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The article is devoted to construction and research of dynamic stochastic model of park of aircrafts. A stochastic is enclosed in all of natural characteristic exploitations of this set of apparatuses: times of flight and landing, possibility of receipt of damage on flight, including the past recovery air apparatus; times of repair. The estimations of total possible flights are got for the any fixed interval of time.

In this paper a problem of a construction of accuracy and asymptotic formulas for a reliability of internet type networks is solved. Analogously to [1] such network is defined as a tree where each node is connected directly with a circle scheme on a lower level with n>0 nodes. A construction of accuracy and asymptotic formulas for probabilities of an existence of working ways between each pair of nodes of the internet type network is based on a recursive definition of these networks and on asymptotic formulas for a reliability of a random port. This asymptotic formula represents the port reliability as a sum of probabilities of a work for all ways between initial and final nodes of this port. An estimate of a relative error and a complexity of these asymptotic calculations for a radial-circle scheme are shown.

The overall objective of the maintenance process is to increase the profitability of the operation and optimize the availability. However, the availability of a system is described according to lifetime and downtime. It is often assumed that these durations follow the exponential distribution. The work presented in this paper deals with the problem of availability modeling when the failure and repair rates are variable. The lifetime and downtime were both governed by models of Weibull (the exponential model is a particular case). The differential equation of the availability was formulated and solved to determine the availability function. An analytical model of the asymptotic availability was established as a theorem and proved. As results deduced from this study, a new approach of modeling of the asymptotic availability was presented. The developed model allowed an easy evaluation of the asymptotic availability. The existence of three states of availability for a system has been confirmed by this evaluation. Finally, these states can be estimated by comparing the shape parameters of the Weibull model for the failure and repair rates.

The main idea of Universal Generating Function is exposed in reliability applications. Some commonalities in this approach and the C++ language are discussed.

In this paper we construct effective single sampling plans for reliability inspection, when the distribution of failure times of underlying objects obey a Weibull law. To this purpose we use the index average lifetime (E (T)/testing time (T) for two values of E(T) - acceptable and non acceptable ones - and known shape parameter (K) of the Weibull cdf. We derive also a relationship between this index and reliability function R(t) of the assumed statistical law. A numerical illustrations is provided in the case of Rayleigh cdf - that is for a Weibull shape k = 2.

In this paper a problem of asymptotic and numerical estimates of relative errors for different asymptotic formulas in the reliability theory are considered. These asymptotic formulas for random networks are similar to calculations of Feynman integrals. A special interest has analytic and numerical comparison of asymptotic formulas for the most spread Weibull and Gompertz distributions in life time models. In the last case it is shown that an accuracy of asymptotic formulas is much higher.

NATURAL DISASTERS AND STRUCTURAL SURVIVABILITY

V. Raizer

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San Diego, CA

1. PROBLEM OF DISASTER'S PREDICTION

The term "disaster" is known to denote any environmental changes putting human lives under treat or materially deteriorating living conditions. A considerable part of disasters comprises natural calamities. These disasters can originate inside Earth (earthquakes, volcanic processes), near or on its surface (disturbance of slope stability, karsts, considerable changes in soil conditions and ground's settlements). The causes of disasters can as well be associated with a water, either at a liquid (flood, tsunami) or at a frozen state (complex or glacier avalanches), and, finally with atmospheric conditions. In many cases successions of interdependent disasters are possible, including these occurring in different media (earthquake-tsunami, earthquake-landslide, and lands-flood etc.).

The analysis of conditions associated with the onset and the development of the dangerous natural processes becomes at present the subject of both the natural research and the engineering analysis. New cities, industrial, power and other facilities are mostly erected in areas where natural calamities emerge. Environmental changes of natural or man-caused origin lead to disastrous effects in areas developed earlier, too.

It is always that the mechanisms of the dangerous natural phenomena can be represented by the direct cause-and –the effect relations. A prediction of the type, the time and the size of the expected disaster, even if practicable, can only be probabilistic. Therefore, for the analysis of the structures for the areas where natural calamities can take place the probabilistic approach and the use of the reliability theory can prove to be more efficient and necessary than in regular cases.

The level of the development of many problems concerning the comprehension of natural calamity's origination and hence, the level of the efficiency in predicting their time, conditions and the character of manifestation, as well as the development of measures for their prevention and mitigation of losses, leg behind with the practical needs of the national economy. To a certain extent, it can be accounted for by absence of common approaches to the constructing models of some natural disasters and the methods of their prediction.

To predict future events using statistical methods, we should dispose of information for rather a long time period. Practically, however, the prediction is based on limited information, due to which it is often imprecise and sometimes merely incorrect.

Prediction accuracy, however, fluctuates within a certain range, if the prediction is based on statistics alone. It implies that different methods should also be employed in prediction. For sufficiently substantiated prediction the following methods are generally used [2-6]: malty-dimensional regression analysis, theory of quantitative analysis, graph theory for error analysis, Delphi method (method of expert evaluation), and statistical analysis.

The latest research in the field of forecasting disastrous events and preventing the maximum risk and losses due to abnormal actions have shown that ever more widespread, together with the foregoing five methods, is becoming the approach based on the theory of fuzzy sets [7]. This can be accounted for by the fact that any classification, any algorithm, any rule of decision making, any model (theoretical or calculated) can be correlated with its fuzzy analogue. For example, classification implies the breakdown of a totality of elements into classes or groups of similar elements. Rigorous classification refers each element to a single definite class, whereas according to fuzzy classification it can belong to different classes depending to on certain conditions. The fuzzy classification is generally more realistic than the rigorous one. The use of the theory of fuzzy sets permits to elaborate, basing on fuzzy input data, a certain optimum solution setting applicability borders.

2. METHODOLOGICAL ASPECTS OF THE ANALYSIS

An engineering analysis proper is not aimed at evaluating of the probabilistic parameters that represent natural processes and, in theory; the engineer should obtain from experts in natural sciences properly represented statistical information. The task of the engineer is to assess, using this information, a risk associated with a particular structure, and to device measures of disaster protection of human life and property, efficient terms of the data available. In practice, however, similarly to the case of estimating disastrous wind's speed or water's pressure parameters, for example, when designing safe structures or estimating a stressed state of undisturbed soil's mass, engineers dealing with the theory of a structural analysis cannot count on obtaining the foregoing information "from the outside". Hence, an independent statistical analysis of available information is required, so that the data based on it should correspond to peculiarities involved in the engineering analysis. Moreover, sometimes it becomes necessary to describe, in terms of these peculiarities, mechanisms of natural phenomena and to reveal their quantitative characteristics determining the extent of a structural damage.

Another moment that should be born in mind is the comprehension that for not all natural disastrous effects structures can and must be designed and it is not always that engineering measures aimed to mitigating of the destructive effect of disasters can be designed and implemented. Design procedures envisaged in disregarding disastrous effects of an artificial origin. Similarly, when, for example, developing the code of design with due regard for the natural disasters one should not tackle an unsolvable problem of an analysis for all types or levels of the foregoing effects. In fact, there is nothing new about it: the same idea is employed in specifying the "assumed" seismicity for which the structures in the area are to be designed, whereas a higher-level earthquake motion is considered a "beyond-design" occurrence. Here the expected events can be classified as "design" or "beyond-design", according to the level of motion. Meanwhile, referred to "beyond-design" cases are, sometimes, entire types of events hard to predict or even quite unpredictable occurrences, as mentioned above. It needs to be said that the formal division of seismic effects upon structures and occurrences associated with them into "design" and "beyond-design" cannot be accepted, unless their consequences will be taken into account.

