following parameters:

$$\alpha(1) = 3, \ \ \beta(1) = 0.00001,$$

$$\alpha(2) = 2.5, \quad \beta(2) = 0.0001,$$

$$\alpha(3) = 2, \quad \beta(3) = 0.0016,$$

$$\alpha(4) = 1, \quad \beta(4) = 0.05.$$

Hence it follows that the pipeline system exact reliability function is given by

$$\boldsymbol{R}_{3,100}(t,\cdot) = [1,1 - [1 - \exp[-0.001t^3]]^3, 1 - [1 - \exp[-0.01t^{5/2}]]^3, 1 - [1 - \exp[-0.16t^2]]^3,$$

$$1 - [1 - \exp[-5t]]^3] \text{ for } t \geq 0. \tag{17}$$

By (17), the expected values $M(u)$, $u = 1,2,3,4$, of the system sojourn times in the state subsets in years, calculated on the basis of the approximate formula are:

$$M(1) = \Gamma(4/3)[3(0.001)^{-1/3} - 3(0.002)^{-1/3} + (0.003)^{-1/3}] \cong 11.72,$$

$$M(2) = \Gamma(7/5)[3(0.01)^{-2/5} - 3(0.02)^{-2/5} + (0.03)^{-2/5}] \cong 7.67,$$

$$M(3) = \Gamma(3/2)[3(0.16)^{-1/2} - 3(0.32)^{-1/2} + (0.48)^{-1/2}] \cong 3.23,$$

$$M(4) = \Gamma(2)[3(5)^{-1} - 3(10)^{-1} + (15)^{-1}] \cong 0.37.$$

Hence, the system mean lifetimes $\overline{M}(u)$ in particular states are:

$$\overline{M}(1) \cong 4.05, \ \overline{M}(2) \cong 4.44, \ \overline{M}(3) \cong 2.86, \ \overline{M}(4) \cong 0.37.$$

If the critical state is $r = 2$, then the system risk function, according (9), is given by

$$\boldsymbol{r}(t) = [1 - \exp[-0.01t^{5/2}]]^3.$$

The moment when the system risk exceeds an admissible level $\delta = 0.05$, from (10), is

$$\tau = \boldsymbol{r}^{-1}(\delta) = [-100\log(1 - \sqrt[3]{\delta})]^{2/5} \cong 4.62.$$

The behaviour of the system risk function is presented in Table 2 and **Figure 4**.

Table 2. The values of the piping system risk function

t	$\boldsymbol{r}(t)$
0.0	0.000
1.5	0.000
3.0	0.003
4.5	0.043
6.0	0.201
7.5	0.485
9.0	0.758
10.5	0.918

12.0	0.980
13.5	0.996
15.0	1.000

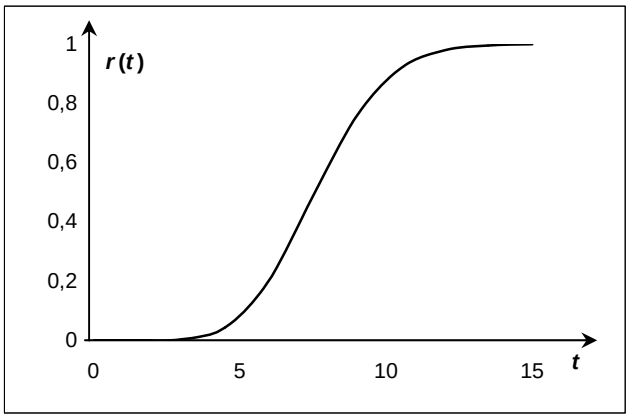

Figure 4. The graph of the piping system risk function

3.5. Multi-state parallel-series system

Definition 9. A multi-state system is called parallel-series if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \min_{1\le i\le k}\{\max_{1\le j\le l_i}\{T_{ij}(u)\}\}, \; u = 1,2,...,z.$$

Corollary 10. The reliability function of the multi-state parallel-series system is given by

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,1),..., \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,u) = \prod_{i=1}^{k}[1-\prod_{j=1}^{l_i}F_{ij}(t,u)] \text{ for } t\in<0,\infty), \; u = 1,2,...,z,$$

where k is the number of its parallel subsystems linked in series and l_i are the numbers of components in the parallel subsystems.

Corollary 11. If the multi-state parallel-series system is homogeneous, i.e.

$$R_{ij}(t,u) = R(t,u) \text{ for } t\in<0,\infty), \; u = 1,2,...,z, \; i = 1,2,...,k, j = 1,2,...,l_i,$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,\cdot) = [1, \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,1),..., \overline{\boldsymbol{R}}_{k,l_1,l_2,...,l_k}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l_1,l_2,\ldots,l_k}(t,u)=\prod_{i=1}^{k}[1-[F(t,u)]^{l_i}] \text{ for } t\in<0,\infty),\ u=1,2,\ldots,z,$$

where k is the number of its parallel subsystems linked in series and l_i are the numbers of components in the parallel subsystems.

Corollary 12. If the multi-state parallel-series system is homogeneous, i.e.

$$R_{ij}(t,u)=R(t,u) \text{ for } t\in<0,\infty),\ u=1,2,\ldots,z,\ i=1,2,\ldots,k,\ j=1,2,\ldots,l_i,,$$

and regular, i.e.

$$l_1=l_2=\ldots=l_k=l,\ l\in N.$$

then its reliability function is given by

$$\overline{\boldsymbol{R}}_{k,l}(t,\cdot)=[1,\overline{\boldsymbol{R}}_{k,l}(t,1),\ldots,\overline{\boldsymbol{R}}_{k,l}(t,z)],$$

and

$$\overline{\boldsymbol{R}}_{k,l}(t,u)=[1-[F(t,u]^{l}]^{k} \text{ for } t\in<0,\infty),u=1,2,\ldots,z,$$

where k is the number of its parallel subsystems linked in series and l is the number of components in the parallel subsystems.

Example 4 (*an electrical energy distribution system*). Let us consider a model energetic network stretched between two poles and composed of three energetic cables, six insulators and two bearers and analyze the reliability of all cables only. Each cable consists of 36 identical wires. Assuming that the cable is able to conduct the current if at least one of its wires is not failed we conclude that it is a homogeneous parallel-series system composed of $k = 3$ parallel subsystems linked in series, each of them consisting of $l = 36$ basic components. Further, assuming that the wires are four-state components, i.e. $z = 3$, having Weibull reliability functions with parameters

$$\alpha(u)=2,\ \beta(u)=(7.07)^{2u-8},\ u=1,2,3.$$

According to *Corollary 12*, we obtain the following form of the system multi-state reliability function

$$\overline{\boldsymbol{R}}_{3,36}(t,\cdot)\cong[1,[1-[1-\exp[-0.000008007t^2]]^{36}]^3,[1-[1-\exp[-0.000400242t^2]]^{36}]^3,$$

$$[1-[1-\exp[-0.20006042t^2]]^{36}]^3] \text{ for } t\in<0,\infty). \qquad (18)$$

By (18), the values of the system sojourn times $T(u)$ in the state subsystems in months, after applying (4), are given by

$$E[T(u)]\cong\int_0^{\infty}[1-[1-\exp[-(7.07)^{2u-8}t^2]]^{36}]^3\,dt$$

for $u = 1,2,3$, and particularly

$$M(1) \cong 650,\ M(2) \cong 100, M(3) \cong 15.$$

Hence, from (8), the system mean lifetimes in particular states are:

$$\overline{M}(1) \cong 550,\ \overline{M}(2) \cong 85,\ \overline{M}(3) \cong 15.$$

If the critical reliability state of the system is $r = 2$, then its risk function, according to (9), is given by

$$\boldsymbol{r}(t) \cong 1 - [1 - [1 - \exp[-0.000400242t^2]]^{36}]^3.$$

The moment when the system risk exceeds an admissible level $\delta = 0.05$, calculated due to (10), is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 76 \text{ months}.$$

4 CONCLUSION

In the paper the multi-state approach to the reliability evaluation of systems with degrading components have been considered. Theoretical results presented in have been illustrated by examples of their application in reliability evaluation of technical systems. These evaluations, despite not being precise may be a very useful, simple and quick tool in approximate reliability evaluation, especially during the design of large systems, and when planning and improving their safety and effectiveness operation processes.
The results presented in the paper suggest that it seems reasonable to continue the investigations focusing on:

- methods of improving reliability for multi-state systems,
- methods of reliability optimisation for multi-state systems related to costs and safety of the system operation processes,
- availability and maintenance of multi-state systems.

