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Part 1

A quantitative risk analysis (QRA) should provide a broad, informative and balanced picture of risk, in order to support decisions. To achieve this, a proper treatment of uncertainty is a prerequisite. Most approaches to treatment of uncertainty in QRA seem to be based on the thinking that uncertainty relates to the calculated probabilities and expected values. This causes difficulties when it comes to communicating what the analysis results mean, and could easily lead to weakened conclusions if large uncertainties are involved. An alternative approach is to hold uncertainty, not probability, as a main component of risk, and regard probabilities purely as epistemic-based expressions of uncertainty. In the paper the latter view is taken, and we describe what should be the main components of a risk description when following this approach. We also indicate how this approach relates to decision-making. An important issue addressed is how to communicate the shortcomings and limitations of probabilities and expected values. Sensitivity analysis plays a key role in this regard. Examples are included to illustrate ideas and findings.

In the paper an analytical model of port transportation systems environment and infrastructure influence on their operation processes is constructed and presented in an example of shipyard rope transportation systems in Naval Shipyard in Gdynia. A general semi-markov model of a system operation process is proposed and the methods of its parameters statistical identification are presented. Further, the shipyard rope transportation system and the ship rope elevator operation processes are analyzed and their operation states are defined. A preliminary collection of statistical data necessary to the ship transportation systems' operation processes identification is included.

This paper presents a work on the study of importance measures in presence of uncertainties originating from the lack of knowledge and information on the system (epistemic uncertainties). A criterion is proposed for ranking the risk contributors in presence of uncertainties described by probability density functions.

The paper is concerned with the analysis of simulation research results of newly design Piast ferry entering to modernized Ystad Port. The ship simulation model is described. After execution of real time simulations the Monte Carlo method of underkeel clearance evaluation is applied to asses the probability of ferry touching the bottom. The results could be used in risk assessment of ships entering to the ports.

The paper presents methods and models used nowadays for risk assessment of ship-bridge collisions.

In this paper, we introduce our newly created DEAR (an abbreviation of Differential Equation Associated Regression) theory, which merges differential equation theory, regression theory and random fuzzy variable theory into a new rigorous small sample based inferential theoretical foundation. We first explain the underlying idea of DEAR modelling, its classification, and then the M-estimation of DEAR model. Furthermore, we explore the applicability of DEAR theory in the analysis in system dynamics, for example, repairable system analysis, quality dynamics analysis, stock market analysis, and ecosystem analysis, etc.

Part 2

As we use complex systems with one shot items in many technical applications we need to know basic characteristics of such system. Performance, safety and other are as much important as dependability measures. In real applications we have to take into account a related distribution of an observed variable. In terms of complex systems with one shot items it is a discrete random variable related to one shot item. The whole system and its failures (unexpected and inadvertent events) may have two typical types of distributions and their characteristics. We either consider a continuous variable (such as time, mileage, etc.) or a counting variable (such as number of cycles, sequences, etc.) regarding to a failure occurrence. As the one shot items is supposed to back up the main system function the total reliability of the system should be higher than. The main issue regarding the system using one shot items in their construction is to determine the probability of the task (mission) success. The paper presents both theoretical approach and practical example of the solution.

There is a model of transport system presented in the paper. The possible semi - Markov process definitions are included. The system is defined by semi – Markov processes, while functions distributions are assumed. There are attempts to assess factors for other than exponential functions distributions. The paper consist discussion on Weibull and Gamma distribution in semi – Markov calculations. It appears that some forms of distribution functions makes computations extremely difficult.

This paper discusses the problem of the optimization of maintenance threshold and inspection period for a continuously deteriorating system with the influence of covariates. The deterioration is modeled by an increasing stochastic process. The process of covariates is assumed to be a temporally homogeneous finite-state Markov chain. A model similar to the proportional hazards model is used to represent the influence of the covariates. Parametric estimators of the unknown parameters are obtained by using Least Square Method. The optimal maintenance threshold and the optimal inspection interval are derived to minimize the expected average cost. Comparisons of the expected average costs under different conditions of covariates and different maintenance policies are given by numerical results of Monte Carlo simulation.

An object ability to realise tasks may be restored by repairing only failed components. This is called imperfect repair as the object is not as good as new after such a repair. Preventive replacement is an example of imperfect repair as well. The advantage of such maintenance is that it enables controlling a reliability level of a system. Sets of objects' components which should be replaced are derived on a basis of statistical diagnosing with use of data about components failures. The acceptable level of a failure risk while executing transportation tasks has been taken as a criterion of choosing elements to be replaced. An algorithm of selecting components for preventive replacement has been developed. It was shown that a level of a system reliability can be controlled by changing an order of a quantile function in coordination and a number of redundant objects. A computer simulation model of the system was used to illustrate derived dependencies.

The paper demonstrates the comparison of Monte Carlo simulation algorithm with neural network enhancement in the reliability case study. With regard to process dynamics, we attempt to evaluate the tank system unreliability related to the initiative input parameters setting. The neural network is used in equation coefficients calculation, which is executed in each transient state. Due to the neural networks, for some of the initial component settings we can achieve the results of computation faster than in classical way of coefficients calculating and substituting into the equation.

This paper presents a method for the quantification of the effects of measures of risk prevention of the frequency for rupture of pipework. Some methodologies, given in the literature for this purpose, assume that each plant under analysis is characterized by the same combinations of causes of failure and prevention mechanisms but this assumption is not always true. The approach suggested here is based on the methodology proposed in 1999 by Papazoglou for the quantification of the effects of organizational and managerial factors. Taking advantage of this methodology the objective of the assessment of the influence of measures of risk prevention in pipework has been achieved through the definition of the links between the causes of failure and the measures adopted by the company in order to prevent and/or to mitigate them.

In the paper a probabilistic model of industrial systems environment and infrastructure influence on their operation processes is proposed. Semi-markov processes are used to construct a general model of complex industrial systems' operation processes. Main characteristics of this model are determined as well. In particular case, for a port oil transportation system, its operation states are defined, the relationships between them are fixed and particular model of its operation process is constructed and its main characteristics are determined. Further, the joint model of the system operation process and the system reliability is defined sand applied to the reliability evaluation of the port oil transportation system.

ASYMPTOTIC APPROACH TO RELIABILITY EVALUATION OF LARGE "M OUT OF L"- SERIES SYSTEM IN VARIABLE OPERATION CONDITIONS

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ABSTRACT

The semi-markov model of the system operation process is proposed and its selected parameters are defined. There are found reliability and risk characteristics of the multi-state "m out of l"-series system. Next, the joint model of the semi-markov system operation process and the considered multi-state system reliability and risk is constructed. The asymptotic approach to reliability and risk evaluation of this system in its operation process is proposed as well.

1 INTRODUCTION

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and complicated operating processes. This complexity very often causes evaluation of systems reliability to become difficult. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet these systems, for instance, in piping transportation of water, gas, oil and various chemical substances or in transport using belt conveyers and elevators.

Taking into account the importance of safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis (Kolowrocki 2004). The assumption that the systems are composed of multi-state components with reliability state degrading in time without repair gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability characteristics is often very difficult to fix and to analyse. A convenient tool for solving this problem is semi-markov modelling (Grabski 2002, Kolowrocki & Soszynska 2005, Soszynska 2006 a, b, Soszynska 2007 a, b, c) of the systems operation processes which is proposed in the paper. In this model, the variability of system components reliability characteristics is pointed by introducing the components' conditional reliability functions determined by the system operation states. Therefore, the common usage of the multi-state system's limit reliability functions in their reliability evaluation and the semi-markov model for system's operation process modelling in order to construct the joint general system reliability model related to its operation process is proposed. On the basis of that joint model, in the case, when components have exponential reliability functions, unconditional multi-state limit reliability functions of the "*m* out l_n"-series system are determined.

2 SYSTEM OPERATION PROESS

We assume that the system during its operation is operating in $v, v \in N,$ different operation states. After this assumption we can define the system operation process $Z(t),\ t \in < 0,+\infty >,$ with discrete states from the set of states

$$Z = \{z_1, z_2, \ldots, z_v\}.$$

In practice a convenient assumption is that $Z(t)$ is a semi-markov process (Grabski 2002, Kolowrocki & Soszynska 2005, Soszynska 2006 a, b, Soszynska 2006 a, b, c) with its conditional sojourn times θ_{bl} at the operation state z_b when its next operation state is z_l, $b, l = 1,2,...,v,\ b \neq l.$ In this case this process may be described by:
- the vector of probabilities of the initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of its transitions between the states $[p_{bl}]_{vxv}$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the sojourn times $\theta_{bl},\ b \neq l.$

If the sojourn times θ_{bl}, $b,\ l = 1,2,...,v,\ b \neq l,$ have Weibull distributions with parameters α_{bl}, β_{bl}, i.e., if for $b, l = 1,2,...,v,\ b \neq l,$

$$H_{bl}(t) = P(\theta_{bl} < t) = 1 - \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0,$$

then their mean values are determined by

$$M_{bl} = E[\theta_{bl}] = \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}),\ b, l = 1,2,...,v,\ b \neq l. \tag{1}$$

The unconditional distribution functions of the process $Z(t)$ sojourn times θ_b at the operation states z_b, $b = 1,2,...,v,$ are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl}[\ 1 \text{ - } \exp[\text{-}\ \alpha_{bl} t^{\beta_{bl}}\ t]],\ = 1 - \sum_{l=1}^{v} p_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}],\ t > 0,\ b = 1,2,...,v, \tag{2}$$

and, considering (1), their mean values are

$$M_b = \mathrm{E}[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl} = \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{1}{\beta_{bl}}} \Gamma(1 + \frac{1}{\beta_{bl}}),\ b = 1,2,...,\mathrm{v}, \tag{3}$$

and variances are

$$D_b = D[\theta_b] = E[(\theta_b)^2] - (M_b)^2, \tag{4}$$

where, according to (2),

$$E[(\theta_b)^2] = \int_0^\infty t^2 dH_b(t) = \sum_{l=1}^{v} p_{bl} \int_0^\infty t^2 \alpha_{bl}\beta_{bl} \exp[-\alpha_{bl} t^{\beta_{bl}}] t^{\beta_{bl}-1} dt = \sum_{l=1}^{v} p_{bl} \alpha_{bl}^{-\frac{2}{\beta_{bl}}} \Gamma(1+\frac{2}{\beta_{bl}}),\ b = 1,2,...,v.$$

Limit values of the transient probabilities

$$p_b(t) = P(Z(t) = z_b),\ t \ge 0,\ b = 1,2,...,v,$$

at the operation states z_b are given by

$$p_b = \lim_{t\to\infty} p_b(t) = \pi_b M_b / \sum_{l=1}^{v} \pi_l M_l,\ b = 1,2,...,v, \tag{5}$$

where M_b are given by (3) and the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases}$$

3 MULTI STATE " M OUT OF L"- SERIES SYSTEM

In the multi-state reliability analysis to define systems with degrading components we assume that all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \ge 1$, the reliability states are ordered, the state 0 is the worst and the state z is the best and the component and the system reliability states degrade with time t without repair. The above assumptions mean that the states of the system with degrading components may be changed in time only from better to worse ones. The way in which the components and system states change is illustrated in Figure 1.
One of multi-state reliability structures with components degrading in time (Kolowrocki 2004, Kolowrocki et. al 2005) are "m out of l_n"- series systems.

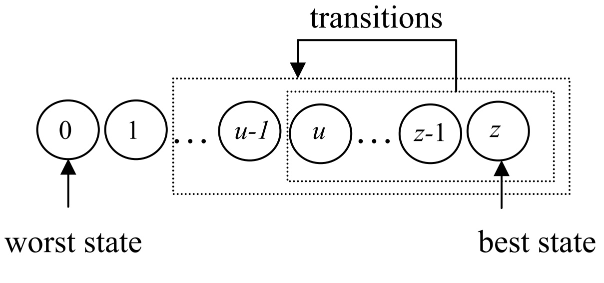

Figure 1. Illustration of states changing in system with ageing components.

To define them, we additionally assume that E_{ij}, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, l_2,..., l_{k_n}, $n \in N$, are components of a system, $T_{ij}(u)$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, k_n, l_1, l_2,..., l_{k_n}, $n \in N$, are independent random variables representing the lifetimes of components E_{ij} in the state subset $\{u,u+1,...,z\}$, while they were in the state z at the moment $t = 0$, $e_{ij}(t)$ are components E_{ij} states at the moment t, $t \in <0,\infty)$, $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u,u+1,...,z\}$ while it was in the reliability state z at the moment $t = 0$ and $s(t)$ is the system reliability state at the moment t, $t \in < 0, \infty)$.

Definition 1. A vector

$$R_{ij}(t,\cdot) = [R_{ij}(t,0), R_{ij}(t,1),..., R_{ij}(t,z)],\ t \in < 0, \infty),$$

where

$$R_{ij}(t,u) = P(e_{ij}(t) \geq u \mid e_{ij}(0) = z) = P(T_{ij}(u) > t)$$

for $t \in < 0, \infty)$, $u = 0,1,...,z$, $i = 1,2,...,k_n$, $j = 1,2,...,l_i$, is the probability that the component E_{ij} is in the reliability state subset $\{u, u+1,..., z\}$ at the moment t, $t \in < 0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_{ij}.

Definition 2. A vector

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,0), \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1),..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)],$$

where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t)$$

for $t \in < 0, \infty)$, $u = 0,1,...,z$, is the probability that the system is in the reliability state subset $\{u, u+1,..., z\}$ at the moment t, $t \in < 0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a system.

It is clear that from *Definition 1* and *Definition 2*, for $u = 0$, we have $R_{ij}(t,0) = 1$ and $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,0) = 1$.

Definition 3. A multi-state system is called "m out of l_n"- series if its lifetime $T(u)$ in the state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq k_n} T_{(l_i - m_i + 1)}(u), \quad m_i \leq l_i, ,\ u = 1,2,..., z,$$

where $T_{(l_i - m_i + 1)}(u)$ is m_i-th maximal statistics in the random variables set

$$T_{i1}(u), T_{i2}(u), ..., T_{il_i}(u), \quad i = 1, 2, ..., k_n,\ u = 1,2,..., z.$$

Definition 4. A multi-state "m out of l_n"- series system is called regular if $l_1 = l_2 = \ldots = l_{k_n} = l_n$ and $m_1 = m_2 = ... = m_{k_n} = m$, $l_n, m \in N$, $m \leq l_n$.

Definition 5. A multi-state "m out of l_n"- series system is called homogeneous if its component lifetimes $T_{ij}(u)$ have an identical distribution function, i.e.

$$F(t,u) = P(T_{ij}(u) \le t),\ t \in <0,\infty),\ \ u = 1,2,...,z,\ i = 1,2,...,k_n,\ \ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function, i.e.

$$R(t,u) = 1 - F(t,u),\ t \in <0,\infty),\ u = 1,2,...,z.$$

From the above definitions it follows that the reliability function of the homogeneous and regular "m out of l_n"- series system is given by (Kolowrocki 2004, Kolowrocki et al 2005)

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1, \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)],$$

(6)

where

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = [1 - \sum_{i=0}^{m-1}\binom{l_n}{i}[R(t,u)]^i\ \ [1 - R(t,u)]^{l_n - i}]^{k_n},\ \mathrm{t} \in <0,\infty),\ u = 1,2,...,z, \tag{7}$$

or by

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot) = [1, \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1), ..., \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)], \tag{8}$$

where

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = [\sum_{i=0}^{l_n - m}\binom{l_n}{i}[1 - R(t,u)]^i[R(t,u)]^{l_n - i}]^{k_n},\qquad t \in <0,\infty),\qquad u = 1,2,...,z,$$

(9)

where k_n is the number of "m out of l_n" subsystems connected series and l_n is the number of components of the "m out of l_n" subsystems.

Under these definitions, if $\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z,$ or $\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = 1$ for $t \le 0$, $u = 1,2,...,z,$ then

$$M(u) = \int_0^\infty \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)dt,\ \ u = 1,2,..., z, \tag{10}$$

or

$$M(u) = \int_0^\infty \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)dt,\ \ u = 1,2,..., z, \tag{11}$$

is the mean lifetime of the multi-state non-homogeneous regular "m out of l_n"- series system in the reliability state subset $\{u, u+1,..., z\}$, and the variance is given by

$$D[T(u)] = 2\int_0^\infty t\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)dt - E^2[T(u)], \tag{12}$$

or by

$$D[T(u)] = 2\int_0^\infty t \ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)dt - E^2[T(u)]. \tag{13}$$

The mean lifetime $\overline{M}(u),\ u = 1,2,...,z,$ of this system in the particular states can be determined from the following relationships

$$\overline{M}(u) = M(u) - M(u+1),\ u = 1,2,...,z-1,\ \ \overline{M}(z) = M(z). \tag{14}$$

Definition 6. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t),\ t \in <0,\infty),$$

that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the reliability state z at the moment $t = 0$ is called a risk function of the multi-state homogeneous regular "m out of l_n"- series system.

Considering *Definition 6* and *Definition 2*, we have

$$\boldsymbol{r}(t) = 1 - \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,r),\ t \in <0,\infty), \tag{15}$$

and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{16}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

4 MULTI STATE " M OUT OF L"- SERIES SYSTEM IN ITS OPERATION PROCESS

We assume that the changes of the process $Z(t)$ states have an influence on the system components E_{ij} reliability and the system reliability structure as well. Thus, we denote the conditional reliability function of the system component E_{ij} while the system is at the operational state z_b, $b = 1,2,...,v,$ by

$$[R^{(i,j)}(t,\cdot)]^{(b)} = [1,\ [R^{(i,j)}(t,1)]^{(b)}, ...,\ [R^{(i,j)}(t,z)]^{(b)}],$$

where for $t \in <0,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$

$$[R^{(i,j)}(t,u)]^{(b)} = P(T^{(b)}_{ij}(u) > t | Z(t) = z_b)$$

and the conditional reliability function of the system while the system is at the operational state z_b, $b = 1,2,...,v,$ by

$$[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot)]^{(b)} = [1,\ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1)]^{(b)},...,\ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)]^{(b)}] \text{ for } t \in <0,\infty),\ u = 1,2,...,z,\ \ b = 1,2,...,\nu,$$

where according to (7), we have

$$[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} = P(T^{(b)}(u) > t \,|\, Z(t) = z_b) = [1 - \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(t,u)]^{(b)}]^i \ \ [1-[R(t,u)]^{(b)}]^{l_n - i}]^{k_n}$$

$$\text{for } t \in <0,\infty),\ \ u = 1,2,...,z,\ \ b = 1,2,...,\nu,$$

or by

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot)]^{(b)} = [1,\ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1)]^{(b)},...,\ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)]^{(b)} \text{ for } t \in <0,\infty),\ u = 1,2,...,z,\ \ b = 1,2,...,\nu,$$

where according to (9), we have

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} = P(T^{(b)}(u) > t \,|\, Z(t) = z_b) = [\sum_{i=0}^{l_n - m}\binom{l_n}{i}[1-[R(t,u)]^{(b)}]^i \ \ [[R(t,u)]^{(b)}]^{l_n - i}]^{k_n}$$

$$\text{for } t \in <0,\infty),\ \ u = 1,2,...,z,\ \ b = 1,2,...,\nu.$$

The reliability function $[R^{(i,j)}(t,u)]^{(b)}$ is the conditional probability that the component E_{ij} lifetime $T^{(b)}_{ij}(u)$ in the reliability state subset $\{u, u+1,...,z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. Similarly, the reliability function $[\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)}$ or $[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,...,z\}$ is not less than *t*, while the process *Z*(*t*) is at the operation state z_b. In the case when the system operation time is large enough, the unconditional reliability function of the system

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,1),...,\ \overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,z)\ ],$$

where

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = P(T(u) > t) \text{ for } u = 1,2,...,z,$$

or

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,\cdot) = [1,\ \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,1),...,\ \overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,z)\ ],$$

where

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u) = P(T(u) > t) \text{ for } u = 1,2,...,z,$$

and $T(u)$ is the unconditional lifetime of the system in the reliability state subset $\{u, u+1, ..., z\}$, is given by

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p_b \ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)}, \tag{17}$$

or

$$[\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} \cong \sum_{b=1}^{v} p_b \ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} \tag{18}$$

for $t \geq 0$ and the mean values and variances of the system lifetimes in the reliability state subset $\{u, u+1, ..., z\}$ are

$$M(u) \cong \sum_{b=1}^{v} p_b M_b(u) \quad \text{for} \quad u = 1,2,...,z, \tag{19}$$

where

$$M_b(u) = \int_0^\infty [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt, \tag{20}$$

or

$$M_b(u) = \int_0^\infty [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt, \tag{21}$$

and

$$D[T^{(b)}(u)] = 2\int_0^\infty t \ [\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)], \tag{22}$$

or

$$D[T^{(b)}(u)] = 2\int_0^\infty t \ [\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}}(t,u)]^{(b)} dt - E^2[T^{(b)}(u)] \tag{23}$$

for $b = 1,2,...,v,$ $t \geq 0,$ and p_b are given by (5).
The mean values of the system lifetimes in the particular reliability states $u,$ by (14), are

$$\overline{M}(u) = M(u) - M(u+1), \ u = 1,2,...,z-1, \ \overline{M}(z) = M(z). \tag{24}$$

5 LARGE MULTI STATE " M OUT OF L"- SERIES SYSTEM IN ITS OPERATION PROCESS

Definition 7. A reliability function

$$\boldsymbol{\mathfrak{R}}(t,\cdot) = [1, \boldsymbol{\mathfrak{R}}(t,1), ..., \boldsymbol{\mathfrak{R}}(t,z)],\ t \in (-\infty, \infty),$$

where

$$\boldsymbol{\mathfrak{R}}(t,u) = \sum_{b=1}^{v} p_b [\boldsymbol{\mathfrak{R}}(t,u)]^{(b)},$$

is called a limit reliability function of a multi-state homogeneous regular "m out of l_n"- series system in its operation process with reliability function

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1,\ \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ...,\ \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)],$$

or

$$\overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(t,\cdot) = [1,\ \overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(t,1), ...,\ \overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(t,z),$$

where $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)$, $\overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(t,u)$, $u = 1,2,...,z$, are given by (17) and (18) if there exist normalising constants

$$a_n^{(b)}(u) > 0,\ \ b_n^{(b)}(u) \in (-\infty, \infty),\ \ b = 1,2,...,v,\ u = 1,2,...,z,$$

such that for $t \in C_{[\boldsymbol{\mathfrak{R}}(u)]^{(b)}}$, $u = 1,2,...,z,\ b = 1,2,...,v,$

$$\lim_{n\to\infty} [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = [\boldsymbol{\mathfrak{R}}(t,u)]^{(b)},$$

or

$$\lim_{n\to\infty} [\overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = [\boldsymbol{\mathfrak{R}}(t,u)]^{(b)}.$$

Hence, the following approximate formulae are valid

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p^b [\boldsymbol{\mathfrak{R}}(\frac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)}, u)]^{(b)},\ u = 1,2,...,z, \tag{25}$$

or

$$\overline{\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p^b [\boldsymbol{\mathfrak{R}}(\frac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)}, u)]^{(b)},\ u = 1,2,...,z. \tag{26}$$

The following auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 1. If

(i) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 0$, $m = \text{const}$, as $n \to \infty$,

(ii) $\overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,u) = \sum_{b=1}^{v} p_b [1 - \sum_{i=0}^{m-1} \frac{[[V(t,u)]^{(b)}]^i}{i!} \exp[-[V(t,u)]^{(b)}]]^k$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "*m* out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \, [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)}, \; t \in (-\infty,\infty),$$

where

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(t,u)]^{(b)}]^i \, [1 - R(t,u)]^{(b)}]^{l_n - i}]^{k_n}, \; t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v, \tag{27}$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}^{(m)}}(t,z)], \; t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if (Kolowrocki et al 2005)

$$\lim_{n\to\infty} n[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}] = [V(t,u)]^{(b)}, \; t \in C_{[V(u)]^{(b)}}, \; u = 1,2,...,z, \; b = 1,2,...,v. \tag{28}$$

Proposition 1. If components of the multi-state homogeneous, regular "*m* out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (29)

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 0$, $m = \text{const}$, as $n \to \infty$,

(iii) $a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}$, $b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)} \log n$, $u = 1,2,...,z$, $b = 1,2,...,v$, (30)

then

$$\overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}_3^{(m)}}(t,z)], \quad t \in (-\infty,\infty), \tag{31}$$

where

$$\overline{\mathfrak{R}_3^{(m)}}(t,u) = \sum_{b=1}^{v} p_b \big[1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]]\big]^k \quad \text{for } t \in (-\infty,\infty)\ u = 1,2,...,z, \tag{32}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \big[1 - \sum_{i=0}^{m-1} \frac{\exp[-it\lambda^{(b)}(u) + i\log n]}{i!} \exp[\exp[-t\lambda^{(b)}(u) - \log n]]\big]^k \tag{33}$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t + \log n}{\lambda^{(b)}(u)} \to \infty \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),\ b = 1,2,...,v,\ u = 1,2,...,z,$$

then, according to (29) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-t - \log n] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ \ b = 1,2,...,v.$$

Hence, considering (28), it appears that

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \lim_{n\to\infty} n\exp[-t - \log n] = \exp[-t]$$

for $t \in (-\infty,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

which means that according to *Lemma 1* the limit reliability function of that system is given by (31)-(32). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 2. If

(i) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(ii) $\overline{\mathfrak{R}^{(\eta)}}(t,u) = \sum_{b=1}^{v} p_b \big[- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-[v(t,u)]^{(b)}} e^{-\frac{x^2}{2}} dx\big]^k$ is a non-degenerate reliability function, where $[v(t,u)]^{(b)}$ is a non-increasing function

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1-\sum_{i=0}^{m-1}\binom{l_n}{i}[[R(t,u)]^{(b)}]^i\ [1-[R(t,u)]^{(b)}]^{l_n-i}]^{k_n} \quad t\in(-\infty,\infty), b=1,2,...,v,\ \ u=1,2,...,z, \tag{12}$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}^{(\eta)}}(t,z)],\ t\in(-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty}\frac{(n+1)[R(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}-m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = [v(t,u)]^{(b)} \quad \text{for} \quad t\in C_{[v(u)]^{(b)}},\quad u=1,2,...,z,\quad b=1,2,...,v.$$

(35)

Proposition 2. If components of the multi-state homogeneous, regular "*m* out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$$[R(t,u)]^{(b)} = 1 \text{ for } t<0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t] \text{ for } t\geq 0,\ \ u=1,2,...,z,\ b=1,2,...,v, \tag{36}$$

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to \eta$, $0<\eta<1$, as $n\to\infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}\sqrt{\frac{n-m+1}{(n+1)m}}, \tag{37}$$

$$b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)}\log\frac{n+1}{m},\quad u=1,2,...,z,\ b=1,2,...,v, \tag{38}$$

then

$$\overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,z)],\ t\in(-\infty,\infty), \tag{39}$$

where

$$\overline{\boldsymbol{\mathfrak{R}}_1^{(\eta)}}(t,u) = \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp[-\frac{x^2}{2}]\,dx\big]^k \text{ for } t\in(-\infty,\infty), u=1,2,...,z, \tag{40}$$

is the multi-state limit reliability function of that system , i.e. for *n* large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{t-b_n^{(b)}(u)}{a_n^{(b)}(u)}}\exp[-\frac{x^2}{2}]\,dx\big]^k \cong \sum_{b=1}^{v} p_b\big[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{t\lambda^{(b)}(u)-\log\frac{n+1}{m}}{\sqrt{\frac{n-m+1}{(n+1)m}}}}\exp[-\frac{x^2}{2}]\,dx\big]^k \tag{41}$$

for $t\in(-\infty,\infty)$, $u=1,2,...,z$.

Proof. For n large enough we have

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)}\sqrt{\frac{n-m+1}{(n+1)m}} + \frac{1}{\lambda^{(b)}(u)}\log\frac{n+1}{m} > 0 \text{ for } t \in (-\infty,\infty),\ u = 1,2,\dots,z,\ b = 1,2,\dots,v.$$

Therefore, according to (37)-(38) for n large enough we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-t\sqrt{\frac{n-m+1}{(n+1)m}} - \log\frac{n+1}{m}] = [1 - t\sqrt{\frac{n-m+1}{(n+1)m}} + o(\frac{1}{\sqrt{n}})]\frac{m}{n+1}$$

for $t \in (-\infty,\infty),\ u = 1,2,\dots,z,\ b = 1,2,\dots,v.$

Hence, considering (35), it appears that

$$[v(t,u)]^{(b)} = \lim_{n\to\infty}\frac{(n+1)[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} - m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = \lim_{n\to\infty}[-t + o(\frac{1}{\sqrt{n}})\sqrt{\frac{(n+1)(n-m+1)}{m}}] = -t$$

for $t \in (-\infty,\infty),\ u = 1,2,\dots,z,\ b = 1,2,\dots,v,$

which means that according to *Lemma 2* the limit reliability function of that system is given by (39)-(40). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).

Lemma 3. If

(i) $k_n \to k = \text{const},\ l_n = n,\ m/n \to 1,\ (n-m) \to \bar{m} = \text{const},$ as $n \to \infty$,

(ii) $\overline{\boldsymbol{\mathfrak{R}}}^{(\bar{m})}(t,u) = \sum_{b=1}^{v} p_b[\sum_{i=0}^{\bar{m}}\frac{[[\bar{V}(t,u)]^{(b)}]^i}{i!}\exp[-[\bar{V}(t,u)]^{(b)}]]^k$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,\cdot) = [1,\ \overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,1),\ \dots,\ \overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}}_{k_n,l_n}^{(m)}(t,u)]^{(b)} = [\sum_{i=0}^{l_n-m}\binom{l_n}{i}[1-[R(t,u)]^{(b)}]^i\ [1-[R(t,u)]^{(b)}]^{l_n-i}]^{k_n},\ t\in(-\infty,\infty), u = 1,2,\dots,z,\ b = 1,2,\dots,v, \quad (42)$$

is its reliability function at the operational state z_b,

then

$$\overline{\overline{\boldsymbol{\mathfrak{R}}}^{(\overline{m})}}(t,\cdot) = [1, \overline{\overline{\boldsymbol{\mathfrak{R}}}^{(\overline{m})}}(t,1), ..., \overline{\overline{\boldsymbol{\mathfrak{R}}}^{(\overline{m})}}(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = [\overline{V}(t,u)]^{(b)} \text{ for } t \in C_{[\overline{V}(u)]^{(b)}},\ u = 1,2,...,z,\ b = 1,2,...,v. \tag{43}$$

Proposition 3. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (44)

(ii) $k_n \to k = \text{const}$, $l_n = n$, $m/n \to 1$, $(n - m) \to \overline{m} = \text{const}$, as $n \to \infty$

(iii) $a_n^{(b)}(u) = \dfrac{1}{n\lambda^{(b)}(u)}$, $b_n^{(b)}(u) = 0$, $u = 1,2,...,z$, $b = 1,2,...,v$, (45)

then

$$\overline{\overline{\boldsymbol{\mathfrak{R}}}_2^{(\overline{m})}}(t,\cdot) = [1, \overline{\overline{\boldsymbol{\mathfrak{R}}}_2^{(\overline{m})}}(t,1), ..., \overline{\overline{\boldsymbol{\mathfrak{R}}}_2^{(\overline{m})}}(t,z)],\ t \in (-\infty,\infty), \tag{46}$$

where

$$\overline{\overline{\boldsymbol{\mathfrak{R}}}_2^{(\overline{m})}}(t,u) = \begin{cases} 1 & \text{for } t < 0, \\ \sum_{b=1}^{v} p_b\ [\sum_{i=0}^{\overline{m}} \frac{t^i}{i!} \exp[-t]]^k & \text{for } t \ge 0,\ u = 1,2,...,z, \end{cases} \tag{47}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\overline{\mathbf{R}}_{k_n,l_n}^{(m)}}(t,u) \cong \begin{cases} 1 & \text{for } t < 0, \\ \sum_{b=1}^{v} p_b\ [\sum_{i=0}^{\overline{m}} \frac{(n\lambda^{(b)}(u)t)^i}{i!} & \text{for } t \ge 0, \\ \exp[-n\lambda^{(b)}(u)t]]^k & . \end{cases} \tag{48}$$

$u = 1,2,...,z.$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{n\lambda^{(b)}(u)} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{n\lambda^{(b)}(u)} \ge 0 \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then according to (44) we obtain

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$
$$= 1 - \exp[-\frac{t}{n}] \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Hence, considering (43), it appears that

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[V(t,u)]^{(b)} = \lim_{n\to\infty} n[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = \lim_{n\to\infty} n[1 - \exp[-\frac{t}{n}]] = \lim_{n\to\infty} n[\frac{t}{n} + o(\frac{1}{n})] = t$$

for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

which means that according to *Lemma 3* the limit reliability function of that system is given by (46)-(47). □

The next auxiliary theorem is proved in (Kolowrocki et al 2005).
Lemma 4. If

(i) $k_n = n$, $l_n \to l = \text{const}$, $m \le l_n$, as $n \to \infty$

(ii) $\overline{\boldsymbol{\mathfrak{R}}}(t,u) = \sum_{b=1}^{v} p_b \exp[-\overline{[V(t,u)]^{(b)}}]$ is a non-degenerate reliability function,

(iii) $\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,\cdot) = [1, \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,1), ..., \overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "*m* out of l_n"- series system, in variable operation conditions, where

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u)]^{(b)} = [1 - \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(t,u)]^{(b)}]^i\ [1-[R(t,u)]^{(b)}]^{l_n - i}]^{k_n}\ \ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$
(49)

is its reliability function at the operational state z_b,

then

$$\overline{\boldsymbol{\mathfrak{R}}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} k_n \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i \ [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = [\overline{V}(t,u)]^{(b)} \quad (50)$$

for $t \in C_{[\overline{V}(u)]^{(b)}}$, $u = 1,2,...,z,\ b = 1,2,...,v.$

Proposition 4. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$$[R(t,u)]^{(b)} = 1 \text{ for } t < 0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t] \text{ for } t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (51)$$

(ii) $k_n = n,\ l_n \to l,\ l \in (0,\infty), m = \text{const},\ m \le l_n$,

(iii) $$a_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n - m+1)}}, \quad (52)$$

$$b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (53)$$

then

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}_2(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}_2(t,z)],\ t \in (-\infty,\infty), \quad (54)$$

where

$$\boldsymbol{\mathfrak{R}}_2(t,u) = 1 \text{ for } t < 0,\ u = 1,2,...,z, \quad (55)$$

$$\boldsymbol{\mathfrak{R}}_2(t,u) = \sum_{b=1}^{v} p_b \exp[-t^{l-m+1}] \text{ for } t \ge 0,\ u = 1,2,...,z, \quad (56)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) = 1 \text{ for } t < 0,\ u = 1,2,...,z, \quad (57)$$

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}]^{l-m+1}] \text{ for } t \ge 0,\ u = 1,2,...,z. \quad (58)$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} \geq 0 \text{ for } t \geq 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

then, according to (51), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 1$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 0 \text{ for } t < 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]$$

$$= \exp[-\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}}] = 1 - o(1),$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u), u)] = 1 - \exp[-\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}}]$$

$$= \frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} - o(\frac{1}{n^{1/(l_n-m+1)}}) \text{ for } t \geq 0, \ u = 1,2,...,z, \ b = 1,2,...,v.$$

Then for each $i = 0,1,...,m-1$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i = 1$$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n-i} = 0 \text{ for } t < 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^i = [1 - o(1)]^i \to 1 \text{ as } n \to \infty \text{ for } t \geq 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n-i} = [\frac{t}{[n\binom{l_n}{m-1}]^{1/(l_n-m+1)}} - o(\frac{1}{n^{1/(l_n-m+1)}})]^{l_n-i}$$

$$= \frac{t^{l_n-i}}{[n\binom{l_n}{m-1}]^{(l_n-i)/(l_n-m+1)}} [1 - o(\frac{1}{[n\binom{l_n}{m-1}]^{(l_n-i)/(l_n-m+1)}})]^{l_n-i} \text{ for } t \geq 0, \ u = 1,2,...,z, \ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n-i} = o(1) \text{ for } i = 0,1,...,m-2, \ t \geq 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = \frac{t^{l_n-m+1}}{n\binom{l_n}{m-1}}[1-o(1)] \text{ for } i = m-1,\ t \ge 0,\ \ u = 1,2,...,z,\ b = 1,2,...,v.$$

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = \lim_{n\to\infty} n\cdot 0 = 0 \text{ for } t<0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = \lim_{n\to\infty} n\binom{l_n}{m-1}[1-o(1)]^{m-1}\frac{t^{l_n-m+1}}{n\binom{l_n}{m-1}}[1-o(1)] \ = t^{l-m+1}$$

for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

which means that according to *Lemma 4* the limit reliability function of that system is given by (54)-(56). □

Proposition 5. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t<0,\ [R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (59)

(ii) $k_n = n$, $c \ll l_n$, $c \log n - l_n \gg s$, $c > 0$, $s > 0$, m = constant ($m/l_n \to 0$, as $n \to \infty$) or $m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\lambda^{(b)}(u)(l_n - m + 1)}, \tag{60}$$

$$b_n^{(b)}(u) = -\frac{1}{\lambda^{(b)}(u)}\log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}],\ \ u = 1,2,...,z,\ b = 1,2,...,v, \tag{61}$$

then

$$\overline{\mathfrak{R}}_3(t,\cdot) = [1, \overline{\mathfrak{R}}_3(t,1), ..., \overline{\mathfrak{R}}_3(t,z)], \qquad t \in (-\infty,\infty),$$
(62)

where

$$\mathfrak{R}_3(t,u) = \sum_{b=1}^{v} p_b \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty), u = 1,2,...,z, \quad (63)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[t[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\ \lambda^{(b)}(u)(l_n - m + 1)\,]$$

$$+ \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}]\ [[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1](l_n - m + 1)] \quad \text{for} \quad t \in (-\infty,\infty), \quad u = 1,2,...,z.$$

(64)

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) > 0 \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) \to 0 \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (59) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty),$$

$$u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - o(\frac{1}{a_n^{(b)}(u)t + b_n^{(b)}(u)}) \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and considering

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),$$

we obtain

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty, \infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Hence, for each $i = 0,1,...,m-1$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - o(\frac{1}{a_n^{(b)}(u)t + b_n^{(b)}(u)})]^i \to 1$$

as $n \to \infty$ for $t \in (-\infty, \infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$,

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+ \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})]^{l_n - i}$$

$$[1 + \frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n - i}$$

$$= [n\binom{l_n}{m-1} + o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n - i}{l_n - m + 1}} [1 + \frac{t}{(l_n - m + 1)[1 + o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n - i} \text{ for } t \in (-\infty, \infty), \ u = 1,2,...,z, \ b = 1,2,...,v.$$

.