We know that in structural design for regular loads the term "failure" is generally used to denote a random event of realization of one of its damage states. The aim of a competent design consists in specifying of the structural parameters in a way that would exclude such failures due to design loads. In the design for natural disasters, however, the requirement of the inadmissibility of the failure in the foregoing sense can hardly be fulfilled and it should therefore be replaced by the requirement of the structural non-destructibility. Non-destructibility would imply the preservation of the main structure's member that would permit to retrofit the whole structure (building, for example). There are some types of structures or buildings, however, for which the foregoing consideration doesn't seem to be important. As far as structures whose failure presents a global threat to the environment are concerned, non-destructibility means, in this case, the prevention from the failure of structural members that contain or emit substances containing environment. This, however, applies to a design situation. As regards "beyond-design" situation, special engineering solutions are seemingly required for the above structures. The solutions should ensure, even in the case of the most improvable and unpredictable effects, spontaneous deviation from hazardous production processes and self-isolation of units containing detrimental or hazardous components.

3. STATITICAL EVALUATION OF NATURAL DISASTERS

The probabilistic approach proper employed in evaluating a possible level of any disastrous phenomenon in a particular area can also prove to be efficient and useful when the structure or soil are not supposed to be analyzed for the mentioned phenomenon. Therefore, when elaborating a probabilistic concept for natural disasters one should primarily consider in a general form the feasibility of using the statistical approach for representing the disastrous effects.

In principle the aim of the statistical analysis in terms of the problem being considered is the probabilistic prediction of the time and the place of a natural disaster or, on the contrary, for the given place and the service life of the structure – the probability of occurrence for the given period of a certain disaster's type.

Generally speaking, besides probabilistic prediction, direct forecasting based on warning signs can be used. Reliable warning signs, however, are often detected just before the disaster and cannot be taken into account in long-term prediction influencing engineering solution.

To have a prior notion of the frequency and the extent of disasters possible in a particular region is the reason, for which statistical methods are to be used. The analysis of observations for previous years can give the information of the frequency and parameters of disasters in the past. Assuming the probability of such events to be invariable in time, the same frequency that was in the past should be predicted for the future. This extrapolation, however, can prove to be rather conventional, since data obtained generally refer to a limited time range alone. For this reason the processing of the available data should be based on specially developed statistical models whose physical correspondence to the phenomena under consideration make the extrapolation trustworthy.

Since natural disasters are, this way or other, extreme occurrences (earthquake or/and tsunami of high intensity, landslide of a great amount of soil, karsts crater of a large diameter), their statistics has the character of "statistics of rare phenomena". The Poisson's distribution can be proposed in this case, and the time character of the disasters manifestations can be represented by the Poisson's process.

The specificity of the probabilistic approach to extreme values of the parameter referred to disastrous manifestations of the natural processes the Poisson's or other distributions that represent the statistics of the extremes take place [8]. The necessity in the accounting and description of the parameters of three-dimensional variability, as well as in the study of this variability at different scale levels is essential in terms of the determining, on the basis of observations, regions, where this danger should be allowed in the practical engineering analysis, i.e. solving the task of micro-zoning. For this purpose, as well as for a more detailed prediction of threatening occurrences, methods for optimum prediction of random fields should be employed.

Areas where dangerous phenomena can occur at intensity levels not yet realized (earthquake exceeding the design level, karsts crater over allowed dimensions), can be determined and assessed be the test's observations of the similar occurrences, however, of lower, pre-ultimate, intensity.

Meanwhile, to say nothing of the abovementioned incomplete trustworthiness of extrapolation, the notion of a somewhat mass scale of occurrences, though less intensive, but in any case, similar to "disasters", is far for being always correct.

There are, certainly, other types of dangerous phenomena, too, whose uniform realizations in the given area are of rather a mass scale; such are natural landslides or stonewalls on different slopes in mountainous areas or rock bursts in mining working; statistical data of these can also be obtained. Natural disasters of geotechnical origin, however, can be "unique"; hence, we must not rely upon full-scale data selection and processing, i.e. upon the so-called "objective analysis".

A specific feature of natural disasters (and man-caused disasters, too) is that they are practically in avoidable. Natural disasters are characterized by power and uncontrollability. Typical of man-cased events is that they result from the speedy development of super-modern technologies and a production whose management contains a weak link, that is, a man able to make with tragic consequences (Chernobyl, for example). The main task here is to predict possible disasters, localizing them and mitigating possible losses. The design of any structure should be preceded by the analysis of all possible types of natural or man-caused disasters in terms of the probability of occurrence, of the practicability of initiation, of some secondary disasters, of the practicability of the localization, of the preventive measures not connected with design methods, and, at last, of the damage in the case of occurrence.

4. SAFETY CRITERIA OF UNIQUE STRUCTURES

Before dealing with safety criteria we should clarify the notion of a unique structure and natural or other effects that, determining its vulnerability, are detrimental for human health. The notion of the structural uniqueness and that of the treat of the natural or other phenomena are interconnected. Considering the structural safety in terms of the treat to human life and health, we should not connect the uniqueness of the structure with its cost or with the expected material losses alone. The uniqueness should as well be linked with the level of the treat for people, irrespective of its probability and of factors causing it, such as: the function and the size of the considered building, the character of productions, the presence of the radioactive products, etc. Hence, unique structures are those whose damage or collapse, no matter how long their probability could be, threaten the life and the health of people, either inside or, which is more often, outside the building.

The foregoing definition of the structural uniqueness permits to refer to refer to such buildings projects of national economy (industry, energy, transport and others) and those of a social sphere, whose damage and collapse would entail threat to human life and health. Vulnerability of unique buildings exposed to disastrous natural effects and possibility of their damage or collapse depend on:

- The extent to which loads due to disastrous natural phenomena exceed standard loads.
- The influence of secondary factors (explosions, fires) due to disastrous natural phenomena.
- The errors involved in the design, analysis and the choice of location of a building and those made at the stage of maintenance.
- Poor workmanship, the discrepancy between the strength characteristics of building materials and the standards, strength degradation in the course of the maintenance.

Analyzing structural vulnerability or safety it is expedient to single out the so-called “critical” elements on which structural safety mostly depends. For many structures such are the bearing members of the buildings that determine their strength and stability (foundation, columns, floors, joints, supports, ets.). For other buildings “critical” elements will be those able to resist explosion or fire caused by natural cataclysms, ensuring a reliable operation of safety systems. For a number of unique buildings “critical” elements are associated with the radioactivity or with the insurance of radiation safety.

Differences in the character of the critical elements require performing, when choosing safety criteria of unique units, a systematic analysis in order to find these elements and to assess the consequences of their failure. The systematic analysis of structural safety should include the elaboration of the scenario of a natural effect, taking into account the specificity of the latter, the structure of the unique building, the presence and the character of the “critical” elements, the consequences of their failure, the nature of unit’s damage or collapse and their influence on the safety of people inside or outside the building and on the environment.

Generally speaking, every natural phenomenon and every unique building require a scenario permitting to take their specificity into account and to obtain statistical data for generalizing the consequences. The elaboration and the analysis of the scenarios require a great professional effort of people acquainted with the specificity of the branch and the particular unique building.
To specify qualitative and quantitative safety criteria of unique buildings exposed to any types of natural effects, an integrated approach should be recommended as based on:

- Systematic deterministic analysis of scenarios of the influence of natural disastrous factors on concrete unique buildings revealing particular quality criteria.
- Probabilistic risk analysis determining particular and general probabilistic safety criteria that include those for limit states representing the extent of the failure, and criteria for the personnel and other people in terms of the threat for human life and health (individual risk, collective risk, etc.).
- "Cost-benefit" analysis to define more exactly safety basing on optimization of investments for protection against unfavorable effects with due regard for socio-economic factors.

5. COMMENTS TO CODIFIED PROCEDURES

Among the codes on design of unique structures there are no codes of environment protection and the boundaries of homeostasis[1]of a living system as predominant in the process of determining the basis and analyzing structural strength, stability, durability. This kind of code should specify a limit state in terms of environment protection: in the result of investigation, construction and maintenance of structures the interface in the space of environmental parameters separating their domain, wherein a living system can exist, from the rest part of the space, should not surpass the boundary of the living system's homeostasis.