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TRELIABILITY ANALYSIS OF TWO-STATE CONSECUTIVE "*M* OUT OF *L*: F"-SERIES SYSTEMS

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ABSTRACT

A non-stationary approach to reliability analysis of two-state series and consecutive "*m* out of *k*: F" systems is presented. Further, the consecutive "*m* out of *k*: F"-series system is defined and the recurrent formulae for its reliability function evaluation are proposed.

1 INTRODUCTION

The basic analysis and diagnosis of systems reliability are often performed under the assumption that they are composed of two-state components. It allows us to consider two states of the system reliability. If the system works its reliability state is equal to 1 and if it is failed its reliability state is equal to 0. Reliability analysis of two-state consecutive "*k* out of *n*: F" systems can be done for stationary and non-stationary case. In the first case the system reliability is the independent of time probability that the system is in the reliability state 1. For this case the main results on the reliability evaluation and the algorithms for numerical approach to consecutive "*k* out of *n*: F" systems are given for instance in Antonopoulou & Papstavridis (1987), Barlow & Proschan (1975), Hwang (1982), Malinowski & Preuss (1995), Malinowski (2005). Transmitting stationary results to non-stationary time dependent case and the algorithms for numerical approach to evaluation of this reliability are presented in Guze (2007a, b). Other more complex two-state systems are discussed in Kołowrocki (2004). The paper is devoted to the combining the results on reliability of the two-state series and consecutive "*m* out of *n*: F" system into the formulae for the reliability function of the consecutive "*m* out of *l*: F"-series systems with dependent of time reliability functions of system components (Guze 2007a, b, c).

2 RELIABILITY OF A SERIES AND CONSECUTIVE "*M* OUT OF *N*: F" SYSTEMS

In the case of two-state reliability analysis of series systems and consecutive "*m* out of *n*: F" systems we assume that (Guze 2007b):

- n is the number of system components,
- $E_i, i = 1,2,...,n,$ are components of a system,
- T_i are independent random variables representing the lifetimes of a components $E_i,\ i = 1,2,...,n,$
- $R_i(t) = P(T_i > t), t \in < 0,\infty),$ is a reliability function of a component $E_i,\ i = 1,2,...,n,$
- $F_i(t) = 1 - R_i(t) = P(T_i \le t), t \in < 0,\infty),$ is the distribution function of a component E_i lifetime $T_i,\ i = 1,2,...,n,$ also called an unreliability function of a component $E_i,\ i = 1,2,...,n.$

In further analysis we will use one of the simplest system structure, namely a series system.

Definition 1 A two-state system is called series if its lifetime *T* is given by

$$T = \min_{1 \le i \le n} \{T_i\}.$$

The scheme of a series system is given in Figure 1.

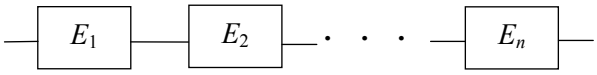

Figure 1. The scheme of a series system

The above definition means that the series system is not failed if and only if all its components are not failed or equivalently the system is failed if at least one of its components is failed. It is easy to motivate that the series system reliability function is given by

$$\overline{\mathbf{R}}_n(t) = \prod_{i=1}^{n} R_i(t), \ t \in <0,\infty). \tag{1}$$

Definition 2. A two-state series system is called homogeneous if its component lifetimes T_i have an identical distribution function

$$F(t) = P(T_i \le t), \ t \in <0,\infty), \ i = 1,2,...,n,$$

i.e. if its components E_i have the same reliability function

$$R(t) = 1 - F(t), \ \ t \in <0,\infty).$$

The above definition results in the following simplified formula

$$\overline{\mathbf{R}}_n(t) = [R(t)]^n, \ t \in <0,\infty), \tag{2}$$

for the reliability function of the homogeneous two-state series system.

Definition 3. A two-state system is called a two-state consecutive "*m* out of *n*: F" system if it is failed if and only if at least its *m* neighbouring components out of *n* its components arranged in a sequence of $E_1, E_2, ..., E_n$, are failed.

After assumption that:
- T is a random variable representing the lifetime of the consecutive "*m* out of *n*: F" system,
- $\boldsymbol{CR}_n^{(m)}(t) = P(T > t), t \in <0,\infty),$ is the reliability function of a non-homogeneous consecutive "*m* out of *n*: F" system,
- $\boldsymbol{CF}_n^{(m)}(t) = 1 - \boldsymbol{CR}_n^{(m)}(t) = P(T \le t), t \in <0,\infty),$ is the distribution function of a consecutive "*m* out of *n*: F" system lifetime T,

we can formulate the following auxiliary theorem (Guze 2007c).

Lemma 1. The reliability function of the two-state consecutive "*m* out of *n*: F" system is given by the following recurrent formula

$$\boldsymbol{CR}_n^{(m)}(t) = \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n} F_i(t) & \text{for } n = m, \\ R_n(t)\boldsymbol{CR}_{n-1}^{(m)}(t) + \sum_{j=1}^{m-1} R_{n-j}(t)\boldsymbol{CR}_{n-j-1}^{(m)}(t) & \\ \cdot \prod_{i=n-j+1}^{n} F_i(t) & \text{for } n > m, \end{cases} \tag{3}$$

for $t \in <0, \infty)$.

Definition 4. The consecutive "*m* out of *n*: F" system is called homogeneous if its components lifetimes T_i have an identical distribution function

$F(t) = P(T_i \le t)$, $i = 1,2,\ldots, n, t \in <0, \infty)$,

i.e. if its components E_i have the same reliability function

$R(t) = 1 - F(t)$, $t \in <0, \infty)$.

Lemma 1 simplified form for homogeneous systems takes the following form.

Lemma 2. The reliability of the homogeneous two-state consecutive "*m* out of *n*: F" system is given by the following recurrent formula

$$\boldsymbol{CR}_n^{(m)}(t) = \begin{cases} 1 & \text{for } n < m \\ & , \\ 1 - [F(t)]^n & \text{for } n = m, \\ R(t)\boldsymbol{CR}_{n-1}^{(m)}(t) & \\ + R(t)\sum_{j=1}^{m-1} F^{j-1}(t) & \\ \cdot \boldsymbol{CR}_{n-j-1}^{(m)}(t) & \text{for } n > m, \end{cases} \tag{4}$$

for $t \in <0, \infty)$.

3 RELIABILITY OF TWO-STATE CONSECUTIVE "*M* OUT OF *L*: F"-SERIES SYSTEM

To define a two-state consecutive "*m* out of *l*: F"-series systems, we assume that

E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$,

are two-state components of the system having reliability functions

$$R_{ij}(t) = P(T_{ij} > t),\ t \in <0,\infty),$$

where

T_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$,

are independent random variables representing the lifetimes of components E_{ij} with distribution functions

$$F_{ij}(t) = P(T_{ij} \le t),\ t \in <0,\infty).$$

Moreover, we assume that components E_{i1}, $E_{i2}, \ldots, E_{il_i}$, i=1, 2,… , k, create a consecutive "m_i out of l_i: F" subsystem S_i, i=1, 2,… , k and that these subsystems create a series system.

Definition 5. A two-state system is called a consecutive "m_i out of l_i: F"-series system if it is failed if and only if at least one of its consecutive "m_i out of l_i: F" subsystems S_i, i = 1, 2,… , k, is failed.