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = o(1) \text{ for } i = 0,1,...,m-2, \ t \in (-\infty, \infty), \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = n[n\binom{l_n}{m-1}]^{-1} [1 + \frac{t}{(l_n - m + 1)}]^{l_n - m + 1} [1 - o(1)]$$

for $i = m-1$, $t \in (-\infty, \infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n \to \infty} k_n \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n-m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} [n\binom{l_n}{m-1}]^{-1} [1 + \frac{t}{(l_n - m + 1)}]^{l_n-m+1} = \exp[t] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (62)-(64). □

Proposition 6. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0$, $u = 1,2,...,z,\ b = 1,2,...,v,$ (65)

(ii) $k_n = n$, $l_n - c \log n \sim s$, $c > 0$, $s \in (-\infty,\infty)$, $m =$ constant ($m/l_n \to 0$, as $n \to \infty$) or $m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $$a_n^{(b)}(u) = \frac{1}{[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\lambda^{(b)}(u)(l_n - m + 1)},\qquad (66)$$

$$b_n^{(b)}(u) = -\frac{1}{\lambda^{(b)}(u)} \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}],\ u = 1,2,...,z,\ b = 1,2,...,v,\qquad (67)$$

then

$$\overline{\mathfrak{R}}_3(t,\cdot) = [1, \overline{\mathfrak{R}}_3(t,1), ..., \overline{\mathfrak{R}}_3(t,z)],\quad t \in (-\infty,\infty),\qquad (68)$$

where

$$\overline{\mathfrak{R}}_3(t,u) = \sum_{b=1}^{v} p_b \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty),\qquad (69)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[t[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1]\ \lambda^{(b)}(u)(l_n - m + 1) + \log[1 - [n\binom{l_n}{m-1}]^{-\frac{1}{l_n-m+1}}]$$
$$[[n\binom{l_n}{m-1}]^{\frac{1}{l_n-m+1}} - 1](l_n - m + 1)] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z.\qquad (70)$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) > 0 \text{ for } t \in (-\infty,\infty),\ b = 1,2,...,v,\ u = 1,2,...,z,$$

and

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty, \; b = 1,2,...,v, \; u = 1,2,...,z,$$

then, according to (65) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; b = 1,2,...,v, \; u = 1,2,...,z,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,..,z, \; b = 1,2,..,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})] \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n(u)})] \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v.$

Hence, for each $i = 0,1,...,m-1$ for n large enough, we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i = [1 - \lambda^{(b)}(u)a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})]^i \exp[-i\lambda^{(b)}(u)b_n^{(b)}(u)] \to 1$$

$$\text{as } n \to \infty \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+ \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n - i}$$

$$= [1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})]^{l_n-i}\ [1+\frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n-i}$$

$$= [n\binom{l_n}{m-1} + o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n-i}{l_n-m+1}}\ [1+\frac{t}{(l_n-m+1)[1+o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n-i}$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

From last equation we obtain

$$[[F(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = o(1)$$

for $i = 0,1,...,m-2,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$

and

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = n[n\binom{l_n}{m-1}]^{-1}\ [1+\frac{t}{(l_n-m+1)}]^{l_n-m+1}\ [1-o(1)]$$

for $i = m-1,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1}\binom{l_n}{i}[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = \lim_{n\to\infty} n\binom{l_n}{m-1}\ [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n-m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1}\ [n\binom{l_n}{m-1}]^{-1}\ [1+\frac{t}{(l_n-m+1)}]^{l_n-m+1} = \exp[t] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (68)-(69). □

Proposition 7. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions,

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (71)

(ii) $k_n = n,\ \ l_n - c\log n >> s,\ c > 0,\ s > 0,\ m =$ constant ($m/l_n \to 0$, as $n \to \infty$) or $m/l_n \to \eta$, $0 < \eta < 1$, as $n \to \infty$,

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)\log n\binom{l_n}{m-1}}$, (72)

$$b_n^{(b)}(u) = \frac{1}{\lambda^{(b)}(u)} \log(\frac{l_n - m + 1}{\log n\binom{l_n}{m-1}}, \; u = 1,2,...,z, \; b = 1,2,...,v, \quad (73)$$

then

$$\overline{\boldsymbol{\mathfrak{R}}}_3(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}_3(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}_3(t,z)], \quad t \in (-\infty,\infty),$$

(74)

where

$$\overline{\boldsymbol{\mathfrak{R}}}_3(t,u) = \sum_{b=1}^{v} p_b \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \quad (75)$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-\exp[\lambda^{(b)}(u) t \log n\binom{l_n}{m-1} - \log[\frac{l_n - m + 1}{\log n\binom{l_n}{m-1}} \log n\binom{l_n}{m-1}]]]$$

(76)

for $t \in (-\infty,\infty), \; u = 1,2,...,z.$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) \to +\infty \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty), \; b = 1,2,...,v, \; u = 1,2,...,z,$$

and

$$a_n^{(b)}(u) \to 0 \text{ as } n \to \infty,$$

then, according to (71) for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v.$$

Moreover for n large enough, we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= [1 - \lambda^{(b)}(u) a_n^{(b)}(u)t + o(\frac{1}{a_n^{(b)}(u)})] \exp[-\lambda^{(b)}(u) b_n^{(b)}] \text{ for } t \in (-\infty,\infty), \; u = 1,2,...,z, \; b = 1,2,...,v,$$

and

$$[F(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)} = 1-\exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t+b_n^{(b)}(u))]$$

$$= 1-[1-\lambda^{(b)}(u)a_n^{(b)}(u)t+o(\frac{1}{a_n^{(b)}(u)})]\ \exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]$$

$$= 1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)}) + \lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, for each $i = 0,1,...,m-1$ for n large enough, we have

$$[[R(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^i = [1-\lambda^{(b)}(u)a_n^{(b)}(u)t+o(\frac{1}{a_n^{(b)}(u)})]^i\ \exp[-i\lambda^{(b)}(u)b_n^{(b)}(u)] \to 1$$

$$\text{as } n\to\infty \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[F(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = [1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)] + o(\frac{1}{a_n^{(b)}(u)})$$

$$+\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]t]^{l_n-i}$$

$$= [1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]+o(\frac{1}{a_n^{(b)}(u)})]^{l_n-i}\ [1+\frac{\lambda^{(b)}(u)a_n^{(b)}(u)\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]}{1-\exp[-\lambda^{(b)}(u)b_n^{(b)}(u)]+o(\frac{1}{a_n^{(b)}(u)})}t]^{l_n-i}$$

$$= [n\binom{l_n}{m-1}+o(\frac{1}{a_n^{(b)}(u)})]^{-\frac{l_n-i}{l_n-m+1}}\ [1+\frac{t}{(l_n-m+1)[1+o(\frac{1}{a_n^{(b)}(u)})]}]^{l_n-i}$$

for $t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = o(1) \text{ for } i = 0,1,...,m-2,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t+b_n^{(b)}(u),u)]^{(b)}]^{l_n-i} = n[n\binom{l_n}{m-1}]^{-1}\ [1+\frac{t}{(l_n-m+1)}]^{l_n-m+1}\ [1-o(1)]$$

for $i = m-1,\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$

Hence, considering (50), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n \sum_{i=0}^{m-1} \binom{l_n}{i} [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = \lim_{n\to\infty} n\binom{l_n}{m-1} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - m+1}$$

$$= \lim_{n\to\infty} n\binom{l_n}{m-1} [n\binom{l_n}{m-1}]^{-1} [1 + \frac{t}{(l_n - m + 1)}]^{l_n - m+1} = \exp[t] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v.$$

which means that according to *Lemma 4* the limit reliability function of that system is given by (74)-(75). □

The next auxiliary theorem is proved in (Kolowrocki 2005).

Lemma 5. If
(i) $k_n = n$, $l_n \to l = \text{const}$, $m \le l_n$, as $n \to \infty$
(ii) $\overline{\boldsymbol{\mathfrak{R}}}(t) = \sum_{b=1}^{v} p_b \exp[-[\overline{V}(t)]^{(b)}]$ is a non-degenerate reliability function,
(iii) $\overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,\cdot) = [1, \overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,1), ..., \overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,z)]$, $t \in (-\infty,\infty)$, is the reliability function of a homogeneous regular multi-state "m out of l_n"- series system, in variable operation conditions, where

$$\overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,u) \cong \sum_{b=1}^{v} p_b\ [\overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,u)]^{(b)},\ t \in (-\infty,\infty),$$

and

$$[\overline{\overline{\boldsymbol{R}}}{}^{(m)}_{k_n,l_n}(t,u)]^{(b)} = [\sum_{i=0}^{l_n - m} \binom{l_n}{i} [1 - [R(t,u)]^{(b)}]^i [[R(t,u)]^{(b)}]^{l_n - i}]^{k_n},\ t \in (-\infty,\infty),\ u = 1,2,...,z,\ b = 1,2,...,v, \quad (77)$$

is its reliability function at the operational state z_b,
then

$$\overline{\boldsymbol{\mathfrak{R}}}(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state limit reliability function of that system if and only if

$$\lim_{n\to\infty} k_n [1 - \sum_{i=0}^{l_n - m} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}] = [\overline{V}(t,u)]^{(b)} \quad (78)$$

for $t \in C_{[\overline{V}(u)]^{(b)}}$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Proposition 8. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b
(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (79)

(ii) $k_n = n$, $c << l_n$, $c \log n - l_n >> s$, $c > 0$, $s > 0$, $(l_n - m) = \overline{m}$ = const, ($m/l_n \to 1$ as $n \to \infty$),

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}$, $b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (80)

then

$$\overline{\mathfrak{R}}_2(t,\cdot) = [1, \overline{\mathfrak{R}}_2(t,1), ..., \overline{\mathfrak{R}}_2(t,z)],\ t \in (-\infty,\infty), \tag{81}$$

where

$$\overline{\mathfrak{R}}_2(t,u) = 1 \text{ for } t < 0, \tag{82}$$

$$\overline{\mathfrak{R}}_2(t,u) = \sum_{b=1}^{v} p_b \ \exp[-t^{\overline{m}+1}] \text{ for } t \geq 0,\ u = 1,2,...,z, \tag{83}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) = 1 \text{ for } t < 0, \tag{84}$$

$$\overline{\boldsymbol{R}_{k_n,l_n}^{(m)}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\binom{l_n}{\overline{m}+1}^{1/\overline{m}+1}]^{\overline{m}+1}] \quad \text{for} \quad t \geq 0, \quad u = 1,2,...,z. \tag{85}$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} \geq 0 \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (79), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v.$$

Next, for each $i = \overline{m}+1, \overline{m}+2, ..., l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = 1 \text{ for } t < 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = 0 \text{ for } t < 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{l_n - i} \to 1 \quad \text{as } n \to \infty$$

for $t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = [\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n(\frac{l_n}{\overline{m}+1})]^{i/(\overline{m}+1)}} [1 - o(1)]^i \text{ for } t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = o(\frac{1}{n(\frac{l_n}{\overline{m}+1})}) \text{ for } i = \overline{m}+2, \overline{m}+3, ..., l_n, \ t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = \frac{t^{\overline{m}+1}}{n(\frac{l_n}{\overline{m}+1})} [1 - o(1)] \text{ for } i = \overline{m}+1, \ t \ge 0, \ u = 1,2,...,z, \ b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$

then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \ \text{ for } \ t < 0,\ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \quad [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \ \text{ for } \ t \geq 0,\ \ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v.$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (81)-(83). □

Proposition. 9 If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \geq 0,\ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$ (86)

(ii) $k_n = n,\ l_n - c\log n \sim s,\ c > 0,\ s \in (-\infty,\infty), (n - m) = \overline{m} = \text{const}, (m/l_n \to 1 \text{ as } n \to \infty)$

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}},\ b_n^{(b)}(u) = 0,\ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$ (87)

then

$$\bar{\mathfrak{R}}_2(t,\cdot) = [1, \bar{\mathfrak{R}}_2(t,1), ..., \bar{\mathfrak{R}}_2(t,z)]\,,\ t \in (-\infty,\infty), \tag{88}$$

where

$$\bar{\mathfrak{R}}_2(t,u) = 1 \ \text{for}\ t < 0, \tag{89}$$

$$\bar{\mathfrak{R}}_2(t,u) = \sum_{b=1}^{v} p_b\ \exp[-t^{\bar{m}+1}]\ \text{for}\ t \geq 0,\ \ u = 1,2,...,z, \tag{90}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) = 1 \ \text{for}\ t < 0, \tag{91}$$

$$\overline{\boldsymbol{R}^{(m)}_{k_n,l_n}}(t,u) \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\left(\frac{l_n}{\bar{m}+1}\right)^{1/\bar{m}+1}]^{\bar{m}+1}]\ \text{for}\ t \geq 0,\ u = 1,2,...,z. \tag{92}$$

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}} < 0 \ \text{for}\ t < 0\,,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}} \geq 0 \ \text{for}\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (86), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \ \text{for}\ t < 0\,,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \ \text{for}\ t < 0\,,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}) \ \text{for}\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u))]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\bar{m}+1})]^{1/(\bar{m}+1)}})$$

for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$

Next, for each $i = \overline{m}+1, \overline{m}+2, ..., l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n(\binom{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{l_n - i} \to 1 \quad \text{as } n \to \infty$$

$$\text{for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = [\frac{t}{[n(\binom{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\binom{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n(\binom{l_n}{\overline{m}+1})]^{i/(\overline{m}+1)}}\,[1 - o(1)]^i \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = o(\frac{1}{n(\binom{l_n}{\overline{m}+1})}) \text{ for } i = \overline{m}+2, \overline{m}+3, ..., l_n,\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i} = \frac{t^{\overline{m}+1}}{n(\binom{l_n}{\overline{m}+1})}\,[1 - o(1)] \text{ for } i = \overline{m}+1,\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{b)}]\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)}]^{l_n - i}$$

$u = 1,2,...,z,\ b = 1,2,...,v,$ then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \ \text{ for } \ t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^i \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n\binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \ \text{ for } \ t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (88)-(90). □

Proposition 10. If components of the multi-state homogeneous, regular "m out of l_n"-series system at the operational state z_b

(i) have exponential reliability functions

$[R(t,u)]^{(b)} = 1$ for $t < 0$, $[R(t,u)]^{(b)} = \exp[-\lambda^{(b)}(u)t]$ for $t \ge 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (93)

(ii) $k_n = n,\ \ l_n - c \log n >> s,\ c > 0,\ s > 0,\ (n - m) = \overline{m} = \text{const},\ (m/l_n \to 1 \text{ as } n \to \infty)$

(iii) $a_n^{(b)}(u) = \dfrac{1}{\lambda^{(b)}(u)[n\binom{l_n}{\overline{m}+1}]^{1/(\overline{m}+1)}},\ b_n^{(b)}(u) = 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$ (94)

then

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,\cdot) = [1, \overline{\boldsymbol{\mathfrak{R}}}_2(t,1), ..., \overline{\boldsymbol{\mathfrak{R}}}_2(t,z)],\ t \in (-\infty,\infty), \tag{95}$$

where

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = 1 \text{ for } t < 0, \tag{96}$$

$$\overline{\boldsymbol{\mathfrak{R}}}_2(t,u) = \sum_{b=1}^{v} p_b \ \exp[-t^{\overline{m}+1}] \text{ for } t \ge 0, \tag{97}$$

is the multi-state limit reliability function of that system , i.e. for n large enough we have

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}(t,u)} = 1 \text{ for } t < 0, \tag{98}$$

$$\overline{\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}(t,u)} \cong \sum_{b=1}^{v} p_b \exp[-[t\lambda^{(b)}(u)n\left(\frac{l_n}{\overline{m}+1}\right)^{1/\overline{m}+1}]^{\overline{m}+1}] \quad \text{for} \quad t \geq 0, \quad u = 1,2,...,z.$$

(99)

Proof. Since

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} < 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$a_n^{(b)}(u)t + b_n^{(b)}(u) = \frac{t}{\lambda^{(b)}(u)[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} \geq 0 \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, according to (93), we obtain

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = 1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

$$[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} = 1 - \exp[-\lambda^{(b)}(u)(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]$$

$$= 1 - \exp[-\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}] = \frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}}) \text{ for } t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v.$$

Next, for each $i = \overline{m}+1, \overline{m}+2, ..., l_n$ we have

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = 1 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = 0 \text{ for } t < 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

and

$$[[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i} = [1 - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{l_n - i} \to 1 \quad \text{as} \quad n \to \infty$$

for $t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = [\frac{t}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}} - o(\frac{1}{[n(\frac{l_n}{\overline{m}+1})]^{1/(\overline{m}+1)}})]^{i}$$

$$= \frac{t^i}{[n(\frac{l_n}{\overline{m}+1})]^{i/(\overline{m}+1)}}\ [1 - o(1)]^i \ \text{for} \ t \geq 0,\ \ u = 1,2,...,z,\ b = 1,2,...,v.$$

From last equation we obtain

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = o(\frac{1}{n(\frac{l_n}{\overline{m}+1})}) \ \text{for} \ i = \overline{m}+2, \overline{m}+3,...,l_n,\ t \geq 0,\ u = 1,2,...,z,\ b = 1,2,...,v,$$

$$[[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i} = \frac{t^{\overline{m}+1}}{n(\frac{l_n}{\overline{m}+1})}\ [1 - o(1)] \ \text{for} \ i = \overline{m}+1,\ t \geq 0,\ \ u = 1,2,...,z,\ b = 1,2,...,v.$$

Since

$$1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - \sum_{i=0}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= 1 - [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)} + [R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n}$$

$$+ \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}],\ u = 1,2,...,z,\ b = 1,2,...,v,$$

then, considering (78), it appears that

$$[\overline{V}(t,u)]^{(b)} = \lim_{n \to \infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n \to \infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{i}\ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n \cdot 0 = 0 \text{ for } t < 0,\ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$$

and

$$[\overline{V}(t,u)]^{(b)} = \lim_{n\to\infty} k_n [1 - \sum_{i=0}^{\overline{m}} \binom{l_n}{i} [[F(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} k_n \sum_{i=\overline{m}+1}^{l_n} \binom{l_n}{i} [[F^{(b)}(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^i \ \ [[R(a_n^{(b)}(u)t + b_n^{(b)}(u),u)]^{(b)}]^{l_n - i}]$$

$$= \lim_{n\to\infty} n\binom{l_n}{\overline{m}+1} \frac{t^{\overline{m}+1}}{n\binom{l_n}{\overline{m}+1}} [1 - o(1)] = t^{\overline{m}+1} \quad \text{for } t \ge 0,\ u = 1,2,\ldots,z,\ b = 1,2,\ldots,v,$$

which means that according to *Lemma 5* the limit reliability function of that system is given by (95)-(97). □

6 CONCLUSION

The purpose of this paper is to give the method of reliability analysis of multi-state "*m* out of *l*"- series systems in variable operation conditions. Their exact and limit reliability functions, in constant and in varying operation conditions, are determined. The paper proposes an approach to the solution of practically very important problem of linking the systems' reliability and their operation processes. To involve the interactions between the systems' operation processes and their varying in time reliability structures a semi-markov model of the systems' operation processes and the multi-state system reliability functions are applied. This approach gives practically important in everyday usage tool for reliability evaluation of the large systems with changing their reliability structures and components reliability characteristic during their operation processes. The results can be applied to the reliability evaluation of real technical systems.

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ASYMPTOTIC DEPENDENCE OF AVERAGE FAILURE RATE AND MTTF FOR A RECURSIVE, MESHED NETWORK ARCHITECTURE

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ABSTRACT

The paper is concerned with the exact and asymptotic calculations of the availability, average failure rate and MTTF (Mean Time To Failure) for a recursive, meshed architecture proposed by Beichelt and Spross. It shows that the asymptotic size dependences of average failure rate and MTTF are different, but not inverse of each other, as is unfortunately assumed too frequently. Besides, the asymptotic limit is reached for rather small networks.

1 INTRODUCTION

Network availability and reliability have long been a practical issue in telecommunication networks, among others. Quality of Service (QoS) requirements imply high availabilities A, but also a good knowledge of the failure frequency $\overline{\nu}$ – and of the average failure rate $\overline{\lambda} = \overline{\nu} \,/\, A$ – of (for instance) point-to-point connections, when the system is repairable. When the system is not repairable, an important parameter is the MTTF (Mean Time To Failure). As explained in many textbooks (Shooman 1968, Singh & Billinton 1977, Kuo & Zuo 2003), a system whose failure rate λ is constant over time has a reliability described by the exponential distribution $R(t) = \exp(-\lambda\, t)$, so that the MTTF, defined by

$$\mathrm{MTTF} = \langle t \rangle = \int_0^{\infty} t\ (-R'(t)\,dt\,) = \int_0^{\infty} R(t)\,dt\,, \tag{1}$$

is in this case $\mathrm{MTTF}_{\exp} = 1/\lambda$. This may lead to confusions in repairable systems, where it may still be legitimate to consider constant failure rates for each element of the system, and yet obtain an average failure rate $\overline{\lambda}$. If A is the availability of a system made of m elements – whose failures are assumed to be statistically independent – having individual availabilities p_i ($1 \leq i \leq m$) and constant failure rates λ_i, then (Buzacott 1967, Singh & Billinton 1974, Schneeweiss 1981, 1983, Shi 1981, Hayashi 1991, Druault-Vicard & Tanguy 2006)

$$\overline{\lambda} = \frac{1}{A}\sum_{i=1}^{m} \lambda_i\, p_i\, \frac{\partial A}{\partial p_i}\,. \tag{2}$$

Most results of the literature are devoted to series-parallel systems, where all components are identical, with the same (constant) failure rate λ. For n components in series, $A(p) = p^n$ so that the aggregate failure rate is equal to $\overline{\lambda} = \lambda \frac{p}{A}\frac{\partial A}{\partial p} = n\,\lambda$. The reliability of n components in series is $R(t) =$

$[\exp(-\lambda\, t)]^n = \exp(-\, n\, \lambda\, t)$, which gives MTTF = $1/(n\, \lambda)$. For n components in parallel however, $R(t) = 1\text{-}(1\text{-}\exp(-\lambda\, t))^n$, which leads to

$$\mathrm{MTTF}_{parallel} = \frac{1}{\lambda}\sum_{i=1}^{m}\frac{1}{i} = \frac{1}{\lambda}\left(\ln n + \mathrm{C} + \frac{1}{2n} + \ldots\right) \tag{3}$$

for n large (Shooman 1968, Kuo 2003) and where C = 0.577216... is the Euler constant. Quite generally, it is therefore important to estimate the reliability and related parameters of large systems in order to get a better understanding of key issues (Kołowrocki 2004).

In this work, we consider a recursive, meshed – not series-parallel – network configuration first considered by Beichelt and Spross (Beichelt & Spross 1989) as well as Prékopa and collaborators (Prékopa et al. 1991). For the repairable case we shall use the availability $A(p)$, and in the non-repairable case the reliability $R(p(t))$, even though $A(p)$ and $R(p)$ are formally identical for the same network made of identical components. We show in detail that when such a system is large, knowledge of the generating function of the reliability/availability allows us find the analytic, asymptotic expressions for $\overline{\lambda}$ and MTTF. These expressions, which both have simple n-dependences, are *not* the inverse of each other: while for $\overline{\lambda}$, we find again a linear dependence in n (Druault-Vicard & Tanguy 2006), we obtain a $n^{-1/5}$ dependence for the MTTF. Besides, they are in very good agreement with the exact values even when n remains relatively small.

2 NETWORK ARCHITECTURE: A CASE STUDY

2.1 Description

The network configuration defined by Beichelt and Spross (Beichelt & Spross 1989) is represented in Figure 1. They wanted to estimate the two-terminal reliability between the endpoints of the structure (in the original paper, the destination point was S_6).

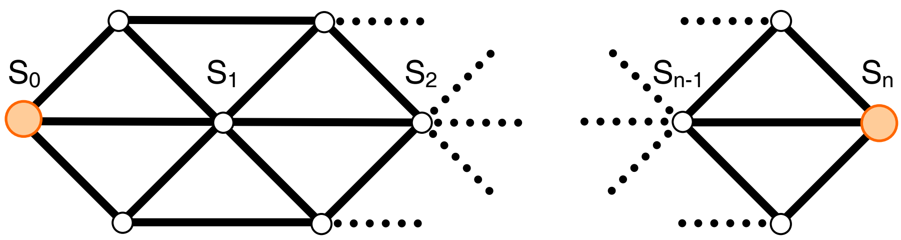

Figure 1. Recursive network architecture (Beichelt and Spross 1989). The source is S_0 and the destination is S_n.

Following the method developed in (Tanguy 2007), we have been able to show that the two-terminal reliability between S_0 and S_n may be expressed as a product of transfer matrices, in which each edge or link probability of functioning is arbitrary. It turns out that this transfer matrix is 15×15. However, if nodes are perfect and if links have the same reliability/availability p, things are much simpler, because a single transfer matrix needs be considered, the successive powers of which are to be calculated. Fortunately, these necessarily obey a recursion relation of finite order stemming from the characteristic polynomial of the transfer matrix. When dealing with $\mathrm{Rel}_2(S_0 \rightarrow S_n) \equiv \mathrm{Rel}_2^{(n)}$, a very useful tool is the generating function formalism (Stanley 1997), since it encodes the exact result in a very concise manner.

2.2 Generating function of the reliability/availability

The generating function $G(z) = \Sigma_n \mathrm{Rel}_2^{(n)}(p)\, z^n$ may eventually be written as $G(z) = N(z)/D(z)$ (Tanguy 2007), where

$$
\begin{aligned}
N(z) = {} & 1 - (1-p)^2\, p\,(1+4p+8p^2-20p^3+9p^4)\, z \\
& - (1-p)^3\, p^3\,(-2-7p+13p^2+26p^3-74p^4+38p^5+29p^6-34p^7+9p^8)\, z^2 \\
& - (1-p)^6\, p^5\,(1+6p-17p^2+4p^3+3p^4+5p^5-14p^6+8p^7+5p^8-7p^9+2p^{10})\, z^3 \\
& - (1-p)^9\, p^8\,(-2+16p^2-4p^3-36p^4+34p^5-7p^6-3p^7+p^8)\, z^4 \\
& + (1-p)^{13}\, p^{11}\,(-1+3p-p^2-3p^3+p^4)\, z^5,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
D(z) = {} & 1 - p\,(2+4p-p^2-33p^3+58p^4-38p^5+9p^6)\, z \\
& + (1-p)^2\, p^2\,(1+6p+11p^2-31p^3-44p^4+168p^5-158p^6+20p^7+63p^8 \\
& \qquad -43p^9+9p^{10})\, z^2 \\
& - (1-p)^4\, p^4\,(2+10p-2p^2-73p^3+138p^4-105p^5+41p^6-40p^7+64p^8-41p^9-5p^{10} \\
& \qquad +21p^{11}-11p^{12}+2p^{13})\, z^3 \\
& + (1-p)^8\, p^6\,(1+8p+2p^2-4p^3-30p^4+23p^5+43p^6-76p^7+47p^8-10p^9 \\
& \qquad -2p^{10}+p^{11})\, z^4 \\
& - (1-p)^{12}\, p^9\,(2+6p+p^2-18p^3+13p^4+2p^5-4p^6+p^7)\, z^5 + (1-p)^{16}\, p^{12}\, z^6.
\end{aligned}
\tag{5}
$$

We deduce for $n = 6$

$$
\begin{aligned}
\frac{\mathrm{Rel}_2^{(6)}(p)}{p^6} = {} & 1+42p+328p^2+826p^3-1473p^4-11400p^5-9975p^6+61060p^7+160918p^8 \\
& -153606p^9-1203380p^{10}-101102p^{11}+6957668p^{12}+2134306p^{13}-33913956p^{14} \\
& -11462384p^{15}+179889959p^{16}-49002916p^{17}-965490222p^{18}+2056136956p^{19} \\
& -213511696p^{20}-7360834390p^{21}+19329198282p^{22}-29836117826p^{23} \\
& +33105011509p^{24}-28179232812p^{25}+18911540288p^{26}-10111211062p^{27} \\
& +4305721566p^{28}-1446762862p^{29}+376155108p^{30}-73146582p^{31} \\
& +10029258p^{32}-865872p^{33}+35442p^{34},
\end{aligned}
\tag{6}
$$

so that $\mathrm{Rel}_2^{(6)}(0.9)$ is equal to 0.9974544308852755355007942390030310588362, which is close to the upper bound given by Beichelt and Spross (Beichelt & Spross 1989). A partial fraction decomposition of G(z) gives

$$
G(z) = \sum_{i=1}^{6} \frac{\alpha_i}{1-\varsigma_i\, z}\ .
\tag{7}
$$

There are six eigenvalues ζ_i; a few of them may be pairs of complex conjugate values for some values of p. When the ζ_i's are distinct, (7) immediately gives

$$\mathrm{Rel}_2^{(n)}(p) = \sum_{i=1}^{6} \alpha_i \, \varsigma_i^{\,n} \,. \tag{8}$$

2.3 Asymptotic reliability/availability

In the limit $n \to \infty$, a single eigenvalue will prevail in the sum of (8), that of largest modulus. In the following, we shall name it ζ_+. It is real for the whole range $0 \le p \le 1$ (see Fig. 2), and necessarily goes to 1 when $p \to 1$ because $\mathrm{Rel}_2^{(\infty)}(p = 1) = 1$; all other eigenvalues tend to zero in that limit.

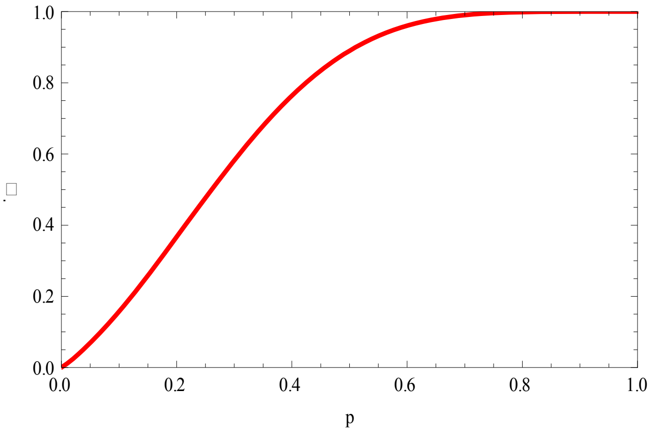

Figure 2. Variation of ζ_+ with p; $\zeta_+(0.9) = 0.9999596999379792$.

Even though it is not possible to get an analytic expression for ζ_+ as a function of p (D(z) is of degree 6 in z), we may compute it numerically very effectively, and also derive the expansion of ζ_+ as a function of $q = 1 - p$ for small q's. Using symbolic software, we deduce from the constraint $D(1/\zeta_+) = 0$

$$\varsigma_+ \to 1 - 4q^5 - 4q^7 + 9q^8 + 9q^9 + 13q^{10} + \ldots \tag{9}$$

When p is close to zero, we have instead:

$$\varsigma_+ \to p + \sqrt{2}\, p^{3/2} + p^2 + \frac{5\sqrt{2}}{4} p^{5/2} + \ldots \tag{10}$$

The prefactor α_+ is deduced from p and ζ_+ because it is closely related to the residue of $G(z)$ at $z = 1/\zeta_+$. The general result is in fact

$$\alpha_+ = \frac{-\varsigma_+ \, N(1/\varsigma_+)}{D'_z\,(1/\varsigma_+)} \tag{11}$$

where $D'_z = \partial D(z)/\partial z$. From the knowledge of p and the numerical value of $\zeta_+(p)$, we simply obtain $\alpha_+(p)$, which is plotted in Figure 3.

Here again, we may consider two limits. For $p \to 1$,

$$\alpha_+ \to 1 - 2q^3 - 4q^4 + 10q^5 - 7q^6 + \ldots \tag{12}$$

while when $p \to 0$,

$$\alpha_{+} \to \frac{1}{2} + \frac{\sqrt{2}}{4} p^{1/2} - \frac{11\sqrt{2}}{16} p^{3/2} + \ldots \tag{13}$$

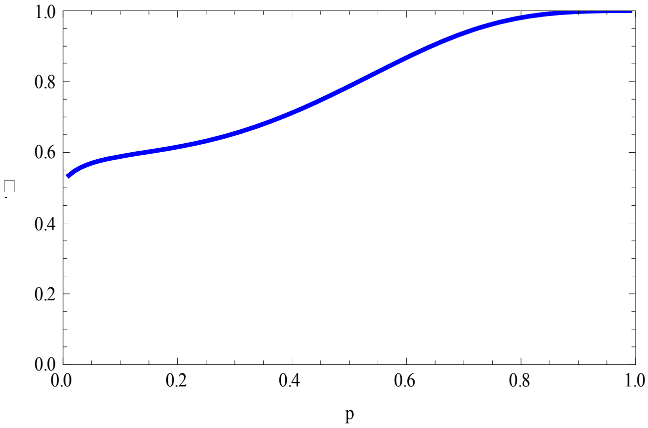

Figure 3. Variation of ζ_+ with p; ζ_+ (0.9) = 09976956497611774972.