The transition from homeostatic domain through its boundary means the termination of the existence of given organism, i.e. the given living system. To ensure homeostasis it is required: to determine its boundaries, to be able to assess the position of the whole living system with respect to the specified homeostasis boundary, e.g. to develop a specific informational system: sensors, gauges, monitoring, decision making procedures.

With codifying boundary protection and homeostatic boundaries of a unique structures living system, particular attention should be paid to geo-pathogenic[2] areas within the limits of design, construction and maintenance.

Geo-pathogenous zones result from the heterogeneous[3] structure of Earth's Crust, that anomalous information fields, detrimental for the energy of bio-systems or objects of inanimate nature. It is not advisable to assembly in the geo-pathogenous zones structures, important in terms of economy and ecology. Codes specifying the contents of designs of unique systems should contain the section of analysis and evaluation of damage or failure probability of the structure being designed. This section should also contain appropriate scenarios for the operation of expert teams trained to eliminate damage, localize ecological losses and to rescue people, animals and the whole animate system in the region of disaster.

As concerns the abovementioned section, national data bank should be complied and constantly replenished; the data bank should contain information on the causes and the physical meaning of failures, systematic analysis, material and other losses and on methods of damage elimination and rescue of the animate system.

[1] Relatively stable state of equilibrium.
[2] This term was coming from the world of Dowsing.
[3] Derved from the Greek, used to describe that has a large amount of variants.

Reliability is determined by the extent of structure's non-exposure to danger (in case under consideration, to elemental natural and elemental man-cause disasters), it being impracticable and inexpedient here to guarantee structural survivability as regards all, including almost improbable dangerous effects.

6. STRUCTURAL SYSTEM'S SURVIVABILITY[4]

Different situations in beyond-design states of structures can appear as a result of applying of natural or man-caused abnormal actions on building, which have not been foreseen in design. These states can be classified according to failure form, degree of damage and final state. The following forms of failure can be considered for ultimate limit state:

- Loss of strength in time of plastic, brittle, ductility or fatigue failure of elements.
- Elastic or inelastic buckling of structures.
- Loss of the stable equilibrium of the whole building.

According to the degree of the intensity it can be:

- Full progressive failure of the whole building. Such form of a failure is typical for brittle structures when a damage of separate elements can arouse dynamic effects in other elements of a structure.
- Little by little growing failure of accidental character as a result of plastic deformations accumulation. This situation will stop exploitation and demands restoration. This form of failure is typical for structures from elastic-plastic materials when failure of separate elements accompanies by growing of large displacements and redistributions of inner forces.

It is useful to denote that failure analysis shows that practically always the process of structural failure is avalanche-like, representing a sequence of failures of the members the is composed of, in which case "failure" means both, partial damage and complete failure. In the overwhelming majority of cases, however, in individual failures do not bad to a total breakdown; in a structure, provided it is redundant, stress redistribution takes place and the structure keeps performing its functions, though, perhaps, not to the full capacity.

This is favorable from the practice point of view; the situation can be accounted for by bearing capacity reserves that the structures posses. At present these margins are envisaged in the design, as based on experience and intuition. For achievement of an expedient reliability level the structure should be designed to bridge over a loss of a supporting member so that the area of damage is limited and localized [9].

It is but natural to use the word "survivability" applicably of the structural system to preserve an ability to carry out the main functions in the period of accidental perturbation and do not permit the progressive collapse or the cascade development of failures. Survivability is quite an important and, applicably to unique and important structure, indispensable property, since reliable performance of structures is only possible if an appropriate level of survivability is ensured.

There arises at once the question of this property's quantitative aspect. At present, conventional is a probabilistic approach to structural reliability evaluation; hence it is natural to employ it when obtaining numerical characteristics of survivability, too. Then, in compliance with the general methods, survivability level will be determined by a probability of some events characterizing the process of failure. It is logical to consider, how some critical state is attained in the process of successive failures of members. This can be the failure of some numbers of members, assigned in advance, and the formation of an instantaneous mechanism, or the failure of some isolated members, etc. Complying with this approach, a structure can be considered to possess

[4] The term integrity can be used too.

survivability if the probability of the above event for damaged structure is not so high; as compared to its undamaged counterpart (other criteria can as well be used).

The index of survivability can be expressed in the following way

$$\eta = \frac{P_f}{P_f^{'}} \tag{1}$$

Where P_f -probability of failure of the designed system; $P_f^{'}$ –probability of failure of the same system when some members failed. Survivability factors η are in [0,1] interval. The more is its value, the larger is the reserve of survivability in structural system. The steel frame is considered in Fig.1.

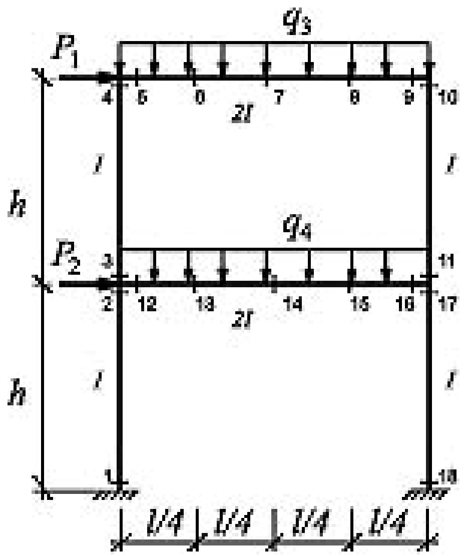

Fig.1 Two-story frame

In the longitudinal direction frame's span is 6m, h = 4m. All members of considered frame have I-sections with aria moments $W = 6.15 \cdot 10^{-5} m^3$ (1st floor column); $W = 8.28 \cdot 10^{-5} m^3$ (2nd floor column); $W = 1.270 \cdot 10^{-4} m^3$ (1st floor girder); $W = 1.098 \cdot 10^{-4} m^3$ (2nd floor girder).

Probabilistic analysis was performed taking into account random nature of applied loads and yield stress of frame's material, with given probability distributions. Table 1 contains parameters of these distributions. Calculations were made on the base of linear programming method (simplex method) with the application of the direct integration of distribution function [10,11]. Probability of failure is $P_f = 5.51 \cdot 10^{-5}$.

Table 1

Random value	Distribution	Mean value	Standard deviation s	Parameters of distribution	Design values
Wind load P_1, P_2	Gumbel	$0.144\, кН/м^2$	$0.037\, кН/м^2$	$u = 0.127\, кН/м^2$ $z = 0.029\, кН/м^2$	$0.2576\, кН/м^2$
Snow load q_3	Gumbel	$1.1418\, кН/м^2$	$0.4681\, кН/м^2$	$u = 0.931\, кН/м^2$ $z = 0.365\, кН/м^2$	$1.6\, кН/м^2$
Load due to use q_4	Gauss	$0.88\, кН/м^2$	$0.21\, кН/м^2$	–	$1.68\, кН/м^2$
				$\beta = 14.3$	

Random value	Distribution	Mean value	Standard deviation s	Parameters of distribution	Design values
Yield point σ_y	Weibul	$305.25 МПа$	$25 МПа$	$\alpha = 316.42\, МПа$ $x_0 = 0$	$245 МПа$

More probable is the partial mechanism of failure when plastic hinges appear in cross-sections 4, 7 and 9 (Fig.1). The values of the failure probabilities of considered frame are listed in Table 2 for different cases of cross-sections weakening.