According to the above definition and formula (4) the reliability function of the subsystem S_i is given by

$$\boldsymbol{CR}_{i,l_i}^{(m_i)}(t) = \begin{cases} 1 & \text{for } l_i < m_i, \\ 1 - \prod_{j=1}^{l_i} F_{ij}(t) & \text{for } l_i = m_i, \\ R_{il_i}(t)\boldsymbol{CR}_{i,l_i-1}^{(m_i)}(t) + \sum_{j=1}^{m_i-1} R_{il_i-j}(t)\boldsymbol{CR}_{i,l_i-j-1}^{(m_i)}(t) & \\ \cdot \prod_{\nu=l_i-j+1}^{l_i} F_{i\nu}(t) & \text{for } l_i > m_i \end{cases} \tag{5}$$

and its lifetime distribution function is given by

$$\boldsymbol{CF}_{l_i}^{(m_i)}(t) = 1 - \boldsymbol{CR}_{l_i}^{(m_i)}(t),\ i = 1,2,... , k.$$

Hence and by (1), denoting by $\overline{\boldsymbol{CR}}_{k,l_1,l_2,\ldots,l_k}^{(m_1,m_2,\ldots,m_k)}(t) = P(T > t),\ t \in <0,\infty),$ the reliability function of the consecutive "*m* out of *l*: F"-series system, we get the next result.

Lemma 3. The reliability function of the two-state consecutive "m_i out of l_i: F"-series system is given by the following recurrent formula

$$\overline{\boldsymbol{CR}}^{(m_1,m_2,\ldots,m_k)}_{k,l_1,l_2,\ldots,l_k}(t)=\prod_{i=1}^{k}\boldsymbol{CR}^{(m_i)}_{i,l_i}(t) \tag{6}$$

$$=\begin{cases}1 & \text{for } l_i = m_i,\\ \prod_{i=1}^{k}[1-\prod_{j=1}^{l_i}F_{ij}(t)] & \text{for } l_i < m_i,\\ \prod_{i=1}^{k}[R_{il_i}(t)\boldsymbol{CR}^{(m_i)}_{i,l_i-1}(t) & \\ +\sum_{j=1}^{m_i-1}R_{il_i-j}(t)\boldsymbol{CR}^{(m_i)}_{i,l_i\text{-}j\text{-}1}(t) & \\ \cdot\prod_{\nu=l_i-j+1}^{l_i}F_{i\nu}(t)] & \text{for } l_i > m_i,\end{cases} \tag{7}$$

for $t \in <0,\infty)$.

Motivation. Assuming in (1) that $R_i(t)=\boldsymbol{CR}^{(m_i)}_{il_i}(t)$, we get (6) and next considering (5), we get the formula (7).

Definition 6. The consecutive "*m* out of *l*: F"-series systems is called regular if

$l_1 = l_2 = \ldots = l_k = l$ and $m_1 = m_2 = \ldots = m_k = m$,

where

$l, m \in N, \quad m \le l.$

Definition 7. The consecutive "m_i out of l_i: F"-series system is called homogeneous if its components lifetimes T_{ij} have an identical distribution function

$F(t) = P(T_{ij} \le t)$, $i = 1,2,\ldots, k$, $j = 1,2,\ldots, l_i$, $t \in <0,\infty)$,

i.e. if its components E_{ij} have the same reliability function

$R(t) = 1 - F(t)$, $i = 1,2,\ldots, k$, $j = 1,2,\ldots, l_i$, $t \in <0,\infty)$.

Under *Definition 6* and *Definition 7* and formula (7), denoting by $\overline{\boldsymbol{CR}}^{(m)}_{k,l}(t)=P(T>t)$, $t \in <0,\infty)$, the reliability function of a homogeneous and regular consecutive "*m* out of *l*: F"-series system, we get following result.

Lemma 4. The reliability function of the homogeneous and regular two-state consecutive "m_i out of l_i: F"-series system is given by

$$\overline{\boldsymbol{CR}}_{k,l}^{(m)}(t)=\begin{cases}1 & \text{for } l<m,\\ [1-F^{l}(t)]^{k} & \text{for } l=m,\\ \prod_{i=1}^{k}[R(t)\boldsymbol{CR}_{i,l-1}^{(m)}(t) & \\ +R(t)\sum_{j=1}^{m-1}\boldsymbol{CR}_{i,l-j-1}^{(m)}(t) & \\ \cdot F^{j-1}(t)] & \text{for } l>m,\end{cases} \tag{8}$$

$t \in <0,\infty)$.

4 CONCLUSIONS

The paper is devoted to a non-stationary approach to reliability analysis of two-state systems. Two recurrent formulae for two-state reliability functions, a general one for non-homogeneous and its simplified form for regular and homogeneous two-state consecutive "*m* out of *l*: F"-series system have been proposed.
The proposed methods and solutions may be applied to any two-state consecutive "*m* out of *l*: F"-series systems.

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MINMAXDM DISTRIBUTION FAMILY FOR TENSILE STRENGTH OF COMPOSITE

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Abstract

Generalization of extended family of weakest-link distributions with application to the composite specimen strength analysis is presented. Composite (specifically, monolayer) specimen for tensile strength is modeled as series system but every "link" of this system is modeled as parallel system. Results of successful attempts of using some specific distribution from this family for fitting of experimental dataset of strength of some carbon fiber reinforced specimens are presented.

1. Introduction

We consider a composite specimen for test of tensile strength as a bundle of n_C longitudinal items (fibers or bundles) immersed into *composite matrix* (CM), which is a composition of the matrix itself and all the layers with stackings different from the longitudinal one. We make very simplified assumption that only longitudinal items (LI) carry the longitudinal load but matrix only redistributes the loads after the failure of some longitudinal items. In fact, therefore, our model is a model of unidirectional (more specifically, monolayer) composite. We divide the composite into n_L parts of the same length l_1 (approximately, this length can be interpreted as the interval in which the load of failed LI is fully transmitted to the adjacent intact LI; the stronger the CM the smaller l_1). The total length of the composite specimens is equal to $l = n_L\, l_1$. We suppose that development of the process of fracture of a specimen takes place in one or in several of these parts ("links"). For simplicity, we call these links as "cross sections" (CS). So using this term we describe the composite as a *series system of CS*. For description of the development of fracture process of the series system it is appropriate to use the ideas on which the extended weakest link distribution family, described in the authors' papers [1-7], is based. Let the process of monotonous tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI) in the specimen cross section) be described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$, and let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of failures of LI in *i*-th CS with n_C initial number of LI at the load x_t. Then the strength of *i*-th CS

$$X_i^* = \max(x_t : n_C - K_{Ci}(t) \ge 0), \qquad (1)$$

but the ultimate strength of the specimen (which is the sequence of n_L CS) is

$$X = \min_{1\le i\le n_L} X_i^* = \min_{1\le i\le n_L} \max(x_t : n_C - K_{Ci}(t) \ge 0). \quad (2)$$

We consider different versions of cumulative distribution function (cdf) calculation methods and their applications to processing results of test of fiber strands (threads) and strip of them (monolayer) [8].

2. Models of failure of a parallel system with redistribution of load after failure of some LI

Statistical description of the development of the process of fracture of one CS (as loose bundle of LI (fibers or strands)) was initially studied by Daniels [9]. The respective model can be described in a following way. Let $(X_1,...,X_n)$ be random strengths of intact LI in some CS and X_j the *j*-th order statistics in this CS. If there is a uniform distribution of load between n LI, and load increases uninterruptedly, then the ultimate strength of this CS

$$X^* = \max_{1\le j\le n} X_j (n-j+1)/n. \quad (3)$$

We consider the case when $n = n_C - K_C$. Daniels studied the case K_C=0. In the general case for random value of K_C, (technological) failure number, there is a priori distribution $\pi_C = (\pi_1, \pi_2, ..., \pi_{n_C+1})$ (here $\pi_k = P(K_C = k-1)$). Then

$$F_{X^*}(x) = \pi_C \vec{F}(x), \quad (4)$$

where vector column $\vec{F}(x) = (F_1(x),...,F_{n_{C+1}}(x))'$, $F_k(x)$, $k = 1,...,n_C$, is cdf of X^* if $n = n_C + 1 - k$, $F_{n_C+1}(x)$ is identical with unity (there are no intact LI).