The essential result is that, when n is large,

$$\mathrm{Rel}_2^{(n)}(p) = \alpha_+ \varsigma_+^{\,n} + (\mathrm{neglig.\,terms}) \tag{14}$$

Basically, it looks as if the recursive network is made of n elements in series, each of which having the reliability/availability ζ_+. The two asymptotic expressions of $\bar{\lambda}$ and MTTF we shall derive as functions of n in the following section are a mere consequence of (14).

3 AVERAGE FAILURE RATE

3.1 Exact expression

In the case of identical links with constant failure rate λ, (2) gives

$$\bar{\lambda}_n = \frac{\bar{\nu}_n}{A_n(p)} = \lambda \frac{p}{A_n(p)} \frac{\partial A_n(p)}{\partial p} \tag{15}$$

Knowing $A_n \equiv \mathrm{Rel}_2^{(n)}(p)$ by recursion (using (4)-(5)), the derivative is easily obtained for arbitrary values $0 \le p \le 1$.

3.2 Asymptotic expression

Because $\mathrm{Rel}_2^{(n)} \approx \alpha_+ \zeta_+^{\,n}$ for n large, we get (Druault-Vicard & Tanguy 2006)

$$\bar{\lambda}_n \approx \lambda \left[n \frac{d \ln \varsigma_+}{d \ln p} + \frac{d \ln \alpha_+}{d \ln p} \right]. \tag{16}$$

Of course, it would be easier to get $d \ln \zeta_+ / d \ln p$ and $d \ln \alpha_+ / d \ln p$ if ζ_+ were known analytically. Still, as in the formal calculation of α_+, $D(1/\zeta_+) = 0$ implies that

$$\frac{\partial D}{\partial p}(1/\varsigma_+)+\left(-\frac{\varsigma_+'(p)}{\varsigma_+(p)^2}\right)\frac{\partial D}{\partial z}(1/\varsigma_+)=0\ , \tag{17}$$

from which we deduce $\zeta_+'(p)$ and then

$$\frac{d\ln\varsigma_+}{d\ln p}=p\,\varsigma_+\,\frac{D'_p\,(1/\varsigma_+)}{D'_z\,(1/\varsigma_+)}\ , \tag{18}$$

$$\frac{d\ln\alpha_+}{d\ln p}=\frac{d\ln\varsigma_+}{d\ln p}\left[1-\frac{N'_z\,(1/\varsigma_+)}{\varsigma_+\,N(1/\varsigma_+)}+\frac{D''_{zz}\,(1/\varsigma_+)}{\varsigma_+\,D'_z\,(1/\varsigma_+)}\right]+p\,\frac{N'_p\,(1/\varsigma_+)}{N(1/\varsigma_+)}-p\,\frac{D''_{zp}\,(1/\varsigma_+)}{D'_z\,(1/\varsigma_+)}\ . \tag{19}$$

Their variations for $0\le p\le 1$ are displayed in Figures 4-5.

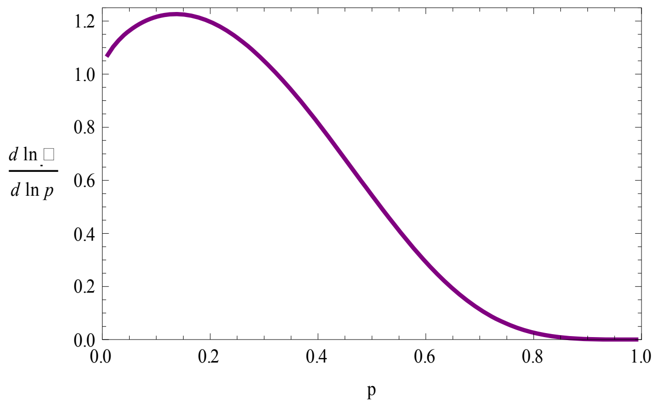

Figure 4. Variation of $d\ln\zeta_+/\mathrm{d}\ln p$ with p.

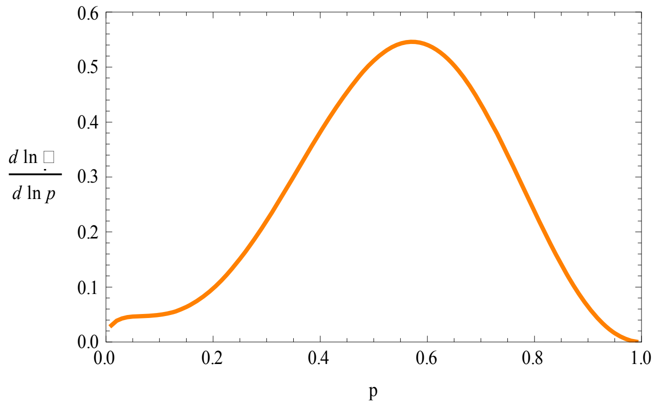

Figure 5. Variation of $d\ln\alpha_+/\mathrm{d}\ln p$ with p.

Note that, unsurprisingly, they exhibit singular behaviors in the vicinity of $p=0$:

$$\frac{d\ln\varsigma_+}{d\ln p}\to 1+\frac{\sqrt{2}}{2}\,p^{1/2}+\frac{11\sqrt{2}}{8}\,p^{3/2}+\dots \tag{20}$$

$$\frac{d\ln\alpha_+}{d\ln p}\to \frac{\sqrt{2}}{4}\,p^{1/2}-\frac{1}{4}\,p-\frac{31\sqrt{2}}{16}\,p^{3/2}+\dots \tag{21}$$

Exact results as well as the linear approximation (see (16)) are displayed in Figure 6 for $p = 0.9$. We see that the agreement is excellent even for $n = 2$.

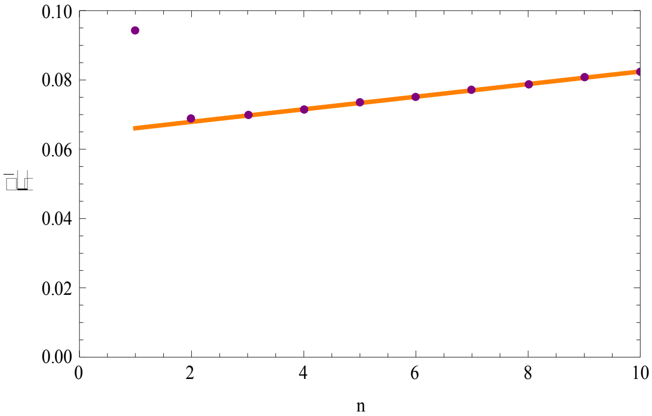

Figure 6. Comparison between exact results (purple) and asymptotic approximation 0.06426+0.0018180 n (orange) for $\overline{\lambda}/\lambda$ and $p = 0.9$.

4 MTTF CALCULATIONS

4.1 Exact expression

We are now considering a non-repairable system, and its reliability $R_n(t)$. Let us recall that

$$\mathrm{MTTF}_n = \int_0^\infty R_n(t)\, dt \ . \tag{22}$$

If each element has reliability $p(t) = \exp(-\lambda\, t)$, we can write $t = (-1/\lambda) \ln p(t)$ and then (22) as

$$\mathrm{MTTF}_n = \frac{1}{\lambda}\int_0^1 \frac{dp}{p}\, \mathrm{Rel}_2^{(n)}(p) \ . \tag{23}$$

We can reuse the results obtained in Section II. Clearly, the exact MTTF_n is obtained from (23), since such an integration is routinely performed by mathematical software.

4.2 Asymptotic expression

The calculation of the asymptotic expansion of MTTF_n is based again on $R_n \approx \alpha_+ \zeta_+^n$ when n is large:

$$\mathrm{MTTF}_n \approx \frac{1}{\lambda}\int_0^1 \frac{dp}{p}\, \alpha_+(p)\, \varsigma_+^n(p) \ . \tag{24}$$

We have plotted ζ_+ and ζ_+^{40} in Figure 7. Because ζ_+ vanishes for $p \rightarrow 0$, the $1/p$ factor does not play a significant role in the integral. As n increases, the essential contribution to the integral will obviously come from the domain "p close to unity".

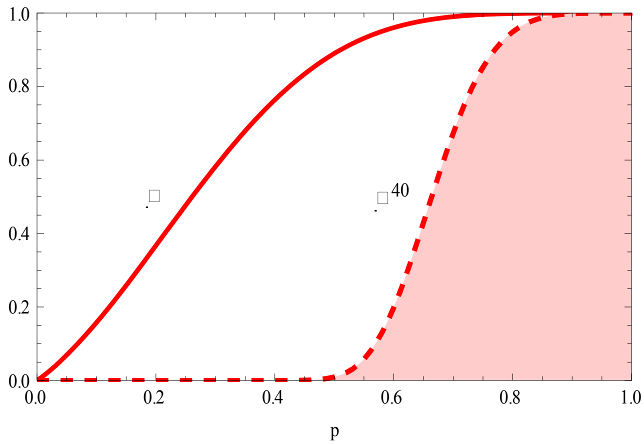

Figure 7. Variation with p of ζ_+ and ζ_+^{40}.

The best approach is therefore to use q as the variable of integration

$$\mathrm{MTTF}_n \approx \frac{1}{\lambda}\int_0^1 \frac{dq}{1-q}\ \alpha_+(1-q)\ \varsigma_+^n(1-q)\ . \tag{25}$$

The gist of the calculation, quite standard in asymptotic expansions, is to extract the prevailing contribution of the integrand when $q \to 0$. We can write

$$\varsigma_+^n = \exp\left(-n(-\ln\ \varsigma_+)\right) \tag{26}$$

and derive the expansion of - ln ζ_+ in q from (9)

$$-\ln\ \varsigma_+ = 4q^5 + 4q^7 - 9q^8 - 9q^9 - 5q^{10} + \ldots\ , \tag{27}$$

so that

$$\varsigma_+^n = e^{-4nq^5}\ \exp\left(-n(4q^7 - 9q^8 - 9q^9 - 5q^{10} + \ldots)\right) \tag{28}$$

This manipulation may seem quite formal, but now we can use a rescaled variable $\tau = 4\ n\ q^5$, or, equivalently, set $q = \tau^{1/5}/(4\ n)^{1/5}$. Equation (28) then gives

$$\varsigma_+^n = e^{-\tau}\ \exp\left[-n\left(4\left(\frac{\tau}{4n}\right)^{7/5} - 9\left(\frac{\tau}{4n}\right)^{8/5} + \ldots\right)\right] = e^{-\tau}\ \exp\left[-\frac{4}{n^{2/5}}\left(\frac{\tau}{4}\right)^{7/5} + \frac{9}{n^{3/5}}\left(\frac{\tau}{4}\right)^{8/5} + \ldots\right]. \tag{29}$$

Equation (25) leads to

$$\mathrm{MTTF}_n \approx \frac{1}{\lambda}\int_0^{4n} \frac{\tau^{-4/5}\ d\tau}{5(4n)^{1/5}}\ \frac{\alpha_+\left(1-\left(\frac{\tau}{4n}\right)^{1/5}\right)}{1-\left(\frac{\tau}{4n}\right)^{1/5}}\ e^{-\tau} \times \exp\left(-\frac{4}{n^{2/5}}\left(\frac{\tau}{4}\right)^{7/5} + \frac{9}{n^{3/5}}\left(\frac{\tau}{4}\right)^{8/5} + \ldots\right). \tag{30}$$

The upper bound of the integral depends on n. However, because of the $e^{-\tau}$ factor, the error made by replacing this upper bound by $+\infty$ vanishes exponentially with n (as also do the already discarded contributions of the eigenvalues different from ζ_+). Consequently, we can merely integrate $\tau^{-4/5}\ e^{-\tau}$

multiplied by an expression admittedly depending on τ and n, but which can be easily expanded in the $n \to \infty$ limit, assuming τ remains finite. For instance, the leading term of MTTF_n is (see (12))

$$\mathrm{MTTF}_n \to \frac{1}{\lambda}\int_0^{\infty} \frac{\tau^{-4/5}\, d\tau}{5(4n)^{1/5}}\, e^{-\tau} = \frac{1}{\lambda}\, \frac{\Gamma(1/5)}{5(4n)^{1/5}} = \frac{1}{\lambda}\, \frac{\Gamma(6/5)}{(4n)^{1/5}}\,, \tag{31}$$

where Γ(x) is the Euler gamma function. Using (30), going beyond the leading term is not difficult, and we find

$$\lambda\, \mathrm{MTTF}_n \to \frac{\Gamma(6/5)}{(4n)^{1/5}} + \frac{1}{5}\frac{\Gamma(2/5)}{(4n)^{2/5}} + \frac{2}{25}\frac{\Gamma(3/5)}{(4n)^{3/5}} - \frac{1}{40n} + \ldots \tag{32}$$

$$\approx \frac{0.6958417869}{n^{1/5}} + \frac{0.2547996219}{n^{2/5}} + \frac{0.05185668604}{n^{3/5}} - \frac{0.025}{n} + \ldots \tag{33}$$

By contrast to the series or parallel cases, the leading term in the asymptotic expansion of the MTTF has a behavior in $n^{-1/5}$, which slowly decreases with n. Each of the following terms of the expansion adds another $n^{-1/5}$ factor.

4.3 Comparison of exact and asymptotic results

We can now compare (33) with the exact values. The results are displayed in Figure 8. Even for $n \approx 10$, the asymptotic expansion gives a very satisfying agreement, despite the limited number of used terms (four).

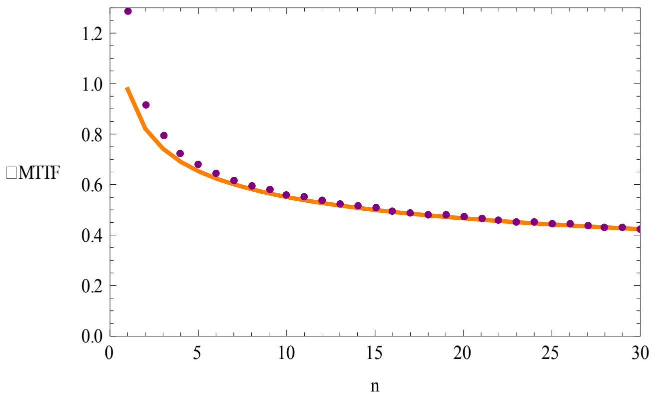

Figure 8. Comparison between exact values (purple) and the four-term asymptotic expansion of (33) (orange) of the MTTF.

5 CONCLUSION AND OUTLOOK

We have calculated the availability of the architecture studied by Beichelt and Spross, and shown that for perfect nodes and identical links with constant failure rate, the asymptotic expansions of the associated average failure rate and MTTF obey quite different power-law behaviors in n (the extension of the network). It could be useful as a reminder that average failure rate and MTTF are not necessarily the inverse of each other.
The present study may be easily generalized to various recursive networks. Actually, it is possible to find the asymptotic expansion of the MTTF for different classes of large, arbitrary recursive networks, even though the exact generating function is not known (Tanguy 2008).

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CONTRIBUTION TO FAILURE DESCRIPTION

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ABSTRACT

In our lives we meet many events which have very diverse causes, mechanisms of development and consequences. We frequently work with the events´ description besides other assessments in safety/risk assessment. In pure technical applications these events are related with the failure occurrence of equipment, a device, a system or an item. The theory speaks about failure itself, its mechanisms, circumstances of occurrence, etc. but at the same time we need appropriate terminology to describe these conditions. Our basic approaches into observing, dealing and handling failure may fall into two groups. We either talk about a probabilistic approach or about a deterministic (logic) approach. As we need to get some information about a failure we need to find it or transfer it from different sources. This contribution can be a complex problem for the term "failure" and its related characteristics. In the paper there are mentioned functions of an object and their description, classification of failures, main characteristics of failure, possible causes of failure, mechanisms of failure and consequences of failure and also other contributions related with failure very closely.

1 INTRODUCTION

Before we introduce the topic of a failure let us ask a simple question. Why do things actually break? Answers can vary. One of the answers might be the following statement which we are going to develop more. Usually the reason for this is that the applied load exceeds the dimension/robustness of the product. The load can be purely mechanical (force, tension, etc.), purely electrical (power, electromagnetic field, etc.), purely chemical (effect of chemical substances, etc.), general physical (warmth, radiation, etc.), or of a totally different nature. Whenever the applied load exceeds the assumed dimension of the item, unwanted (usually irreversible) processes start, and sooner or later a failure occurs. The load can be a one time load or it can be applied a number of times. Concerning the first instance, overload failure will occur and in the second case fatigue failure will occur. As time passes, the product could become weaker for any one of many reasons (unless a failure occurs immediately). One of the basic assumptions dealing with a failure is as follows. Before any failure incurred due to inner cause (e.g. operation or using an item) occurs, it is essential to have a device in operation. Idleness of an item or a system can end in a failure due to natural ageing, but in this case the initial mechanism is not properly understood. A relevant failure occurs mostly only during operation. Some factors and characteristics for describing failures:

Process in time of occurrence and manifestation:

- failure causes;
- failure manifestations ;
- failure consequence;

Failure profile

Failure causes:

- design failures;
- manufacturing failures;
- overstress failures;
- misuse failures;
- degradation failures;

Failure manifestations:

- random failures;
- gradual;
- sudden;
- common caused failures;
- primary and secondary failures;
- intrinsic failures;
- extrinsic failures;

Failure consequences:

- insignificant;
- marginal;
- minor;
- major;
- critical;
- catastrophic;

Failure is a term widely used in technical practice especially concerning dependability theory. For the reliability practitioners failure is a basic term in dependability theory, and it is key and essential for observing stochastic relations of item behavior. It is an event which is used by probability theories on a general level, for they speak about a random event. In dependability theory it is necessary to realize the fact of failure as a stochastic term, to understand its meaning, and to understand other links. And only because of this, mathematical tools, used in dependability, are not only a dead and boring "set" of formulas, relations and graphical expressions.

While observing a technical item we concentrate basically on possible causes of failures, their development over time, their process, mechanism, and of course their impact, effect, or other influences which might result from a failure occurrence. It is inevitable to realize that a failure is of key importance for operation and function of technical items. Theory and practice in particular shows us that failures occur under different situations, various circumstances, different conditions, etc. Theoretically, dealing with failures, we can describe their possible causes, nature of occurrence, process of development, and we are able to model them at the same time. We can see connections between individual groups of failures and their profiles. We can match a range of importance and numerical values with the failures, they can fall into groups, sets, etc. However, our biggest, continual effort is to eliminate failure occurrence, reduce its number (frequency), limit the number of its occurrences over a specified time period or in relation to another observed dependent quantity (mileage, cycles, etc.). Our intention is to be able to determine their occurrence so exactly that we could be prepared to face it as well as possible. Simply our aim is to get a better profile of an observed item from the view of its dependability and related properties.

Furthermore, we would like to describe possible classes of failures, their profiles, courses, development, consequences, and other relations which might be important for dependability theory and especially for this paper itself. The phenomena involved in this article are definitely not an example of a complete and synoptic list of all known and possible events assisting a failure. The aim of this article is to introduce the topic which is usually believed to be obvious, familiar and clear. However, reality need not match our ideas or the ideas of other people in full. The purpose of the paper is also to initiate the reader into the topic of a failure and at the same time to popularize it. Without full understanding we would not like the reader to absorb a piece of scripted information and not to obtain its complex form. A frequently used term might have a totally different meaning then. It would be great while working on it and finding it in a book, using theoretical tools, profiles, graphs, models, and other descriptions and contexts, we would be able to imagine there is definitely something more to the term (Blischke 2000, Elsayed 1996, Meeker & Luis 1998, Modares & Kaminskyi & Krivtsov 1999).

2 CURRENT TERMINOLOGY SITUATION

The following part speaks just briefly about the current terminology situation in the standardization field and especially in the branch of dependability and risk. The situation is caused by the ISO/IEC representatives and national bodies. Failure according to the present version of the IEC 60050-191/1990 is defined as follows: “termination of the ability of an item to perform a required function”.

Note 1. After failure the item has a fault.

Note 2. Failure is an event, as distinguished from fault, which is a state.

Note 3. This concept as defined does not apply to items consisting of software only.

Failure according to the newly upgraded version IEC 60050-191 is defined as follows: “loss of ability to perform as required”

Note 1: When the loss of ability is caused by a pre-existing condition, the failure occurs when a particular set of circumstances is encountered.

Note 2: A failure of an item is an event, as distinct from a fault of an item, which is a state.

Note 3: Qualifiers may be used to classify failures according to the severity of consequences, such as catastrophic, critical, major, minor, marginal and insignificant, the definitions depending upon the field of application.

It results from these definitions and further analysis that the term “failure” will be understood as an event which leads straight to either a partial or complete loss of ability of an item to fulfil a required function. Most terms that are specified in the introduction dealing with the description of failure factors and profiles might also be found in a basic source document mentioned before.

At present it just so happens that because of modification and updating of terminology, an existing view of understanding a failure and relating facts can be changed. Just to demonstrate the complexity of the present state we introduce the following facts. According to the notes of the term failure mentioned above (see IEC 60050-191/1990) an item after failure has a fault. (“An item after failure has a fault”.) Owing to continual discussions about this topic it is impossible to ignore the idea that a fault does not follow a failure but precedes it. This technical incompatibility together with many others has not been solved yet but their form has been very much discussed. A possible decision in favour of a new view will influence radically the existing approach, conception and observation of the failure.

While working with the term failure, as well as with relating states, it is necessary to take the current terminology mismatch into account and to adapt possible decisions to it. The possibility of a realized change has to be accepted along with all the suffered consequences. Unfortunately, this change will violate the understanding of all existing terms/disciplines introduced so far that deal with a proper function/failure and dependability.

3 WHAT MIGHT THE FAILURE AFFECT

In this part it is necessary to draw attention to some relating events. We are dealing with a failure which prevents the items ability from performing a required function (either the main one, the minor one, or some other one as detailed below). It results from all the definitions in the paper that the inability of a system or a product to operate in a required way is a key term determining a failure.

Based on many studies and approaches a factual scale of individual functions description in complex conception was formed for a system. On the basis of these assumptions it is also essential to distinguish the influence of a failure on a function performed by an item. A failure occurrence might affect the range of the function. An outline of item functions is provided to make the understanding much easier, and failures occurrence is not strictly limited to a kind of an item function.

A required function – specifies an item task. A correct, exact and unequivocal definition is a primary, starting point for all dependability definitions as well as for a right failure definition. Operation conditions – affect significantly both dependability and especially possible failure occurrence, hence why they have to be determined very thoroughly.

1. Main function: - an intended (required) or primary function
2. Minor function: - need for providing main function
3. Supporting function: - the aim is to provide protection of people and an environment from potential damage regarding main or minor function failure as well as common support (brakes, circuit breakers, filters, etc.)
4. Information function: - it provides conditions, monitoring, measuring, diagnostics, etc. (it refers to displays, indicators etc.)
5. Interface function: - it provides an interface between an assessed item and other items (cabling, operating elements, switches, breakers, etc.).

The required function and/or operation conditions might be time dependent. In this case a mission profile has to be determined and all dependability viewpoints have to be related to it. A representative mission profile and corresponding dependability targets have to be stated in the item's specification. The mission duration is often/usually considered as a parameter t, that is time. The dependability function – especially the reliability function is designated as $R(t)$. $R(t)$ is the probability that no failure at item level will occur in the interval $(0;t\rangle$, often with the assumption $R(0) = 1$ – it means that at the time $t = 0$ the object was in the state of operation. In order to avoid confusion a distinction between predicted and estimated (assessed) dependability should be made on the basis of a real evaluation during operation or tests. The predicted dependability is calculated on the basis of the item's dependability structure and the failure rate of its component. The estimated dependability is specified on the basis of a statistical evaluation of dependability tests or field data by known operating and environmental conditions.

Failure: - it occurs when an item terminates its ability to perform its required function. However simple the definition might look, it is difficult to apply it to complex items/systems. The basic operating time is generally a random variable. It is often reasonably long but on the other hand it might be very short, caused by systematic failure influence for example. It can also be caused by early failure influence resulting from a transient event at turn-on. A general presumption in investigating failure-free operating times is that at $t = 0$ which means that in an instant $t = 0$ the object is free of defects and systematic failures and therefore it is able to operate one hundred per cent. Besides their relative frequency, failures can be categorized according to one of the views mentioned before (mode, course, cause, consequences, mechanisms, etc.).

Failure profiles:

– critical stage – consequence seriousness

– failure cause
  - misuse failure;
  - mishandling failure;
  - weakness failure;
  - design failure;
  - manufacturing failure;
  - ageing/wearout failure;
  - others (e.g. software).

– failure mode (velocity)
  - sudden;
  - gradual

  degradation

– according to a range of a consequence
  - cataleptic;
  - complete;
  - partial.

– according to a place of occurrence
  - during a test;
  - during operation.

– according to occurrence mechanism
  physical, chemical, or other processes leading to a failure
  - primary;
  - secondary;
  - systematic/reproducible;

– according to verification possibility
  - verified failure;
  - unverified failure.

These are the very basic failures categories and factors they fall into, and this is the common way of how to work and deal with them. Moreover, we can determine some other (supplementary) failure categories but their presence here is not possible due to space limits of the paper. The authors of the paper may provide more information for those who are interested (Elsayed 1996, Meeker & Luis 1998, Modares & Kaminskyi & Krivtsov 1999).

4 FAILURE OCCURRENCE CAUSE

According to the (IEC 60050-191/1990) the circumstances occurring during design, manufacture or use which have resulted in a failure are the cause of a failure. To know the cause of a failure is useful in case we want to decide how to prevent a failure or its reoccurrence. Failure causes can be classified in relation to the life cycle of the system.

Cause – the cause of a failure can be intrinsic, due to weaknesses in the item and/or wearout, or extrinsic, due to errors, misuse or mishandling during the design, production and especially the use itself. Extrinsic causes often lead to systematic failures which are deterministic and might be considered like defects (dynamic defects in software quality). Defects are present at $t=0$, even if they cannot be discovered at $t=0$. Failures always seem to appear in time, even if the time to failure is very short as it can be with systematic or early failures.

1. Design failure - occurs due to inadequate design. It is basically any failure directly related to item design. It means that due to item design a part of the whole degraded or got damaged and this resulted in a failure of the whole.
2. Weakness failure - occurs due to weakness (internal) inherent or induced in the system so that the system cannot stand the stress it encounters in its normal environment.
3. Manufacturing failure - a failure caused by nonconformity during manufacturing and processing. It is basically any failure caused by faulty processing, or inadequate manufacturing, or an error made while controlling the process during manufacturing, tests and repairs.
4. Ageing failure - a failure caused by the effects of usage and/or age.
5. Misuse failure - a failure caused by misuse of the system (operating in environments for which it was not designed).
6. Mishandling failure - a failure caused by incorrect handling and/or lack of care and maintenance.
7. Software error failure - a failure caused by a PC programme error.

5 FAILURE MECHANISM

The failure mechanism is a very complex and extensive passage of the failure profile. It can be sudden or gradual with its relating manifestations.

Failure mechanism - physical, chemical, electrical, thermal or other process that results in failure.

Mode (manifestation, course) – the mode of a failure is a symptom (local effect) by which a failure is observed. For example – opens, shorts, or drifts (for electronic components). Brittle rupture, creep, cracking, seizure, or fatigue (for mechanical components), etc.

A complete and sudden failure is called a catastrophic failure and a gradual and partial failure is designated a gradually degraded failure.

The connections related to these aspects of a failure are shown in the following description:

1. Intermitted (incoherent) failure - a failure which lasts only for a short time. A good example of this is a fault that occurs only under certain conditions occurring intermittently (irregularly).
2. Extended failure - failures that occur until some corrective action rectifies the failure. They can be divided into the following two categories:

a) Sudden failure - a failure which occurs without warning

b) Gradual failure - a failure which occurs with signals to warn of the occurrence. Usually it is a case of significant behaviour changes (decreasing performance, increasing temperature, rising vibrations, etc.) or this style.

We have to distinguish among different failure mechanisms of mechanical, electrical, electronic and hydraulic parts. The differentiation is so complex that it can not be easily presented in this paper. The example of failure mechanism will be given at the section 9 (Blischke 2000, Elsayed 1996, Meeker & Luis 1998, Modares & Kaminskyi & Krivtsov 1999).

6 FAILURE CONSEQUENCES

Many information sources use the term failure consequence. Also many standards define them and work with them differently. The following part should help to clarify the concept of failure consequences, as we also know them from many reliability analyses.

Effect - the effect (consequence) of a failure can be different if considered on the item itself or at a higher level. A usual classification of a failure has usually the following qualitative profile and is: non-relevant, partial, complete, …, critical failure. Since a failure can also cause further failures in an item or a system, a distinction between primary and secondary failure is important.

A classification of the severity of a failure mode in accordance with the MIL-STD 882 is listed:

1. Catastrophic failure - a failure that can lead to death or can cause total system (item) loss.
2. Critical failure - a failure which results in many serious injuries or major system damage. Sometimes we think of it as a failure, or combination of failures, that prevents an item from performing a required mission.
3. Marginal failure - a failure that leads to minor injury or minor system damage.
4. Negligible failure - a failure that leads to less than minor injury of system damage.

Another classification can be found in the RCM approach where the following classes are used:

Failures with safety consequences;
Failures with environmental consequences;
Failures with operational consequences;
Failures with non-operational consequences.

A classification of the failure severity into groups (categories) is given in more standards. Each of them is specific in a way and corresponds with a presupposed application. The IEC 61 882, IEC 60 812, IEC 50 126 and many others are some of the examples. We do not have the ambition to make a complete list of failure consequences and their classification. The issue is to take into account many different approaches and handle with care with them as well as use them with clear intention (Meeker & Luis 1998, Modares & Kaminskyi & Krivtsov 1999).

7 SOURCES FOR FAILURE PROFILE DETERMINATION

We do not want to speak about basic and clear failure measures and characteristics which are obviously well known in our community. Our attempt is to present different sources of failure data/measures/characteristic obtaining. The main sources are:

1. Data on elements' reliability guaranteed by a producer – there is no need to expand on it;
2. Conclusive test results (observation) of the same (comparable) item reliability. It is based on the standardized assessment of reliability tests of technical items. The methods and methodologies of how to conduct tests are standardized for different equipment.
3. Predictions – standardised calculation of item's reliability based on a reliable source (MIL HDBK 217F). This is the American military standard that enables the data on electronic elements' reliability to be estimated. It is commonly used when estimating the elements' failure rate especially in military applications.
4. Specialized information databases on elements' reliability (specialized in terms of elements' profile or conditions of usage). Specialized information databases on elements' reliability are usually established and kept to meet the needs of single industrial branches or technical areas. The data acquired when observing items in operation or the results of specialized dependability tests are collected in the databases. One of the most respectable and frequently used databases on reliability in this area is the database established and kept by the Reliability analyses centre (RAC) which at present distributes three important databases on the commercial basis: (EPRD-97; NPRD-95; FMD-97; SPIDR 2007).
5. General information database on elements' reliability. These databases are usually published as parts of specialized literature in the dependability area. The information put in them is usually very general.

6. Expert estimations. Expert estimations of numerical values of reliability measures might be used only when appropriate values cannot be specified by a different, more reliable method. The authors of the article know from experience that this solution is accepted only as an exception because in most cases the numerical values of reliability measures can be determined by other methods described in this paper.or this style.

8 TYPICAL MEASURES OF A FAILURE OCCURRENCE

Failure rate

Failure rate plays a major role in dependability analyses. It is a numeric value of the measure that describes failure occurrence depending on the measurement of continuous/discrete quantity. It specifies the occurrence of a certain number of events per observed/measured unit.

Factors affecting failure rate:

- Component type;
- Component design;
- Component technology;
- Operational stress (temperature, voltage, pressure, etc.)
- Component quality grade (involving production quality control and post-production screening including burn-in)
- Environmental stress (vibration, shock, humidity)
- Activation and deactivation transients, e.g. voltage spikes, current surges, transient thermal stresses
- Component application;

Failure occurrence probability

This is another measure describing possible phenomenon-failure occurrence in a numeric way. It can be described by a discrete distribution or continuous distribution depending on a kind of variable and provided that it follows a certain level of relevancy which is called a confidence interval.

Mean-time to failure

Another frequently used measure of a continuous random variable (usually time), which specifies assumed mean-time to failure.

9 EXAMPLE OF FAILURE CHARACTERISTICS

The intention of this example is to present some technical parts which are commonly used and to show their typical failure mechanisms, failure modes/causes and the percentage distribution of these characteristics for them. Based on (EPRD-97; NPRD-95; FMD-97; SPIDR 2007) the example of several mechanical parts is shown. The items chosen for the example are the most common mechanical parts which are typically implemented in the systems. This example as well as the guidelines presented in the paper is supposed to contribute to the analyst knowledge and help him to orient while conducting standard analysis (e.g. PHA, FMECA, FTA, OSHA, JSA, etc.).

Example of several mechanical parts:

Statically loaded
Demountable:

– Screw:	Loose (approx 50%) Worn (approx 25%) Induced – vibration/missing (approx 25%)
– Nut:	Bearing failure (approx 50%) Loose (approx 50%)
– Key:	Bent/Dented/Warped (approx 100%)

Non-rewirable:

– Welded joint:	Broken (approx 50%) Workmanship (approx 50%)
– Riveted joint:	Broken (approx 50%) Workmanship (approx 50%)

Dynamically loaded

– Bearing:	Worn (approx 60%) Binding/Sticking (approx 20%) Loss of lubrication (approx 10%) Contaminated (approx 5%) Scored (approx 5%)
– Gear:	Worn (approx 52%) Binding/Sticking (approx 19%) Stripped (approx 10%) Broken (approx 7%) Jammed/Stuck (approx 7%) Displaced (approx 3%) Noisy (approx 2%)

This is only small example of the failure characteristic regarding few typical mechanical parts. The purpose of the example is to extend current lack of information we normally face. Based on the information mentioned in the previous section we frequently do not have such information about failures and their characteristics guaranteed by the producer. We do not have plenty of information from tests either since the tests are not conducted very frequently and in the wide range. Some prediction methods like (MIL HDBK 217F, and others) are not very suitable for every parts prediction and they give only one characteristic of the failure.

Next point which was the purpose of the presentation of the example for was to present also the related characteristics of a failure (mode/cause) apart of the measure. Sometimes if the analyst does not have clear imagination about modes/causes of failure he/she can hardly imagine if the item may fail down or not.

10 CONCLUSION

This contribution is supposed to give a general overview in the area of the basic term “a failure” as described above. As the understanding of all related matters is very complex it is not possible to express complete knowledge and experience here. Some reliability and safety engineers might be confused while beginning with specific analysis (e.g. FMECA, PHA, JSA, OSHA, etc.). The main benefit of this contribution is supposed to be a general and introductive material for understanding a failure its full profile with all related characteristics. The next purpose of the paper is to provide a hand (possibly guide lines) to orient the analyst on the appropriate information sources which are necessary for the analysis. Due to the limited space within the paper, the information provided is not complete, therefore those who are interested we kindly ask to contact the authors.

ACKNOWLEDGEMENTS

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CONTRIBUTION TO AVAILABILITY ASSESSMENT OF COMPLEX SYSTEMS

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ABSTRACT

As we use complex systems with one shot items in many technical applications we need to know basic characteristics of such system. Performance, safety and other are as much important as dependability measures. In real applications we have to take into account a related distribution of an observed variable. In terms of complex systems with one shot items it is a discrete random variable related to one shot item. The whole system and its failures (unexpected and inadvertent events) may have two typical types of distributions and their characteristics. We either consider a continuous variable (such as time, mileage, etc.) or a counting variable (such as number of cycles, sequences, etc.) regarding to a failure occurrence. As the one shot items is supposed to back up the main system function the total reliability of the system should be higher than. The main issue regarding the system using one shot items in their construction is to determine the probability of the task (mission) success. The paper presents both theoretical approach and practical example of the solution.