Table 2

№ section s	Probability of failure P_f Lowering of aria moments W in different sections					
	5%	10%	25%	50%	75%	95%
1	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$7.53 \cdot 10^{-5}$	$8.42 \cdot 10^{-5}$
2	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$7.41 \cdot 10^{-5}$	$8.94 \cdot 10^{-5}$
3	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$
4	$5.83 \cdot 10^{-5}$	$5.96 \cdot 10^{-5}$	0.000101	0.000207	0.000389	0.000570
5	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$8.42 \cdot 10^{-5}$	0.000122	0.000309	0.000547
6	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000107	0.000755	0.004562
7	$6.19 \cdot 10^{-5}$	$7.90 \cdot 10^{-5}$	0.000303	0.001246	0.006322	0.025580
8	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$8.34 \cdot 10^{-5}$	0.000734	0.004771
9	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000137	0.000319	0.000593
10	$5.95 \cdot 10^{-5}$	$6.86 \cdot 10^{-5}$	0.000103	0.000207	0.000392	0.000564
11	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$
12	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000112	0.000265
13	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000224	0.000873
14	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000890	0.001063	0.002327
15	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000229	0.000871
16	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	0.000112	0.000259
17	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$7.30 \cdot 10^{-5}$	$8.27 \cdot 10^{-5}$
18	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-5}$	$7.34 \cdot 10^{-5}$	$8.33 \cdot 10^{-5}$

From Table 2 follows that in the case of a failure of any cross-section, probability of failure for frame will not exceed the value $P_f^{'} = 0.02558$ (the failure of cross-section 7). The failure of cross-section 7 will not lead to the collapse of all structure but essentially decreases its survivability. Even the full failure of cross-sections 2 or 11 has no influence on probability of this frame. The failure of the cross-section 1, 2, 17 or 18 has also no essential influence at this probability. Survivability index of the considered frame with regard to the failure of cross-section 7 constitutes:

$$\eta = \frac{5.51 \cdot 10^{-5}}{0.02558} = 0.00215$$

If in the process of structure exploiting some actions will be ensuring, then the probability of the failure of the whole frame in case when one cross-section failed, can be decreased to the value $P_f^{'} = 0.004771$. Survivability index will be:

$$\eta = \frac{5.51 \cdot 10^{-5}}{0.004771} = 0.0115$$

At Fig. 2 graphs due to dependences between probability of failure and weakening of cross-sections 7, 8 and 3 are presented.

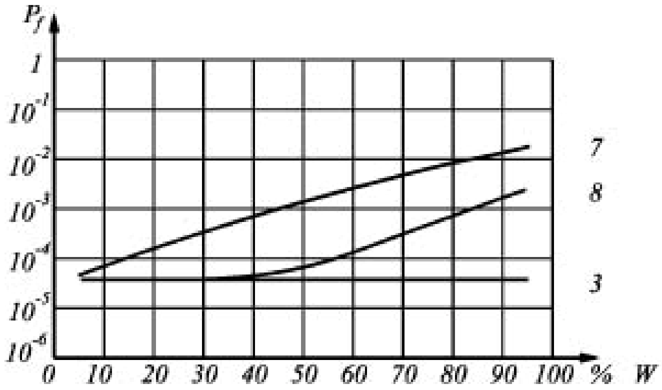

Fig. 2 Dependence between P_f and W

The process of developing and utilizing structures and structural members comprises numerous measures; considered herein, however, are only those ensuring a required reliability level. Different reliability levels are ensured through different cost of construction. For structures in hazardous areas an expedient reliability level should be specified. It should be determined the necessary safety guarantee of the structure and people. The failure criterion assumed in the design of buildings for ordinary performance conditions is mainly that of serviceability.

A reliability level for construction in hazardous areas should be that of failure –free performance. This should be an objective criterion determining the totality of codes, control services and other measures that would ensure an expedient reliability level.

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DYNAMIC MODEL OF AIR APPARATUS PARK

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Abstract

The article is devoted to construction and research of dynamic stochastic model of park of aircrafts. A stochastic is enclosed in all of natural characteristic exploitations of this set of apparatuses: times of flight and landing, possibility of receipt of damage on flight, including the past recovery air apparatus; times of repair. The estimations of total possible flights are got for the any fixed interval of time.

Key Words Flight time, time on the ground, recoverable damage, loss of air apparatus, repair time, generating function, renewal equation.

1. INTRODUCTION

The important problem of management of the park of air apparatuses (PAA in short) maintenance, as stage of their life cycle, is an estimation of ability to provide the necessary amount of flights in given time interval of exploitation. The dynamics of exploitation of every apparatus consists of alternation of times of flight, times of repair and times of stand-down. These times are determined both external requests on flights and different damages during flight or loss of air apparatus (AA in short) on flight. Forecasting of the state of PAA is one of way of control of quality of management. This approach may be realized by modeling [1]. Analysis of literature in this direction shows that mainly authors develop of the models in a few lines. The authors of line [2-4] simulate of control of technical state of PAA with aim the optimization of preventive maintenance with respect to restoration of PAA parameters. The authors of next line [5-7] develop either methodological approach of operation adaptive control of technical state of PAA on basis of using of potential of corporative resources of unit information space (network-center environment) with purpose improving or support on the given level of reliable and durability indexes [5, 6] or task of definition of optimal type of PAA taking into account economical indexes. The authors of another line [8-10] build their own investigations on expert estimations. In this case, experience shows that decisions may be false. Therefore, it is urgency to develop models, which, first, consider of change of state of PAA by different manner.

Namely, which are grounded on the following probability indexes: probability of return of AA from flight without damages, probability of to receive certain damages of AA in flight, probability of to lose of AA in flight. Analysis of interaction of these indexes of random events is not simple process.

And so, it is actually, secondly, a development of such models that are based on analytical dependence with more complex mathematical filling.

In the article approaches are offered to the solution of the following task. We will designate through n_i the amount of AA able to fly up in some i-th moment of time. It is required to estimate of possibility to do given amount of flights Q in times of k successive time starts: j - th , $j+1$-th , $\cdots$, j-th starts. In other words, we must estimate possibility of implementation of relation $Q \le n_j + n_{j+1} + \cdots + n_{j+k}$ at any fixed integer j and k .

2. It is assumed that N units of AA, which are exploited from some initial moment of time. For definiteness we suppose that all (able to fly) AA fly up and land at the simultaneously.

Let us adopt the following notation.

We will denote by τ_k the flight time after k-th takeoff and by ξ_k the time on the ground after k-th landing. Thus moments of takeoffs $\{s_l\}$ are defined recurrently:

$$s_1 = 0, \quad s_2 = \tau_1 + \xi_1, \quad \ldots \quad , s_l = \sum_{k=o}^{l} (\tau_k + \xi_k).$$

The moments of landing $\{t_l\}$ are defined analogy:

$$t_1 = \tau_1, \quad t_2 = \tau_1 + \xi_1 + \tau_2, \quad \ldots \quad , t_l = \tau_l + \sum_{k=o}^{l-1} (\tau_k + \xi_k).$$

Further, we will consider the following probabilities as result of flight of every AA.

Let us denote by $p_i, i = 1,2$ the probabilities to obtain (in flight) eliminated damages;

by p_3 the probability of loss of AA in flight; by p_4 the probability to be safe and sound.

It is assume that $p_1 + p_2 + p_3 + p_4 = 1$.

We will use symbols β_l and α_l to denote the amount of AA at the $l-$ th takeoff ($l-$ th flight) and at the $l-$ th landing respectively.

The time of repair at the $i-$ th eliminated damage of the $k-th$ AA in the $l-th$ flight is equal to a random variables $d_i^{(k,l)}$, $i = 1,2;\ 1 \le k \le \beta_l;\ \ l \ge 1$ with the distribution functions

$$F_1(x) = P\left(d_1^{(k,l)} < x\right), \quad F_2(x) = P\left(d_2^{(k,l)} < x\right)$$

We will introduce sequences of independent events $A_i^{(k,l)}$ $i = 1,2,3,4;\ l \ge 1;\ 1 \le k \le \beta_l$.