Much broader spectrum of models of the considered process can be developed using the theory of Markov chains. We consider the process of accumulation of failures as an inhomogeneous finite Markov chain (MC) with finite state space $I = \{i_1, i_2, ..., i_{n_C+1}\}$. We say that MC is in state i if $(i-1)$ LI have failed, $i = 1,...,n_C+1$. State i_{n_C+1} is an absorbing state corresponding to the fracture of CS (fracture of all LI in this CS). The process of MC state change and the corresponding process $K_{Ci}(t)$ are described by transition probabilities matrix *P*.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1(n_C+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_C+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_C+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{n_C(n_C+1)} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (5.1)$$

At the *t*-thstep of MC matrix *P* is a function of *t*, *t*=1,2,...

The cdf of strength of CS is defined on the sequence $\{x_1, x_2, ..., x_t, ...\}$ by equation

$$F_{X^*}(x_t) = \pi_C \left(\prod_{j=1}^{t} P(j)\right) u, \quad (5.2)$$

where $P(j)$ is the transition matrix for *t=j,* column vector $u = (0,...,0,1)'$.

We consider three main versions (hypotheses) of the structure of matrix *P*, denoted as P_a, P_b and P_c. In the simplest version we assume that in one step of MC only failure of one LI can take place. And for the corresponding matrix P_a we define $p_{ii} = 1 - F_C(x_t)$, where

$F_C(x_t) = (F_0(x_t) - F_0(x_{t-1}))/(1 - F_0(x_{t-1}))$ is conditional cdf of strength of a LI, the failure of which did not take place under load x_{t-1}, $F_0(x)$ is the initial cdf of strength of a LI ; $p_{i(i+1)} = 1 - p_{ii}$, $i = 1,...,n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

It can be assumed also that the number of failures in one step of MC has binomial distribution. Then for the corresponding matrix P_b we have $p_{i(i+r)} = b(r;p,k) = p^r(1-p)^{k-r}k!/r!(k-r)!$, $p = F_C(x_t)$, $k = n_c + 1 - i$, $r = 0,...,k$, $i = 1,...,n_C$; and again $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

For both versions of P described by P_a and P_b we suppose a uniform load distribution between intact LI. The third version corresponds to a transverse crack growth in the monolayer. We suppose that the first failure appears in the boundary of CS and all the following failures can appear only in the adjacent LI. The difference between the second and the third version is illustrated in Fig.1. Let now j be ordernumber of LI in a CS (j=1 for the boundary LI). In this case it is easy enough to take into account the stress concentration next to the tip of the crack. Let the redistribution of CS load $x(t)$ between intact LI be defined by a "stress concentration" function $h(j;i,n_C)$. Then in the corresponding P_c matrix $p_{ij} = \prod_{i+1}^{j} F_C(x_{ij}(t)) \prod_{j+1}^{n_C+1} (1 - F_C(x_{ij}(t)))$ for $j = i+1,...,n_C$; $p_{i(n_C+1)} = \prod_{i+1}^{n_C+1} F_C(x_{ij}(t))$ for $j = n_C$; $p_{ii} = 1 - \sum_{i+1}^{n_C+1} p_{ij}$, $p_{ij} = 0$ for $j < i$, $i = 1,...,n_C$;
where $x_{ij}(t) = h(j;i,n_C)x(t)n_C/(n_C + 1 - i)$ describes stress in *j*-th order LI after failure of *i*-th order LI.

1	0	1	1
0	1	0	1
1	1	1	0
0	0	0	1
1	1	0	1
1	1	1	1

a

0	1	1	1
0	0	1	1
0	1	1	1
0	0	0	1
0	1	1	1
1	1	1	1

b

Fig.1. Failed (0) and surviing (1) longitudinal items (LI) in specimens (under longitudinal load) with six cross sections and four LI; for uniform stress distribution (a) and for the case of transverse crack growth.

3. Models of failure of a series system (chain of links) with damaged items

In the framework of considered problem, there is a special case of n_C=1 (i.e. there is only one fiber, strand or thread). This case was studied in [6]. Below, we remind the main ideas, make the necessary corrections (appropriate for notation of this paper), and provide some generalization. We consider a specimen as a straight binary series system with n_L links of two types. There is a random number of "damaged" links , K_L, $0 \le K_L \le n_L$, with strength cdf $F_Y(x)$ (we say that they are Y-type links), and there are $(n_L - K_L)$ links with strength cdf $F_Z(x)$ (we say they are Z-type links). "Damaged" links appear if stress in LI exceeds *defect initiation stress*. The probability of this event at the load (stress) x is defined by cdf of defect initiation stress $F_K(x)$.

We suppose (see [6]) that the failure process of considered system has two-stages. In the first stage, the process develops along the specimen and damage appear in K_L, $0 \le K_L \le n_L$, links (K_L links of Y-type appear). Then the second stage takes place: the process of accumulation of elementary damages in crosswise direction up to specimen failure. We consider three levels of accuracy of description of the second stage and three corresponding probability models (probability structure). Level A: the development of fracture process takes place in every link (containing or not some initial defects) and the strength of the weakest link defines the strength of the specimen. Level AB: the strength of the link without defects can be (relatively) so high and probability of its fracture before fracture of the damaged link so small that independence of failure probability of any Z-type CS on n_L can be assumed (only the probability that $K_L > 0$ depends on the number of links, n_L). And finally, level B: in addition to the assumption of the level AB it is assumed that the cdf of strength of the critical link does not depend on this number also. Correspondingly we have three probability structures.

A : $X = \min(Y_1,...,Y_{K_L}, Z_1,...,Z_{n_L-K_L})$;

AB: $X = \begin{cases} \min(Y_1,...,Y_{K_L},Z), & K_L > 0, \\ Z, & K_L = 0; \end{cases}$ B : $X = \begin{cases} Y, & K_L > 0, \\ Z, & K_L = 0. \end{cases}$

Two different versions of the first stage can be considered also. First version: (technological) defects appear before the loading and their number does not depend on the subsequent loading. Second version: defects appear during loading (instantly or gradually) and their number depends on the load.

3.1. For "instant fracture" version for structures A, AB, B we have correspondingly

$$F(x) = 1-(1-F_Z(x))^{n_L}\sum_{k=0}^{n_L} p_k \delta^k(x), \quad \delta(x) = (1-F_Y(x))/(1-F_Z(x)), \tag{6}$$

$$F(x) = 1-\sum_{k=0}^{n_L} p_k \left(1-F_Y(x)\right)^k \left(1-F_Z(x)\right) = 1-(1-F_Z(x))\sum_{k=0}^{n_L} p_k \left(1-F_Y(x)\right)^k, \tag{7}$$

$$F(x) = p_Y F_Y(x) + (1-p_Y)F_Z(x), \tag{8}$$

where (in equations (6, 7)) binomial probability mass function (pmf) $p_k = b(k; p_L, n_L) = p_L^k (1-p_L)^{n_L-k} n_L!/k!(n_L-k)!$ is probability that there is k links of Y-type; $p_Y = 1-p_0 = 1-(1-p_L)^{n_L}$ is the probability that there is at least one link of Y-type (in this case, actually, it is enough to know only p_Y; we should not know two parameters n_L and p_0 separately). Binomial or Poisson pmf can be used for random number of links of Y-type, K_L. In the latter case equations (6, 7) (approximately, if n_L is sufficiently large) can be written in the following way

$$F(x) = 1-(1-F_Z(x))^{n_L}\exp(-\lambda(1-\delta(x))), \tag{9}$$

$$F(x) = 1-(1-F_Z(x))\exp(-\lambda F_Y(x)), \tag{10}$$

where $\lambda = n_L p_L$ or it is just independent parameter of Poisson pmf. If initiation of the defects depends on the applied load, then it can be assumed that $p_L = F_K(x)$, where $F_K(x)$ is the cdf of defect initiation load.