1 INTRODUCTION

This paper is supposed to contribute to a solution of dependability qualities of the complex (in this case) weapon system as an observed object. We would like to show one of the ways how to specify a value of single dependability measures of a set. The aim of our paper is to verify the suggested solution in relation to some functional elements which influence fulfilment of a required function in a very significant manner (Koucky & Valis 2007).

The paper contents deals with a weapon set which is a complex mechatronics system, designed and constructed for military purposes. We are talking about a barrel shooting gun – a fast shooting two-barrel cannon. It is going to be implemented in military air force in particular.

Generally speaking the set consists of mechanical parts, electric, power and manipulation parts, electronic parts and ammunition. For the purpose of use in our paper we are not going to deal with isolated functional blocks and ammunition only. In this case we consider the ammunition as the key element in the whole process as recommended standardised rounds and pyrotechnic cartridges.

Single parts of the set can be described with qualitative and most importantly quantitative indices which present their quality. In this paper we are dealing especially with quality in terms of dependability characteristics. We have been working first and foremost with probability values which characterize single indices, and which describe functional range and required functional abilities of the set. We do not focus only on the part handling rounds and pyrotechnic cartridges which are crucial for this case. In order to continue our work it is necessary to define all terms and specify every function.

The main type of data which can be found in the area of dependability statistical analysis is as follows: simple, censored, cut (reduced) data, or the combination of it.

Simple data: It is a basic category in which the established information $t_1, t_2, \ldots, t_n$ is the random sample of probability distribution of time to failure T.

Censored data: The data is designated (t_1, d_1), …, (t_n, d_n), where $t_i = \min(T, C)$, T is a random value determining time to failure, C is censoring time and d_i is an indicator defined by the formula $d_i = 1$, if t_i is time to failure and $d_i = 0$ in other cases. The basic types include censoring by fixed time (C is fixed time) and random time (C is a random variable with given probability distribution). This type of dependability data is frequently used in practice and it can be found in the situations where the observation is terminated after some time, because the system is put out of operation, etc. Concerning laboratory tests these are the so called tests terminated by time.

Cut (reduced) data: This is the data of the failures registered after some time passes. In practice one can come in contact with this sort of data when the information about failures is not put in the early stages.

Classification of statistical methods used in dependability statistical analysis:

Parametric methods: These methods proceed from the assumption that the observed data represent random sample described by a given probability distribution (e.g. exponential, Weibull´s, gama, etc.). The main task then is to determine (estimate) values of unknown parameters based on the observed data.

Non-parametric methods: These methods do not take into account any specific classification of data and they are a “universal” alternative to parametric methods (their main advantage). The main disadvantage is their smaller power (when compared to parametric methods).

Semi-parametric methods: These methods which are a sort of compromise between parametric and non-parametric methods require only a “partial” specification of the distribution. A parametric model is introduced for important variables and a non-parametric one is introduced for these of minor importance.

2 ESENTIAL TERMS, DEFINITIONS AND SIGNS

We are always talking about an object in terms of reliability analyses. The definition for object is the same as the used in IEC 60500-191/1990. Consequently we need to describe the basic object’s measures (Koucky & Valis 2007).

Object’s function:

The main function: The main function of the object is putting into effect a fire from a gun using standard ammunition.

The step function: Manipulation with ammunition, its charging, initiation, detection and indication of ammunition failure during initiation, initiation of backup system used for re-charging of a failed cartridge.

It is expected that the object will be able to work under different operating conditions especially in different temperature spectra, under the influence of varied static, kinetic and dynamic effects, in various zones of atmospheric and weather conditions.

In this case we will not take into account any of the operating conditions mentioned above. However, their influence might be important while considering successful mission completion.

One of the main terms we are going to develop is:

Mission: It is an ability to complete a regarded mission by an object in specified time, under given conditions and in a required quality.

In our contribution it is a case of cannon ability to put into effect a fire in a required amount – in a number of shot ammunition at a target in required time, and under given operating and environmental conditions.

As it follows from the definition of a mission it is a case of a set of various conditions which have to be fulfilled all at once in a way to satisfy us completely. Our object is supposed to be able to shoot a required amount of ammunition which has to hit the target with required accuracy (probability). We will not take into consideration circumstances relating to evaluation of shooting results, weapon aiming, internal and external ballistics, weather conditions and others. We will focus only on an ability of the object to shoot (Koucky & Valis 2007).

As we have stated above we will not deal with isolated function blocks only. We are presuming that these blocks act according to required and determined boundary conditions. In order to understand functional links fully we introduce our way of dividing the object although we will understand the object as a complex system in the paper.

We speak about the following blocks:

Manipulation with ammunition, its charging, initiation, failure detection and indication during initiation, initiation of a backup system in order to recharge a failed cartridge, all mechanical parts, all electric and electronic parts, interface elements with a carrying device - Block A;

Ammunition – Block B;

Pyrotechnic cartridges – Block C.

Symbols used in the text:

Trandom value expressing time to failure,

$t_1, t_2, \dots, t_n$	measured times to failure (that is a random selection of T), or data on possible censoring,
$t_{(1)}, t_{(2)}, \dots, t_{(n)}$	arranged values $t_1, t_2, \dots, t_n$ (including data on censoring),
$t_{[1]}, t_{[2]}, \dots$	arranged random selection of times to failure , that is, without data on censoring,
$\Lambda_T(t), \Lambda_T^*(t)$	cumulative failure rate or its point estimation,
$R_T(t), R_T^*(t)$	probability of reliable operation or its point estimation,
$E[\bullet], E^*[\bullet]$	mean value of the variable or its point estimation,
$var[\bullet], var^*[\bullet]$	dispersion variance, or its point estimation $\bullet$.

3 DESCRIPTION OF THE PROCESS

The process as a whole can be described this way:

From a mathematical and technical point of view it is a fulfilling of requirements´ queue which gradually comes into the service place of a chamber. The requirements´ queue is a countable rounds´ chain where the rounds wait for their turn and are transported from the line where they wait in to a service place (fulfilment of a requirement) of a chamber and there they are initiated. After the initiation the requirement is fulfilled. An empty shell (one of the essential parts of a round) leaves a chamber taking a different way than a complete round. When the requirement is fulfilled, another system which is an integral part of a set detects process of fulfilling the requirement. The process is detected and indicated on the basis of interconnected reaction processes. In this case fulfilling the requirement is understood as a movement of a barrel breech going backwards. Both fulfilling the requirement and its detection are functionally connected with transport of another round waiting in a line to go into a chamber.

Let's presume that rounds are placed in an ammunition feed belt of an exactly defined length. A maximum number of rounds which could be placed in a belt is limited by the length then. The length is given either by construction limitations or by tactical and technical requirements for a weapon set. Let's presume that despite different lengths of an ammunition belt, this will be always filled with rounds from the beginning to the end. Let's also assume that the rounds are not non-standard and are designed for the set.

The process of fulfilling the requirement is monitored all the time by another system which is able to differentiate if it is fulfilled or not. The fulfilment itself means that a round is transported into a chamber, it is initiated, shot, and finally an empty shell leaves a chamber according to a required principle. If the process is completed in a required sequence, the system detects it as a right one.

Because of unreliability of rounds the whole system is designed in the way to be able to detect situations in which the requirement is not fulfilled in a demanded sequence and that is why it is detected as faulty.

Although a round is transported into a chamber and is initiated, it is not fired. A function which is essential for a round to leave a chamber is not provided either, and therefore another round waiting in line cannot be transported into a chamber. That is the reason why fulfilling of the requirement is not detected.

The system is designed and constructed in such a way that it is able to detect an event like this and takes appropriate countermeasures. A redundant system which has been partly described above is initiated. After a round is initiated and the other steps don't carry out (non-fire, non-movement of a barrel breech backwards, non-detection of fulfilling the requirement, non-leaving of a chamber by an empty shell, and non-transport of another round into a chamber) a system of pyrotechnic cartridges is initiated. It is functionally connected with all the system providing mission completion. A pyrotechnic cartridge is initiated and owing to this a failed round is supposed to leave a chamber. A failed functional link is established and another round waiting in line is transported into a chamber.

In order to restore the main function we use a certain number of backup pyrotechnic cartridges. Our task is to find out a minimum number which is essential for completing the mission successfully. Next issue we are supposed to solve is to find out the availability function of the system. We would like to know if the system is capable to carry out next mission with its technical/mission "history". If the operational unit left are much enough to complete the task successfully from the technical point of view without any impact on terms of repair/replacement, etc. As based onto the collected data observed from previous deployment and initial operation period of the system we might use standard mathematical tools for their assessment. Due to specific system construction and specific process procedure it seems to us that another than common methods are to be applied. Following section is the example of our effort (Koucky & Valis 2007).

4 MATHEMATICAL MODEL

Since the data on system operation and process behaviour is available we use two methods while analyzing this. The first one is the Nelson – Altsschuler estimation (Akersten 1987, Crowder & Kimber 1991, Nelson 1990). It is a case of one of the basic non-parametric methods which are used for statistical dependability analysis, especially while estimating instantaneous cumulative failure rate $\Lambda_T(t)$. It is expressed by the equation:

$$\Lambda_T(t) = \int_0^t \lambda_T(u)du \tag{1}$$

where $\lambda_T(t)$ is failure rate at the time t, thus

$$\lambda_T(t) = \lim_{h \to 0+} \frac{P(t \le T < t + h \mid t \le T)}{h} \tag{2}$$

Let us assume that the obtained dependability data $t_1, t_2, \ldots, t_n$ are the information on time to failure or time information about censoring. In this case the Nelson-Altschuler´s (N-A) point estimation $\Lambda^*_T(t)$ of the cumulative failure rate is expressed by

$$\Lambda^*_T(t) = \sum_{\substack{[i] \\ t_{[i]} \le t}} \frac{m_{[i]}}{r_{[i]}} \tag{3}$$

where $t_{[i]}$ is the i-th element of the arranged random selection of times to failure (that is we do not include censoring times in the selection),

$m_{[i]}$ is frequency of the value $t_{[i]}$,

$r_{[i]}$ is number of objects in operation to the time $t_{[i]}$.

If the failure occurs together with the censoring, we assume that the censoring occurs straight after the failure. In order to estimate the dispersion variance $\Lambda^*_T(t)$ we use the asymptomatic formula

$$var^*\left[\Lambda^*_T(t)\right] = \sum_{\substack{[i] \\ t_{[i]} \le t}} \frac{m_{[i]}}{r^2_{[i]}} \tag{4}$$

through which we determine even relevant interval estimation. For $(1-\alpha)\%$ dependability interval of the value $\Lambda^*_T(t)$ we get

$$\left(\Lambda^*_T(t) - u_{1-\alpha/2}\sqrt{\mathrm{var}^*\left[\Lambda^*_T(t)\right]}\,,\ \Lambda^*_T(t) + u_{1-\alpha/2}\sqrt{\mathrm{var}^*\left[\Lambda^*_T(t)\right]}\right) \tag{5}$$

where u_α is $\alpha\%$ a quantile of standard normal distribution.

Of course there is large variety of other non-parametric methods which are suitable for dependability assessment based on operational data. These are for example non-parametric renewal density estimations, renewal functions and non-parametric trend tests.

Another method used for the system assessment is determining the distribution of time to failure and its properties. This is the statistical test TTT (Total Time on Test-plot) which allows us to decide whether distribution of time to failure is of increasing (IFR – Increasing Failure Rate), or decreasing (DFR – Decreasing Failure Rate) failure rate. If $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ is an arranged selection of times to failure, then the test statistic $u_{(i)}$ is defined as follows:

$$u_{(i)} = \frac{T_{i,n}}{T_{n,n}} \tag{6}$$

where $T_{i,n} = t_{(1)} + t_{(2)} + \ldots + t_{(i-1)} + (n-i+1)t_{(i)}$

The testing itself is based on putting the values $u_{(i)}$ and i/n in the graph. In case of the IFR distribution the graph $u_{(i)}$ is convex , concerning the DFR distribution the graph is concave.

5 EXAMPLE OF THE APPLICATION

The assessed failures were as follow:

- only mechanical, software and process ones;
- the failures resulting from shortage of redundant cycles (pyrotechnical cartridges)

The source of the data is operating data – number of cycles (shots) to failure (mechanical, software, process cause not at all due to shortage of redundant cycles that is pyrotechnical cartridge) regarding sixteen observed systems.

Ad – only mechanical, software and process failures:

The data used for the analysis are put in Table 1, the data in red (last column) stand for censoring by time and not the failure. Complete enumeration consists of a number of shots to failure regarding sixteen renewed systems of the same type (cannon). In the paper there is presented only one system how to carry out the method. The data is arranged according to its real occurrence and is essential for quite a few of non-parametric tests. The values are modified owing to industrial protection. The thick blue line (between column 6 and 7) separates the years 2005, 2006.

Table 1. Data from system operation

Canon	t_i – time to failure							
1	201	339	660	512	156	1293	2	798

Table 2 shows the calculation of the Nelson-Altschuler estimation of cumulative failure rate. $\Lambda_T^*(t)$.

Table 2. Table of Nelson-Altschuler Estimation calculation

Event	Nelson-Altschuler Data				
(i)	$\Lambda(t_{[i]})$	$R(t_{[i]})$	D_R	H_R	$u_{(i)}$
1-2	0,033	0,968	0,925	1,000	0,003
3	0,050	0,951	0,899	1,000	0,007

Description of the table:

Event (i) – serial number of an event (failures including possible censoring by time).

Values in the column $\Lambda(t_{[i]})$ are calculated according to the formula (3) and they are point estimation of cumulative failure rate in the interval $\left(t_{[i-1]}, t_{[i]} \right\rangle$.

The values D_R, H_R are relevant lower and upper limits of 95% of the dependability interval.

The column $u_{(i)}$ – the values of test statistic for TTT are counted using the equation (6). Figure 1 and 2 show the course (typically step-wise) of the estimations $\Lambda_T^*(t)$ and $R_T^*(t)$, including relevant 95% of dependability intervals. The course of the estimation of reliable operation probability R1(t) and its 95% of the dependability interval (D(t) – the course of the lower limit H(t) – the course of the upper limit) is put in Figure 1.

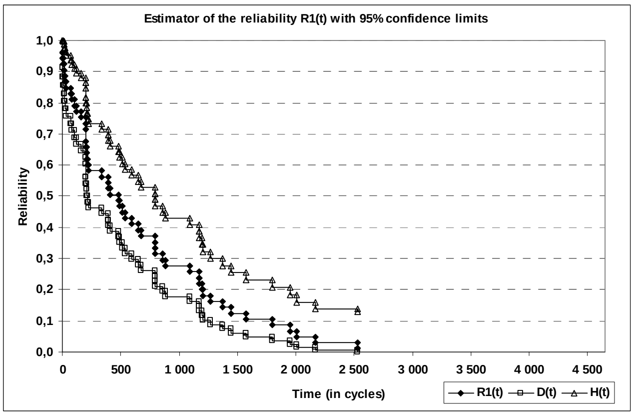

Figure 1. Reliability of the system and its 95% confidence intervals.

Ad – the failures resulting from shortage of redundant cycles (pyrotechnical cartridges:
This time the data in table 3 contains number of cycles to failure owing to shortage of redundant cycles, the data in red (last column) shows the information on censoring by time. By way of demonstration there is also one system only which is supposed to demonstrate how to carry out the method. The values are again modified due to industrial protection.

Table 3. Data from system operation censored by lack of cycles

System	Number of main cycles to failure		
1	1200	668	2299

Even in this case the NA non-parametric estimation of cumulative failure rate $\Lambda_2(t)$ was used in order to estimate reliable operation probability . The example of calculation results is put in table 4.

Table 4. Table of Nelson-Altschuler Estimation calculation

$t_{[i]}$	$\Lambda_2(t_{[i]})$	$D\_\Lambda_2(t_{[i]})$	$H\_\Lambda_2(t_{[i]})$	$R_2(t_{[i]})$	$D\_R_2(t_{[i]})$	$H\_R_2(t_{[i]})$
76	0,0385	0,0000	0,1138	0,9623	0,8924	1,0000
149	0,0785	0,0000	0,1872	0,9245	0,8293	1,0000
236	0,1201	0,0000	0,2561	0,8868	0,7740	1,0000

The course of the estimation of reliable operation probability $R_2(t)$ and its 95% dependability interval ($D(t)$ – lower and $H(t)$ – upper limits) is put in Figure 2.

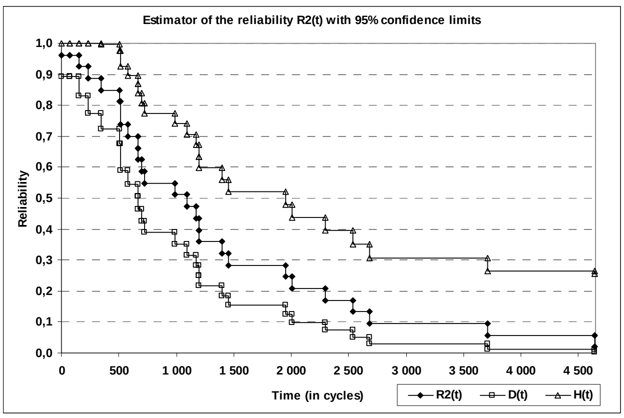

Figure 2. Reliability of the system and its 95% confidence intervals.

Last but not least, it is necessary to carry out the test which shows us whether the courses of $R_1(t)$ and $R_2(t)$.are identical/similar. From the operational point of view it is important to assess the impact of both types of failures (mechanical-software-process, or shortage of redundant cycles) they made on the reliability of the analysed system. The courses of both reliability functions $R_1(t)$ and $R_2(t)$ are put in Figure 3. Mathematically this issue is supposed to result in a statistical test.

$$H_0 : R_1(t) = R_2(t) \quad \times \quad H_1 : R_1(t) \neq R_2(t)$$

With the respect to the nature of the data the non-parametric Mantel´s test (N. Mantel: Evaluation of survival data and two new rank order statistics arising from its consideration. Cancer Chemother. Rep., 50, 163-170) was selected. When we apply the test to the data described above, we come to the conclusion that the impact of mechanical-software-process failures on system reliability is statistically a lot higher than the impact of the failures due to shortage of redundant cycles (pyrotechnical cartridges). This is also the case of the modified data.

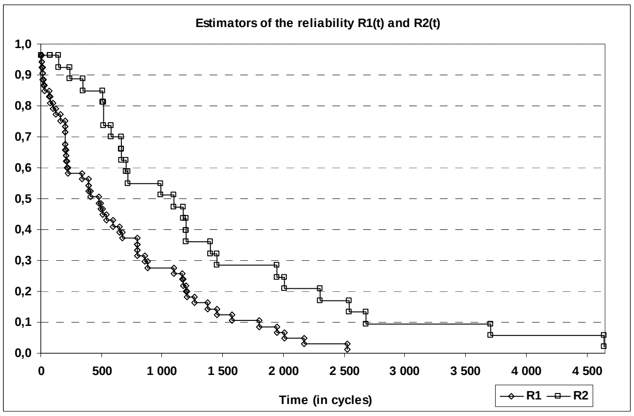

Figure 3. Reliability comparison of the R_1 *(t)* and R_2*(t)* functions.

The calculated parameters from NA test could be also displayed in the following graphical form. We speak about the $u_{(i)}$ value of test statistic for "Total Time on Test-plot" and the *i/n* value which represent the intensity of the event in number of sequences. The Figure 4 represents this dependence.

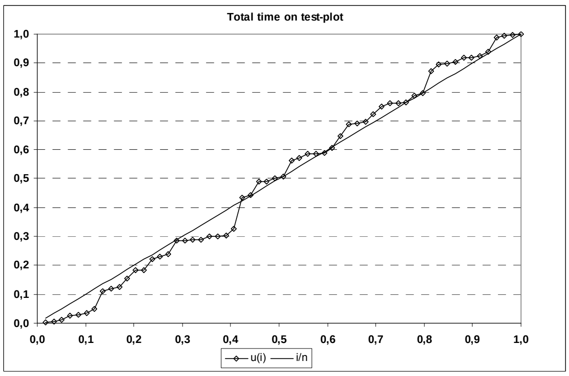

Figure 4. Total Time on Test-plot.

From this diagram it is remarkable that due to its form we can not confirm both the "IFR" (Increasing Failure Rate) and "DFR" (Decreasing Failure Rate) of the system.

6 CONCLUSION

In the paper we wanted to shed light on evaluating quite specific technical systems which, by all means, are present in different processes. The new contribution is in the application itself regarding the system assessed. Since they are specific both by their construction and the way they work, then the analysis of their properties might not be standard either. So far some ways of finding optimum construction arrangements in order to obtain a required level of dependability and function have been shown. The method we chose is aimed at verifiable evaluation of the real data obtained from operation by using appropriate methods. Both the mathematical model and the example of a practical application together with operational data reflect the behaviour of the real system. The graphs covering the courses help us to catch the behaviour of the system even more precisely. On the basis of this information it is quite easy to get reliability measures as well as readiness measures of the system where the parameter is discreet there and it is given by a number of cycles the system performs during its function.

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ON DETERMINATION OF SOME CHARACTERISTICS OF SEMI-MARKOV PROCESS FOR DIFFERENT DISTRIBUTIONS OF TRANSIENT PROBABILITIES

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ABSTRACT

There is a model of transport system presented in the paper. The possible semi - Markov process definitions are included. The system is defined by semi –Markov processes, while functions distributions are assumed. There are attempts to assess factors for other than exponential functions distributions. The paper consist discussion on Weibull and Gamma distribution in semi – Markov calculations. It appears that some forms of distribution functions makes computations extremely difficult.

1 INTRODUCTION

The reliability model of intermodal transport was presented during ESREL'06 conference (Zajac 2006b). The model is described by semi – Markov processes. During the presentation assumed, that, probabilities of transition between states were exponential. Complex technical systems are usually assumed, that probabilities of transition between states or sojourn times' probabilities are exponential. Lack of information, too little number of samples or inaccurate assessment of data may cause that such assumption is abused. In some cases, when exponential distribution is assumed, there is also possibility to assess factors according to different distributions (Weibull, Gamma, etc.). Probabilities of transition between states are one of the fundamental reliability characteristic. The paper includes example of determination of above mention characteristic for one of the phases of combined transportation systems reliability model.

2 TRANSHIPMENT PHASE CHARACTERISTIC

There are three methods to define semi – Markov processes (Grabski 2002, Grabski&Jazwinski 2003):

- by pair ($\boldsymbol{p}$, $\mathbf{Q}(t)$),

when: $\boldsymbol{p}$ – vector of initial distribution, $\mathbf{Q}(t)$ – matrix of distribution functions of transition times between states;

- by threes ($\boldsymbol{p}$, $\boldsymbol{P}$, $\mathbf{F}(t)$),

where: $\boldsymbol{p}$ – vector of initial distribution, $\boldsymbol{P}$ – matrix of transition probabilities, $\mathbf{F}(t)$ – matrix of distribution functions of sojourn times in state *i-th*, when *j-th* state is next;

- by threes ($\boldsymbol{p}$, $\boldsymbol{e}(t)$, $\mathbf{G}(t)$), where: ***p*** – vector of initial distribution, ***e***(t) – matrix of probabilities of transition between *i-th* and *j-th* states, when sojourn time in state *i-th* is x, **G**(t) – matrix of sojourn times distribution functions. For transshipment phase semi – Markov process is defined by (***p***, ***P***, **F**(t)). Phase of transshipment includes following states:

1. standby,
2. dislocation works,
3. transshipment,
4. preventive maintenance,
5. repair (after failure).

Activities which are involved into each of above states are described in papers (Zajac 2006a, Zajac 2007). The graph of state is presented in Figure 1.

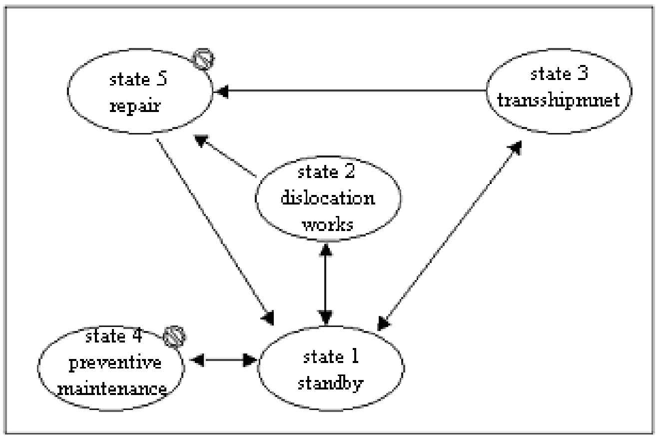


Figure 1. Graph of states in transshipment phase

3 CONDITIONS DETERMINATION FOR TRANSSHIPMENT PHASE RELIABILITY

Transshipment phase elements can stay in reliability states from the set S (0,1), where:
0 – unserviceability state,
1 – serviceability state.
Operation states takes values from the set T (1,2,3,4,5). Cartesian product of both states creates following pairs: (0,1), (1,1), (0,2), (1,2), (0,3), (1,3), (0,4), (1,4), (0,5), (1,5). The model allows for existence of following pairs, only: S_{1p} - (1,1), S_{2p} - (1,2), S_{3p} - (1,3), S_{4p} - (0,4), S_{5p} - (0,5).
Means of transport are in first operation state (standby) during time described by random variable ζ_{p1}. The distribution function of random variable is

$$F_{\zeta p1}(t) = P\{\zeta_{p1} \le t\},\ t \ge 0\,.$$

The time of the second state (dislocation) is described by ζ_{p2}. The distribution function of random variable takes form

$$F_{\zeta p2}(t) = P\{\zeta_{p2} \le t\},\ t \ge 0$$

The time of third state (transshipment) is described by ζ_{p3}, where distribution function of random variable is given by formula:

$$F_{\zeta p3}(t) = P\{\zeta_{p3} \le t\},\ t \ge 0$$

If the time of realization of preventive maintenance is known (and lasts γ_p), than the distribution function of sojourn time in the fourth state (preventive maintenance) is

$$F_{\gamma_p}(t) = P\{\gamma_p \le t\},\ t \ge 0\,.$$

Some of activities can be interrupted by failures. It was assumed, that time of work without failure in states 2-nd and 3-rd is described by η_{pi}, $i = 2,3$. The distribution function is given by formula:

$$F_{\eta pi}(t) = P\{\eta_{pi} \le t\},\ t \ge 0\,,\ i = 2,3.$$

If there is known time, when the system is broken down, and that time is given by χ_p, then the distribution function of state 5-th (repair) is

$$F_{\chi_p}(t) = P\{\chi_p \le t\},\ t \ge 0\,.$$

States 4-th and 5-th are states of unserviceability, however only state 5-th requires repair after failure. We assume that random variables ζ_{pi}, η_{pi} and χ_p are independent.

3.1 Kernel determination and the definition of semi – Markov process in transshipment phase

The phase of transshipment can be described by semi – Markov process $\{X(t): t \ge 0\}$ with the finite set of states $S_p = \{1, 2, 3, 4, 5\}$. The kernel of the process is described by matrix

$$\mathbf{Q_p}(t) = \begin{bmatrix} 0 & Q_{p12} & Q_{p13} & Q_{p14} & 0 \\ Q_{p21} & 0 & 0 & 0 & Q_{p25} \\ Q_{p31} & 0 & 0 & 0 & Q_{p35} \\ Q_{p41} & 0 & 0 & 0 & 0 \\ Q_{p51} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

Transshipments from 1-st state to 2-nd, 3-rd and 4-th can be described by

$$Q_{p12}(t) = p_{12}F_{\zeta_{p1}}(t),$$

$$Q_{p13}(t) = p_{13}F_{\zeta_{p1}}(t),$$

$$Q_{p14}(t) = p_{14}F_{\zeta_{p1}}(t)\,.$$

Transshipments from 2-nd state to 1-st and 5-th:

$$Q_{p21}(t) = p_{21}F_{\zeta_{p2}}(t),$$

$$Q_{p25}(t) = p_{25} F_{\zeta_{p2}}(t).$$

Transshipments from 3-rd state to 1-st and 5-th:

$$Q_{p31}(t) = p_{31} F_{\zeta_{p3}}(t),$$

$$Q_{p35}(t) = p_{35} F_{\zeta_{p3}}(t).$$

Transshipment from 4-th state to 1-st:

$$Q_{p41}(t) = P(\gamma_p < t) = F_{\gamma_p}(t).$$

Transshipment from 5-th state to 1-st:

$$Q_{p51}(t) = P(\chi_p < t) = F_{\chi_p}(t).$$

The vector $\boldsymbol{p} = [p_1, p_2, p_3, p_4, p_5]$ is initial distribution of the process. In this case vector takes values $\boldsymbol{p} = [1, 0, 0, 0, 0]$.
The matrix of transient probabilities is given by

$$\boldsymbol{P} = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} & 0 \\ p_{21} & 0 & 0 & 0 & p_{25} \\ p_{31} & 0 & 0 & 0 & p_{35} \\ p_{41} & 0 & 0 & 0 & 0 \\ p_{51} & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (2)$$

3.2 The transient probabilities

Transient probabilities are one of the most important characteristics of semi – Markov processes. They are defined as conditional probabilities

$$P_{ij}(t) = P\{X(t) = j \mid X(0) = i\}, \; i, j \in S \qquad (3)$$

Above probabilities obey Feller's equations (Grabski 2002, Grabski&Jazwinski 2003)

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \sum_{k \in S_0} \int^{t} P_{kj}(t-x) dQ_{ik}(x) \qquad (4)$$

Solution of that set of equations can be found by applying the Laplace – Stieltjes transformation. After that transformation the set takes form

$$\widetilde{p}_{ij}(s) = \delta_{ij}[1 - \widetilde{g}_i(s)] + \sum_{k \in S} \widetilde{q}_{ik}(s) \widetilde{p}_{kj}(s), \quad i, j \in S. \qquad (5)$$

In matrix notation this set of equation has form

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{g}}(s)] + \tilde{\mathbf{q}}(s)\,\tilde{\mathbf{p}}(s)\,, \tag{6}$$

hence

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{q}}(s)]^{-1}[I - \tilde{\mathbf{g}}(s)]\ . \tag{7}$$

Determination of transient probabilities requires finding of the reverse Laplace – Stieltjes transformation of the elements of matrix $\tilde{\mathbf{p}}(s)$.

4 DATA AND ASSUMPTIONS FOR CALCULATIONS

Data were collected in 2006 in one of the Polish containers terminals. The data includes information about numbers of transient between states during 50 succeeded days. Selected data are presented in Table 1.

Table 1. Selected data about time of states [h]

	state 1 stand by	state 2 dis-location	state 3 trans-shipment	state 4 preventive maintenance	state 5 repair
average	3.610	3.766	3.805	0.538	0.28
variance	0.387	0.022	0.020	0.046	0.287
min. value	1.862	3.482	3.482	0.3	0
max. value	4.411	4.042	4.020	1.1	2
dispersion	2.548	0.560	0.538	0.8	2

Collected data didn't allow for verifying probabilities distribution. The information gave possibility to estimate necessary parameters to assess factors for exponential, Weibull and Gamma distribution functions. Factors are presented in Table 2.

Table 2. Distribution parameters for different distribution function

	state 1	state 2	state 3	state 4	state 5
Parameter of exponential distribution					
λ	0.28	0.27	0.26	1.86	0.86
Parameters of gamma distribution					
α	9.260	152.059	181.992	13.968	4.120
λ	33.208	531.455	665.314	9.129	4.878
Parameters of Weibull distribution					
α	1.020	1.01	1.014	0.985	1.021
λ	0.269	0.26	0.266	1.538	0.833

At first calculation has been done with assumption, that transient probabilities are exponential. The distribution function of sojourn times and their Laplace – Stieltjes transformation respectively, take form

$$F_{w1}(t) = 1 - e^{-0,28t}\ ,\ f_{w1}{}^{*}(t) = \frac{0,28}{s + 0,28}\,,$$

$$F_{w2}(t) = 1 - e^{-0,27t}, \; f_{w2}{*}(t) = \frac{0,27}{s+0,27},$$

$$F_{w3}(t) = 1 - e^{-0,26t}, \; f_{w3}{*}(t) = \frac{0,26}{s+0,26},$$

$$F_{w4}(t) = 1 - e^{-1,86t}, \; f_{w4}{*}(t) = \frac{1,86}{s+1,86},$$

$$F_{w5}(t) = 1 - e^{-0,86t}, \; f_{w5}{*}(t) = \frac{0,86}{s+0,86}.$$

Then, kernel of the process is given by matrix

$$\boldsymbol{Q_p}(t) = \begin{bmatrix} 0 & 0,81(1-e^{-0,28t}) & 0,16(1-e^{-0,28t}) & 0,04(1-e^{-0,28t}) & 0 \\ 0,98(1-e^{-0,27t}) & 0 & 0 & 0 & 0,02(1-e^{-0,27t}) \\ 0,99(1-e^{-0,26t}) & 0 & 0 & 0 & 0,01(1-e^{-0,26t}) \\ (1-e^{-1,86t}) & 0 & 0 & 0 & 0 \\ (1-e^{-0,86t}) & 0 & 0 & 0 & 0 \end{bmatrix} \quad (8)$$

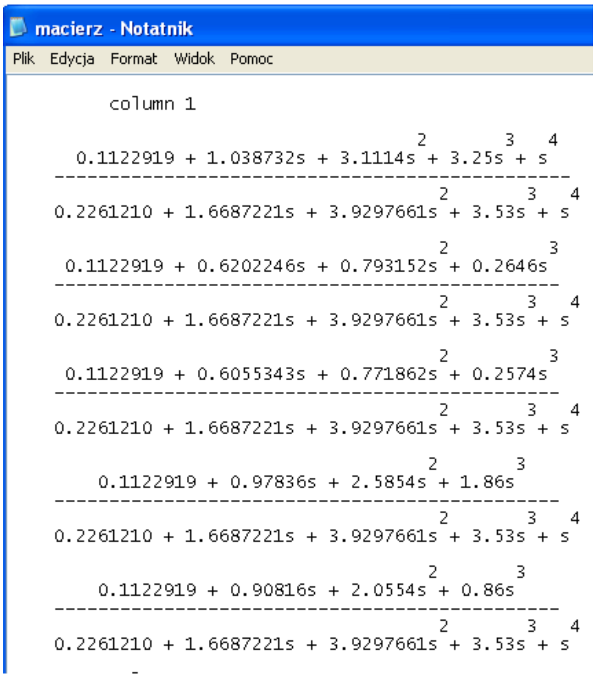
Matrices $\tilde{\mathbf{q}}(s)$ and $\tilde{\mathbf{g}}(s)$ have been determined according to equations (5) – (7). In considered example we obtain

$$\tilde{\mathbf{q}}(s) = \begin{bmatrix} 1 & 0,80\frac{0,28}{s+0,28} & 0,16\frac{0,28}{s+0,28} & 0,04\frac{0,28}{s+0,28} & 0 \\ 0,98\frac{0,27}{s+0,27} & 1 & 0 & 0 & 0,02\frac{0,27}{s+0,27} \\ 0,99\frac{0,26}{s+0,26} & 0 & 1 & 0 & 0,01\frac{0,26}{s+0,26} \\ \frac{1,86}{s+1,86} & 0 & 0 & 1 & 0 \\ \frac{0,86}{s+0,86} & 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

and

$$\tilde{\mathbf{g}}(s) = \begin{bmatrix} \frac{0,28}{s+0,28} & 0 & 0 & 0 & 0 \\ 0 & \frac{0,27}{s+0,27} & 0 & 0 & 0 \\ 0 & 0 & \frac{0,26}{s+0,26} & 0 & 0 \\ 0 & 0 & 0 & \frac{1,86}{s+1,86} & 0 \\ 0 & 0 & 0 & 0 & \frac{0,86}{s+0,86} \end{bmatrix} \tag{10}$$

According to (7), matrix $\tilde{\mathbf{p}}(s)$ is a result of multiplying of two matrices. Elements from first column of obtained matrix $\tilde{\mathbf{p}}(s)$ are shown on Figure 2.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column 1

$$\frac{0.1122919 + 1.038732s + 3.1114s^2 + 3.25s^3 + s^4}{0.2261210 + 1.6687221s + 3.9297661s^2 + 3.53s^3 + s^4}$$

$$\frac{0.1122919 + 0.6202246s + 0.793152s^2 + 0.2646s^3}{0.2261210 + 1.6687221s + 3.9297661s^2 + 3.53s^3 + s^4}$$

$$\frac{0.1122919 + 0.6055343s + 0.771862s^2 + 0.2574s^3}{0.2261210 + 1.6687221s + 3.9297661s^2 + 3.53s^3 + s^4}$$

$$\frac{0.1122919 + 0.97836s + 2.5854s^2 + 1.86s^3}{0.2261210 + 1.6687221s + 3.9297661s^2 + 3.53s^3 + s^4}$$

$$\frac{0.1122919 + 0.90816s + 2.0554s^2 + 0.86s^3}{0.2261210 + 1.6687221s + 3.9297661s^2 + 3.53s^3 + s^4}$$

Figure 2. First column of matrix $\tilde{\mathbf{p}}(s)$

Determination of transient probabilities requires finding of reverse Laplace – Stieltjes transformation of each element of the $\tilde{\mathbf{p}}(s)$ matrix. For elements of the first column of matrix $\tilde{\mathbf{p}}(s)$ reverse transformations are as follow:

$$P_{11} = 0,4966 - 0,0087 \cdot e^{-0,856t} + 0,0002 \cdot e^{-0,262t} + 0,5033 \cdot e^{-0,539t} + 0,0086 \cdot e^{-1,873t} \tag{11}$$

$$P_{21} = 0,4966 - 0,0209 \cdot e^{-0,856t} + 0,006 \cdot e^{-0,262t} - 0,5221 \cdot e^{-0,5390t} - 0,0014 \cdot e^{-1,8734t} \tag{12}$$

$$P_{31} = 0,4966 - 0,0118 \cdot e^{-0,856t} - 0,0301 \cdot e^{-0,262t} - 0,4769 \cdot e^{-0,539t} - 0,0014 \cdot e^{-1,873t} \tag{13}$$

$$P_{41} = 0,4966 - 0,0162 \cdot e^{-0,856t} + 0,0002 \cdot e^{-0,262t} + 0,7087 \cdot e^{-0,539t} - 1,1894 \cdot e^{-1,873t} \tag{14}$$

$$P_{51} = 0{,}4966 - 1{,}1838 \cdot e^{-0{,}856t} + 0{,}0003 \cdot e^{-0{,}262t} + 1{,}3486 \cdot e^{-0{,}539t} + 0{,}0073 \cdot e^{-1{,}873t} \quad (15)$$

All other transient probabilities (for columns 2 – 5) have been calculated similar way. On the basis of above results, characteristics of transient probabilities from state 1-st (standby) to other, both serviceability and unserviceability, states were calculated. The values of those probabilities stabilize after few days of work of the system. The transient probabilities functions to serviceability states are shown on Figure 3, to unserviceability states on Figure 4.