These events are connected with aircraft events in flight so that the following equalities take place $P\left(A_i^{(k,l)}\right) = EI\left(A_i^{(k,l)}\right) = p_i$, here $I(\cdot)$ denotes the indicator of events.

In what follows, we shall be assuming that random variables form ensemble

$$\Pi := \left\{ \tau_i ,\ \xi_i ,\ d_1^{(k,l)} ,\ d_2^{(k,l)} ,\ I\left(A_i^{(k,l)}\right),\ i,k,l \ge 1 \right\}$$

are independent in common.

Put

$$s_2^l = \sum_{k=2}^{l} (\tau_k + \xi_k) \quad , \quad r_1 = E \sum_{i=1}^{2} I\left(A_i^{(k,1)}\right) I\left(d_i^{(k,1)} \in [0\ ,\ \xi_1)\ \right)$$

$$r_l = E \sum_{i=1}^{2} I\left(A_i^{(k,l)}\right) I\left(d_i^{(k,l)} \in [\xi_1 + s_2^{l-1}\ ,\ \xi_1 + s_2^l)\ \right), \quad l \ge 2.$$

here and in the sequel, we assume that $s_2^l = 0$, if $l < 2$.

By hypothesis on independence

$$r_1 = \sum_{i=1}^{2} p_i P\left(d_i^{(1,1)} < \xi_1\right);\ r_l = \sum_{i=1}^{2} p_i r_{li} \ ,\ \ l \ge 2 ,$$

where $r_{1i} = P\left(d_i^{(1,1)} < \xi_1\right)$, $r_{li} = P\left(d_i^{(1,1)} \in [\xi_1 + s_2^{l-1}\ ,\ \xi_1 + s_2^l)\ \right)$, $l \ge 2.$

We introduce the generating functions

$$B(s) = \sum_{m=1}^{\infty} s^m\, b_m \ , \text{where} \quad b_m = E\beta_m . \quad R(s) = \sum_{m=1}^{\infty} s^m\, r_m \ , \quad s \in [0,1] .$$

Theorem 1. *The following formulas take place*

$$B(s) = \frac{sN}{1 - s p_4 - sR(s)} . \tag{1}$$

Proof. We shall establish the stochastic relations for sequences β_m, α_m $m \ge 1$. The designation $\omega \overset{w}{=} \zeta$ means that random variables ω and ζ have the same distribution function.
We will denote by $\overline{A}$ the complement of a set A.

$$\beta_1 = N, \qquad \alpha_1 \overset{w}{=} \sum_{k=1}^{N} I\left(\overline{A_3^{(k,1)}}\right).$$

$$\beta_2 \overset{w}{=} \sum_{k=1}^{\beta 1} I\left(A_4^{(k,1)}\right) + \sum_{k=1}^{\beta 1} \sum_{i=1}^{2} I\left(A_i^{(k,1)}\right) I\left(d_i^{(k,1)} < \xi_1\right), \quad \alpha_2 \overset{w}{=} \sum_{k=1}^{\beta 2} I\left(\overline{A_3^{(k,2)}}\right).$$

$$\vdots$$

$$\beta_m \overset{w}{=} \sum_{k=1}^{\beta m-1} I\left(A_4^{(k,m-1)}\right) + \gamma_m , \quad \alpha_m \overset{w}{=} \sum_{k=1}^{\beta m} I\left(\overline{A_3^{(k,m)}}\right),$$

where $$\gamma_m \overset{w}{=} \sum_{l=1}^{m-1} \sum_{k=1}^{\beta l} \sum_{i=1}^{2} I\left(A_i^{(k,l)}\right) I\left(d_i^{(k,l)} \in [s_{m-1} - t_l \vee 0, s_m - t_l)\right).$$

The random value γ_m is equal to amount of AA, which finished the repairs in the interval of time between $m-1-th$ and $m-th$ takeoffs.

By the construction of β_m, we have the following relations

$$b_1 = N, \quad b_m = p_4 b_{m-1} + \sum_{l=1}^{m-1} b_l \, r_{m-l} , \quad m \geq 2 . \tag{2}$$

We introduce the functions

$$B(s) = \sum_{m=1}^{\infty} s^m \, b_m , \qquad R(s) = \sum_{m=1}^{\infty} s^m \, r_m , \quad s \in [0,1] .$$

From the (1) we obtain

$$s^m b_m = s^m p_4 b_{m-1} + s^m \sum_{l=1}^{m-1} b_l \, r_{m-l} , \quad m \geq 2 . \tag{3}$$

Summarizing left and right parts of (3) yields

$$B(s) - sN = s\, p_4 \, B(s) + s B(s) R(s) .$$

From the latter one we get

$$B(s) = \frac{sN}{1 - s p_4 - s R(s)} .$$

Proof is completed.

Corollary . *Assume that the sequences from* Π *satisfy the conditions*

$\lim_{n\to\infty} P\left(d_i^{(1,1)} < \xi_1 + \sum_{k=2}^{n} (\tau_k + \xi_k) \right) = 1,\ i = 1,2$. *Then the following equality is valid*

$$\sum_{m\ge 1} b_m = \frac{N}{p_3} \tag{4}$$

Proof . Since, random variables from Π are independent, we have that

$$R_n := \sum_{l=1}^{n} r_l = \sum_{i=1}^{2} p_i P\left(d_i^{(1,1)} < \xi_1 + s_2^n \right). \tag{5}$$

Combining (1), (5) and condition from the Corollary, we get

$$R(1) = \lim_{n\to\infty} R_n = \sum_{i=1}^{2} p_i, \quad B(1) = \frac{N}{1 - p_4 - p_1 - p_2} = \frac{N}{p_3}.$$

The proof is completed.

2.

We shall formulate the problems of estimations of b_m in terms of theory of renewal processes.

Let us denote by $\{\kappa_i \in \{1,2,\ldots\}, i \ge 1\}$ the sequence of independent discrete random values with common distribution law $\delta_1 = P(\kappa_1 = 1) = p_4 + r_1$, $\delta_l = P(\kappa_1 = l) = r_l$, $l \ge 2$.

It is well known that if $S_k = \sum_{i=1}^{k} \kappa_i$ and $\eta(m) = \min\{k : S_k \ge m\}$, then $E\eta(m), m \ge 1$ is unique solution of the renewal equation $E\eta(m) = \sum_{l=1}^{m-1} \delta_l E\eta(m-l)$.

Comparing latter one and (2), we conclude that $b_m = E\eta(m)$.

Let $G(m) = P(\kappa_1 \le m)$ and $h(m, M) = \sum_{i=m}^{m+M} b_i$.

Now, we obtain the following upper estimation

$$h(m, M) = \sum_{n=1}^{\infty} \sum_{i=0}^{M} P(S_n \le m+i) = \sum_{i=0}^{M} \sum_{n=1}^{m+i} P(S_n \le m+i) \le \sum_{i=0}^{M} \sum_{n=1}^{m+i} G^n(m+i) \tag{6}$$

Since $G(m) = p_4 + \sum_{i=1}^{2} p_i P\left(d_i^{(1,1)} < \xi_1 + s_2^m\right),\ m \ge 1$, the estimation (6) is well calculated.

3.

Now we will consider the construction of $B(s)$ more detail for special case. We make the following additional assumptions:

– τ_k, $k \ge 1$ have the same distribution function with Laplace transformation $\psi(s) = E\exp\{-s\tau_1\}$, $s > 0$.

– ξ_k, $k \ge 1$ have the same distribution function with Laplace transformation $\varphi(s) = E\exp\{-s\xi_1\}$, $s > 0$.

– $F_i(x) = 1 - \exp(-\lambda_i x)$, $i = 1,2$.

For convenience we put $f(s) = \psi(s)\varphi(s)$.

Now we shall obtain more exact expression for $R(s)$.