In the numerical example considered in this paper it was assumed that the strength of defected link S has Weibull distribution; then $Y = \log(S)$ has the smallest extreme value (sev) distribution

$$F_Y(x) = 1 - \exp(-\exp((x-\theta_{0Y})/\theta_{1Y}))\,. \tag{11}$$

And it was assumed also that for link without defects

$$F_Z(x) = 1 - \exp(-\exp((x-\theta_{0Z})/\theta_{1Z})) \tag{12}$$

but for the logarithm of defect initiation stress

$$F_K(x) = 1 - \exp(-\exp((x-\theta_{0K})/\theta_{1K}))\,. \tag{13}$$

In some numerical examples it was considered that if $\theta_{0Z} = C$, but $\theta_{1Z} \to 0$, then

$$F_Z(x) = \begin{cases} 0, & x < C, \\ 1, & x \geq C. \end{cases} \tag{14}$$

3.2. The process of gradual (during loading) accumulation of defects along the chain of n_L links again can be considered as a Markov chain (MC). In this case MC is in state i if there are $(i-1)$ of Y-type links, i=1,...,n_L+1. State i_{n_L+2} is an absorbing state corresponding to the fracture of specimen. The matrix of transition probabilities has the same form as in (5.1) . The initial distribution of K_L is represented now by some row vector $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L,n+1,}\pi_{L,n+2})$. In the new approach the number of CS of Y-type and the strength of specimens are random functions of time, $K_L(t)$ and $X(t)$. Now the three main structures we denote by MA, MAB and MB. They have the same description but instead of K_L we should write $K_L(t)$. For example, for the MA we have $X(t) = \min(Y_1, Y_2, ..., Y_{K_L(t)}, Z_1, Z_2, ..., Z_{n_L - K_L(t)})$. In similar way $X(t)$ is defined for the other structures.

Now the ultimate strength of specimen is defined again by equations (2) but it is more convenient to write it in new form:

$$X = x_{T^*}, \tag{15}$$

where

$$T^* = \max(t : X(t) > x_t)\,. \tag{16}$$

The cdf of ultimate strength , X, is defined again by an equation similar to equation (5.2):

$$F_X(x_t) = \pi_L(\prod_{j=1}^{t} P(j))u\,.$$

Specifying the matrix *P* for probability structures A and AB. The probability that in some element a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t) = (F_K(x_t) - F_K(x_{(t-1)}))/(1 - F_K(x_{(t-1)}))\,.$$

Consider the case of *s* defects present. The probability that *r* new defects appear, $0 \leq r \leq k = n - s$, and the total number of defects is equal to m=s+r

$$\widetilde{p}_{sm}(t) = (b(t))^r (1-b(t))^{k-r}\, k!/r!(k-r)!$$

Conditional probability of Y-type link fracture at the nominal stress x_t

$$q_Y(t) = (F_Y(x_t) - F_Y(x_{(t-1)}))/(1 - F_Y(x_{(t-1)}))\,.$$

Conditional probability of Z-type link fracture at the nominal stress x_t

$$q_Z(t) = (F_Z(x_t) - F_Z(x_{(t-1)}))/(1 - F_Z(x_{(t-1)}))\,.$$

Corresponding probability that none of the links (of both types) fails when there are defects in *m* links for probability structure MA is

$$u_m(t) = (1-q_Y(t))^m (1-q_Z(t))^{n_L-m},$$

and for probability structure MAB

$$u_m(t) = (1-q_Y(t))^m (1-q_Z(t)).$$

The probability of coincidence of these events, which we consider as independent, and the probability of transition from state $i=s+1$ to state $j=i+r$

$$p_{ij}(t) = \widetilde{p}_{(i-1)(j-1)}(t) u_{j-1}(t),$$

where $i \le j \le (n+1)$.

It is worth to note that if equation (14) is used and C is large enough (this means that only damaged CS define the strength) then it can be assumed that $q_Z(t)=0$.

Conditional fracture probability (for both probability structure MA and MAB) at state i

$$p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t).$$

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

Specifying the matrix *P* for probability structures MB . The corresponding Markov chain has only three states. The first state corresponds to the absence of defective links, the second one means the presence of at least one defective link, and the third, absorbing one, means failure of the specimen. The corresponding probabilities at a t-th step are determined by the formulae

$$p_{11}(t) = [1-b(t)]^{n_L}, \quad p_{12}(t) = (1-p_{11}(t))(1-q_Y(t))(1-q_Z), \quad p_{13}(t) = 1-p_{11}(t)-p_{12}(t),$$
$$p_{21}(t) = 0, \quad p_{22}(t) = (1-q_Y(t))(1-q_Z(t)), \quad p_{23}(t) = 1-p_{22}(t), \quad p_{31}(t) = p_{32}(t) = 0, \quad p_{33}(t) = 1.$$

4. MinMaxDM distribution family

Clearly, all the ideas considered in the previous section can be used also for the series system of CS if instead of the word "link" now we use the word CS. Instead of cdf $F_Y(x)$ and $F_Z(x)$, which were defined by (11-12) now we should use cdf of CS strength of Y-type or Z-type correspondingly. For building these cdf in the following numerical examples we again suppose that logarithm of strength of one LI (in one CS) without defect has the smallest extreme value (sev) distribution: $F_0(x) = 1-\exp(-\exp((x-\theta_{0Z1})/\theta_{1Z1}))$. We use the logarithm scale and in this case the cdf of specimen strength also has location and scale parameters θ_0 and θ_1: $F_X(x) = F_{0_X}((x-\theta_0)/\theta_1))$. Of course it is not the only possible assumption. Different assumptions about the distribution of strength of bundles within the frame of one CS (one "link") , a priori distribution of initial (technological) defects, the influence of length and width of specimens compose a family of the distributions of ultimate composite tensile strength. Taking into account (2) and (3) we denote this family by abbreviation MinMaxD (in memory of Daniels) if the strength $F_{X^*}(x)$ is defined by equation (4) and by abbreviation MinMaxM (because of connection with Markov chain theory), if it is defined by equation (5), and for unified family we suggest an abbreviation MinMaxDM.

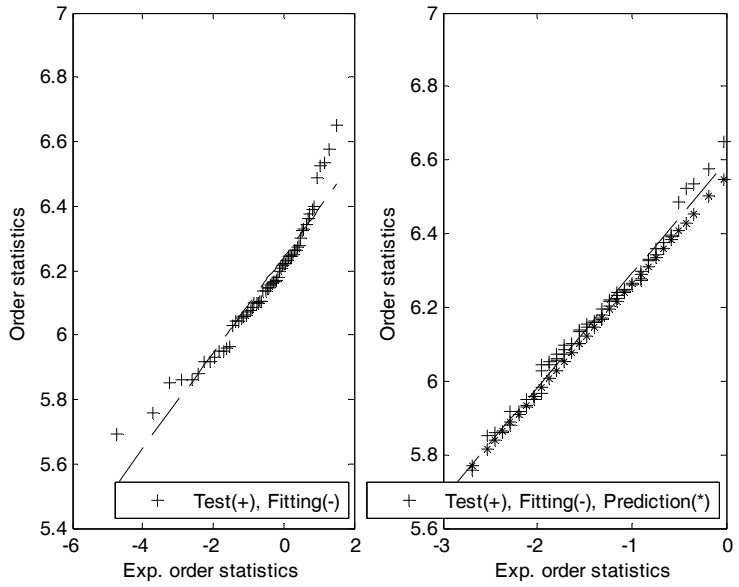
5. Processing of test data

In this paper we consider only the application of B-structure to the test data set processing. In [5] there are the test results of both 64 carbon fiber strands with length 20 mm (data_1) and the same number of strips of 10 strands of the same length (data_2) considered. We attempt to obtain statistical description of data_2 using results of processing of data_1. Let x_i be i-th order statistic, $i = 1,2,...,n$, n is the sample size; $E(X_i)$ is the expected value of ith order statistic, $E(\overset{0}{X}_i)$ is the

same but for θ_0=0 and θ_1=1. Then for estimation of θ_0 and θ_1, if all the other parameters are fixed, we have the following linear regression model: $E(X_i)=\theta_0+\theta_1 E(\overset{0}{X}_i)$. We perform fitting of the data_1 and get linear regression parameter estimates $\hat{\theta}_0$=6.554 and $\hat{\theta}_1$=0.1243 assuming that sev distribution holds (here *x* is logarithm of strength).Then we perform fitting (expected value of "standard" order statistics $E(\overset{0}{X}_i)$ versus order statistics) of the data_2 (+) assuming the same type of distribution (see Fig. 2a). In Fig. 2b we see the fitting of the same data_2 using $E(\overset{0}{X}_i)$ of cdf corresponding to MinMaxMa-Bsev model (for P_a type of matrix P, $F_0(x)$ is sev distribution, structure B (see equation (8) where n_C=5; π_C is a binomial a priori distribution of K_C with p_C=0.01 , $n=n_C$=5; $p_Y=0.9048$). "Regression prediction"(*), $x_i=\hat{\theta}_0+\hat{\theta}_1 E(\overset{o}{X}_i)$, using estimates $\hat{\theta}_0$ and $\hat{\theta}_1$ obtained processing data_1 is shown also. But here we take into account variation of Young's modulus also: Var(E)= 0.03).