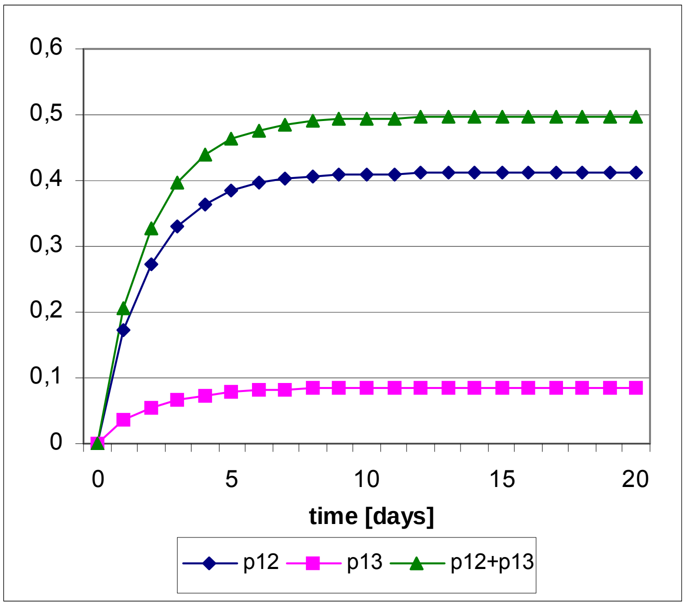

Figure 3. Graph of transient probabilities to serviceability states

For assumed conditions of phase of the system and distribution parameters, transient probability to serviceability states is: $p_{12} + p_{13} = 0.498$. Transient probabilities to unserviceability states achieve stable value for t = 4 days and don't change until t = 300. The calculation hasn't been done for greater values of t.

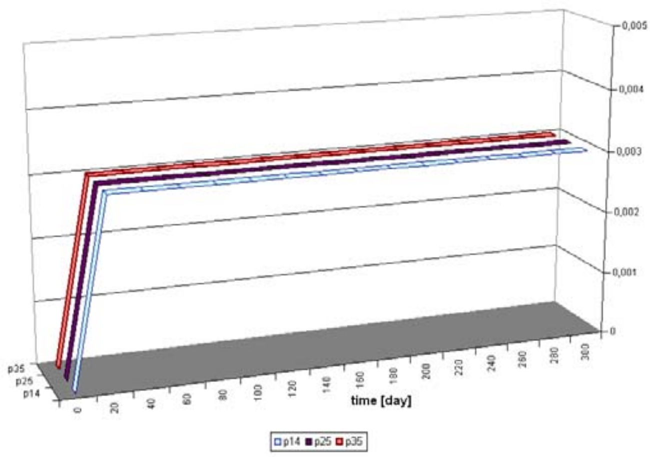

Figure 4. Graph of transient probabilities to unserviceability states

5 METHODS OF DETERMINING OF TRANSIENT PROBABILITIES FOR OTHER DISTRIBUTION FUNCTIONS

Gamma distribution is appropriate for describing age – hardening processes of technical object. There exists an assumption that sum of *n* independent random variables (with exponential distributions), with parameter λ, has two parameters gamma distribution (where α is shape parameter, and λ is scale parameter) (Jazwinski&Fiok 1990). Weibull distribution very often is used to object's durability modeling.
According to Table 2, collected data can be described by Weibull or gamma distributions. In the paper, for those distributions, only Laplace - Stieltjes transformation are presented. Sojourn times for Weibull distribution functions take form

$$F_{b1}(x) = 1 - e^{-0,269t^{1,02}},$$

$$F_{b2}(x) = 1 - e^{-0,26t^{1,01}},$$

$$F_{b3}(x) = 1 - e^{-2,66t^{1,014}},$$

$$F_{b4}(x) = 1 - e^{-1,538t^{0,985}},$$

$$F_{b5}(x) = 1 - e^{-0,833t^{1,021}}.$$

Derivative of Weibull distribution function (i.e. density function) is presented by

$$F'(t) = \lambda\alpha \cdot e^{-\lambda t^{\alpha}} \cdot t^{\alpha-1}$$

(16)

Laplace – Stieltjes transformations of Weibull distribution function can be obtained by using formula

$$f^*(t) = \int_0^\infty e^{-st} \cdot F'(t)\,dt = \int_0^\infty e^{-st}(1-e^{-\lambda t^{\alpha}})'\,dt = s\int_0^\infty e^{-st}(1-e^{-\lambda t^{\alpha}})\,dt - (1-e^{-\lambda\cdot 0^{\alpha}}) \tag{17}$$

Hence

$$f^*(t) = s\int_0^\infty e^{-st}(1-e^{-\lambda t^{\alpha}})\,dt = s\int_0^\infty e^{-st}dt - s\int_0^\infty e^{-st}\cdot e^{-\lambda t^{\alpha}}\,dt = 1 - s\int_0^\infty e^{-st}\cdot e^{-\lambda t^{\alpha}}\,dt \tag{18}$$

Using Maclaurin series for element "exp(-λt$^{\alpha}$)" we obtain Laplace – Stieltjes transformation of the Weibull distribution function

$$f^*(t) = \lambda\frac{\alpha\cdot\Gamma(\alpha)}{s^{\alpha}} - \frac{\lambda^2}{2!}\frac{2\alpha\cdot\Gamma(2\alpha)}{s^{2\alpha}} + \frac{\lambda^3}{3!}\frac{3\alpha\cdot\Gamma(3\alpha)}{s^{3\alpha}} - \ldots = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{n\alpha\cdot\Gamma(n\alpha)}{S^{n\alpha}}$$

(19)

For considered example, Weibull distribution Laplace – Stieltjes transformations take form, respectively

$$f_{b1}*(t)=\frac{0{,}2713}{s^{1,02}}-\frac{0{,}0774}{s^{2,04}}+\frac{0{,}0210}{s^{3,06}}-\frac{0{,}0059}{s^{4,08}}+\dots$$

$$f_{b2}*(t)=\frac{0{,}2611}{s^{1,01}}-\frac{0{,}0689}{s^{2,02}}+\frac{0{,}0183}{s^{3,03}}-\frac{0{,}0049}{s^{4,04}}+\dots$$

$$f_{b3}*(t)=\frac{2{,}6760}{s^{1,014}}-\frac{7{,}2619}{s^{2,028}}+\frac{19{,}845}{s^{3,042}}-\frac{54{,}489}{s^{4,056}}+\dots$$

$$f_{b4}*(t)=\frac{1{,}5284}{s^{0,985}}-\frac{2{,}3013}{s^{1,97}}+\frac{3{,}4391}{s^{2,96}}-\frac{5{,}1139}{s^{3,94}}+\dots$$

$$f_{b5}*(t)=\frac{0{,}8405}{s^{1,021}}-\frac{0{,}7216}{s^{2,042}}+\frac{0{,}6260}{s^{3,063}}-\frac{0{,}5468}{s^{4,084}}+\dots$$

According to equation (7), after determining of reverse Laplace - Stieltjes transformation of elements of the matrix $\tilde{\mathbf{p}}(s)$, transient probabilities can be calculated.
Gamma distribution is given by formula

$$F(t)=\frac{\Gamma_{\lambda t}(\alpha)}{\Gamma(\alpha)}. \tag{20}$$

Density function of gamma distribution has form

$$F'(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}.$$

(21)
Laplace – Stieltjes transformation can be obtain by using formula

$$f*(t)=\int_0^\infty e^{-st}\cdot F'(t)\,dt=\int_0^\infty e^{-st}\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}dt=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-(s+\lambda)t}t^{\alpha-1}dt \tag{22}$$

Taking into account equation

$$\int_0^\infty e^{-st}t^{a}dt=\frac{\Gamma(a+1)}{s^{a+1}}, \tag{23}$$

Laplace – Stieltjes transformation takes form

$$f*(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(s+\lambda)^{\alpha}}=\frac{\lambda^{\alpha}}{(s+\lambda)^{\alpha}}.$$

(24)

In this case sojourn times distribution functions and respective Laplace – Stieltjes transformations are as follows

$$F(t) = \frac{\Gamma_{33,21t}(9,26)}{\Gamma(9,26)},\ f^*(t) = \frac{33,21^{9,26}}{(s+33,21)^{9,26}},$$

$$F(t) = \frac{\Gamma_{531,46t}(152,06)}{\Gamma(152,06)},\ f^*(t) = \frac{531,46^{152,06}}{(s+531,46)^{152,06}},$$

$$F(t) = \frac{\Gamma_{665,31t}(181,99)}{\Gamma(181,99)},\ f^*(t) = \frac{665,31^{181,99}}{(s+665,31)^{181,99}},$$

$$F(t) = \frac{\Gamma_{9,13t}(13,97)}{\Gamma(13,97)},\ f^*(t) = \frac{9,13^{13,97}}{(s+9,13)^{13,97}},$$

$$F(t) = \frac{\Gamma_{4,88t}(4,12)}{\Gamma(4,12)},\ f^*(t) = \frac{4,88^{4,12}}{(s+4,88)^{4,12}}.$$

Using of equation (7) and calculating reverse Laplace - Stieltjes transformations transient probabilities can be obtained. In the case of gamma distribution, there are numerical problems with calculating of incomplete gamma functions values. Moreover, even values of gamma function for arguments larger than 50 cannot be easy obtained. Used software tools (SciLab 4.1.1 and Derive 6.1) don't allow for calculating such great values.

6 CONCLUSIONS

1. Semi - Markov processes allow for estimate basic reliability characteristics like availability or transient probabilities for systems, where distributions functions are discretional.
2. Usages of distribution functions other than exponential in case of semi Markov processes causes that further calculations are very complicated.
3. There is no easy available software which allow for calculations connected with semi – Markov processes. Because of that, profits from usage of semi – Markov processes are limited.
4. Lack of information about type of distribution and routine assessment of exponential distribution can bring not accurate assumptions and consequently false results.

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MAINTENANCE POLICY FOR DETERIORATING SYSTEM WITH EXPLANATORY VARIABLES

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ABSTRACT

This paper discusses the problem of the optimization of maintenance threshold and inspection period for a continuously deteriorating system with the influence of covariates. The deterioration is modeled by an increasing stochastic process. The process of covariates is assumed to be a temporally homogeneous finite-state Markov chain. A model similar to the proportional hazards model is used to represent the influence of the covariates. Parametric estimators of the unknown parameters are obtained by using Least Square Method. The optimal maintenance threshold and the optimal inspection interval are derived to minimize the expected average cost. Comparisons of the expected average costs under different conditions of covariates and different maintenance policies are given by numerical results of Monte Carlo simulation.

1 INTRODUCTION

Optimal replacement problems for deteriorating systems have been intensively studied in the past decades by a number of researchers (for instance, Aven & Jensen (1999), Wang (2002) and Wang & Pham (2006), van Noortwijk (2009)). As far as continuously deteriorating systems are considered, most of the attention has been focused on static environment and on monotonic increasing deterioration systems, with periodic or non-periodic inspection. Various stochastic processes have been proposed to represent the degradation or wear process (e.g. Grall et al. (2002), Bérenguer et al. (2003) and van Noortwijk (2009)). Recently more interest and attention has been given to two approaches. One approach is to deal with degradation models including explanatory variables (covariates). These variables describe the dynamic environment; in the experiments of life science and engineering, they are often expressed by the proportional hazards model (Newby (1994), Singpurwalla (1995), Meeker & Escobar (1998) and Lawless & Crowder (2004)). Bagdonavičius & Nikulin (2000) propose a method to model an increasing degradation by a gamma process which includes time-dependent covariates. Makis & Jardine (1992) consider an optimal replacement problem for a system with stochastic deterioration which depends on its age and also on the value of covariates. Kharoufeh & Cox (2005) deal with a degradation-based procedure to estimate lifetime distribution, where the single-unit system is exposed to a stochastically evolving environment characterized by a stationary continuous-time Markov chain. Meeker et al. (1998)

describe a degradation reliability model, where the dynamical temperature is represented by an accelerated model. The other approach is to consider a non-monotonic deteriorating system with increasing tendency (Newby & Dagg (2002), Newby & Dagg (2003), Newby & Barker (2006), Barker & Newby (2009)). Barker and Newby (2009) study the problem of optimal inspection and maintenance policy for a non-monotonic system. They use the last exiting time from a critical set instead of the first hitting time to determine the optimal policy.

In this paper we focus on the optimal policy of periodic inspection/replacement for a monotonic deteriorating system with explanatory variables (covariates), in which the covariate process is supposed to be a temporally homogeneous Markov chain. The influence of the covariates on degradation is considered by a multiplicative exponential function. The system is supposed to be failed when the system state crosses a fixed threshold known as failure threshold. The purpose is to propose an optimal maintenance policy for the considered system in order to minimize the global long-run expected average maintenance cost per time unit.

The other particularity of this paper is that we compare the maintenance cost under following cases: (1) the optimization when the covariates are defined as a Markov chain; (2) the optimization when the covariates $Z_n = i\ (i = 1,2,3)$ are fixed; (3) the weighted mean of the optimal costs for each $Z_n = i\ (i = 1,2,3)$ weighted by the steady-state probabilities. All results are illustrated by a Monte Carlo study.

The structure of the paper is as follows. In Section 2 we model the degradation process by a stochastic process, where the influence of the covariates is modeled by a multiplicative exponential function. In Section 3 we study the maintenance optimization problem. Finally, we compare the expected average maintenance costs per unit time for the different cases mentioned above.

2. STOCHASTIC DETERIORATION PROCESS

In this section, we consider a single-unit replaceable system in which an item is replaced with a new one, either at failure or at preventive replacement.

2.1. Deterioration model without covariates

The degradation of the system is represented by a continuous-state stochastic process $D(t)$ with initial degradation level $D(0) = 0$. We also suppose that the increment of the system can be modeled by a continuous nonnegative-valued process $X(t)$ with exponential distribution, that is, the random increment $D(s) - D(t)$ subjects to an exponential distribution with mean $\lambda(s-t)$.

Suppose that the deterioration can be observed at each time unit t_k ($k = 1,2,\cdots$), the discrete observed stochastic processes are defined as follows: $D_k = D(t_k)$ and $X_k = X(t_k)$.
The process D_n is defined as:

$$D_n = D_{n-1} + X_n \,, \tag{1}$$

where X_n are random variables of exponential distribution with mean μ_n, denoted by $X_n \sim \varepsilon(1/\mu_n)$.

Denote $\lambda_i = \dfrac{1}{\mu_i}$, it can be proved (see Appendix) that if $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \varepsilon(\lambda_i)$ are independent, then:

(1) If $\lambda_i = \lambda$ are the same parameters, then X will be an Erlang distributed variable with parameters (n, λ) (Soong (2004));

(2) If $\lambda_i \neq \lambda_j$ for $i \neq j$, when $y > 0$, the density probability function will be:

$$f_X(y) = \left(\prod_{i=1}^{n} \lambda_i\right) \sum_{i=1}^{n-1} \frac{\exp(-\lambda_i y) - \exp(-\lambda_n y)}{\prod_{1 \le j \le n; j \ne i} (\lambda_j - \lambda_i)} .$$

Since the degradation is calculated as $D_n = \sum_{i=1}^{n} X_i$ with independent exponentially distributed increment X_i, then we can obtain the distribution function, the density function of the deteriorating process D_n by the above results.

2.2. Modeling the influence of covariates on degradation

The covariate process $Z = \{Z(t), t \ge 0\}$ is assumed to be a temporally homogeneous discrete Markov process with finite states $S = \{1, 2, \cdots, K\}$, here S describes the states of the dynamic environment. Suppose that covariates are available only at each time unit t_k $(k = 1, 2, \cdots)$, and the covariates at time t_k are defined by $Z_k = Z(t_k)$

Let $P_{ij}(k) = P(Z_{k+1} = j \mid Z_k = i)$ be the transition probabilities of process $\{Z_k, k = 1, 2, \cdots\}$. The filtration $\Im_t = \sigma\{Z_s : s \le t\}$ denotes the history of the covariates. Since the process Z is a finite temporally homogeneous Markov process, so $P_{ij}(k) = P_{ij}$ does not depend on k for all $i, j \in S$. We denote by $P = (P_{ij})$ the transition matrix.

We assume that the increment of the degradation at time t_n depends only on the covariates at that time. We shall denote by D_n the observed process at time t_n, defined as:

$$D_n = D_{n-1} + X_n(Z_n) , \tag{2}$$

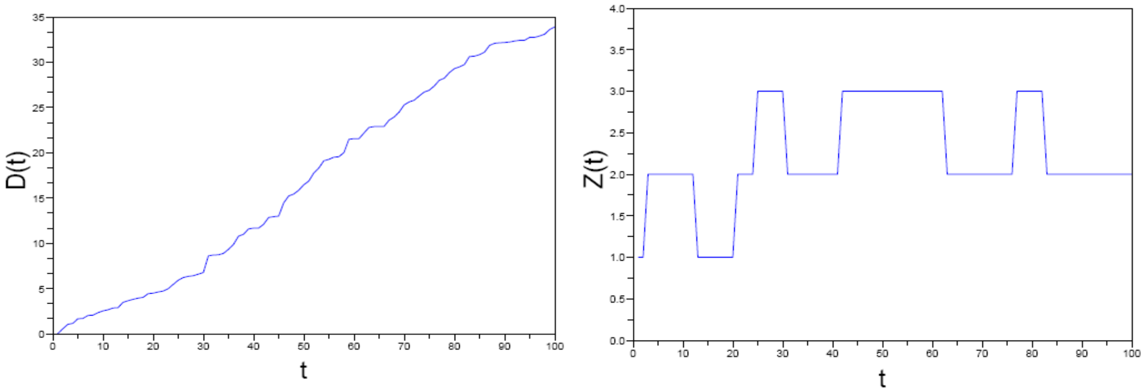
where $X_n(Z_n)$ are exponential distributed with mean parameters $\mu_n(Z_n)$. So $\{D_n, Z_n\}$ is a non-homogeneous Markov process in the sense that the transition probabilities satisfy the following equality:

$$\begin{aligned} &P(D_n \le y, Z_n = j \mid D_{n-1} = x, D_{n-2} = x_{n-2}, \cdots, \quad D_1 = 0; Z_{n-1} = i, Z_{n-2} = z_{n-2}, \cdots, Z_1 = z_1) \\ &= P(D_n \le y, Z_n = j \mid D_{n-1} = x, Z_{n-1} = i) . \end{aligned}$$

To describe precisely the influence of the covariates $Z_n = z_n$ on X_n, similar to the proportional hazards model proposed by Cox (1972), we suppose that the parameters $\mu_n(Z_n)$ depend on Z_n as follows:

$$\mu_n(Z_n) = \mu_0 \exp(\beta_1 1_{\{Z_n = 1\}} + \cdots + \beta_k 1_{\{Z_n = K\}}) = \mu_0 \exp(\beta_{Z_n}) , \tag{3}$$

where $\beta = (\beta_1, \cdots, \beta_K)$ is a regression parameter. Considering the symmetrical property of β, without loss of generality, in what follows, we assume that $\beta_1 \le \cdots \le \beta_K$.



(a) (b)

Fig.1 An example of the non-maintained degradation process (a) and the corresponding covariates process (b)

The distribution function and the density function of the increment under the condition of $Z_n = z_n$ are calculated in the same way as before. Then the distribution F_n of $D_n = \sum_{i=1}^{n} X_i(Z_i)$ can be derived using the method of convolution and the total probability formula.

Example 1 An example of degradation for 100 days is given in Figure 1, where Z_n is a 3-state Markov chain with transition matrix $P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0.02 & 0.95 & 0.03 \\ 0.00 & 0.05 & 0.95 \end{pmatrix}$ (corresponds to a steady-state distribution (0.3, 0.5, 0.2)), initial state $Z_0 = 1$, $\beta = (0.2, 0.5, 1)$, the baseline mean parameters $\mu_0 = 0.2$.

For the covariates with initial state $Z_0 = 1$, denoted by $\pi^n = (\pi_1^n, \pi_2^n, \cdots, \pi_K^n)$ the distribution of the covariates Z_n with $\pi_i^n = P(Z_n = i \mid Z_0 = 1)$ the conditional distribution of Z_n under the condition of $Z_0 = 1$. We have

$$(\pi_1^n, \pi_2^n, \cdots, \pi_K^n) = (1, 0, \cdots, 0) P^n,$$

and $\lim_{n \to +\infty} \pi_i^n = \pi_i$, where π_i is the steady-state distribution of the Markov chain.

In this case, the distribution F_n of $D_n = \sum_{i=1}^{n} X_i(Z_i)$ will be:

$$F_{n+1}(x) = \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} P\left(\sum_{k=1}^{n+1} X_k(i_{k-1}) \le x \right) P_{1 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}.$$

When the covariates form a steady-state Markov chain, each replacement makes the system restart from its new state $D_0 = 0$ and the covariates Z_n follow their trajectory. Let us denote T_n the instant of replacement (preventive or corrective), then the variables (D_t, Z_t) and (D_{t+T_n}, Z_{t+T_n}) have the same distribution, therefore the trajectory of the degradation does not depend on the history before the replacement. Henceforth, the deterioration process is a renewal process.

2.3 Parametric estimation using least square method

In this section, we use the least square method to estimate the unknown parameters. The data sample is all the degradation data observed before failure, i.e., before the beyond of the critical threshold L.

Since in general case, the mean degradation at time t_k is equal to

$$E(D_n) = \sum_{i=1}^{n} E(\Delta D_i) = \sum_{i=1}^{n}\sum_{j=1}^{K} E(\Delta D_i | Z_i = j) P(Z_i = j) = \sum_{i=1}^{n}\sum_{j=1}^{K} E(\Delta D_i | Z_i = j)\pi_j^i$$

Because of the difficulty of calculating the distribution π_j^i of Z at time t_i, and the case that π_j^i can be approximated by π_j when i is large enough, we can approximate the degradation mean as follows:

$$E(D_n) = \sum_{i=1}^{n}\sum_{j=1}^{K} E(\Delta D_i | Z_i = j)\pi_j = \sum_{i=1}^{n}\sum_{j=1}^{K} \Delta m_i \exp(\beta_j)\pi_j = \left(\sum_{j=1}^{K} \exp(\beta_j)\pi_j\right)\sum_{i=1}^{n} \Delta m_i$$

Therefore the Least Square Estimator $\hat{\theta} = (\mu_0, \beta_1, \beta_2, \beta_3)$ is defined by

$$\hat{\theta} = \arg\min_{\theta} Q_n(\theta), \tag{4}$$

where $Q_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} (D_i - E(D_i))^2$.

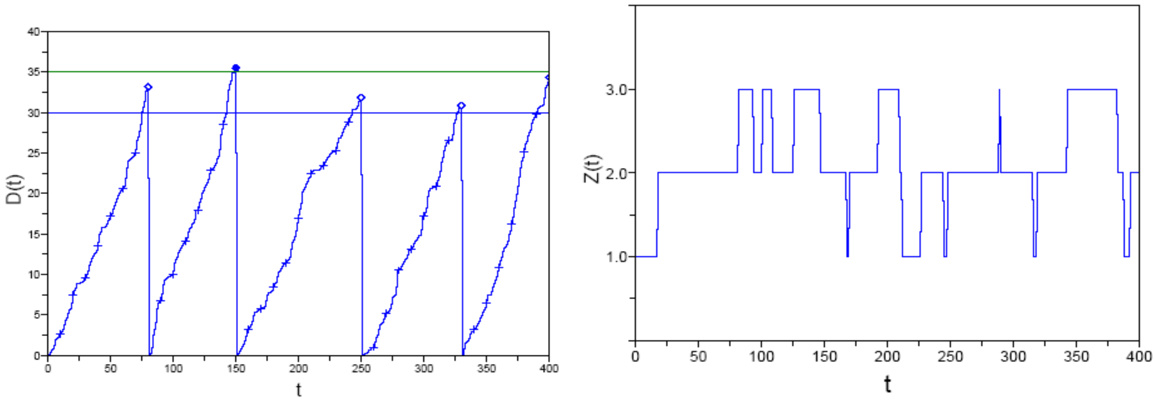
3. CONDITION-BASED PERIODIC MAINTENANCE MODEL

In this section, we study the optimal periodic maintenance policy for the deteriorating system described in Section 2.

Suppose that the system is a monotonically deteriorating stochastic system with initial state $D_0 = 0$, and the state can exclusively be monitored by inspections at the periodic times $T_k = k\tau$, where $\tau \in \aleph$ is the inspection interval. We now give some assumptions under which the model is studied.

(1) Inspections are perfect in the sense that they reveal the true state of the system and the explanatory variables.
(2) The system state is only known at inspection times and all the maintenance actions take place only at inspection times and they are instantaneous.
(3) Two maintenance operations are available only at the inspection time: preventive replacement and corrective replacement.
(4) The maintenance actions have no influence on the covariate process.

3.1 Maintenance decision



(a) (b)

Fig.2 An example of maintained deterioration system (a) and the corresponding covariate process (b)

Suppose that the system starts with $D_0 = 0$, and is perfectly inspected at periodic times $\Pi = \{\tau, 2\tau, \cdots\}$, ($\tau \in \aleph$), the states are only known at inspection times, and maintenance actions are instantaneous. We define a failure threshold L and a preventive maintenance threshold L_p ($L_p \le L$).

If at inspection time $T_k = k\tau$ we have $D_{k\tau} < L_p$, then three exclusive events may occur at time T_{k+1}:

E1: $D_{(k+1)\tau} \ge L$: which means that the system fails at time $t \in (k\tau, (k+1)\tau]$ and it will be correctively replaced at time $(k+1)\tau$. Costs of corrective replacement C_F as well as a cumulative cost $C_d \times d$ corresponding to the 'inactivity' time have to be considered, where $d = (k+1)\tau - t$ is the cumulated 'inactivity' time.

E2: $D_{(k+1)\tau} \in [L_p, L)$: means that there is no failure in interval $t \in [k\tau, (k+1)\tau]$, however the degradation level is greater than the preventive threshold L_p at time $(k+1)\tau$. So a preventive replacement action takes place at $(k+1)\tau$ which induces a preventive maintenance cost.

E3: $D_{(k+1)\tau} < L_p$: means that the degradation level is always lower than L_p, so there is no replacement action at $(k+1)\tau$, we only have to take into account an inspection cost and the decision time is postponed to $(k+1)\tau$.

An example of a maintained system is given in Figure 2, where the preventive threshold $L_p = 30$, the corrective threshold $L = 35$, and $\tau = 5$, other parameters are the same as in Example 1.

3.2 Calculation of the maintenance cost

Each action of inspection and replacement results in a unit cost. Let C_i, C_p, C_F denote respectively the unit cost of inspection, preventive replacement and corrective replacement. We also consider the cost for 'inactivity' with per unit time cost C_d.

Then the cumulative maintenance cost in $(0,t]$ is:

$$C(t) = C_i N_i(t) + C_p N_p(t) + C_F N_F(t) + C_d d(t), \tag{5}$$

where $N_i(t)$ (respectively $N_p(t), N_F(t)$) is the number of inspections (respectively number of preventive replacements, number of corrective replacements) from 0 to t.

The expected average cost is calculated as follows:

$$EC_\infty = \lim_{t\to\infty} \frac{EC(t)}{t} = \lim_{t\to\infty}\left(\frac{C_i E(N_i(t)) + C_p E(N_p(t)) + C_F E(N_F(t)) + C_d E(D(t))}{t} \right). \tag{6}$$

When the stochastic process (D,Z) is a regenerative process as we stated above, we can calculate the expected cost per time unit as follows:

$$EC_\infty(Z) = \frac{E(V(L(Z)))}{E(L(Z))} \ , \tag{7}$$

where $E(V((Z))$ and $E(L((Z))$ are respectively the expected cost and expected length of a renewal cycle.

Considering the three above exclusive events E1, E2, E3, denote by V_k (respectively L_k) the total cost (respectively length) from time T_k to the time when the system is replaced.

Since the total cost V_k (respectively the total length L_k) is a combination of the cost (respectively the length) in time interval $[T_k, T_{k+1})$ and the cost (respectively the length) after T_{k+1}, we calculate the total maintenance cost $V = V(Z)$ and the length of a renewal cycle $L = L(Z)$ by following iterate method:

$$V_k = (C_i + C_F + C_d d(t))1_{\{E_1\}} + (C_i + C_p)1_{\{E_2\}} + (C_i + V_{k+1})1_{\{E_3\}}, \tag{8}$$

$$L_k = \tau 1_{\{E_1\}} + \tau 1_{\{E_2\}} + (\tau + L_{k+1})1_{\{E_3\}}, \tag{9}$$

and the expectation will be

$$v_k = E(V_k) = (C_i + C_F)P(E_1) + C_d E(d(t)1_{\{E_1\}}) + (C_i + C_p)P(E_2) + C_i P(E_3) + E(V_{k+1}1_{\{E_3\}}), \tag{10}$$

$$l_k = \tau\,(P(E_1) + P(E_2) + P(E_3)) + E(L_{k+1}1_{\{E_3\}}). \tag{11}$$

The optimization problem is to find the value of τ^* and L_p^* minimizing the expected long-run average maintenance cost:

$$(L_p^*, \tau^*) = \arg\min_{(L_p,\tau)} EC_\infty(Z). \tag{12}$$

3.3 Description of the optimization procedurec

We now give a formal description of the optimization procedure.

For a given L_p and τ, we estimate the expected maintenance cost as follows.

Step 0: Initialization.

At time $t_0 \equiv 0$, let $D_0 \equiv 0, Z_0 \equiv 1$.

Step 1: Generation of the trajectory of the degradation process

Table1. Estimation of the parameters: mean and standard deviation (within parentheses)

Sample size n	$\hat{\mu}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
100	0.180 (0.046)	0.186 (0.031)	0.480 (0.048)	0.985 (0.058)
200	0.182 (0.039)	0.189 (0.022)	0.485 (0.040)	0.989 (0.039)
500	0.183 (0.030)	0.190 (0.011)	0.488 (0.038)	0.993 (0.040)
1000	0.184 (0.031)	0.196 (0.012)	0.492 (0.043)	0.996 (0.042)

(1) Simulate a trajectory of the covariate process $\{Z_n\}$with the initial state $Z_0 \equiv 1$ and transition matrix P.
(2) Generate a trajectory of the degradation process conditional upon the trajectory $\{Z_n\}$.

Step 2: Estimation of the maintenance cost conditional upon covariates above

Estimate the total maintenance cost and the total length based on N renewal cycles (N large enough). In each renewal cycle, the maintenance decision is taken according to the three exclusive events (E1)-(E3) mentioned above, the maintenance cost and the maintenance length are calculated as (8) and (9).

Step 3: Estimation of the expected average cost for a stationary Markov chain.

Repeat Step 0-Step 2 to derive the total maintenance cost and the total length for a stationary Markov chain, then calculate the expected average maintenance cost as (6) or (7) indicated. The repetition does not be stoped until the convergence of the expected average maintenance cost.

After the calculation of the expected average maintenance cost by the procedure above for each L_p and τ, we obtain a maintenance cost matrix with respect to L_p and τ, then the optimal decision $\left(L_p^*, \tau^*\right)$ can be derived based on the criteria (12)

4. NUMERICAL RESULTS

4.1 Numerical results for parametric estimators

We apply the least square estimator for a degradation sample described in Section 2. The estimator is defined by (4).

We simulate $N = 1000$ samples with various sample size n. For each sample we give the estimator of the unknown parameter $\theta = (\mu_0, \beta_1, \beta_2, \beta_3)$ for $\theta_0 = (0.2, 0.2, 0.5, 1)$. In Table 1 we summarized the results for Least Square Estimation. For each estimator we give the empirical mean and the empirical standard deviation based on the N estimators we obtained.

The results in Table 1 show that the least square method has a good behavior to estimate the unknown parameters.

4.2 Numerical results for optimal periodic maintenance

In this section we give numerical results of our maintenance optimization problem. The deteriorating system is the system defined in Example 1. We consider four different cases of unit maintenance cost:

Table 2. The optimal preventive threshold, the optimal inspection period and the expected average maintenance cost with periodical inspection

Covariates	(L_p^*, τ^*, C^*) (Case 1)	(L_p^*, τ^*, C^*) (Case 2)	(L_p^*, τ^*, C^*) (Case 3)	(L_p^*, τ^*, C^*) (Case 4)
Z general	(12, 60, 1.0607)	(12, 54, 1.1238)	(19, 63, 1. 5923)	(11, 80, 2. 6891)
Z=1	(21, 120, 0.5158)	(21, 114, 0. 5263)	(23, 123, 0. 9292)	(21, 120, 1. 3016)
Z=2	(20, 87, 0.7183)	(18, 81, 0.7901)	(19, 90, 1. 2955)	(19, 90, 1.7511)
Z=3	(18, 51, 1.2509)	(16, 48, 1.3437)	(17, 51, 2.2431)	(19, 54, 2. 9740)
Mean cost	0.8376	0.90344	1.50657	2.028111

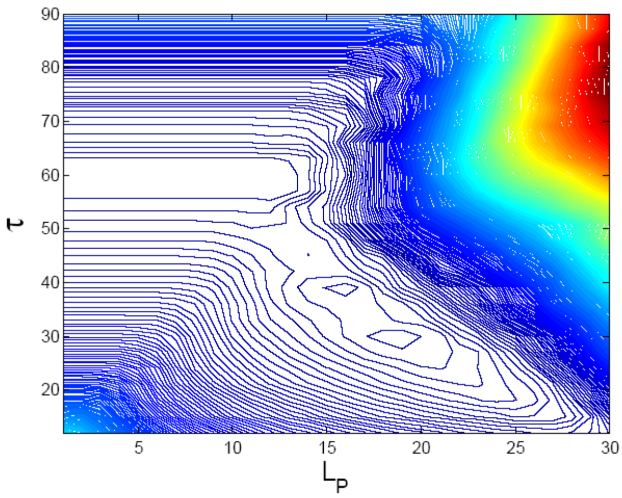


Fig.3 The iso-level curves of EC_∞ for $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for a deteriorating system.