By induction, we shall calculate the r_{li} for $l = 1,2,\dots$.

$$r_{li} = \int_0^\infty P(d_i^{(1,1)} < x)P(\xi_1 \in dx) = \int_0^\infty (1 - \exp(-\lambda_i x))P(\xi_1 \in dx) = 1 - \varphi(\lambda_i);$$

$$r_{2i} = \int_0^\infty\int_0^\infty (P(d_i^{(1,1)} < x + y) - P(d_i^{(1,1)} < x))P(\xi_1 \in dx)P(\tau_2 + \xi_2 \in dy) = \varphi(\lambda_i) - \varphi(\lambda_i)f(\lambda_i);$$

$$r_{3i} = \int_0^\infty\int_0^\infty\int_0^\infty (P(d_i^{(1,1)} < x + y + z) - P(d_i^{(1,1)} < x + y))P(\xi_1 \in dx)P(s_2^{l-1} \in dy)P(\tau_l + \xi_l \in dz) =$$

$$= \varphi(\lambda_i)f(\lambda_i) - \varphi(\lambda_i)f^2(\lambda_i) = \varphi(\lambda_i)f(\lambda_i)(1 - f(\lambda_i));$$

$$\vdots$$

$$r_{li} = \varphi(\lambda_i)f^{l-2}(\lambda_i)(1 - f(\lambda_i)), \quad l \ge 4.$$

After routine calculations we get

$$R(s)=\sum_{i=1}^{2}p_i\sum_{m=1}^{\infty}s^m\,r_{mi}=\sum_{i=1}^{2}p_i\left\{s(1-\varphi(\lambda_i))+\frac{s^2\varphi(\lambda_i)(1-f(\lambda_i))}{1-s\,f(\lambda_i)}\right\}.$$

Thus, we have the following expression for this case

$$B(s)=N\frac{s-(f_1+f_2)s^2+f_1f_2s^3}{1-a_1s+a_2s^2-a_3s^3}\ , \tag{7}$$

where for convenience, we introduced notation $f_i:=f(\lambda_i),\quad \varphi_i:=\varphi(\lambda_i)$,

$$a_1:=p_4(1+f_1+f_2)+\sum_{i=1}^{2}p_i(1-\varphi_i),$$

$$a_2:=f_1f_2+p_4(f_1+f_2)-\sum_{i=1}^{2}p_i(\varphi_i-f_i-f_{3-i}(1-\varphi_i)),$$

$$a_3:=p_4f_1f_2-\sum_{i=1}^{2}p_i(\varphi_i-f_i)f_{3-i}\ .$$

Thus, in this case the term b_m poses no problem because expression (7) can be expanded into the convergent power series about s.

Further, it is easy to check that under such special assumptions the function $G(m)$ from Section 2 has the following form

$$G(m)=p_4+p_1+p_2-\sum_{i=1}^{2}p_i\varphi^m(\lambda_i)\psi^{m-1}(\lambda_i)\ .$$

Remark. It is clear, that restriction on number of different types of eliminated damages (only two) is not essentially. The proved formulas are transformed for more number of types easy.

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AN ASYMPTOTIC ANALYSIS OF A RELIABILITY OF INTERNET TYPE NETWORKS

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Introduction

In this paper a problem of a construction of accuracy and asymptotic formulas for a reliability of internet type networks is solved. Analogously to [1] such network is defined as a tree where each node is connected directly with a circle scheme on a lower level with n>0 nodes. A construction of accuracy and asymptotic formulas for probabilities of an existence of working ways between each pair of nodes of the internet type network is based on a recursive definition of these networks and on asymptotic formulas for a reliability of a random port. This asymptotic formula represents the port reliability as a sum of probabilities of a work for all ways between initial and final nodes of this port. An estimate of a relative error and a complexity of these asymptotic calculations for a radial-circle scheme are shown.

1. An asymptotic formula for a reliability calculation of a port and its accuracy

An asymptotic formula for a reliability of the general type port with low reliable arcs. Consider the no oriented graph Γ with the final nodes set *U*, the arcs set *W*, the fixed initial and final nodes *u*, *v* and the set of the acyclic ways $\{R_1,...,R_n\}$ between *u*, *v*. Suppose that the probability p_w of the arc $w\in W$ work depends on the parameter h > 0: $p_w = p_w(h)$ and $p_w(h)\to 0,\ \ h\to 0.$ Denote $P(U_p)$ - the probability of the event U_p that all arcs $w_1^p,...w_{m_p}^p$ of the way R_p work. Then the reliability of the port Γ is $P_\Gamma = P\left(\bigcup_{p=1}^{n} U_p\right)$, denote $P_\Gamma^* = \sum_{p=1}^{n} P(U_p).$

Remark that for $p\neq q$ the arcs sets $\{w\in R_p\}$, $\{w\in R_q\}$ can not satisfy the inclusion $\{w\in R_p\}\subseteq\{w\in R_q\}$. In an opposite case there is the node u_* in which the ways R_p, R_q diverge by the arcs (u_*,u_p), (u_*,u_q). But as the arc $(u_*,u_p)\in\{w\in R_q\}$ so there is a circle in the way R_q. This statement contradicts with a suggestion that the way R_q. is acyclic. As the inclusion $\{w\in R_p\}\subseteq\{w\in R_q\}$ is not true for $p\neq q$ so the way R_p contains the arc $\overline{w}\notin R_q$ and consequently $P(U_pU_q) = o(P(U_p)),\ \ h\to 0,\ \ p\neq q.$ An induction by *n* gives the inequality

$$P_\Gamma^* - \sum_{1\le p<q\le n} P(U_pU_q) \le P_\Gamma \le P_\Gamma^*. \tag{1}$$

But

$$\sum_{1\le p<q\le n} P(U_pU_q) \le n\max_{w\in W} p_w(h) P_\Gamma^*$$

and consequently from the formula (1) we obtain

$$P_\Gamma \sim P^*_\Gamma. \tag{2}$$

Denote by $A = | P^*_\Gamma / P_\Gamma - 1|$ the relative error of the asymptotic formula (2). It is obvious that

$$A(h) \le n \max_{w \in W} p_w(h) = \Phi(h) \to 0, h \to 0. \tag{3}$$

Assume that $\varphi(h) \to 0,\ h \to 0$ then for the replacement of h by $\varphi(h)$ the upper bound $\Phi(h)$ of the relative error is to be replaced by $\Phi(\varphi(h)) = o(\Phi(h))$.

Radial-circle scheme. Consider the radial-circle scheme represented on the fig. 1. This scheme has the center 0 connected with the nodes $1,\ldots,n$ arranged on the circle.

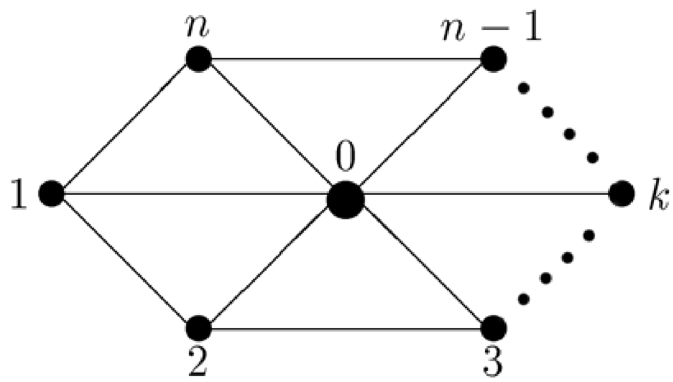

Fig.1 Radial-circle scheme

Each acyclic way from the node i, $1 \le i \le n$, on the circle (the circle node) to the center 0 of this scheme consists of a peace along the circle and a transition to the center 0. A way from the circle node i to the circle node j, $1 \le i \ne j \le n$, has a peace from the node i along the circle, a transition to the center 0, a transition to the circle and a peace along the circle to the node j.