Let us make additional explanations. For "fitting" of data_2 we have used parameters, found by processing of the same data. For "Regression prediction" we have used estimates $\hat{\theta}_0$ and $\hat{\theta}_1$ obtained processing data_1, which are parameters of component of monolayer (as if we did not get the parameter estimates of data_2 while fitting these data). However it is not PREDICTION but "PREDICTION", because in fact we have used also the estimates of "structure parameters" p_C, n_C and p_Y which was found processing data_2. It would be real prediction if n_C and p_Y are **parameters of technology** and they **are nearly the same for different specimens with the same type of technology and are known in advance .**

Unfortunately, it is only hope, but it is not the fact.



a b

Fig. 2. Fitting (expected value of "standard" order statistics $E(\overset{0}{X}_i)$ versus order statistics) and "prediction" of results of tensile strength test of carbon fiber strip of 10 strands using sev distribution (a) and MinMaxMa-Bsev model (b) (see explanation in text).

The statistic $OSPPt=(\sum_{i=1}^{n}(x_i-\hat{x}_i)^2/\sum_{i=1}^{n}(x_i-\bar{x})^2)^{1/2}$, where $\bar{x}=\sum_{i=1}^{n}x_i/n$ [4] , as the measure of fitting for Fig.1a is equal to 0.267 (for sev distribution) and as the measure of fitting and

prediction quality for Fig. 1b (for MinMaxMa.sev-B structure model) is equal to 0.161 and 0.192 correspondingly.

Examples of processing data of strength of fibers of different type are given in [6].

Here we consider processing of the test results of carbon reinforced composite specimens ($(0_6^o/+-45_4^o/90_3^o)_s$, length : 250 mm, width : 38 mm, thickness : 1.7 mm) which are given in [8]. In Fig. 3a we see fitting of these data (+) using sev distribution (statistics OSPPt=0.2504). In Fig. 3b we see fitting of the same data using MinMaxMa-Bsev model (statistics OSPPt=0.1548). "Prediction" of these data using MinMaxMa-Bsev model (*) and linear regression parameter estimates $\widehat{\theta}_0$=6.554 and $\widehat{\theta}_1$=0.1243 of data_1 (statistics OSPPt=0.1879) is shown also . This time n_C=50 was used; π_C is a binomial priori distribution of K_C with p_C=0.325 , $n = n_C$=50; $p_Y = 1$.

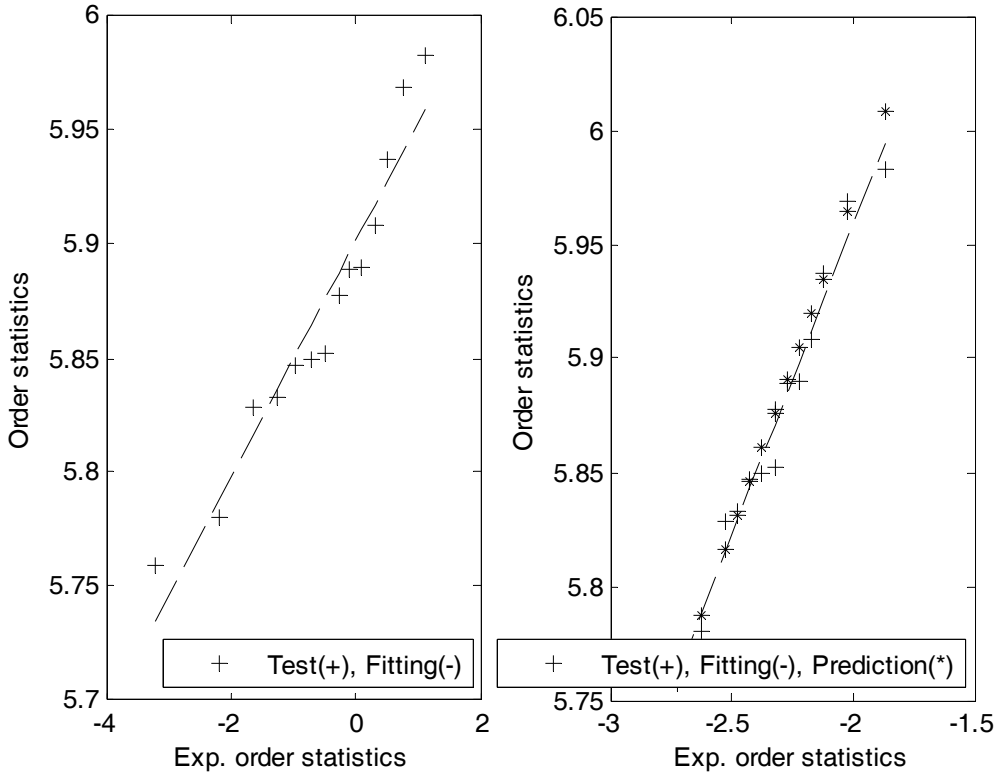


Fig. 3. Fitting (expected value of "standard" order statistics $E(\overset{0}{X}_i)$ versus order statistics) and "prediction" of the tensile strength of carbon reinforced composite specimens test results (+) using sev distribution (a) and MinMaxMa-Bsev model (b) (see explanation in text).

Conclusions

We see that MinMaxMa-Bsev model provides better (than sev distribution) fitting of results of tensile strength test of carbon fiber strip of 10 strands (but only if we assume that in CS there are only 5 strands instead of 10 and taking into account variation of Young's modulus!). It is not surprising, of course, because for MinMaxMa-Bsev we have much more parameters. Nearly the same can be said about processing the specimen data. This time $n_C = 50$ appears much more appropriate. The values $n_C = 5$ and $n_C = 50$ can be interpreted as the numbers of failures of LI which are sufficient to provoke the catastrophic failure of the specimens. Very large value of p_C=0.325 for specimen data set can be explained by the small relative value of ratio of longitudinal layer number to the total number of layers (6/(6+4+3)= 0.4615). There is a

temptation to use the coefficient of filling. However there is a large ambiguity of calculation of this value.

As a whole, it seems that MinMaxDM distribution family deserves to be studied much more thoroughly using much more test data. Interpretation of parameters of a corresponding model allows comparison of different composite structures and explanation of some specific features of failure process of composite . For example, the value p_C=0.325 indicates that at least 32.5% of the critical cross section does not carry the longitudinal load.

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SOME COMMENTS ON STATISTICAL RISKS

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Abstract

In this work we make a detailed analysis of the concept of risk, the stress being focused then on various kinds of statistical risks: producer and consumer risks, technical risk, Taguchi`s risk (making a connection with C_{pm} capability index) and a risk arising in SPC practice.

Key words: statistical risk, error, hazard rate, Taguchi`s loss function, Taguchi`s risk, SPC - Statistical Process Control.

1. Preliminaries: a discussion on the concept of risk

The notion of risk covers a broad area of interpretations. As in many cases, there is a man-in-the street approach and a scientific one which tries to offer quantitative measures of the underlying term.