Case 1 (Inexpensive unavailability): $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$;

Case 2 (Expensive unavailability): $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 150$;

Case 3 (Expensive PR): $C_i = 10$, $C_p = 100$, $C_F = 100$ and $C_d = 50$;

Case 4 (Expensive inspection): $C_i = 100$, $C_p = 50$, $C_F = 100$ and $C_d = 50$.

For each case of maintenance cost, we compare the following three values.

(1) Optimal maintenance cost when Z_n come from a general Markov chain;

(2) Optimal maintenance cost when Z_n is fixed to $Z_n = i$ $(i = 1,2,3)$;

(3) Weighted mean of the optimum cost for $Z_n = i$ $(i = 1,2,3)$ with weight given by the steady-state probability:

$$\overline{EC}_\infty = \sum_{k=1}^{3} EC_\infty^*(Z = k)\pi_k$$

Results in Table 2 summarize the results of optimization for a deteriorating system with different maintenance costs. The iso-level curves of expected long-run average cost EC_∞ with $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for such a deteriorating system is depicted in Figure 3,

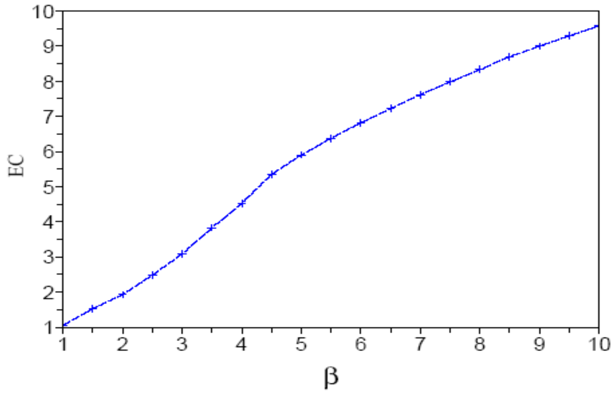


Fig.4 The curve of $EC^*_\infty(\beta)$ for $C_i = 10$, $C_p = 50$, $C_F = 100$ and $C_d = 50$ for a deteriorating system.

where the optimal parameter values are $L^*_p = 12$, $\tau^* = 60$. These optimal values lead to the optimal expected average cost $EC^* = 1.0607$.

In all cases of the different unit maintenance cost (expensive or inexpensive), the optimal expected average cost under the condition of $Z = 1$ $(\beta = \beta_1)$ are the smallest one. Indeed, for $Z = 1$, the degradation increments are smaller in comparison with other cases. The cost for $Z = 2$ $(\beta = \beta_2)$ is higher than that of $Z = 1$, and cost obtained for $Z = 3$ $(\beta = \beta_3)$ is the highest one. As a consequence, the parameter β can be used to express the influence of the dynamic environment on the deteriorating system.

In order to reveal the way that maintenance cost is influenced by the system parameters β, using the symmetrical property, the optimal expected average cost is computed for various value of β_3 with fixed β_1 and β_2. The result appears in Fig 4. We see that the optimal expected average maintenance cost is an increasing function of the system parameter β_3. In fact, since the regression parameter β expresses the influence of the dynamic environment, the expected average maintenance cost under the worst environment has higher cost than that of better environment.

The expected average maintenance cost for system with a Markov chain is always greater than the weighted mean of the optimal costs for the three static environments, since we have less information for the deteriorating system under a Markov chain than under static environment. The weighted mean of the optimal costs gives the lower bound for the cost of a deteriorating system.

5. Conclusion

This paper deals with the periodic inspection/replacement policy for a monotonic deteriorating system with covariates, where the covariates form temporally homogenous finite states Markov chain. We use a method similar to the proportional hazards model to induce the influence of dynamic covariates on the degradation of the system. Expected average cost is estimated and optimum periodic inspection/replacement policies are derived for different maintenance cost per unit. The numerical results show that the optimal average cost is an increasing function of the regression parameters β. Therefore the parameters β can be used to express the effect of the environment. The relationship between the optimal cost in the case of a covariates Markov chain and a combination of fixed covariates (with stead-state distribution) shows that the first is greater than the later. It will be interesting to apply the methods exposed in this paper on non-monotonic systems.

Appendix: The distribution of the increments of increasing degradation system

We prove the conclusion by mathematical induction. For $n = 2$, we have

$$\begin{aligned} f_2(y) &= f_{X_1+X_2}(y) \\ &= \int_0^y \lambda_1 \exp(-\lambda_1 x)\lambda_2 \exp(-\lambda_2(y-x))dx \\ &= \lambda_1\lambda_2 \frac{\exp(-\lambda_1 y)-\exp(-\lambda_2 y)}{\lambda_2-\lambda_1} \end{aligned}$$

Suppose that for $y > 0$, $f_n(y) = \left(\prod_{i=1}^{n}\lambda_i\right)\sum_{i=1}^{n-1}\frac{\exp(-\lambda_i y)-\exp(-\lambda_n y)}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}$

then

$$f_{n+1}(y) = \int_0^y f_n(x) f_{X_{n+1}}(y-x)dx$$

$$= \left(\prod_{i=1}^{n}\lambda_i\right)e^{-\lambda_{n+1}y}\int_0^y\left(\sum_{i=1}^{n-1}\frac{e^{-(\lambda_i-\lambda_{n+1})x}-e^{-(\lambda_n-\lambda_{n+1})x}}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}\right)dx$$

$$= \left(\prod_{i=1}^{n+1}\lambda_i\right)\left[\sum_{i=1}^{n-1}\frac{\exp(-\lambda_i y)-\exp(-\lambda_{n+1}y)}{\prod_{1\le j\le n+1; j\ne i}(\lambda_j-\lambda_i)}+\frac{\exp(-\lambda_n y)-\exp(-\lambda_{n+1}y)}{\lambda_{n+1}-\lambda_n}\sum_{i=1}^{n-1}\frac{1}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}\right].$$

Setting $A = \sum_{i=1}^{n-1}\frac{1}{\prod_{1\le j\le n; j\ne i}(\lambda_j-\lambda_i)}$, we have

$$A = \frac{\sum_{i=1}^{n-1}(-1)^{i+1}\prod_{k>j\ne i}(\lambda_k-\lambda_j)}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)} = \frac{1}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)}\left(\begin{vmatrix} 1 & & 0 & 0 \\ 1 & & 1 & 1 \\ \lambda_1 & & \lambda_{n-1} & \lambda_n \\ & \ddots & & \\ \lambda_1^{n-2} & & & \lambda_n^{n-2} \end{vmatrix} + \cdots + \begin{vmatrix} 0 & & 1 & 0 \\ 1 & & 1 & 1 \\ \lambda_1 & & \lambda_{n-1} & \lambda_n \\ & \ddots & & \\ \lambda_1^{n-2} & & & \lambda_n^{n-2} \end{vmatrix}\right)$$

$$
= \frac{(-1)^{n+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ \lambda_1 & \lambda_{n-1} & \lambda_n \\ & \ddots & \\ \lambda_1^{n-2} & & \lambda_n^{n-2} \end{vmatrix}}{\prod\limits_{1 \le i < j \le n} (\lambda_j - \lambda_i)} = \frac{1}{(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n) \cdots (\lambda_{n-1} - \lambda_n)},
$$

so we obtain the conclusion:

$$
f_{n+1}(y) = \left(\prod_{i=1}^{n+1} \lambda_i \right) \sum_{i=1}^{n} \frac{\exp(-\lambda_i y) - \exp(-\lambda_n y)}{\prod\limits_{1 \le j \le n+1;\, j \ne i} (\lambda_j - \lambda_i)}.
$$

□

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PREVENTIVE MAINTENANCE WITH IMPERFECT REPAIRS OF A SYSTEM WITH REDUNDANT OBJECTS

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ABSTRACT

An object ability to realise tasks may be restored by repairing only failed components. This is called imperfect repair as the object is not as good as new after such a repair. Preventive replacement is an example of imperfect repair as well. The advantage of such maintenance is that it enables controlling a reliability level of a system. Sets of objects' components which should be replaced are derived on a basis of statistical diagnosing with use of data about components failures. The acceptable level of a failure risk while executing transportation tasks has been taken as a criterion of choosing elements to be replaced. An algorithm of selecting components for preventive replacement has been developed. It was shown that a level of a system reliability can be controlled by changing an order of a quantile function in coordination and a number of redundant objects. A computer simulation model of the system was used to illustrate derived dependencies.

1 INTRODUCTION

Preventive replacements of objects' components are used to maintain demanded reliability of system of objects. This way of avoiding failures of individual components in a system was presented in some surveys (McCall 1965, Pierskalla et al. 1976, Valdez-Flores et al. 1989). There are some policies of applying preventive replacements as age replacement, block replacement, imperfect maintenance, corrective maintenance. The latter can be made as perfect repair, minimal repair, imperfect repair or general repair. A component of an object maintained under an age replacement policy is replaced after failure or at a specified operational age. The time required to replace the failed component is often considered negligible and, after replacement, the component is assumed to be "as good as new". Moreover, if repair and replacement times are considered non-negligible, it is possible to construct models to determine the optimal replacement age in order to maximize the component availability (Cassady et al. 1998).

Another case is when a component or system that is maintained under a block replacement policy is replaced at regular time intervals, regardless of age (Shaked et al. 1992). The block replacement policy is easier to administer than the age replacement policy because only the elapsed time, rather than the operational time, since the last replacement must be monitored,. However, a component that was just replaced after failure may be replaced again as a part of the planned block replacement. It can be shown that the age replacement policy is preferable to the block replacement policy.

Corrective maintenance actions are those actions that are necessary to restore objects to an operational state after failure, and can be categorized as follows: perfect repair, minimal repair, imperfect repair and general repair. In the above categories, repair may be used interchangeably with replacement. The issue is not whether a repair or replacement takes place. Rather, the issue is the relative age of the component after repair or replacement. For example, if a failed component is replaced with a new one, it is considered the same as if the component was repaired to an "*as good as new*" condition. Perfect repair models assume that after a corrective maintenance action the component is rendered "*as good as new*". The perfect repair assumption is reasonable if failed components are replaced with new and identical ones or if the repair procedure is thorough enough to negate nearly all of the aging effects. There is an optimal balance between preventive maintenance actions and corrective maintenance actions. In the relevant literature, the term imperfect repair has taken a broad meaning from minimal repair to some mixture of minimal repair and perfect repair (Brown et al. 1983). More recently, general repair models have been discussed as the most generally applicable corrective maintenance model that includes perfect repair and minimal repair as special cases (Pham et al. 1996). Very rare imperfect repair models have attempted to use component availability as a performance measure instead of cost.

Commonly, the effects of applied maintenance actions are modeled through changes in the failure rate of the component. If replacements are made according to a block replacement policy and repair actions bring the state of the component to a value somewhere between that applicable to completely new state and that just prior to failure, this can be interpreted as changes in the chronological age of the object, creating the so called virtual age (Kijima 1989).

Independently of applied preventive policy, the need for high reliability of such a system being used can result in great number of components replaced during preventive actions. As it cannot be considered full restoration of object reliability after maintenance, only components of the object should be replaced. This is a case of imperfect repair of the object.

High reliability is achieved in practice by replacing specific components with new ones. If they are negligible, a criterion of selecting components may depend on level of reliability that is expected.

It is obvious that a range of prophylactic activities depends not only on a reliability level of a system but also on its reliability structure. If there are some redundant objects, they can replace failed objects enabling execution of the planned tasks. A number of redundant objects also depend on the acceptable probability of failure during the task implementation period.

Instead of a method of replacing object at a given rate known from the literature (Wang 2002), the method of block replacement of sets of chosen components is proposed. This enables achieving demanded level of the set reliability. The method uses statistical characteristics of the objects instead of applying measurable parameters of their components.

2. A SYSTEM WITH REDUNDANT OBJECTS

Let us assume that n objects are essentially required for carrying out the planed tasks. If the entire set consists of n objects, then an assumption can be made that reliability structure of the system is in series. This imposes high requirements on reliability of each object, which is often not achievable. Then, in order to keep reliability of the set at its required level, redundant objects can be introduced. Adding *k* redundant objects allows for considering the system reliability structure as a threshold structure, in this case "*n out of n+k*".

The model of system reliability depends on the way the redundant objects are operating in it. They may play a role of the "cold reserve" (standby system), that is, they passively wait for one of the objects to fail, or the "hot reserve" (parallel system), thus increasing the whole system capacity until one of the objects has failed.

In case of the system "*n out of n+1*" with the cold reserve, the reliability function $R_{n+1}(t)$ will be a sum of probabilities for occurrence of the following situations:

1) until moment t no object will fail out of *n* objects in a series system,
2) at any moment $\tau < t$ one out of *n* objects shall fail and will be replaced with a reserve object that will not fail along with the remaining objects at an interval (τ, t).

Probabilities for occurrence of the above situations are as follows, respectively:

$$P_1 = R^n(t),$$

$$P_2 = \int_0^t f_n(\tau) R_n(\tau,\ t) d\tau, \tag{1}$$

where:

$$R_n(\tau, t) = \left[\frac{R(t)}{R(\tau)}\right]^{n-1} R(t-\tau), \tag{2}$$

$$f_n(\tau) = \frac{d}{d\tau}\left[1 - R^n(\tau)\right] = n\, R^{n-1}(\tau) f(\tau). \tag{3}$$

Substituting (2) and (3) for (1) yields

$$P_2 = n \cdot R^{n-1}(t) \int_0^t f(\tau) R(t-\tau) d\tau \cdot$$

Hence

$$R_{n+1}(t) = P_1 + P_2 = R^{n-1}(t)\left[R(t) + n\int_0^t f(\tau) R(t-\tau) d\tau\right]$$

where $R_{n+1}(t)$ = reliability function of "*n out of n+1*" system, $R(t)$ = reliability function of an object, $R_n(\tau, t)$ = probability of a non-failure in the interval (τ, t) of the set consisting of (*n-1*) objects aged τ and one new object, $f(t)$ = probability density function of an object's failure, and $f_n(\tau)$ = probability density function of a failure of one out of *n* identical objects in a series system.

Probability density function of a failure of the system "*n out of n+1*" with the cold reserve is expressed by the following relation and no recurrence formulas are known:

$$f_{n+1}(t) = n\, R^{n-1}(t) \int_0^t \left[f(t - \tau) + (n-1)\frac{f(t)}{R(t)} R(t-\tau) \right] dF(\tau) \cdot$$

In case of the system "*n out of n+2*", the analytical description becomes more complex, as there is the second reserve object. This means that in the set, established at the moment τ and consisting of (*n-1*) objects aged τ and one new object, one of the objects may fail and be replaced with the second reserve object before the moment t.

In case of the system "*n out of n+k*" of identical objects with the hot reserve, we may use the following relation:

$$R_{(n,n+k)} = \sum_{i=n}^{n+k} \binom{n+k}{i} R^i (1-R)^{n+k-i} \tag{4}$$

and the recurrence formula:

$$R_{(n,n+k)} = R \cdot R_{(n-1,n+k-1)} + (1-R) \cdot R_{(n,n+k-1)}, \tag{5}$$

where R = reliability of a single object.

Complexity of the analytical description, regardless of simplifying assumptions that have been made (i.e. identical objects, omission of the reliability structure of objects alone), indicates that there is a need for using a computer simulation for issues being considered here.

3. PREVENTIVE REPLACEMENTS WITH STATISTICAL DIAGNOSIS

A method that is known from literature and used for defining a scope and deadlines of preventive replacements is to include the costs of attentive replacements and the costs generated by the occurring failures (Barlow et al. 1965, Smalko 1991). As a result of application of this method, minimum average costs per unit of time related to maintained objects in a proper reliability status are achievable. However, in order to benefit from that effect there is a need to replace individual components at various time intervals, usually uncoordinated with the objects' operations, which may wipe out advantages resulting from the implemented optimisation. Therefore, a possibility should be considered to make preventive replacements of selected components of objects at the assumed time intervals. Its scope can be defined on the basis of assessment of reliability of the components and the assumed reliability level of the entire set (Okulewicz et al. 2006). The system maintained in such a way preserves its ability to carry out the planned tasks with a given probability.

A series system in case of complex objects can be considered. Thus, a failure appears whenever any component has failed. A repair usually involves a replacement of the component with a brand new one.

However, the replacement of the damaged component with the new one does not result in recovery of such a reliability level as that before occurrence of the failure. This is because the value of the reliability function of the damaged component before the failure was less than 1, and following the replacement it was equal to 1. In effect, the condition of the object after the repair is – and must be – slightly better than that before the failure. So, practically there are no possibilities to recover such a status of the object following the repair, as the one right before the failure.

Both the objects and their components are considered when developing the preventive replacements policy. Properties of the components are more predicable than those of objects which they are part of. Dynamic determination of a scope of preventive replacements could be based on a statistical assessment of present status of objects' components.

The term is widely used to describe a situation when decision about the system state is taken on the basis of a statistical analysis of data. In this case the statistical analysis gives distribution function of lifetime of the object. On this basis a mean time to failure is calculated. In order to do that, data are required about a distribution of time to failure and its parameters as well as about its operational use so far (since being new or from the moment of its replacement).

The problem is in determining a moment when working object should be replaced to prevent its failure. This decision should be made according to a particular object on the basis of statistical data concerning the whole population of objects. So data from the past – i.e. gathered in a computerized system – should be used to calculate parameters of a distribution function of lifetime of the objects. They concern failures, repairs and replacements of object components. Alternative way is relying upon experts' opinions at the start. Next, the probability distribution function of time to failure for each of these components is determined. Then a procedure of selecting objects to preventive replacement is used. Thus, it could be called as preventive maintenance on the basis of statistical data.

The statistical diagnosis is a maintenance methodology in the area of maintaining objects with non-exponential distributions. It identifies preventive maintenance actions to realise the inherent reliability of equipment at a minimum expenditure of resources. Because of statistical parameters of objects it can be performed at any moment. It could be done either in a constant period of time or during planned service or during running repair. Also the distribution parameters are modified when either repair or replacement of the component has been done.

This way the actual technical condition of the object is not taken into consideration as that would require for the object to be excluded from its operational use. Having data, reliability characteristics of components, updated working time of individual components, and a period for execution of the task, it is possible to define components that require preventive replacement in order for the project implementation probability not to decline below its assumed value. The procedure statistically predicts failures at part level by calculating the mean residual lifetime to failure (MRL).

Parameters of distribution for all chosen components are kept in the computer system. When time comes for diagnosing the MRL for each of all chosen components is calculated according to the formula:

$$r(t) = \frac{1}{R(t)} \int_t^{\infty} R(x)dx$$

where r(t) = mean residual lifetime function, R(x) = reliability function, and t = time from previous replacement.

However, the MRL compared to required work period results in that about half of objects would undergo services before failure and the rest would fail without any treatment. Thus, instead of the MRL, it would be better to apply a quantile function of residual lifetime to enlarge the probability of preventive maintenance. This measure directly relates to predicted work period and the reliability of the system. For any moment **t** the following conditions have to be met:

$$q_p(t) \geq d\,, \tag{6}$$

where d = tasks implementation period, and $q_p(t)$ = quantile of residual lifetime function, order p.

Function $q_p(t)$ shall be defined as in (Joe et al. 1983):

$$q_p(t) = F_t^{-1}(p) = \inf\{x : F_t(x) \geq p\},$$

where $F_t(x)$ = cumulative distribution function of the residual lifetime, $R_t(x)$ = conditional reliability function, and

$$1 - F_t(x) = R_t(x) = \frac{R(t+x)}{R(t)}, \qquad x, t \geq 0\,.$$

It is also true that:

$$R(t + q_p(t)) = (1 - p)\, R(t).$$

The statistical diagnosing can be applied both to components and to complex objects. In a case of complex object, its reliability structure as well as special procedure of choosing components to replace would be considered, which enables achieving demanded probability of proper work of the object (Salamonowicz 2005, Okulewicz et al. 2006). Probability of a failure during a task period can be determined in both cases, that is, when the replacements either have or have not been made. Additionally, the assessment may refer to the entire set of objects that have been assigned for execution of the tasks.

If k objects work as the hot reserve, it is the system "*n out of n+k*" and the order p represents demanded level of reliability. However, in the case of k redundant objects as the cold reserve, n objects present a series system. On the basis of the formula (5) it is possible to calculate a new value for the lower level of demanded reliability, with the formula:

$$1 - \alpha_k = \frac{1-p}{(1-R)^k\left[\binom{n+k}{n} + \sum_{i=1}^{k}\binom{n+k}{n+i}\left(\frac{R}{1-R}\right)^i\right]} \tag{7}$$

where α_k = probability of failure of one of n objects (α_0 = p), p = acceptable probability of system failure, R = reliability of a single object, n is a number of objects needed for the tasks execution, and k = number of redundant objects.

Procedural way of pointing out the new order α_k is presented in Figure 1. First, a quantile for the system "*n out of n+k*" is calculated for the order p. Then the reliability R of a single object for the same quantile is calculated for the system "*1 out of 1*". With use of these values the order α_k can be calculated, which is the order for the structure "*n out of n*".

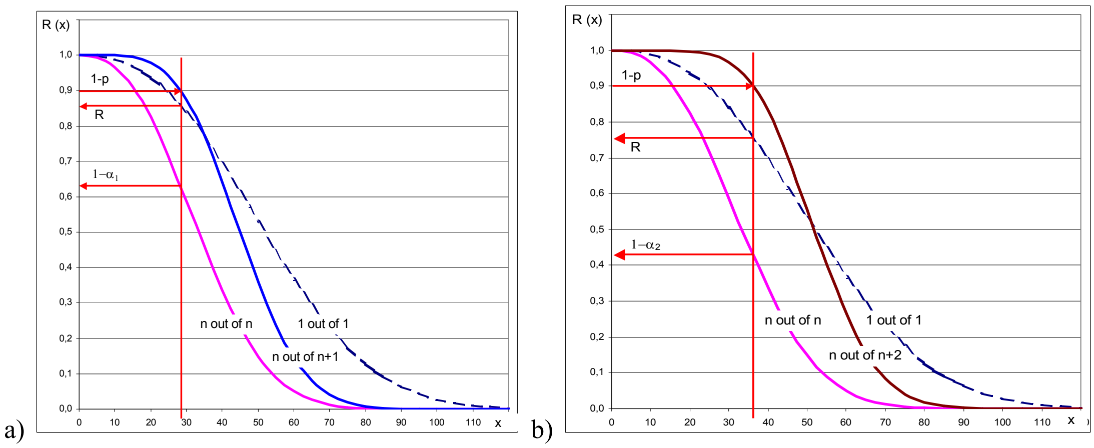

Figure 1. Graphical interpretation of orders calculation on the basis of structures "*n out of n+k*", "*n out of n*" and "*1 out of 1*": a) "*n out of n+1*", b) "*n out of n+2*".

4. IMPERFECT REPAIR

Majority of theoretical conclusions concerning maintenance are derived from assumption of perfect object restoring. However, such processes with use of models of full renewal are adequate only when an object is replaced with a new one or in a case of a general repair. In the case of corrective repairs made after failing of any object component, a model of minimal repair is often used (Barlow et al. 1996). This means that the object is to be restored to the condition just before failure. However, it is practically not possible, as object reliability status after repair of its component is better than that before failure. Those are reasons that theoretical models of either perfect or minimal repairs have limited applicability. Real repair restores object reliability to an intermediate value, and it is called an imperfect repair. However, a degree of object restoration by replacing one or more its components can be estimated only after repair. Modelling of the exploitation process with use of the imperfect repairs means defining characteristics of random variable X_k concerning time of proper work after (k-1)th repair. Object's reliability function after the first repair at moment t is given by the following formula (Salamonowicz 2005):

$$R_2(x) = [R_1(x)]^{\alpha}\left[\frac{R_1(t+x)}{R_1(t)}\right]^{1-\alpha},$$

where $R_1(x)$, $R_2(x)$ = reliability functions of the object before and after the repair, respectively, α = degree of the object restoration, and t = moment of the repair.

The formula for the failure rate function relation before and after the repair is as follows:

$$\lambda_2(x) = \alpha\lambda_1(x) + (1-\alpha)\lambda_1(t+x).$$

Hence

$$\alpha = \frac{\lambda_1(t+x) - \lambda_2(x)}{\lambda_1(t+x) - \lambda_1(x)},$$

where $\lambda_1(x)$, $\lambda_2(x)$ = failure rate functions before and after the repair, respectively.

The preventive replacements of components of complex objects are made if the value of function (6) – calculated for the objects – is lower than the duration of the scheduled task planned. The appropriate algorithm is presented in Figure 2.

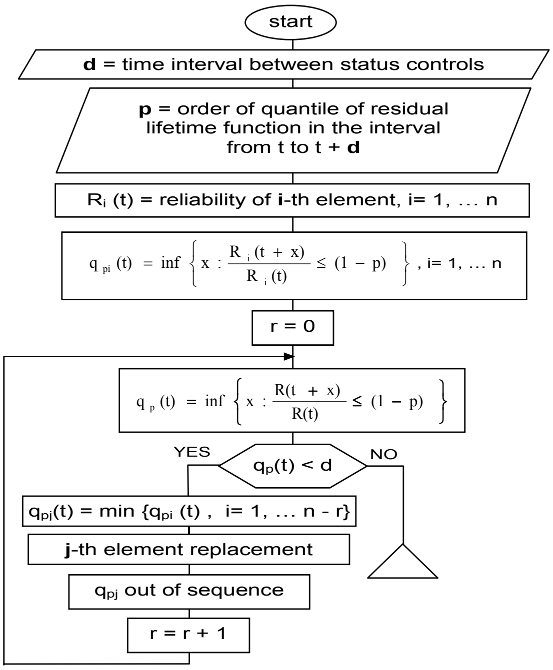

Figure 2. Algorithm for selecting components for preventive replacement.

In order to select such a subset of components to be replaced at a given moment, an updated value of the reliability function is calculated, including operational time of each and every one of them. Then a quantile of a given order is calculated for a distribution of the residual lifetime of each component.

The components are put in order according to the growing quantile value. Then subsequent components are assigned for replacement, starting from a component of the lowest quantile value until the quantile of the entire set of objects – calculated by having included the replacement of assigned components with brand new ones – is not lower than the duration of the scheduled task. The replacement of components that have been assigned in that way ensures the assumed probability that the object will not fail during implementation of the task.

5. SIMULATION EXPERIMENTS

The above consideration was confirmed with use of a computer simulation. A system "*n out of n+k*", for k=0, 1, 2 was considered as an example. In this model, objects were applied, that were partially replaced at steady intervals of time, according to results of statistical diagnosis. The planned process of replacements was combined with random process of failures and repairs.

The set of *n* objects was used for execution of tasks in the model. Each object is composed of three groups of different components. The time to failure of a single group was Weibull distribution with a reliability function $R(x) = \exp[-(x/b)^a]$.

Parameters of the model were as follows: $n = 50$, $p = 0.1$, $d = 2.5$, $a_1 = 2.5$, $b_1 = 65$, $a_2 = 2.5$, $b_2 = 80$, $a_3 = 2.5$, $b_3 = 100$. The acceptable probability of the set unavailability was *p*. The required reliability was maintained by preventive replacements of objects components. Statistical diagnosing was done at intervals of length *d*. A graph of the model states is presented in Figure 3.

The initial state of all objects in the model is "work". After predefined interval d the statistical diagnosis is done. This means that for the set of objects there are chosen these elements of objects which after replacement will cause increasing of the probability of this set of objects to the demanded level. This is done according the algorithm in Figure 2.

Then all of these elements are replaced with new ones and objects after such an imperfect repair come back to the initial state. Objects that have not elements to be replaced are going back to the initial state immediately after the statistical diagnosis. For some of elements the real residual time to failure can be less than the interval *d*. Such an element causes a failure of the object to which it belongs.

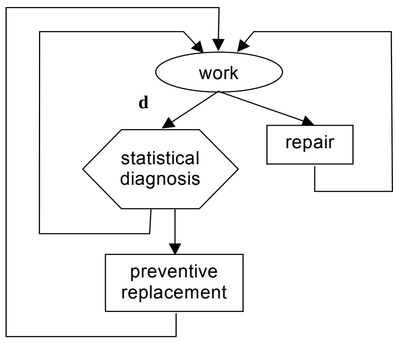

Figure 3. Graph of model states (*work* – working of a system, *statistical diagnosis* – selecting a set of components, *preventive replacement* – replacement of selected components with new ones, *repair* – replacement of a failed component with a new one).

Depending on the reliability structure of the system it may cause the failure of this system or not. In the case of k redundant objects, the *k+1* failure will result in the failure of the whole system. After this it undergoes the repair. This means replacing all failed elements with new ones. Such replacements restore the system to a little better state than that before the failure, as some of its elements are new and the rest remain unchanged.

The range of simulation was $T = 1000$, and experiments were repeated 10 times. As a result of simulation, numbers of replacements, failures of objects and unavailability of the whole set were estimated.

First of all, the number of failures was estimated for two cases: in the system "*n out of n*" without any prophylaxis and for the system with statistical diagnosing with $p = 0.1$ and $d = 2.5$. The mean number of object failures (and the same of the system unavailability) without preventive replacements and with perfect object repair was 1102. Then this number was increased to 2065 after applying the preventive replacements together with imperfect object repairs. After applying preventive replacements, the mean number of system unavailability was decreased to 42, according to low probability of failure allowed for that system. But 33568 components have to be preventively replaced in order to sustain the appropriate reliability of objects. Such a great number of replacements in this case were a result of rather low reliability of object components.

The empirical reliability of a single object was estimated by the following formula:

$$R = 1 - \frac{N_u \cdot d}{T \cdot (n + k)} \tag{8}$$

where N_u = number of object failures, d is a interval of statistical diagnosing, T = time of simulation, n = number of objects, and k = number of redundant objects.

Using this formula, the reliability of the system was $R_{50} = R^{50} = 0.998^{50} = 0.900$, so the probability of failure did not exceed the demanded $p = 0.1$. Such results show that it is possible to achieve the demanded reliability with significant decrease in the number of random brakes in system work but with a very big number of preventive replacements of components.

Imperfect repairs result in a greater number of system unavailability states than compared to perfect object repairs. This is obvious because repairing only components the object resources are not fully restored, and the object after the repair is not as good as new. Only for exponential distribution numbers of system unavailability with and without perfect repairs are equal to each other.

The aim of preventive replacements is to decrease number of random object failures by avoiding them with assumed probability, as they break system operation and bring many unpredictable consequences. Adding redundancy to the system also results in enlarging its reliability. According to formula (7) the result of this is analogical to appropriate increasing of the quantile order of the system "*n out of n*". The modified orders of quantile calculated using formula (7) for systems with *n+1* and *n+2* objects are as follows: $\alpha_1 = 0.410$, $\alpha_2 = 0.660$, and interpreted in Figure 4.

Such a result is only valid for a perfect repair after every statistical diagnosing, i.e. each object is replaced with the new one. For complex objects, i.e. composed of some components this condition could be fulfilled when the interval of statistical testing is long enough. However, imperfect repairs – done by replacing selected components of maintained objects – are useful only when the interval between statistical diagnosing is shorter than the initial quantile at t = 0. After a number of such replacements of objects' components, the system does not consist of new objects. So the probability of tasks fulfilling by a single object cannot be calculated based on reliability function of a new object. Instead of this there should be used systems "*n out of n+1*" and "*n out of n+2*" separately on the basis of appropriate experiments from Table 1. Then in both cases the modified order for system "*n out of n*" should be calculated with use of formula (7). They are, of course, less than those in the previous case.

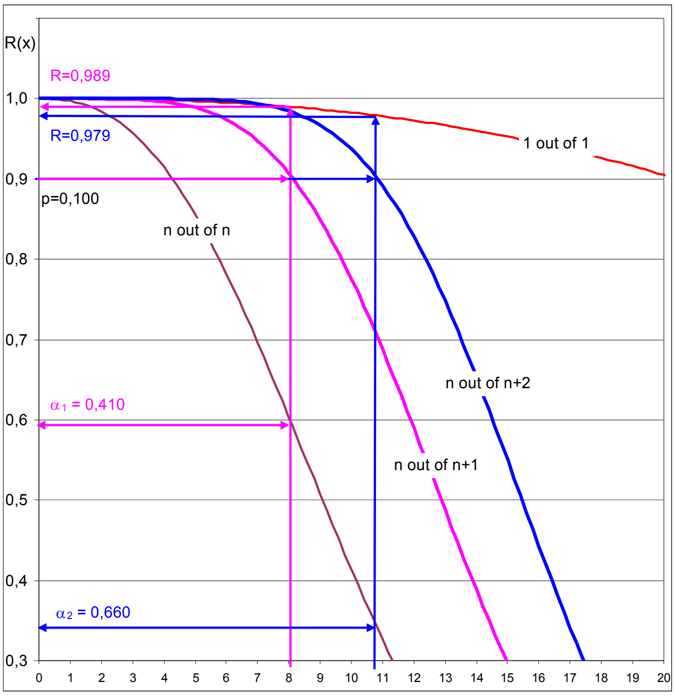

Figure 4. Graphical interpretation of calculating new quantile orders.

Simulation results in Table1 showed that the numbers of replaced components in systems "*n out of n+1*" with $p = 0.1$ and "*n out of n*" with $p = \alpha_1$ are similar, as well as in systems "*n out of n+2*" with $p = 0.1$ and "*n out of n*" with $p = \alpha_2$

Table 1. Simulation experiments results (for $d = 2,5$).

p = 0.10	n out of n+k;			
k	1	0	2	0
α	-	0.346	-	0.482
Number of:				
– preventive components replacements	14405	14638	10886	9807
• group 1	7434	10213	5340	6582
• group 2	4516	2982	3435	2175
• group 3	2455	1443	2112	1050
– system unavailability	23	174	16	256
– object failures	154	174	238	256
Reliability of a single object	0.992	0.991	0.989	0.987

The results of simulations also confirmed a natural supposition, that components of the lowest reliability constitute a dominating group of replaced components. The share of such components in the total number is greater than without statistical diagnosing. These components were recognized and such a shifting was done by the algorithm assuring the demanded level of the system reliability.

The reliability of a single object according to formula (8) for the system "*50 out of 51*" was R = 0.992. So the reliability of the system "*50 out of 51*" – according to formula (4) – was $R_{50,51}$ = 0.943. The reliability of the system "*50 out of 50*" estimated with use of these data was $R_{50,50}$ = 0.647 and it was appropriate to demanded probability of failure $\alpha_1 = 0,346$.

As is shown in *Table 1*, the mean numbers of replaced objects in systems "*n out of n*" with $p = \alpha_1$ and "*n out of n+1*" with $p = 0.1$ are similar, as well as in systems "*n out of n*" with $p = \alpha_2$ and "*n out of n+2*" with the same $p = 0.1$.