Define the connection matrix $\mathbf{P} = \| P_{ij} \|_{i,j=0}^n$ of the radial-circle scheme in which P_{ij} is the probability that there is a working way between the nodes i, j of this scheme. Represent the results of the matrix $\mathbf{P}$ calculation with n=6 and

$p_{01} = 0.0471595$	$p_{02} = 0.0469944$
$p_{03} = 0.0287418$	$p_{04} = 0.0499121$
$p_{05} = 0.0135117$	$p_{06} = 0.00822811$
$p_{12} = 0.0490761$	$p_{23} = 0.0340865$
$p_{34} = 0.0442866$	$p_{45} = 0.0004677$
$p_{56} = 0.00818179$	$p_{16} = 0.0173955$

Here the matrix $\mathbf{P}$ is calculated by the Monte-Carlo method with 1000000 realizations during 14 hours. Denote by $\mathbf{P^*} = \| P^*_{ij} \|_{i,j=0}^n$ the connection matrix with elements calculated by the asymptotic formula (2). The matrix $\mathbf{P^*}$ have been calculated during one minute that is approximately 1000 times faster. As $\mathrm{P_{ij}} = \mathrm{P_{ji}}$ and $\mathrm{P_{ii}} = 1$ we show only the elements P_{ij}, P^*_{ij} with $1 \le i < j \le n$.

P*=

–	0.0496627	0.050371	0.0326335	0.0512658	0.0136101	0.00920024
–	–	0.0523323	0.00549431	0.00262581	0.000817883	0.0178084
–	–	–	0.0378183	0.00408933	0.000753723	0.00131303
–	–	–	–	0.0458115	0.000532299	0.00112028
–	–	–	–	–	0.00123165	0.00129662
–	–	–	–	–	–	0.00912706
–	–	–	–	–	–	–

P=

–	0,049758	0,049859	0,03268	0,051263	0,013703	0,009279
–	–	0,052157	0,005359	0,002637	0,000844	0,017839
–	–	–	0,037725	0,004073	0,000743	0,001327
–	–	–	–	0,045997	0,000555	0,001108
–	–	–	–	–	0,001253,	0,001301
–	–	–	–	–	–	0,009229
–	–	–	–	–	–	–

The matrix of the relative errors $\mathbf{A} = \| A_{ij} \|_{0 \le i < j \le 6}$ satisfies the equality:

A=

–	0.00191948	0.0101643	0.00142467	0.0000537835	0.00682727	0.00856111
–	–	0.00335063	0.0246274	0.0042632	0.0319321	0.00171756
–	–	–	0.00246654	0.00399357	0.0142263	0.0106371
–	–	–	–	0.00404871	0.0426474	0.0109635
–	–	–	–	–	0.0173307	0.00337921
–	–	–	–	–	–	0.0111695
–	–	–	–	–	–	–

Remark. *Analogously it is possible to obtain asymptotic formulas for a general type network or a radial circle scheme with high reliable arcs. But in this case it is necessary to replace a work probability by a failure probability and a way by a cross section.*

2. Recursively defined networks

A calculation of the connection matrix in recursively defined networks. Suppose that D_* is the set of networks Γ with no intersected sets of arcs. Define recursively the networks class D, $D_* \subset D$ by the condition

$$\Gamma_1 = \{U_1, W_1\} \in D \ , \ \Gamma_2 = \{U_2, W_2\} \in D_*, \ W_1 \cap W_2 = \text{Ш}, \\ U_1 \cap U_2 = \{z\}, \text{ (z is a single node)} \to \Gamma_1 \cup \Gamma_2 \in D. \tag{4}$$

Analogously to [2] in this paper we calculate $\{P_\Gamma, \ u, v \in U, \ u \ne v\}$, not its single element. These calculations are based on the recursive formulas: if $\Gamma' \in D, \ \Gamma'' \in D_*, \ U' \cap U'' = \{z\}$, then

$$P_{\Gamma' \cup \Gamma''} = \begin{cases} P_{\Gamma'}, \ u, v \in U', \\ P_{\Gamma''}, \ u, v \in U'', \\ P_{\Gamma'} P_{\Gamma''}, \ u \in U', v \in U''. \end{cases} \tag{5}$$

In the last equality the quantity $P_{\Gamma'}$, characterizes the connection between the nodes *u, z* and the quantity $P_{\Gamma''}$ – the connection between the nodes *z, v.* The number of arithmetical operations $n(P_\Gamma)$ necessary to calculate $\{P_\Gamma, u,v\in U, u\neq v\}$, by the recursive formulas (5) is characterized by the following statement.

Theorem. *Suppose that* $\Gamma_1,...,\Gamma_l$ *is the sequence of networks with the no intersected sets of arcs. If* D_* *consists of sequences of independent probability copies of* $\Gamma_1,...,\Gamma_l$, *then for each* $\Gamma\in D$ *the inequalities*

$$\frac{l(\Gamma)(l(\Gamma)-1)}{2}\le\sum_{u,v\in U,u\neq v}n(P_\Gamma)\le\frac{l(\Gamma)(l(\Gamma)-1)}{2}+\sum_{i=1}^{l}\sum_{u,v\in U_i,\,u\neq v}n(P_{\Gamma_i}) \tag{6}$$

are true with $l(\Gamma)$ *the number of nodes in the graph* Γ.

From the inequalities (6) obtain that

$$\lim_{l(\Gamma)\to\infty}\frac{2\sum\limits_{u,v\in U,u\neq v}n(P_\Gamma)}{l(\Gamma)(l(\Gamma)-1)}=1.$$

So asymptotically when $l(\Gamma)\to\infty$ to calculate a connection probability for a single pair of nodes it is necessary a single arithmetical operation.

Proof. Suppose that the inequality (6) is true for Γ' then from the recursive formulas (5) and the equality $l(\Gamma'\cup\Gamma'')=l(\Gamma')+l(\Gamma'')-1$ we have

$$\sum_{u,v\in U'\cup U'',u\neq v}n(P_{\Gamma'\cup\Gamma''})\le\sum_{i=1}^{l}\sum_{u,v\in U_i,\,u\neq v}n(P_{\Gamma_i})+\frac{l(\Gamma_1)(l(\Gamma_1)-1)}{2}+\frac{l(\Gamma_2)(l(\Gamma_2)-1)}{2}+$$
$$+(l(\Gamma_1)-1)(l(\Gamma_2)-1)=\sum_{i=1}^{l}\sum_{u,v\in U_i,\,u\neq v}n(P_{\Gamma_i})+\frac{l(\Gamma_1\cup\Gamma_2)(l(\Gamma_1\cup\Gamma_2)-1)}{2}.$$

A calculation of the transition matrices in the internet type networks. Analogously to [1] define the class of the internet type networks as the recursively defined class of networks D with the set of originating schemes D_* which consists of radial-circle schemes and in the formula (4) the node *z* is the center of the radial-circle scheme Γ_2.

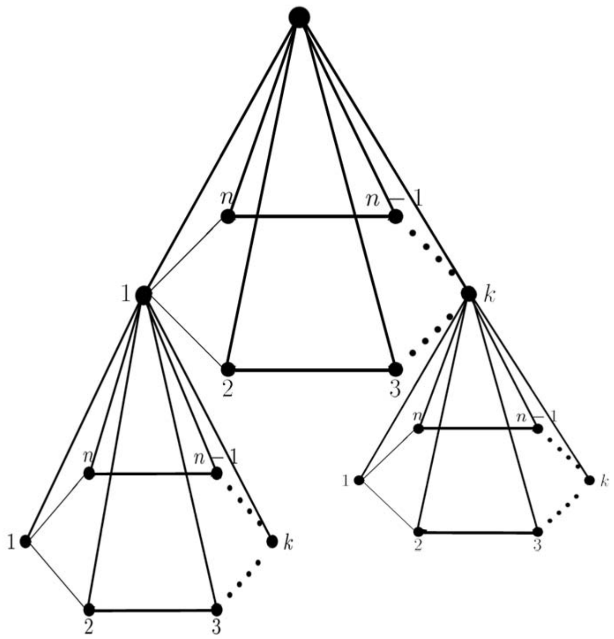

Fig.2. The internet type network

So if we have the transition matrix for the radial-circle schemes it is possible to calculate the transition matrix of the internet type network by the formula (5). This algorithm is significantly faster than general type algorithm from [1]. It contains fast algorithm to calculate the transition matrix in the radial-circle scheme and practically optimal algorithm to calculate the transition matrix for the internet type networks.