Let us visit first some usual dictionaries. For instance, BBC English - Romanian Dictionary (Editura CORESI, Bucureşti, 1998, page 966(, risk is assimilated to a **danger**: if there is a risk of something, it might have unpleasant or even dangerous consequences (results).

We seize here the potentiality of such kind of results, which may or may not occur. Therefore, it is a suggestion that risk is associated with uncertainty: it might happen, but we do not know for sure if it will indeed happen.

Merriam - Webster`s Collegiate Dictionary (Tenth Edition M. W. Incorporated, Springfield, Mass, U.S.A., 1996, page 1011) is more generous and specific: **possibility** of loss or injury, a peril but also **the degree of probability** of such loss (this is a new element in the usual definitions).

The very recent „Illustrated Oxford Dictionary of English Language" (2008, Dorling/Oxford Univ. Press, Litera International, Bucureşti - Chişinău, page 709) defines it as a chance or possibility of danger, loss, injury etc.

The term „chance" is straightforwardly linked with that of uncertainty. Some authors consider that risk is characterized by possibility to be described by the aid of probability laws (see Bârsan-Pipu and Popescu, 2003 [2, page 2]). Uncertainty can be described also by quantitative measures - if we regard it from the metrological point of view (see Petrescu et al., 2006 [12]).

Webster`s Unabridged Dictionary of English Language (edited 2002) advances the concept **risk management** (RM) and also that of risk manager. This RM is viewed as a technique of estimation, prevention and minimization the accidental losses which could appear in a business by taking some safety measures (insurances - for instance).

Risk appears therefore as an uncertain event which may take place if some **risk factors** actually act.

On the other hand, the risk is always associated with the anthropical element - that human factor which finally will suffer eventual bosses of its „risky decisions".

2. Kinds of risks

Generally speaking, there are several types of risk - depending on the domain we consider to be of interest. Isaic-Maniu et al (1999, page 492) [7] believe that the so-called economic risk is of great importance. This risk is defined as the incapacity (or just impossibility!) of a given organization to survive in a business environment: this means that its managers have no the skills (and knowledge) to adapt the economical policy of the company to variations (sometimes unexpected and unfriendly) of the social-economical reality at a specified moment.

This economic risk (quite general) has some components such as „bankruptcy subrisk" which seems to be essential: if an organization cannot pay its bills for current utilities, cannot reimburse its loans, cannot pay its subcontractors, suppliers etc. - all these are the signs that the above risk has already implemented its destructive effects.

Since the risk is regarded as a probability, therefore it is worth to investigate the nature of what is called the **statistical risk**. It plays an important role in the framework of statistical inference. One problem which has been not very deeply investigated is the following: how to manipulate (or to manage) this in order to minimize it, in the sense that the decision taken in an uncertain/risky situation, to be „the best" one?

3. Various types of statistical risks

Usually, in the theory of statistical hypotheses, founded mainly by the British School of Statistics (see Stoichițoiu - Vodă, 2002 [15]) we deal with the so-called **errors** we make as regards the decisions about the underlying hypotheses.

As it is well-known, a hypothesis (in general) is simply defined as a statement/assumption/supposition about a certain phenomenon, process, situation etc. This assumption may be true (that is in accordance with the real status of the entity considered) or may be not. For a **scientific** hypothesis it is sufficient to provide a counterexample, in order to reject **as false** the proposed hypothesis.

Since in statistical analysis we work with **samples** (assumed to be obtained randomly), the conclusions will depend entirely on the sample (or samples) we have at hand. The sample could support the advanced hypothesis (called null-hypothesis, H_0) or it could sustain the alternative one (H_1). Therefore, we say that the couple (H_0, H_1) is accompanied by **two kinds of errors**, namely

$$\alpha = \text{Prob}\{\text{reject } H_0 \mid \text{if } H_0 \text{ is true}\} \quad (1)$$

and

$$\beta = \text{Prob}\{\text{accept } H_0 \mid \text{if } H_1 \text{ is true}\} \quad (2)$$

They are called respectively: error of the first Type (α) and error of the second Type (β) - see for details Blischke and Murthy, 2000 [3] page 157 - 162.

These authors draw the attention that Type I and Type II **errors rates** are the probabilities of making these kind of „mistakes" - namely „do reject H_0" (when H_0 is true) and „do not reject H_0" (when H_1 - the alternative is true).

In fact, $\alpha = \alpha(n;\theta_t)$ and $\beta = \beta(n:\theta_t)$ - that is they depend on the size of the sample we employ and on the **true value** of the parameter (θ) on which the hypothesis is made.

In SQC - Statistical Quality Control - especially in sampling inspection of batches, where practical procedures have been standardized (see American Standards MIL STD 105 D and MIL STD 414 - or their ISO equivalents, ISO 2859 and ISO 3951, α and β are called „producer risk" (α) and „consumer risk" (β) - respectively. In the above documents α and β are taken at fixed levels (α = 5% and β = 10%) and hence there is no possibility to modify these values if in practice we use these standards.

What we can do is diminish **the risk of non-acceptance** of a given lot/batch. This risk is expressed as $1 - P_a(p)$, where $P_a(p)$ is the probability of acceptance of the lot which depends on its defective (or nonconforming) fraction (p). If p is larger then the accepted value (AQL - Acceptable Quality Level), then the risk of non-acceptance is higher.

Other important element of the above mentioned documents is the so-called LQ - Limiting Quality - that is that value of p which we are ready to accept with a small probability (in 10% of the case at most).

If $p = p_0 > AQL$, the risk of non-acceptance increases as long as (p) approaches the LQ value.

It follows that the management of this risk has to be directed to those measures which can lead to a decline of the fraction defective (see Isaic-Maniu and Vodă, 1997, [6]).

3.1. Error of the Third Type?

In [14] has been discussed an argumentation of Malița and Zidăroiu (1980, [10]) in favor of a Raiffa`s idea (1970, [13]) regarding the existence of a Type III error. This last author claims that if an experimenter (or an analyst) tries to solve a **false problem**, then he commits an error of the third kind! Raiffa did not establish clearly what he understands by a „false" problem: is it an ill - posed problem (improperly/wrongly formulated) or the falsity refers to the goal/purpose stated by the responsible authority?

Malița and Zidăroiu tried to justify Raiffa`s proposal by linking it to the Type I and Type II errors, claiming that this Type III error „is expected to weight in a specific manner, the previous two classical type errors". They say also that the main source of Type III error is the lack of communication between the analyst and the decisional factor. This communication must act in both directions: from the decision unit to the experimenter/analyst and conversely, in order to check/verify that indeed we detected the right problem!

Such an argumentation seems to be at most at a metaphoric level: nobody will ask himself or someone else if the problem he solves is false …

We shall mention the Cambridge Dictionary of Statistics, Cambridge University Press, 1998 (author B. S. Everitt) where he draws the attention to not confound this risk with Type III error – term used for identifying the poorer of two treatments as the better (pages 116 and 338).

3.2. Technical risk

Irina Isaic-Maniu (see [8 , page 51 - 65], 2003) gives a „risk interpretations" for the main indicators used in reliability theory; in fact, the distribution function F(t) of a continuous and positive random variable (T) which describes the failure behavior of a given entity may be viewed as a „technical risk" - that is the complement of the reliability function:

$$\text{Prob}\{T < t_0\} = F(t_0) = 1 - R(t_0) = 1 - \text{Prob}\{T \geq t_0\} \tag{3}$$

Here $F(t_0)$ is hence the probability that the system operates less than a desired time t_0. If the reliability $R(t_0)$ is low, consequently this technical risk is high.

More adequate to define this technical risk seems to be the hazard rate (or failure rate) function which may be called also „the danger of failure":

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{d\ F(t)}{R(t)} = -\frac{d\ R(t)}{R(t)} \tag{4}$$

A high value of h(t) means a low level of reliability (h(t) is expressed usually in failures/hour).