6. CONCLUSIONS

The imperfect repairs are a natural way of maintaining objects ability to perform given tasks. They better fit with real situations, since the perfect repair policy is quite unrealistic in case of objects. Preventive replacement of object's components is a kind of imperfect repair as it restores the object capacity partially. This way a considerable reduction in a number of incidental failures of objects, compared to a use without any prophylaxis, is achievable through application of the statistical control. However, maintaining a high reliability of a set of objects is accompanied by a great number of preventive replacements of objects' components. This means that there are many more preventive replacements than random failures of objects because of relatively low reliability of a single object.

Thus in such a situation, it would be easier to achieve the required availability of the system by adding redundant objects that replace the damaged ones than to maintain a high reliability of that system without redundancy.

Required level of system reliability could be achieved by adding surplus objects and properly matching them with the quantile order applied to the main part of the set of objects. So, by adding redundant objects, more failures of objects can be accepted as well as a number of preventive replacements is reduced. It would be useful to combine redundancy and preventive replacements based on statistical diagnosing.

By those two measures, random failures of the system are significantly reduced in number of replaced components being much lower than those without redundancy.

The hereto presented method for setting a scope of preventive replacements, based on reliability properties of individual objects being used, allows for matching the parameters of replacements for applied reliability parameters of the objects.

Reliability analysis with respect to preventive replacements can also be performed with reference to objects' components being of critical importance to tasks that are executed. This analysis can be carried out for any system of complex objects that will jointly be used for execution of the tasks.

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RBF NETWORKS FOR FUNCTION APPROXIMATION IN DYNAMIC MODELLING

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ABSTRACT

The paper demonstrates the comparison of Monte Carlo simulation algorithm with neural network enhancement in the reliability case study. With regard to process dynamics, we attempt to evaluate the tank system unreliability related to the initiative input parameters setting. The neural network is used in equation coefficients calculation, which is executed in each transient state. Due to the neural networks, for some of the initial component settings we can achieve the results of computation faster than in classical way of coefficients calculating and substituting into the equation.

1 INTRODUCTION

Let us have the model of a dynamic system, in which the temperature is evolving according to the time and initial component settings. The target is to specify the probability of a system failure, which is defined as exceeding the temperature bounds. We are also interested in the time necessary for computing the result. It is proposed to enhance the simulation algorithm with neural network tools which will be used in calculating the differential equation coefficients *a* and *b* (chap. 3. relation (4)) being changed according to k_i component states (on/off). After each k_i switching, which is invoked by either passing the temperature transition state or failure of k_i component, we must calculate new values of parameters *a* and *b* in equation (4) according to (2).
As a solution, it is appropriate to apply neural networks for the approximation of parameters (2) dependent on the k_1, k_2 and k_3 component settings.
Optimal tool for constructing the simulation algorithm is the Monte Carlo (MC) method. This paper is derived from (Nedbálek 2007, Pasquet et al. 1998).

2 THE BENCHMARK PROCESS DESCRIPTION

We dispose of the tank with warmed water, which temperature is kept in the specific maximal or minimal bounds – in this range, we consider system as stable and reliable. The system also contains two electric components, responsible for water heating, and security valve, which decreases the temperature. In the bottom of the tank, there is a faucet for water supplying. We suppose, the volume of water in the tank is constant during our experiment.

Let us define variables:
T(t) – temperature of water at the time t;
Tempmax – maximal temperature of water in the tank;
Tempmax = 368,15K

Tempmin – minimal temperature of water, for T< Tempmin failure occurs
Tempmin = 338,15K
Tempbas – security level for the minimal temperature
Tempbas = 343,13K
Temphau – security level for the maximal temperature
Temphau = 363,15K
Secu – reserve for the maximum temperature, for T> (Tempmax + Secu) failure occurs, Secu = 2 K
M – water weight, M = 500kg
Te – external temperature, Te = 293K
A – tank surface, A = $6m^2$
h – thermal exchange coefficient, h = 6 $WK^{-1}m^{-2}$
c_p – measure heat capacity, c_p = 4184 $Jkg^{-1}K^{-1}$
$W_1 = W_2$ – heating power, W = 5000W
tm – process duration, tm = 720 h
hazard rate – transition to on-state
$\lambda_{W1on} = \lambda_{W2on} = 6.10^{-4}\ h^{-1}$
hazard rate – transition to off-state
$\lambda_{W1off} = \lambda_{W2off} = 4.10^{-4}\ h^{-1}$
hazard rate – transition to on or off-state
$\lambda_{Vson} = \lambda_{Vsoff} = 1.10^{-3}\ h^{-1}$

3 THE EQUATION SOLUTION

To evaluate the probability failure, we need to write the differential equation, describing our system evolution. The equation obviously reflects the following points:

1. Decreasing the initial temperature due to heat penetration through the tank wall.
2. Increasing the water temperature caused by two heating components, if activated.
3. The water temperature decrease invoked by the security valve activation.

Our equation comes from (Pasquet et al. 1998) but it is altered for the behaviour of the system

$$\frac{dT}{dt} = aT + b \tag{1}$$

where

$$a = -\left(\frac{A \cdot h}{M \cdot c_p} + \frac{Q_s \cdot c_p \cdot k_3}{M \cdot c_p}\right) \quad b = \left[\frac{1}{M \cdot c_p} \cdot \left(A \cdot h \cdot T_e + Q_s \cdot c_p \cdot T_e \cdot k_3 + W_1 \cdot k_1 + W_2 \cdot k_2\right)\right] \tag{2}$$

and

$$Q_s = \frac{\dfrac{W_1 + W_2}{Temp\max - T_e} - A \cdot h}{c_p} \tag{3}$$

The solution of (1) follows the equation

$$T = \left(T_0 + \frac{b}{a}\right)\exp^{a \cdot t} - \frac{b}{a} \tag{4}$$

where T_0 is the starting simulation temperature.

The k_1, k_2 and k_3 coefficients equals 1 or 0 (the specific component is either on or off). For $k_1 = k_2 = 1$ the heating components are active and temperature of water in the tank is increasing, for $k_3 = 1$ the vent is unclosed and the temperature is decreasing, etc. We watch the process along the period of tm = 720 h. The initial temperature is set between Tempmin and Tempmax, that is – T_0 = 353.15 K.

4 CREATING AN ALGORITHM

To construct the correct algorithm for our test case simulation, we take into account following points:

1. As mentioned before, for T< Tempmin and also for T> (Tempmax + Secu) failure occurs
2. The temperature passes by 5 stages generally – see the diagram:

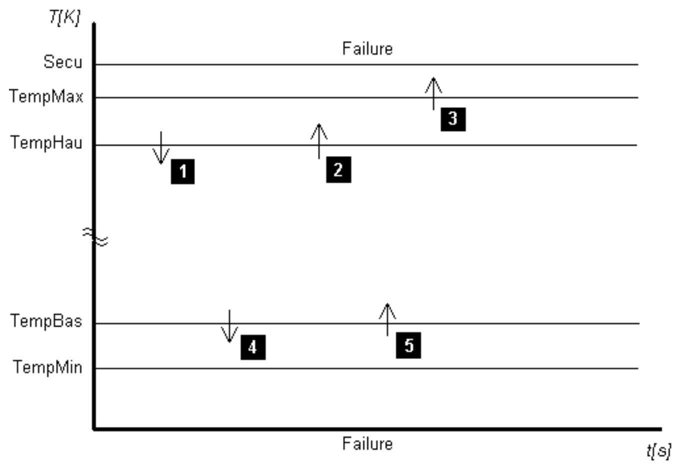

Figure 1. Dynamic rules of the system

For each of the temperature stages, the change (switch) of the specific component to the opposite state, that causes the required temperature turnover (see (1)) and stabilization in tolerable bounds. In case of random failure of the k_i component, we keep on monitoring evolution of the temperature, until it exceeds limits – we consider the system as disfunctional. (In the terms of the k_i failure definition, the whole system does not have to be failed yet. The temperature of water in the tank could be still between bounds.)

3. There are following rules for components changes at temperature borders crossing:

State 1: If T(t-1)>= Temphau and T(t) <= Temphau, then $k_3 = 0$ (vent will be closed)
2: If T(t-1) <= Temphau and T(t) >= Temphau, then $k_1 = 0$ (heating component num. 1 will be cut off)
3: If T(t-1) <= Tempmax and T(t) >= Tempmax, then $k_3 = 1$ (vent will be opened)

4:If T(t-1) >= Tempbas and T(t) <= Tempbas, then $k_1 = 1$ & $k_2 = 1$ (both heating components are active)
5: If T(t-1) <= Tempbas and T(t) >= Tempbas, then $k_2 = 0$ (heating component num. 2 will be cut off)

4. Time step option-considering fact, that we present the evolution of (4) at time *t* during the period of tm, it is necessary to select an appropriate time to explore all detail changes of the temperature bahaviour and also to reduce the inadequate number of cycles of numerical simulation. The optimal solution seems to be the one minute pattern, which reflects suitably all changes at temperature borders crossing. Longer patterns do not suit our solution due to inaccuracies – a “jump-over” of some of the states mentioned in 3. occurs sometimes.

5. Switching the component to the opposite state could happen at any time in the simulation due to random failure.

6. Period of the process is set for 720 hours.

5 APPLICATION OF THE RBF

Our simulation algorithm contains cycle, running over the process duration, in which (4) evolves according to time. This equation has coefficients *a* and *b*, that depend on k_i component states (on/off) – see (2). In the simulation, the k_i state is influenced by either passing the temperature transition state (see Fig. 1) or failure of component itself. It means, that we must recalculate the *a* and *b* whenever the temperature transition or failure of the k_i occurs. Simply, we are able to write lines of code to enumerate new values of the *a* and *b* right in the body of process duration cycle, whenever it is necessary to do so. The second possibility is to apply the Radial Basis Function (RBF) neural network to approximate the function of *a* and *b* coefficients depending on k_i component states.

It is acceptable to use other types of neural network, nevertheless the RBF is obviously the best to solve the problem. This is the result of two main facts, firstly, we are not urged to design the network architecture (RBF has two layers standardly) and secondly, the RBF can not be trapped in a local minimum during training phase (Chen et al. 1991). RBF complies our requirements on the function approximation (Yee & Haykin, 2001). Applying other types of neural network to unriddle this case study and to compare them with the used RBF network is the matter of a future research.

At the beginning, we need to find out the convenient training set. This is obtained by simple computation of (2) for all combinations of the k_i states (see Tab. 3). Then, before the process duration cycle, we are ready to create and train the standard RBF architecture – there are several implementations and function support of the RBF in programming languages – for example, the Matlab software provides large neural network toolbox.

Consequently, the *a* and *b* parameters in (4) everywhere in the cycle are replaced with the callback function of the RBF network.

We can generally summarize, that the main modification consist in using the RBF as an auxiliary tool for working with equation (4) during the time of a simulation cycle. In any case, the MC construction of the algorithm remains the same for both cases.

6 THE RESULTS PRESENTATION

Table 1. contains the distribution function of failure probability value averages for each initial components settings. The results were obtained for 10^5 Monte Carlo simulations (1- the comp. active, 0 – comp. inactive at the beginning). The fifth column shows the computational time. All results are obtained in the state of tm = 720 h. The simulation was implemented in the Matlab software.

Table 1. The results for 10^5 cycles of Monte Carlo

k1	k2	k3	$\overline{F(tm)}$	$\bar{t}$ [s]
0	0	0	0.3517	2315.0
0	0	1	0.5303	2174.0
0	1	0	0.5567	1928.6
0	1	1	0.5312	2170.7
1	0	0	0.3518	2332.1
1	0	1	0.5306	2194.0
1	1	0	0.5580	1920.4
1	1	1	0.5602	1915.6
Average			0.4963	2118.8
Sigma			0.0901	174.1

Table 2. The results for the same Monte Carlo algorithm with RBF neural network enhancement

k1	k2	k3	$\overline{F(tm)}$	$\bar{t}$ [s]
0	0	0	0.3510	2383.1
0	0	1	0.5305	2037.1
0	1	0	0.5574	2002.9
0	1	1	0.5325	2042.2
1	0	0	0.3506	2390.6
1	0	1	0.5305	2040,5
1	1	0	0.5578	1975.2
1	1	1	0.5593	1968.9
Average			0.4962	2105.1
Sigma			0.0906	176.2

From comparison of Table 1. with Table 2., we can see the results of simulation at the time of 720 hours are very close – the RBF neural network is able to approximate with good accuracy (that was tested in the simulation code itself).

The results of computing time look more interesting – the average time necessary to simulate 720 hours long process is shorter by roughly 10 sec. This value seems to be neglectable, nevertheless the differences in results between the MC and the modification with RBF are larger when we look at the specific initial component settings.

Generally, we can express the presumption, that if the vent is opened and maximally one heating spiral is activated, it is more useful to enhance the MC algorithm with RBF network (the result is reached by 2 – 2.5 min faster). In other cases, the Monte Carlo itself is faster (1 min. advance).

In this place, we should stress out the information, that the comparison test on the MC and RBF network enhancement was executed on the computer, which had all applications, including hidden ones, and non-operation system processes not pertaining to simulation itself, halted. This measure is needed in order to provide the simulation the similar computing system capacity along the whole processing time and avert the distortion in result time values (operating system sometimes allocates the memory to other running applications, as consequently leads to Matlab processing slow down).

With respect to the length of algorithm, the MC enhanced with the RBF is larger in creation and training of the network. In the simulation itself, the length of code remains the same.

In Table 2., we also considered time necessary to train the RBF network.

The results from Table 1. and 2. are presented in the figures. The x-axis denotes possible component states according to binary code, as it is shown in Table 3.

Table3. The k_i component states combination
(1 -on, 0 -off)

x axis	k1	k2	k3
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
… etc.			

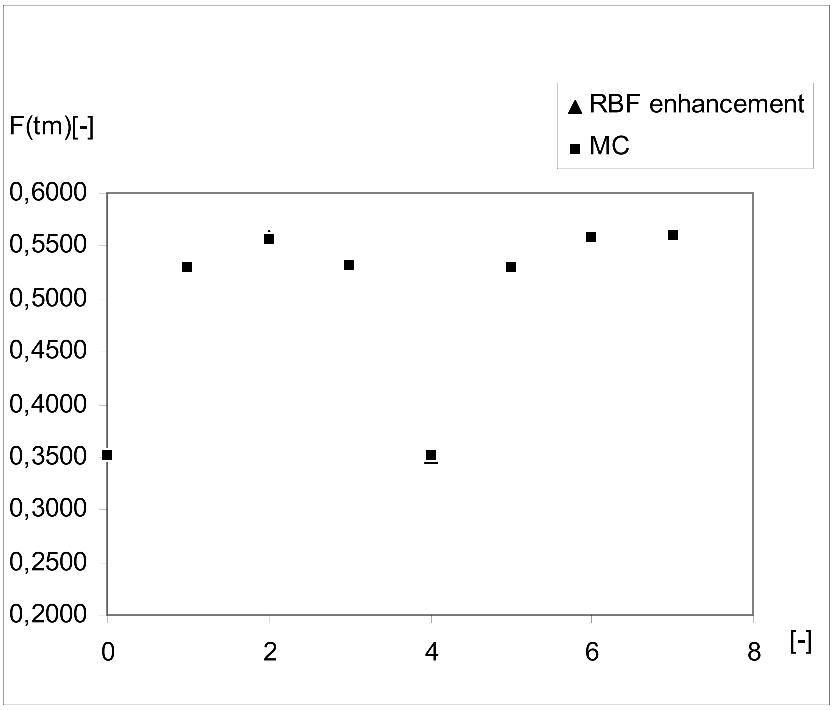

Figure 2. Failure probability comparison of the MC and the RBF neural network enhancement at time tm

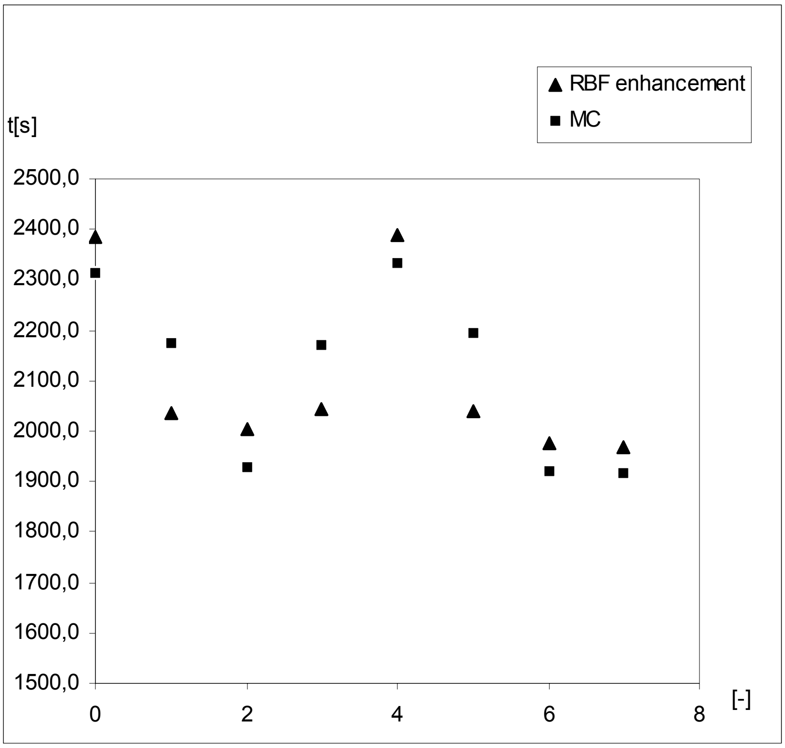

Figure 3. Computing time comparison of the MC and the RBF neural network enhancement

7 CONCLUSION

For 10^5 cycles, the failure probability at time t = 720hrs equals to the value $\overline{F}_{(720)}$= 0.4963 ± 0.0901 (MC) or 0.4962 ± 0.0906 (RBF enhancement). The algorithm in chapter 4 is implemented in the Matlab software.

Out of the comparison of the Figure 1. and 2. follows, that the failure probability values are similar for both methods.

The whole computing time needed to obtain results for each initial component settings is shorter by approx. 10 sec. when we use enhancement with RBF network. The greater differences in time consumption are evident for specific settings – we can state, that if the security vent is opened and maximally one heating spiral is activated than it is preferable to add the RBF in algorithm (the result is known by 2 – 2.5 min faster), in all other cases, the plain Monte Carlo method is more suitable (faster by about 1 min). This piece of knowledge was verified on the 498 MHz and 256MB RAM computer. The computation on the stronger machine – 3.1 GHz and 1GB RAM – took less time and the RBF enhancement method was still faster than the plain MC.

Application of the RBF neural network can sometimes lead to obtain results faster. This information is likely to be applicable in other, not only dynamic simulation, test cases. Participating of the RBF neural network in some computation problems is the matter of future research.

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FREQUENCY ASSESSMENT OF LOSS OF CONTAINMENT INCLUDING THE EFFECTS OF MEASURES OF RISK PREVENTION

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ABSTRACT

This paper presents a method for the quantification of the effects of measures of risk prevention of the frequency for rupture of pipework. Some methodologies, given in the literature for this purpose, assume that each plant under analysis is characterized by the same combinations of causes of failure and prevention mechanisms but this assumption is not always true. The approach suggested here is based on the methodology proposed in 1999 by Papazoglou for the quantification of the effects of organizational and managerial factors. Taking advantage of this methodology the objective of the assessment of the influence of measures of risk prevention in pipework has been achieved through the definition of the links between the causes of failure and the measures adopted by the company in order to prevent and/or to mitigate them.

1 INTRODUCTION

Accidental analyses in chemical plants have shown that the main causes of incidents are often due to deficiencies in the corporate structure, which can influence the safety of these installations. The use of appropriate risk analysis techniques permits the identification of the cause and evolution of accidents and the calculation of the frequencies of top events associated with process anomalies and loss of containment.

The likelihood of an accident is a function of various parameters such as components failure rates, probabilities of human error, etc. The availability of general values for these parameters from literature data simplifies risk analysis, unfortunately, it is also obvious that the use of such information provides standardized results which do not permit taking into consideration plant specific managerial and organizational factors. If managerial and organizational factors are neglected, the risk analysis for two identical establishments, characterized by totally different management systems, gives the same results and this appears unacceptable especially when risk analysis is used as a tool for risk-based decisions.

In the nuclear field, several quantitative studies (Izquiedo-Rocha & Sanchez-Perea, 1994; Montmayeul et al., 1994), have been performed to approach management-related safety problems. In recent years, also in the chemical industry, great attention has been paid to the study of the relationship between the managerial system and the safety level of chemical plants (Papazoglou & Aneziris, 1999; Thomas, 1980).

The main object of this work is to study the influence of measures of risk prevention on the frequency of loss of containment in pipework. In order to achieve this aim it has been necessary to define the relationship between the measures of prevention of the risk adopted by a company and the causes of failure in piping. Once the relationship between the individual causes of failure and the measures of risk prevention have been established and after the estimation of the weight coefficients for the causes of failure, it has been possible to modify the frequencies taking account managerial and organizational factors.

In the first part of the paper, some methodologies for the quantification of measures of risk prevention, currently available in the literature, will be discussed. Finally in the second part the proposed approach and its application will be described.

2 ACCIDENT FREQUENCIES IN THE CHEMICAL INDUSTRY

Frequency calculation for incidents in the chemical industry and consequence quantification of the associated accidental scenarios are fundamental steps for quantitative risk analysis.

The general procedure for frequency evaluation comprises the definition of the top events, *risk identification*, and then the application of appropriate techniques and equations derived from probability theory.

Risk identification is the most critical step in the overall analysis, in this phase it is important to consider all the initial causes of incidents. The available techniques for risk identification are historical-statistical methods, based on examination of incidents happened in the chemical industry and recorded in databases, and/or analytical methods, such as PHA (Preliminary Hazard Analysis), HAZOP (Hazard and Operability Analysis), FMEA (Failure Modes and Effects Analysis).

Analytical methods, mentioned above, are applied only for the identification of top events associated with process deviations or failures of components. The assessment of loss of containment requires a specific approach.

Events related to loss of containment, sometimes called *random ruptures*, are caused by accidental phenomena such as uncontrolled wearing, anomalous corrosion, pipe defects, etc. These events are not associated with process anomalies, but, as already mentioned, are often due to deficiencies in the corporate structure.

In this study attention has been paid to the *random rupture* of pipework because incident analysis, reported in Lees (1996), shows that loss of containment in the chemical industry, frequently, does not occur from vessels but from pipework and associated fittings.

3 LOSS OF CONTAINMENT EVENTS

The pipework-fittings system includes the piping itself, flanges and joints, and fittings, such as the many types of valves, bellows, etc, together with the pipe supports. As already mentioned, a large proportion of failures of containment in process plants occurs in the pipework. For this reason suggestions for reducing piping failures have been given in the past.

Kletz (1994) has given information on some 50 major pipe failures and associated fittings in process plants and some suggestions for preventing each failure. He also made proposals for the reduction of pipework failures by improved design and inspection of the pipework.

In order to decrease the number of loss of containment events, a proper design of even small bore pipework is recommended. Thus the pipework should be designed for ease of maintenance and, in the case of rupture, there should be easy access to the point where the failure has occurred.

Safety in piping systems has been the object of a study by the Institution of Chemical Engineers (IChemE), reported in Lees (1996), where the main features considered were: layout, quality control, construction, pipe supports, and vibration. The study has also concluded that the main causes of failure are vibration, external corrosion, inadequate temporary supports, blocked-in liquids, water hammer, steam hammer, cavitation and pressure surge.

There is a considerable amount of data available on pipework failures (Lees, 1996), but the range of values quoted is wide and tends to be confusing. There are several important distinctions to be made concerning the type of failure and the pipe size. Based on these considerations, complete pipe breaks, or guillotine fractures, constitute only a small proportion of failures and the breakdown rate tends to be higher for small than for large diameter pipes.

A survey of pipework failures in plant in the nuclear, chemical and other industries had been described by Blything and Parry (1986) quoted in Lees (1996). The data were analysed as *causes of failure* and *root causes*. Essentially, causes of failure are the mechanical causes, such as corrosion, fatigue and water hammer, and root causes are activities such as error in design, operation and maintenance. Results are given as cause of failure vs. root cause and have been identified for chemical plants, refineries and for nuclear and steam power plants. Data for chemical plants are summarized in Table 1, where the percentage of incidents is separated for direct causes and main activities (Lees, 1996). Table 2 gives the direct causes of failure obtained from two different sources (Lees, 1996; Thomas, 1980).

Finally, incidents can also be classified under three headings: direct cause, origin of failure or underlying cause and recovery from failure or preventive mechanism. Table 3 shows the underlying causes vs. the recovery from failure, this distribution is the result of a study of pipework failures in process plants made by Bellamy and co-workers (Lees, 1996). This study reviewed 921 incidents from incident databases such as HSE MARCODE, MHIDAS and FACTS.

Table 1. Causes of failure in chemical plants and refineries: – *cause of failure* vs. *root cause*(Lees, 1996)

	Design	Installation	Design /Installation	Operation	Maintenance	Manufacture	Unknown	Unspecified	Total
Corrosion									
- external	18	8	-	2	4	-	-	1	33
- internal	56	1	2	1	1	1	-	3	65
- stress	15	-	1	-	-	-	-	-	16
Erosion	2	1	-	-	1	-	-	-	4
Restrain	1	2	4	-	-	-	-	-	7
Vibration	9	1	3	1	-	-	-	1	15
Mechanical	28	10	5	11	12	18	2	21	107
Material	5	7	10	-	4	2	-	21	49
Freezing	13	1	-	2	-	-	-	1	17
Thermal fatigue	2	1	-	2	-	1	-	1	7
Water hammer	2	1	1	4	-	-	-	-	8
Work system	6	4	36	47	49	-	-	2	144
Unknown	-	-	-	-	-	-	29	1	30
Unspecified	1	1	13	3	3	-	-	33	54
Total	158	38	75	73	74	22	31	85	556

Table 2. Causes of failures in pipework: direct causes

Failure causes	Percentage of incidents	
	P.F.Lees (1989)	H.Thomas (1980)
Manufacture & fabrication:	31.9	
- base materials (defects)		9.6
- welding		11.8
Material selection	--	28.8
Corrosion	9.3	24.6
Erosion	0.8	
External load	3.0	--
Impact	4.8	--
Thermal shock	3.8	1.3
Mechanical shock	12.1	
Fatigue	1.5	
- low cycle		7.8
- vibration		4.3
Expansion & Flexibility	--	2.7
Wrong or incorrectly located in-line equipment	4.0	--
Operator error	18.2	7.0
Unknown	1.5	--
Other	9.1	7.0
Total	100.0	100.0

Table 3. Causes of failures in pipework: *underlying causes* vs. *recovery failure* (Lees, 1996)

Underlying cause	Recovery failure						
	Not recoverable	Hazard study	Human factors review	Task checking	Routine checking	Unknown recovery	Total
Natural causes	1.8	-	-	0.2	-	-	2.0
Design	-	24.5	2.0	-	0.2	-	26.7
Manufacture	-	-	-	2.4	-	-	2.4
Construction	0.1	0.2	1.9	7.5	0.2	0.4	10.3
Operation	-	0.1	11.0	1.6	0.2	0.8	13.7
Maintenance	-	0.4	14.5	12.7	10.3	0.8	38.7
Sabotage	1.2	-	-	-	-	-	1.2
Domino	4.5	0.2	-	-	0.3	-	5.0
Total	7.6	25.4	29.5	24.4	11.1	2.0	100.0

4 FREQUENCY OF LOSS OF CONTAINMENT

The analysis of loss of containment events permits a complete description of all the potential incidental events, which are the initial causes for the release of hazardous substances.

Their identification consists of the following steps:

– identification of the process and stored dangerous substances inside the establishment;
– characterization of the pipework and equipment and definition of the operating conditions;
– identification of the units of the plant, which have the same operating conditions;
– definition of representative causes of leakage for each unit.

The calculation of accidental frequencies can be made using the Fault Tree method for events deriving from process deviations, while the analysis of the loss of containment events requires a specific approach. A commonly adopted method for calculating the frequency of occurrence of these events is the API 581 Methodology, other similar methods are available based on the use of statistical leak frequency data specific to "random ruptures".

The estimation of the frequency of loss of containment events must include the quantification of the influence of measures of risk prevention. Some methodologies are given in the literature (API 581; Papazoglou & Aneziris, 1999; Thomas, 1980) for this purpose, these assume that each part of the plant under analysis is characterized by the same percentage of causes of failure. This consideration is not acceptable.

4.1 The API 581 methodology

The method proposed in the standard API 581 *Risk Based Inspection Guideline*, , supplies a generic value of frequency of release from pipes and other main process equipment, this is a statistical average value. The standard provides a way to correct this value, depending on the specific characteristics of the system under examination and using appropriate correction factors based on the complexity of the system (number of flanges, valves, etc.).

The generic frequency is calculated using literature or incidental data for similar systems. In the API 581 standard frequencies of release are given for four diameters of hole: 1/4", 1", 4" and full bore (hole dimension equal to the pipe diameter). These are calculated assuming a log-normal distribution of the data, generic frequencies of release are the mean values.

The API 581 methodology defines a modification factor for the frequencies for each type of equipment, *equipment modification factor*, based on its complexity and its location. In order to take into account differences in the safety management system of an establishment, the method also defines an adjustment factor, *management systems evaluation factor*.

4.2. Quantification of the influence of management and organizational factors

The most common methodologies for the quantification of the influence of organizational and managerial factors on the frequency of release from pipework and vessels are the methods of Thomas (1980), developed in 1980, and the approach of Papazoglou & Aneziris, proposed in 1999. Both methods are based on the analysis of incidental data in the chemical industry.

The approach of Thomas to the estimation of frequency of leakage and rupture for piping and vessels is based on a statistical analysis of failures. The total frequency is initially identified through a global estimation based simply on the size, shape, welds and the age of the equipment. Subsequently, the results can be modified using specific factors for the type of equipment and the influence of the curves of learning for technology and design. Unfortunately these graphical correlations are based on obsolete data, new technologies are currently available, the factors evaluation requires valid data.

The method of Papazoglou permits the quantification of the effects of organizational and managerial factors on the frequency of leakage from vessels and pipes defining a link between an audit of the safety management system (SMS) and a quantitative risk analysis (QRA).

Another approach for the quantification of the influence of management and organizational factors has been proposed by Maschio et al. (2006), which is based on the methodology proposed by Papazoglou. First of all the method permits the exclusion of the causes of failure that can be prevented through the adoption of appropriate prevention measures. Thus it is possible to apply the method of Papazoglou using realistic values of the percentage of the causes of failure.

As mentioned in the previous section, in order to take into account the influence of managerial and organizational factors, also the API 581 methodology for the evaluation of the accidental frequencies uses the *management systems evaluation factor*. This adjustment factor is estimated on the base of the percentage of failure causes.

4.3 The method of Papazoglou

The *Papazoglou method* aims to quantify the organizational and managerial factors based on an audit of the safety management system (SMS). A Safety Management Audit, SMA, allows to verify the compliance of the safety management system with an ideal scheme. This can be made by analysing a number of combinations of causes of failure and mechanisms for the prevention of accidents. A number of important areas of concern are identified and each area is assessed from the SMS point of view through the audit as being GOOD, AVERAGE or POOR.

As mentioned above, the method proposed by Papazoglou is able to link the results of a management audit with the QRA model. This is possible by defining a factor modifying the average frequencies, which is calculated on the basis of the relative importance of each area of audit and the corresponding assessment. A QRA gives quantitative indexes which define the risk level of a plant taking into account its specific structure and its potential modes of failure, etc. The results of a QRA can help in the identification of optimum risk reduction actions by reducing the incidental frequencies and/or mitigating the consequences of the undesired events. For this reason the QRA represents a basic support for risk based decision making.

By means of the combination of generic causes of failure categories (underlying causes categories) and prevention mechanism (recovery mechanism), 54 audit areas of the SMS are defined but only 8 of them, indicated as main audit areas (MAAs), are meaningful from the point of view of the numbers of incidents.

The underlying cause of failure categories are: design (DES), maintenance activities (MAINT), operations during normal activities (OP), construction/installation (CON), manufacture/assembly (MANU), natural causes (NAT), domino (DOM), sabotage (SAB) and unknown origin (UO).

The recovery or preventive mechanisms are the mechanisms that theoretically could have recovered or prevented the failure. The categories of mechanism are appropriate hazard study of design as-built, e.g. HAZOP (HAZ), human factors review (HF), task-driven recovery activities (CHEC), routine, regular, recovery activities (ROUT), not recoverable (NR) and unknown recovery (UR).

The Papazoglou methodology consists of the following phases:

- in the first step of this approach the results of the audit are grouped through a subjective expert judgement into eight qualitative factors, one for each MAA. This is done by translating the results of the audit into an assessment of the elements of each MAA, then each area is assessed as GOOD, AVERAGE or POOR.
- the second step consists of grouping the eight assessments into a single number.

The method is based on an analysis of the frequencies of incidents that have occurred in the chemical industry, in particular Papazoglou found that the analysis of the loss of containment data, reported in the RIDDOR database, indicates that the frequencies of release for various plants spans two orders of magnitude and shows a certain symmetry around their mean values.

According to this observation the following equation was proposed, which can be used for the modification of the frequency of release:

$$\log f_{\mathrm{mod}} = \log f_{md} + \sum_i a_i \cdot x_i / 100 \qquad (1)$$

where f_{mod} = modified frequency; f_{md} = mean frequency of failure based on world-wide experience; a_i = weight coefficient for audit area i; and x_i = parameter indicating the judgement of the MAA i of the SMS following the audit.

Concerning x_i, it can assume the following values:

- -1 if the plant is judged GOOD
- 0 if the plant is judged AVERAGE
- +1 if the plant is judged POOR

The analysis of observed incidents in the chemical industry has assessed the relative frequency of occurrence for the eight MAAs, the normalization of these frequencies has provided the a_i values indicating the importance of each SMS area in terms of the likelihood of accidents in pipework, connections and vessels (Papazoglou & Aneziris, 1999).

5 THE PROPOSED METHOD FOR THE ESTIMATION OF THE FREQUENCY OF LOSS CONTAINMENT EVENTS

The methodology proposed by Papazoglou permits the evaluation of each part of the SMS, for this reason this is an excellent way to evaluate the organization and managerial factors. This method implies that each installation under analysis is characterized by the same combinations of origins of failure and mechanisms to prevent and/or mitigate them and thus by the same percentage of failure causes. It is known that different plants are characterized by differences in construction, operation and maintenance procedures and practices, thus they differ from the point of view of the percentage of causes of failure. In order to take into account these differences, in this work, a modified approach for the Papazoglou method for frequencies evaluation is proposed.

The modified method is based on an examination of the whole plant and permits to define how the measures of risk prevention adopted inside the establishment can influence the frequencies of rupture of pipework. In order to make this a detailed analysis for each unit of the plant is necessary and, from this, it will be possible to identify the causes of failure which can occur in each unit and the measures which can prevent them.

The proposed method is innovative since through its application it is possible to identify and exclude from the analysis all those causes of failure that are not present in the establishment because of the adoption of appropriate measures to prevent them. The adjustment of the percentages of causes of failure allows the use of realistic coefficients, this is fundamental for the application of the method.

The analysis of the overall plant allows a complete identification and quantification of the relationships between measures of risk prevention and causes of failure in piping. These permit the incorporation in the final results of a great number of plant-specific characteristics concerning the design, operational and maintenance aspects of the installation.

The proposed method modifies the frequency of release using equation 1, whose application depends on the definition of the weight coefficients a_i. The approach aims at the estimation of the influence on f_{md} of prevention measures, which have been judged *a priori* "GOOD", thus the problem is to determine which causes of failure can be prevented by a certain measure adopted by the Company.

In order to identify which measure can prevent or mitigate a failure and its effectiveness, also in this case an audit is necessary. After the definition of the relationship between the causes of failure and the measures of prevention, the weight coefficients for the causes of failure are estimated and, then, it is possible to modify the frequencies taking into account the prevention measures.