4. Taguchi`s risk

Genichi Taguchi (see Alexis, 1999, [1]) revitalized Gauss` quadratic function $f_{(x)} = a(x - x_0)^2, \quad a > 0, x \geq x_0 \geq 0$ and associated it to the so-called quality loss

$$\mathrm{L}(x_0; \mathrm{T}) = k(x_0 - \mathrm{T})^2, \quad k > 0, x_0, \mathrm{T} \in \mathrm{R} \tag{5}$$

where x_0 is the measured value of the quality characteristic (X) and (T) is it target value (k is a constant depending on the specific case at hand).

If $f(x;\theta)$ is the density of X $(x \in \mathrm{D}, \ \mathrm{D}$ being a part of R, $\theta \in \mathrm{R})$ then the average value

$$\mathrm{E}[\mathrm{L}(x;\mathrm{T})] = \int_{\mathrm{D}} \mathrm{L}(x;\mathrm{T}) f(x;\theta) dx \tag{6}$$

is called Taguchi type risk (see Kackar, 1986, [9]).

Taking into account (5), we may write (6) as

$$\mathrm{E}[\mathrm{L}(x;\mathrm{T})] = k\left[\mathrm{Var}(\mathrm{x}) + (\mathrm{E}(\mathrm{x}) - \mathrm{T})^2\right] \tag{7}$$

and if X is normally distributed $\mathrm{N}(\mu;\delta^2)$, we have.

$$\mathrm{E}[\mathrm{L}(x;\mathrm{T})] = k\left[\delta^2 + (\mu - \mathrm{T})^2\right] \tag{8}$$

The empirical risk (denoted $\hat{\mathrm{R}}_{\mathrm{T}}(x)$) is therefore

$$\hat{\mathrm{R}}_{\mathrm{T}}(x) = k\left[\mathrm{s}^2 + (\bar{x} - \mathrm{T})^2\right] \tag{9}$$

where $\bar{x}$ and s are the well-known sample statistics.

There is a straightforward link between Taguchi`s risk and his own process capability index $\hat{\mathrm{C}}_{\mathrm{pm}}$ (see Chan et al, 1988, [4]):

$$\hat{\mathrm{C}}_{\mathrm{pm}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6 \cdot \sqrt{\mathrm{s}^2 + (\bar{x} - \mathrm{T})^2}} \tag{10}$$

where USL = Upper Specified Limit and LSL = Lower Specified Limit of the given quality characteristic $\mathrm{X} \sim \mathrm{N}(\mu, \delta^2)$ with T as its target value.

We may write hence immediately

$$\hat{\mathrm{R}}_{\mathrm{T}}(x) = k\left(\frac{\mathrm{USL} - \mathrm{LSL}}{6} \cdot \frac{1}{\hat{\mathrm{C}}_{\mathrm{pm}}}\right)^2 \tag{11}$$

If USL - LSL = 6s - that is the minimal level for admissible process capability, we get:

$$\widehat{R}_T(x;T) = k\left(\frac{s}{\widehat{C}_{pm}}\right)^2 \tag{12}$$

and we draw the conclusion that the Taguchi`s risk can be regarded as a function of the length of the specified interval USL - LSL measured in standard deviations units.

The theoretical Taguchi risk corresponding to (12) is

$$R_T(x) = k\left(\frac{\delta}{C_{pm}}\right)^2 = \frac{k\delta^2}{C_{pm}^{\ 2}}$$

Denoting $k\delta^2 = M$ and $C_{pm} = X$, we shall have a hyperbolic dependence of the type $R = M/X^2$. If in C_{pm}, the true mean-value μ is just the target T, then C_{pm} becomes the classical potential index of a process namely $C_{pm} = (USL - LSL)/6\delta$ (see figure 1).

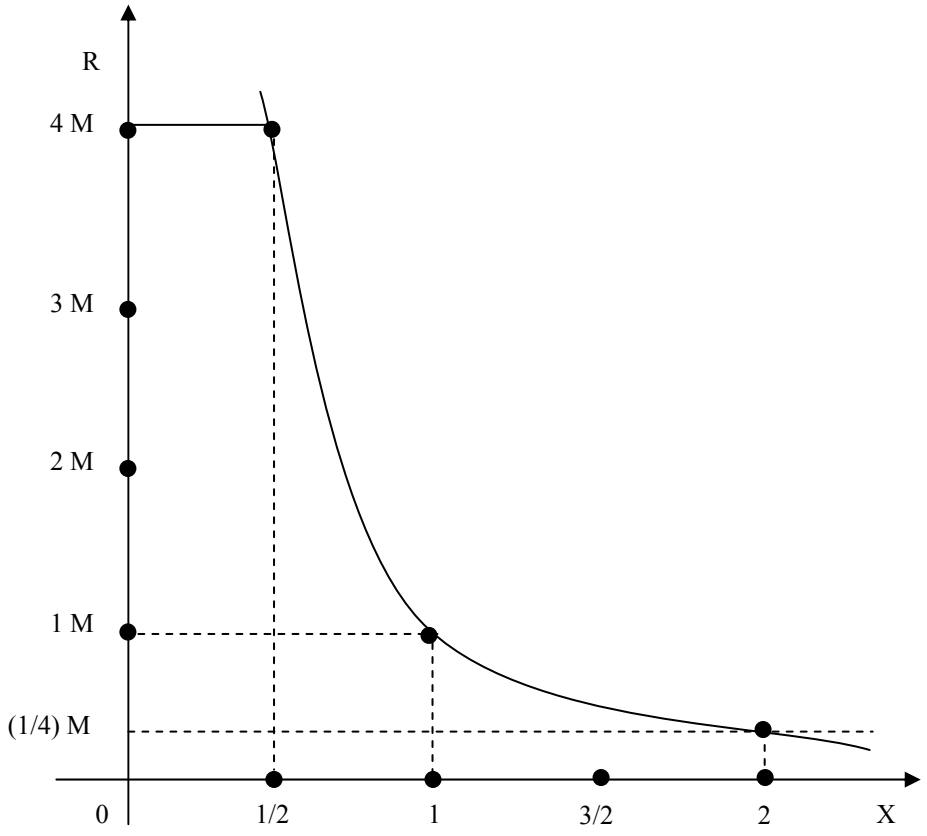

Fig. 1. The relationship between Taguchi`s risk and C_{pm}

5. Risk in SPC practice

SPC - or Statistical Process Control is mainly based on the theory and practice of Shewhart control charts (see ISO document ISO 8258 „Shewhart control charts", 1991 or Petrescu-Vodă, 2002 [11]).

From a statistical point of view, Shewhart control charts can be viewed as a continuously testing of by hypothesis $H_0 : \text{Mean} = \mu$ versus the alternative $H_1 : \text{Mean} \neq \mu$ at the significance level $\alpha = \text{Prob}\{|Z| > 3\} = 0.0027$ (see Derman and Ross, [5]).

From a practical perspective, this means that even when a certain process is in the state of statistical stability (remains in control) there is a chance - a risk (0.0027) - that a subgroup average will fall outside the control limits $UCL = \mu + 3\delta/\sqrt{n}$, $LCL = \mu + 3\delta/\sqrt{n}$ and the experimenter would incorrectly take the „risky decision" to correct the process that is to dig for an illusory cause of trouble.

Numerical example: Consider a measurable characteristic for which two specified limits are fixed, namely LSL = 263.48 c.u. (c. u. = conventional units) and USL = 263.68 c. u. The target value is T = 263.58 c. u. If we ask a performance level for C_p to be 2 and if from data we get the mean value $\bar{x}$ = 263.58 c. u. and standard deviation s = 0.011 c. u., we shall get $\hat{C}_p$ approximately 0.40 – that is a very weak potential index of the process. The estimated Taguchi risk is therefore $\hat{R}_T$ (x) ≈ 0.007 k and this risk is expressed in monetary units. This values shows that if the defective unit is cheap, then the risk is small. For such low production cost items it is not necessary to impose a performance at the level of SIX SIGMA (see the excellent monograph of Praveen Gupta "The Six Sigma Performance Handbook. A Statistical Guide to Optimizing Results", McGraw-Hill Book Co., 2005, New York

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