It is obvious that the a priori exclusion of some causes of failure requires modification of the mean frequency of failure obtained from the literature, f_{md}. This value will be reduced by a percentage equal to the excluded causes of failure. The methodology proposed by Papazoglou will then be applied to the *a priori* modified frequency.

5.1 Weight coefficients for the causes of failure

Many data regarding the main causes of release from piping is available in the literature, this information is summarized in Table 2 (direct causes). The evaluation of the weight coefficients can be made using the failure data of Lees, because these percentages are relatively more recent and specific for piping in the chemical industry. Nevertheless it is necessary to verify the consistency between the data reported by Lees and the causes of failure evidenced inside the examined establishment.

In some cases the values of the weight coefficients need to be corrected taking into account that modern design and manufacture and the use of new materials might reduce the number of failures due to certain causes. The correction of the data of Table 2 can be made using specific correction factors defined in agreement with the plant management.

5.2 Weight coefficients for corrosion phenomena

Concerning pipework (Lees, 1996), phenomena such as corrosion and mechanical causes of failures have been analysed in detail, thus the analysis of incidental data allows the distribution of these failure modes as shown in Table 4. General causes of failure, which are emphasized in Table 4, are general corrosion, stress corrosion cracking and fatigue. The number of failures caused by brittle fracture is small.

Using the data of Table 4, it is possible to detail the causes of failure for corrosion and mechanical failures, thus the single values of table 2 can be split into the contributions associated with each type of corrosion and/or mechanical failure. In order to divide the data of Table 2, a detailed analysis of the fluid flowing in the pipework and the process conditions is necessary. The analysis allows the definition of which types of corrosion can occur in the equipment.

5.3 Weight coefficients for human error

In order to improve the quantification of the influence of the measures of risk prevention on the frequency of rupture of pipework, a more detailed analysis of human errors has been carried out.

The literature on human error in process plants shows that a large proportion of serious incidents is attributable to errors in maintenance work, while the most frequent human error in pipework failures concerns the installation. A study of human error as cause of piping failures has been made by Bellamy and co-workers (Lees, 1996). Incidents have been classified as direct causes, origin of failures and recovery mechanisms. This data showed that operator error contributed 18 % to the direct causes of pipework failure, whilst defective pipe or equipment contributed 32 % and unknown causes 9 %.

Table 5 gives the distribution of human errors in underlying causes, it is possible to see that the predominant errors are in maintenance. As shown operator error usually can be disguised as other causes of failure (eg. impact, corrosion, etc). Therefore, using this table, human error can be split into the failure modes of Table 2 and then included in the other types of failure. The distribution of operator error in the other causes of failure provides the correct values for the weight coefficients a_i.

Table 4. Causes of corrosion and mechanical failure (Lees, 1996)

Corrosion	%	Mechanical failure	%
Cavitation	0.3	Abrasion, erosion or wear	5.4
Cold wall	0.4	Blister, plating	0.1
Cracking, corrosion fatigue	1.5	Brinelling	0.1
Cracking, stress corrosion	13.1	Brittle fracture	1.2
Crevice	0.9	Cracking, heat treatment	1.9
Demetallification	0.6	Cracking, liquid metal pen	0.1
End grain	0.4	Cracking, plating	0.6
Erosion-corrosion	3.8	Cracking, thermal	3.1
Fretting	0.3	Cracking, weld	0.6
Galvanic	0.4	Creep or stress rupture	1.9
General	15.2	Defective material	1.6
Graphitizzation	0.1	Embrittlement, sigma	0.3
High temperature	1.3	Embrittlement, strain age	0.4
Hot wall	0.1	Fatigue	14.
Hydrogen blistering	0.1	Galling	0.1
Hydrogen embrittlement	0.4	Impact	0.1
Hydrogen grooving	0.3	Leaking through defects	0.4
Intergranular	5.6	Overheating	1.9
Pitting	7.9	Overload	5.4
Weld corrosion	2.5	Poor welds	4.4
		Warpage	0.4
Subtotal	55.2	Subtotal	44.8

Table 5. Human error distribution in *underlying causes*, (Lees, 1996)

Underlying causes	%
Design	8
Manufacture	2
Construction	8
Operation	22
Maintenance	59
Sabotage	1
Total	100

The distribution of the human error in the causes of failure of Table 2 is possible defining the links between *failure causes* and *underlying causes* categories through a subjective expert judgement.

The links *failure causes/underlying causes* are given in Table 6.

Table 6. Links between *cause of failure/underlying cause*

Cause of failure	Underlying cause
Manufacture/fabrication	Manufacture/Construction
Corrosion & Erosion	Maintenance
External load	Maintenance
Impact	--
Thermal shock	Design/Operation
Mechanical shock	Design/Operation
Fatigue (vibration)	Design/Operation
Fatigue (low cycles)	Design/Operation
Wrong or incorrectly located in-line equipment	--
Operator error	--
Creep	Maintenance
Other and unknown	Sabotage

6 APPLICATION TO A CASE-STUDY

The methodology described by Maschio et al. (2006) for the quantification of the influence of the measures of prevention of risk on the frequencies of release from vessels and pipework has been tested by its application to a real industrial plant. This approach has been used for the calculation of the frequencies of the random events which can potentially occur in pipework. The case study presented here is for a petrochemical plant (confidential).

In this case the initial frequencies of failure have been collected from the Safety Report of the establishment, then the influence of the measures of risk prevention on the causes of failure have been discussed and defined in agreement with the plant management. In order to define which causes of failure can be prevented by a certain measure adopted by the Company, an audit has been made.

An example is described. The rupture of a pipe coming from a vessel containing a flammable and toxic liquid has been hypothesized. Two dimensions of leakage have been assumed, 5% and 20% of the pipe diameter, and then the modification of the frequencies of release has been made using both the proposed methodology and the method of the direct reduction of the percentage of the failure causes.

Using the data of Table 2 and 4, the percentages of causes of failure have been corrected as discussed. The corrected values have been normalized. Before the application of the modified procedure, the initial value of frequency has been reduced by the percentage of excluded causes of failure.

6.1 The modified procedure for the quantification of frequency

Application of the modified method can be made using the following steps:

- Definition of the weight-coefficients a_i, which comprise a detailed analysis both for human error and corrosion phenomena;
- Formulation of judgements on the inspection methods;
- Calculation of the frequencies using equation 1.

In order to estimate the effect on f_{md} of measures of risk prevention through equation 1, it is necessary to formulate a judgement x_i for each preventive measure. The attribution of the judgements has been made by analysing each pipe-line and defining which causes of Table 2 and sub-causes of Table 4 can be detected by each measure.

6.2 The method of the direct reduction of the percentage of the failure causes

Using the data of Table 2, it is also possible to modify the frequency of rupture reducing its value by the percentage of the failure causes (P_i) prevented using this measure. Also in this case the *a priori* exclusion of some causes of failure modifies the value of the initial frequency that will be reduced by the percentage of excluded causes of failure.

A more complete quantification of the influence of the routine inspections must take into account also their effectiveness. The effectiveness represents the percentage of failures identified during these inspections.

In order to apply this method it has been necessary to identify which causes of Table 2 and sub-causes of Table 4 can be detected using a certain inspection technique, then the value of P_i will be equal to the corrected value of a_i multiplied by the effectiveness of the measure. In this case the use of the effectiveness classes defined in *API 581 Risk-Based Inspection Base Resource Document* and shown in Table 7 has been adopted.

Table 7. Qualitative Inspection Effectiveness Category

Highly Effective: Inspection methods that correctly identify the anticipated in-service damage in nearly every case. (90%)
Usually Effective: The inspection methods will correctly identify the actual damage state most of the time. (70%)
Fairly Effective: The inspection methods will correctly identify the true damage state about half of the time. (50%)
Poorly Effective: The inspection methods will provide little information to correctly identify the true damage state. (40%)
Ineffective: The inspection methods will provide no or almost no information that will correctly identify the true damage state. (33%)

6.3 Results

Table 8 shows the frequencies of loss of containment modified by the application of the method described. A number of applications of the method shows that generally the frequency of the random event decreases by about an order of magnitude or more in some cases.

The method of the direct reduction of the percentage of the failure causes, has been applied in order to compare the methodologies and to verify the consistency of the proposed procedure with a conservative method. Comparison of the results demonstrates the validity of the proposed methodology.

The entity of the risk reduction can be visualized using risk matrixes (Figs. 1-2).

Table 8. Results

Leak dimension	Frequency	Modified frequencies	
		Proposed method	Direct reduction method
5%	1.93E-03	1.13E-04	4.47E-04
20%	1.25E-04	7.35E-06	2.90E-05

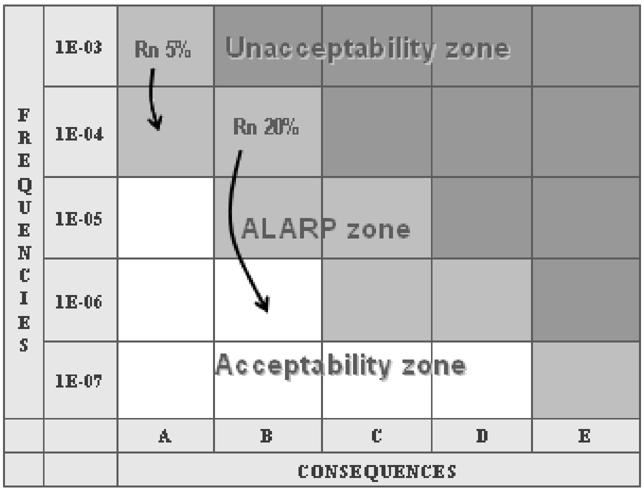

Figure 1. Risk matrix: results of modified frequencies (proposed method).

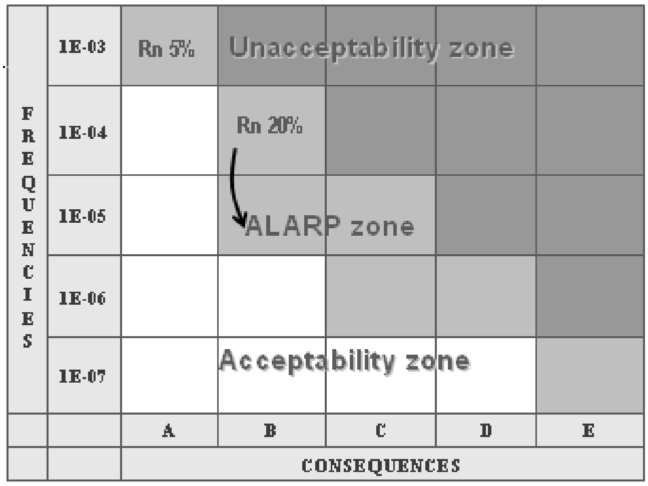

Figure 1. Risk matrix: results of modified frequencies (direct reduction).

The matrix provides a useful tool in order to define the acceptability of the risk associated with an industrial activity, for this reason the identification of three levels of risk, acceptable, ALARP and unacceptable, is necessary in particular for the risk-based decisions. Using the matrices it has been possible to verify if the adoption of certain preventive measures of the risk can move an event from a critical zone to the acceptability zone.

7. CONCLUSIONS

The objective of this work has been the definition of an approach for the calculation of loss of containment frequencies taking into account managerial and organizational factors. This necessity is due to the observation that the main cause of accidents are events that are often due to deficiencies in the corporate structure.

Furthermore the use of common risk analysis techniques does not allow taking account management and organizational factors which are of primary importance in defining the real risk level of a chemical plant and therefore for planning the resources and procedures for emergencies.

The approach suggested in this paper for the quantification of the effects of organizational and managerial factors, has been based on the methodology proposed by Papazoglou (1999). Taking advantage of this method the object of assessing the frequency correction factors has been achieved through the definition of the relationship between the causes of failure and the measures adopted by the company in order to prevent and/or mitigate them.

Regarding the application of the method it has been possible the validation of the procedure using a comparison with the method of the direct reduction of the percentages of causes of failure.

The proposed method appears to be innovative because of the *a priori* exclusion of all the causes of failure that are not present in the establishment because of the adoption of appropriate measures to prevent them. The correction of the percentages of breakdown allows the use the real weight-coefficients.

The approach proposed in this work is suitable to the various high risk industrial activities. It requires for each case the modification of the weights coefficients associated with the single causes of failure and to formulate judgment on the measures of prevention of the risk adopted by the company, in this way it is possible to reproduce the plant-specific characteristics concerning the design, operation and maintenance of the installation.

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MODELLING ENVIRONMENT AND INFRASTRUCTURE INFLUENCE ON RELIABILITYAND OPERATION PROCESSES OF PORT OIL TRANSPORTATION SYSTEM

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ABSTRACT

In the paper a probabilistic model of industrial systems environment and infrastructure influence on their operation processes is proposed. Semi-markov processes are used to construct a general model of complex industrial systems' operation processes. Main characteristics of this model are determined as well. In particular case, for a port oil transportation system, its operation states are defined, the relationships between them are fixed and particular model of its operation process is constructed and its main characteristics are determined. Further, the joint model of the system operation process and the system reliability is defined sand applied to the reliability evaluation of the port oil transportation system.

1 INTRODUCTION

Most real transportation systems are very complex and it is difficult to analyse their reliability and availability. Large numbers of components and subsystems and their operating complexity cause that the evaluation and optimisation of their reliability and availability is complicated. The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability characteristics is often very difficult to fix and to analyse. A convenient tool for solving this problem is semi-markov modelling of the systems operation processes proposed in the paper. Therefore, the common usage of the system's reliability evaluation methods and semi-markov modelling the system's exploitation process in order to construct a general system reliability model related to its operation process is proposed in the paper. Statistical methods of the general model unknown parameters estimation are proposed. Main characteristics of this model are determined. Computer programme for determining these all values is shortly described. The way of its application to reliability evaluation of a port grain transportation system is illustrated.

2 MODELLING OF SYSTEM OPERATION PROCESS

Usually the system environment and infrastructure have either an explicit or implicit strong influence on the system operation process. As a rule some of the initiating environment events and infrastructure conditions define a set of different operation states of the industrial system. Thus, we assume that the system during its operation is operating in $\nu, \nu \in N,$ different operation states. After this assumptions, we can define the system operation process $Z(t)$, $t \in < 0,+\infty >,$ with discrete states from the set of states $Z = \{z_1, z_2, \ldots, z_\nu\}$. If the system operation process $Z(t)$ is semi-markov

(Grabski 2002), (Soszynska 2006a,b, Soszynska 2007) with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b, l = 1,2,...,v$, $b \neq l$, then it may be described by:
- the vector of probabilities of the system operation process initial states

$$[p_b(0)]_{1\times v} = [p_1(0), p_2(0),..., p_v(0)],$$

where

$$p_b(0) = P(Z(0) = z_b) \text{ for } b = 1,2,...,v,$$

- the matrix of probabilities of the system operation process transitions between the operation states

$$[p_{bl}]_{v\times v} = \begin{bmatrix} p_{11} \ p_{12} \cdots p_{1v} \\ p_{21} \ p_{22} \cdots p_{2v} \\ \cdots \\ p_{v1} \ p_{v2} \cdots p_{vv} \end{bmatrix},$$

where $p_{bb} = 0$ for $b = 1,2,...,v$,
- the matrix of the system operation process conditional sojourn times θ_{bl} distribution functions

$$[H_{bl}(t)]_{v\times v} = \begin{bmatrix} H_{11}(t) \ H_{12}(t) \ldots H_{1v}(t) \\ H_{21}(t) \ H_{22}(t) \ldots H_{2v}(t) \\ \cdots \\ H_{v1}(t) \ H_{v2}(t) \ldots H_{vv}(t) \end{bmatrix},$$

where

$$H_{bl}(t) = P(\theta_{bl} < t) \text{ for } b, l = 1,2,...,v, \ b \neq l,$$

and

$$H_{bb}(t) = 0 \text{ for } b = 1,2,...,v.$$

Under these assumptions, the mean values of the system operation process conditional sojourn times θ_{bl} are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t), \ b, l = 1,2,...,v, \ b \neq l. \tag{1}$$

By the formula for total probability the unconditional distribution functions of the sojourn times θ_b of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,...,v$, are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \ b = 1,2,...,v. \tag{2}$$

Hence, the mean values $E[\theta_b]$ of the system operation process unconditional sojourn times θ_b in the particular operation states are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}, \; b = 1,2,...,v, \tag{3}$$

where M_{bl} are defined by (1).
Moreover, it is well known (Grabski 2002) that the limit values of the system operation process transient probabilities at the particular operation states

$$p_b(t) = P(Z(t) = z_b), \; t \in < 0,+\infty), \; b = 1,2,...,v,$$

are given by

$$p_b = \lim_{t \to \infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \; b = 1,2,...,v,$$

(4)

where M_b, $b = 1,2,...,v,$ are defined by (3), whereas the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases}$$

(5)

Other interesting characteristics of the operation process $Z(t)$ possible to obtain are its total sojourn times $\hat{\theta}_b$ in the particular operation states z_b, $b = 1,2,...,v.$ It is well known (Grabski 2002) that the system operation process total sojourn times $\hat{\theta}_b$ in the particular operation states z_b, for sufficiently large operation time θ, have approximately normal distribution with the expected value given by

$$E[\hat{\theta}_b] = p_b \theta, \; b = 1,2,...,v, \tag{6}$$

where p_b are given by (4).

3 STATISTICAL IDENTYFICATION OF SYSTEMOPERATION PROCESS MODEL

In order to estimate parameters of the operation process model the following step should be done:
- to fix the number of states v of the system operation process $Z(t)$ and to define the operation states z_1, z_2, …, z_v of the set Z,
- to fix the vector of realisations

$$[n_b(0)] = [n_1(0), n_2(0), ..., n_\nu(0)],$$

of the numbers $n_b(0)$, $b = 1,2,...,\nu$, of the system operation process $Z(t)$ transients in the particular states z_b at the initial moment $t = 0$

- to fix the matrix of realisations

$$[n_{bl}] = \begin{bmatrix} n_{11} \ n_{12} \ \dots n_{1\nu} \\ n_{21} \ n_{22} \ \dots n_{2\nu} \\ \dots \\ n_{\nu 1} \ n_{\nu 2} \ \dots n_{\nu\nu} \end{bmatrix},$$

of the numbers n_{bl}, $b, l = 1,2,...,\nu$, of the system operation process $Z(t)$ transitions from the state z_b into the state z_l during the experiment time Θ,

- to fix the vector of realisations

$$[p(0)] = [p_1(0), p_2(0), \dots, p_\nu(0)],$$

of the initial probabilities $p_b(0)$, $b = 1,2,...,\nu$, of the system operation process $Z(t)$ transients in the particular states z_b at the moment $t = 0$, according to the formula

$$p_b(0) = \frac{n_b(0)}{n(0)} \text{ for } b = 1,2,...,\nu,$$

where

$$n(0) = \sum_{b=1}^{\nu} n_b(0),$$

is the total number of the system operation process $Z(t)$ realisations at $t = 0$,

- to fix the matrix of realisations

$$[p_{bl}] = \begin{bmatrix} p_{11} \ p_{12} \ \cdots p_{1\nu} \\ p_{21} \ p_{22} \ \cdots p_{2\nu} \\ \cdots \\ p_{\nu 1} \ p_{\nu 2} \ \cdots p_{\nu\nu} \end{bmatrix},$$

of the transition probabilities p_{bl}, $b, l = 1,2,...,\nu$, of the system operation process $Z(t)$ from the operation state z_b into the operation state z_l during the experiment time Θ, according to the formula

$$p_{bl} = \frac{n_{bl}}{n_b} \text{ for } b, l = 1,2,...,\nu, \ b \neq l,$$

$$p_{bb} = 0 \quad \text{for } b = 1,2,...,\nu,$$

where

$$n_b = \sum_{b \neq l}^{\nu} n_{bl} \ , \ b = 1,2,...,\nu,$$

is the realisation of the total number of the system operation process $Z(t)$ transitions from the operation state z_b during the experiment time Θ,

- to formulate and to verify the hypotheses about the conditional distribution functions $H_{bl}(t)$ of the system operation process $Z(t)$ sojourn times θ_{bl}, $b,l = 1,2,...,\nu$, $b \neq l$, in the state z_b while the next transition is the state z_l on the base of their realisations θ_{bl}^k, $k = 1,2,...,n_{bl}$ during the experiment time Θ.

4 APPLICATION

As an example we will analyse the reliability of the port oil transportation system in its operation process (Kolowrocki & Soszynska 2006, - Kolowrocki & Soszynska 2007, Soszynska 2006). The considered system is composed of three terminal parts *A*, *B* and *C*, linked by the piping transportation systems.

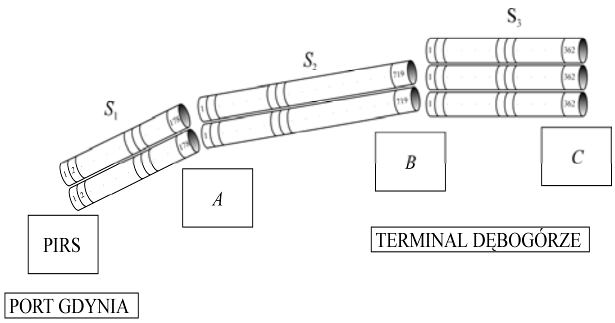

Figure 1. The scheme of port oil transport system.

The Oil Terminal in Dębogórze is designated for the reception from ships, the storage and sending by carriages or cars the oil products. It is also designated for receiving from carriages or cars, the storage and loading the tankers with oil products such like petrol and oil.

The unloading of tankers is performed at the piers placed in the Port of Gdynia. The piers is connected with terminal part *A* through the transportation subsystem S_1 built of two piping lines composed of steel pipe segments with diameter of 600 mm. In the part *A* there is a supporting station fortifying tankers pumps and making possible further transport of oil by the subsystem S_2 to the terminal part *B*. The subsystem S_2 is built of two piping lines composed of steel pipe segments of the diameter 600 mm. The terminal part *B* is connected with the terminal part *C* by the subsystem S_3. The subsystem S_3 is built of one piping line composed of steel pipe segments of the diameter 500 mm and two piping lines composed of steel pipe segments of diameter 350 mm. The terminal part *C* is designated for the loading the rail cisterns with oil products and for the wagon sending to the railway station of the Port of Gdynia and further to the interior of the country.

The Port Oil Transportation system consists of three subsystems S_1, S_2, S_3.

Subsystem S_1 consist of k_n = 2 two identical pipelines, each composed of l_n = 178 elements. In each pipeline there are:

- 176 pipe segments,

- 2 valves.

Subsystem S_2 consist of k_n = 2 two identical pipelines, each composed of l_n = 719 elements. In each pipeline there are:
- 717 pipe segments,
- 2 valves.

Subsystem S_3 consist of two pipelines of the first type and one second type, each composed of l_n = 362 elements. In each pipeline of the first type there are:
- 360 pipe segments (Ø=350mm),
- 2 valves.

In pipeline of the second type there are:
- 360 pipe segments (Ø=500mm),
- 2 valves.

Taking into account the operation process of the considered transportation system we distinguish the following as its five operation states:

• an operation state z_1 – transport of two different kinds of medium from the terminal part *B* to part *C* using two out of three pipelines in part S_3, with the structure given in Figure 2,

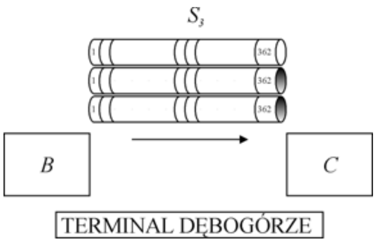

Figure 2. The scheme of port oil transport system at the operation state z_1

• an operation state z_2 – transport of one kind of medium from the terminal part *C* (from carriages) to part *B* using one out of three pipelines in part S_3, with the structure given in Figure 3,

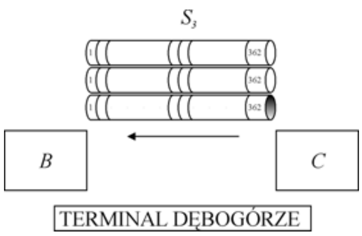

Figure 3. The scheme of port oil transport system at the operation state z_2

• an operation state z_3 – transport of one kind of medium from the terminal part *B* through part *A* to the piers using one out of two pipelines in part S_2 and one out of two pipelines in part S_1, with the structure given in Figure 4,

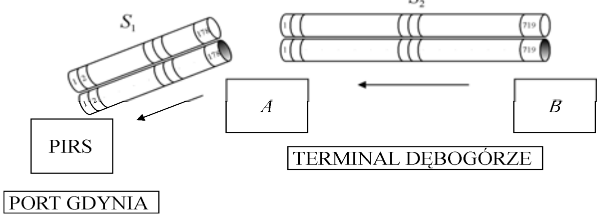

Figure 4. The scheme of port oil transport system at the operation state z_3

• an operation state z_4 – transport of two kinds of medium from the piers through parts A and B to part C using both pipelines in part S_1, both in part S_2 and two out of three pipelines in part S_3, with the structure given in Figure 5,

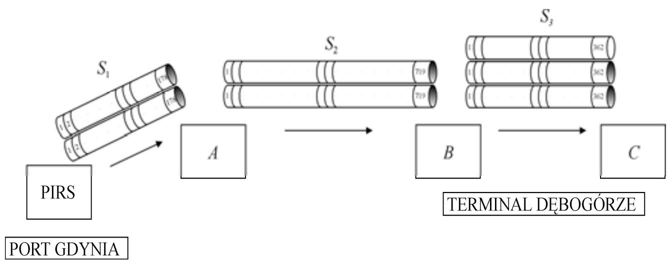

Figure 5. The scheme of port oil transport system at the operation state z_4

• an operation state z_5 – transport of one kind of medium from the piers through part A and B to part C using one out of two pipelines in parts S_1 and S_2 and one out of three pipelines in part S_3, with the structure given in Figure 6 .

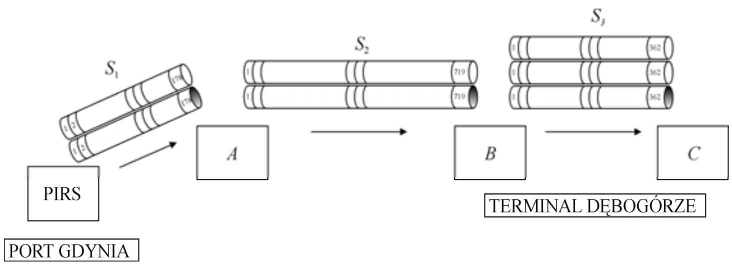

Figure 6. The scheme of port oil transport system at the operation state z_5

Moreover, we almost arbitrarily, i.e. slightly using an expert opinion, assume the following matrix of the conditional distribution functions

$$[H_{bl}(t)] = \begin{bmatrix} 0 & 0 & 0 & 1-e^{-37117.4t^2} & 0 \\ 0 & 0 & 1-e^{-19174.9t^2} & 0 & 0 \\ 1-e^{-148469.5t^2} & 1-e^{-107737.1t^2} & 0 & 0 & 0 \\ 0 & 1-e^{-969634.1t^2} & 0 & 0 & 1-e^{-969634.1t^2} \\ 0 & 0 & 0 & 1-e^{-29.1t^2} & 0 \end{bmatrix}.$$

of the oil terminal operation process sojourn times θ_{bl}, $b,l = 1,2,3,4,5$, in the distinguished operation states z_1, z_2, z_3, z_4, z_5, and the matrix of the probabilities of transitions between the operation states

$$[p_{bl}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.11 & 0.89 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Further, according to (2), the unconditional distribution functions of the oil terminal operation process $Z(t)$ sojourn times θ_b in the states z_b, $b = 1,2,3,4,5,$ are given by

$$H_1(t) = 1 - \exp[-37117.4t^2],$$

$$H_2(t) = 1 - \exp[-19174.9t^2],$$

$$H_3(t) = 1 - 0.11 \cdot \exp[-148469.5t^2] - 0.89 \cdot \exp[-107737.1t^2],$$

$$H_4(t) = 1 - 0.5 \cdot \exp[-969634.1t^2] - 0.5 \cdot \exp[-969634.1t^2],$$

$$H_5(t) = 1 - \exp[-29.1t^2],$$

and their mean values, from (3), are

$$M_1 = E[\theta_1] = 1 \cdot 0.005 = 0.005,$$

$$M_2 = E[\theta_2] = 1 \cdot 0.006 = 0.006,$$

$$M_3 = E[\theta_3] = 0.11 \cdot 0.002 + 0.89 \cdot 0.003 = 0.003,$$

$$M_4 = E[\theta_4] = 0.5 \cdot 0.001 + 0.5 \cdot 0.001 = 0.001,$$

$$M_5 = E[\theta_5] = 1 \cdot 0.164 = 0.164.$$

Since from the system of equations

$$\begin{cases} [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.11 & 0.89 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \end{cases}$$

we get

$$\pi_1 = \frac{1}{22},\ \pi_2 = \frac{9}{22},\ \pi_3 = \frac{9}{22},\ \pi_4 = \frac{2}{22},\ \pi_5 = \frac{1}{22},$$

then the limit values of the transient probabilities $p_b(t)$ at the operational states z_b, according to (4), are

$$p_1 = 0.018,\ \ p_2 = 0.228,\ \ p_3 = 0.095,\ \ p_4 = 0.007,\ \ p_5 = 0.652.$$

And, by (6), the expected values of the total sojourn times $\hat{\theta}_b$, $b = 1,2,3,4,5,$ in particular operation states for the oil terminal system operation time $\theta = 1$ year $= 365$ days are given by

$$E[\hat{\theta}_1] = 0.018 \cdot 365 \cong 6.6 \text{ days},$$

$$E[\hat{\theta}_2] = 0.228 \cdot 365 \cong 83.2 \text{ days},$$

$$E[\hat{\theta}_3] = 0.095 \cdot 365 \cong 34.7 \text{ days},$$

$$E[\hat{\theta}_4] = 0.007 \cdot 365 \cong 2.6 \text{ days},$$

$$E[\hat{\theta}_5] = 0.652 \cdot 365 \cong 238.0 \text{ days}.$$

5 RELIABILITY OF SYSTEMS IN VARIABLE OPERATION PROCESS

We assume that the changes of the system operation process $Z(t)$ states have an influence on the system components E_i, $i = 1,2,...,n,$ reliability and the system reliability structure as well. Thus, we denote the conditional reliability function of the system component E_i while the system is at the operational state z_b, $b = 1,2,...,v,$ by

$$R_i^{(b)}(t) = P(T_i^{(b)} > t | Z(t) = z_b) \text{ for } t \in < 0, \infty), \; i = 1,2,...,n, \; b = 1,2,...,\nu,$$

and the conditional reliability function of the system while the system is at the operational state z_b, $b = 1,2,...,\nu,$ by

$$\boldsymbol{R}_n^{(b)}(t) = P(T^{(b)} > t | Z(t) = z_b) \text{ for } t \in < 0, \infty), \; b = 1,2,...,\nu, \; n \in N,$$

where

$$T^{(b)} = T(T_1^{(b)}, T_2^{(b)}, ..., T_n^{(b)}) \text{ for } t \in < 0, \infty), \; b = 1,2,...,\nu, \; n \in N,$$

and

$$\boldsymbol{R}_n^{(b)}(t) = \boldsymbol{R}_n(R_1^{(b)}(t), R_2^{(b)}(t), ..., R_n^{(b)}(t)) \text{ for } t \in < 0, \infty), \; b = 1,2,...,\nu, \; n \in N.$$

The reliability function $R_i^{(b)}(t)$ is the conditional probability that the component E_i lifetime $T_i^{(b)}$ is greater than t, while the process $Z(t)$ is at the operation state z_b. Similarly, the reliability function $\boldsymbol{R}_n^{(b)}(t)$ is the conditional probability that the system lifetime $T^{(b)}$ is greater than t, while the process $Z(t)$ is at the operation state z_b. In the case when the system operation time θ is large enough, the unconditional reliability function of the system

$$\boldsymbol{R}_n(t) = P(T > t) \text{ for } t \in < 0, \infty),$$

where T is the unconditional lifetime of the system is given by

$$\boldsymbol{R}_n(t) \cong \sum_{b=1}^{v} p_b \ \boldsymbol{R}_n^{(b)}(t) \text{ for } t \geq 0 \tag{8}$$

and the mean value of the system lifetime is

$$\mu \cong \sum_{b=1}^{v} p_b \mu_b, \tag{9}$$

where

$$\mu_b = \int_0^\infty \boldsymbol{R}_n^{(b)}(t)dt,$$

and p_b are given by (4), and the variance of the system lifetime is

$$\sigma^2 = 2\int_0^\infty t \ \boldsymbol{R}_n(t)dt - [\mu]^2. \tag{10}$$

6 RELIABILITY OF PORT OIL TRANSPORTATION SYSTEM IN VARIABLE OPERATION PROCESS

Using the model considering in section 6, the results of section 5 and the results given in [Kolowrocki & Soszynska 2006, Soszynska 2006, Soszynska 2007) by (7) and (8), we have

$$\boldsymbol{R_n}(t) \cong 0.018 \cdot \boldsymbol{R_n}^{(1)}(t) + 0.228 \cdot \boldsymbol{R_n}^{(2)}(t) + 0.095 \cdot \boldsymbol{R_n}^{(3)}(t) + 0.007 \cdot \boldsymbol{R_n}^{(4)}(t) + 0.652 \cdot \boldsymbol{R_n}^{(5)}(t) \tag{11}$$

where $\boldsymbol{R}_n^{(1)}(t)$, $\boldsymbol{R}_n^{(2)}(t)$, $\boldsymbol{R}_n^{(3)}(t)$, $\boldsymbol{R}_n^{(4)}(t)$, $\boldsymbol{R}_n^{(5)}(t)$ are the system reliability functions in particular operation states determined by (24), (28), (32), (36) and (40) given in (Soszynska 2007).
Since according to the results given in (Soszynska 2007), the mean values of the conditional system lifetimes in years are

$$\mu_1 = \int_0^\infty \boldsymbol{R}_n^{(1)}(t)dt = 0.364 \text{ year},$$

$$\mu_2 = \int_0^\infty \boldsymbol{R}_n^{(2)}(t)dt = 0.807 \text{ year},$$

$$\mu_3 = \int_0^\infty \boldsymbol{R}_n^{(3)}(t)dt = 0.307 \text{ year},$$

$$\mu_4 = \int_0^\infty \boldsymbol{R}_n^{(4)}(t)dt = 0.080 \text{ year},$$

$$\mu_5 = \int_0^\infty \boldsymbol{R}_n^{(5)}(t)dt = 0.275 \text{ year}.$$

then applying (11), (9) and (10), we get the mean value and the standard deviation of the system unconditional lifetime given by

$$\mu \cong 0.018 \cdot 0.364 + 0.228 \cdot 0.807 + 0.095 \cdot 0.307 + 0.007 \cdot 0.080 + 0.652 \cdot 0.275 \cong 0.40 \text{ year}$$

$$\sigma \cong 0.37 \text{ year.}$$

7 CONCLUSIONS

The paper proposes an approach to the solution of practically very important problem of linking the systems' reliability and their operation processes. To involve the interactions between the systems' operation processes and their varying in time reliability structures and components' reliability characteristics a semi-markov model of the systems' operation processes and system conditional reliability functions are used. This approach gives practically important in everyday usage tool for reliability evaluation of the systems with changing reliability structures and components' reliability characteristics during their operation processes. Application of the proposed method is illustrated in the reliability evaluation of the port oil transportation system. The reliability input data concerned with the operation process and reliability functions of the components of the port oil transportation system are not precise. They are coming from experts and are concerned with the mean lifetimes of the system components and with the conditional sojourn times of the system in the operation states under arbitrary assumption that their distributions are Weibull. Thus, the final results obtained in the system reliability characteristics evaluation are not precise as well and should be treated as an example of the proposed model possible application. In further developing of the proposed methods it seem to be possible to obtain the results useful in the complex technical systems related to their operation processes reliability evaluation, improvement and optimisation.

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