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A STUDY OF ASYMPTOTIC AVAILABILITY MODELING FOR A FAILURE AND A REPAIR RATES FOLLOWING A WEIBULL DISTRIBUTION

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Abstract:

The overall objective of the maintenance process is to increase the profitability of the operation and optimize the availability. However, the availability of a system is described according to lifetime and downtime. It is often assumed that these durations follow the exponential distribution. The work presented in this paper deals with the problem of availability modeling when the failure and repair rates are variable. The lifetime and downtime were both governed by models of Weibull (the exponential model is a particular case). The differential equation of the availability was formulated and solved to determine the availability function. An analytical model of the asymptotic availability was established as a theorem and proved. As results deduced from this study, a new approach of modeling of the asymptotic availability was presented. The developed model allowed an easy evaluation of the asymptotic availability. The existence of three states of availability for a system has been confirmed by this evaluation. Finally, these states can be estimated by comparing the shape parameters of the Weibull model for the failure and repair rates.

Keywords: Availability function, asymptotic availability, failure rate, repair rate, Weibull distribution

1. Introduction

The last two decades witnessed major progress in the development of new maintenance strategies [1]. The primary objectives of these strategies are to reduce equipment downtime, also increase reliability and availability of the equipment which at the same time optimizes the life-cycle costs [2]. The need for high reliability and availability is not just restricted to safety-critical systems [3]. In general, current technology has ensured that the equipments for industrial application, for example, telephone switches, airline reservation systems, process and production control, stock trading system, computerized banking etc. all require very high availability [2]. Reliability is generally described in terms of the failure rate or mean time between failures (MTBF), while availability is normally associated with total downtime [2]. There is some research on increasing system availability [4]. Goel and Soenjoto proposed a generalized model [4]. Markov models are also implemented to analyze the system availability, which combines both software and hardware failures and maintenance processes [4]. Khan and Haddara [1] proposed a methodology for risk-based maintenance to increase availability of a heating, ventilation and air-conditioning (HVAC) system. Garg S. et al. [3] developed a model for a transactions based software system which employs preventive maintenance to maximize availability, minimize probability of loss, minimize response time or optimize a combined measure. The steady state availability can be modelled using standard formulae from Markov regenerative process (MRGP) theory. The Service rate and failure rate are assumed to be functions of real time (Weibull distribution) [3]. The failure and repair rates are supposed constant (λ and μ respectively), so that system availability can be modeled using a Markov chain in Refs. [5,7]. But, Khan and Haddara [1] considered that the Weibull model is more robust than the other models. Dai et al [4] studied the availability of the centralized heterogeneous distributed system (CHDS) and developed a general model for the analysis. The repair time was exponentially distributed. For the failure intensity function (failure rate), the G.O model presented by Goel and Okumoto was used [4]. Some other research considered that the availability depends on both reliability and maintainability and is defined as the ratio of requested service time to practical service time [6, 7]

Nomenclature	
A(t)	Availability function
$\mathbf{A}_\infty$	Asymptotic availability
λ(t)	Failure rate
μ(t)	Repair rate
β	Shape parameter of Weibull distribution for Failure rate
η	Scale parameter of Weibull distribution for Failure rate
α	Shape parameter of Weibull distribution for repair rate
θ	Scale parameter of Weibull distribution for repair rate

Review of the literature indicates that there is a new trend to use availability and reliability modeling as a criterion to plan maintenance tasks. However, most of the previous studies assumed the failure and/or repair rates are constant. It seems that there is a need for a more generalized methodology that can be applied for variable rates. The present study adopts a new fundamental approach for the asymptotic availability modeling where the failure and repair rates were governed by the Weibull distribution.

This paper is organized as follows. In Section 2, the differential equation of the availability is established. Section 3 is dedicated to the resolution of the differential equation to determine the instantaneous availability. The model of the asymptotic availability is developed in Section 4. Finally, in Section 5, the conclusions along with future research directions are presented.

2. The mathematical formulation of the availability differential equation

According to the standard "Association Française de Normalisation - AFNOR X 06-503" [8, 9] , in order to have a system available at time $t+dt$, there are two possibilities:

- the first is that the system is available at time t and does not have breakdown between t and $t+dt$
- the second is the system is unavailable at time t but it is repaired between t and $t+d$.

These expressions are transformed by the following probabilities:

- A(t+dt): The probability that the system is available at time (t+dt),
- *A*(t): the probability that the system is available at time t,
- 1-□(t)dt : The probability that the system does not have breakdown between t and $t+dt$, knowing that it had already functioned until the time t,
- 1-*A*(t): The probability that the system is unavailable at time t

- □(t)dt: The probability that the system is repaired between t and (t+dt), knowing that it was already failing until the time t.

With:

- □(t) : Instantaneous failure rate
- □(t) : Instantaneous repair rate

Fig. 1 shows the state diagram of the system.

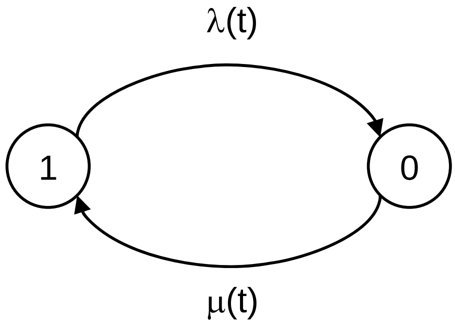

Fig. 1. State transition diagram

A(t+dt)= probabilities (that the system is up at t **and** is no break down between t and (t+dt)**)+ probabilities (**the system to be down at time t and it is repaired between t and (t+dt)**)** [8, 9].

$$A(t+dt) = A(t)[1-\lambda(t)dt] + [1-A(t)]\mu(t)dt \quad (1)$$
$$A(t+dt) - A(t) = [-\lambda(t)A(t) + \mu(t) - A(t)]dt \quad (2)$$
$$\frac{A(t+dt)-A(t)}{dt} = \mu(t) - [\lambda(t)+\mu(t)]A(t) \quad (3)$$

Then:

$$\frac{dA(t)}{dt} = \mu(t) - [\lambda(t)+\mu(t)]A(t) \quad (4)$$

This expression represents the differential equation of first order of the availability. [4,8, 9].

3. The availability function

For t > 0, the failure and repair rates, which are modeled using a Weibull distribution, are given by :

- $\lambda(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$ (5)
- $\mu(t) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}$ (6)
- Eq. (4) can be solved by "Mathematica software", by taking account of the initial conditions A(0)=0 if the system is in the failure state and A(0)=1 if the system is in the functioning state and can be obtained the following solutions:

- If A(0) = 0 then,

$$A(t) = A_0(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\left(\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^{\beta}+\left(\frac{u}{\theta}\right)^{\alpha}} u^{\alpha-1}du\right) \quad (7)$$

- If A(0) = 1 then

$$A(t) = A_1(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\left(1+\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^{\beta}+\left(\frac{u}{\theta}\right)^{\alpha}}u^{\alpha-1}du\right) \tag{8}$$

It can be deduced that:

$$A_1(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} + A_0(t) \tag{9}$$

There are four parameters in the availability functions (7), (8), β, η, α, and θ. The sensitivity of different parameters is described in Figures 2, 3, and 4.

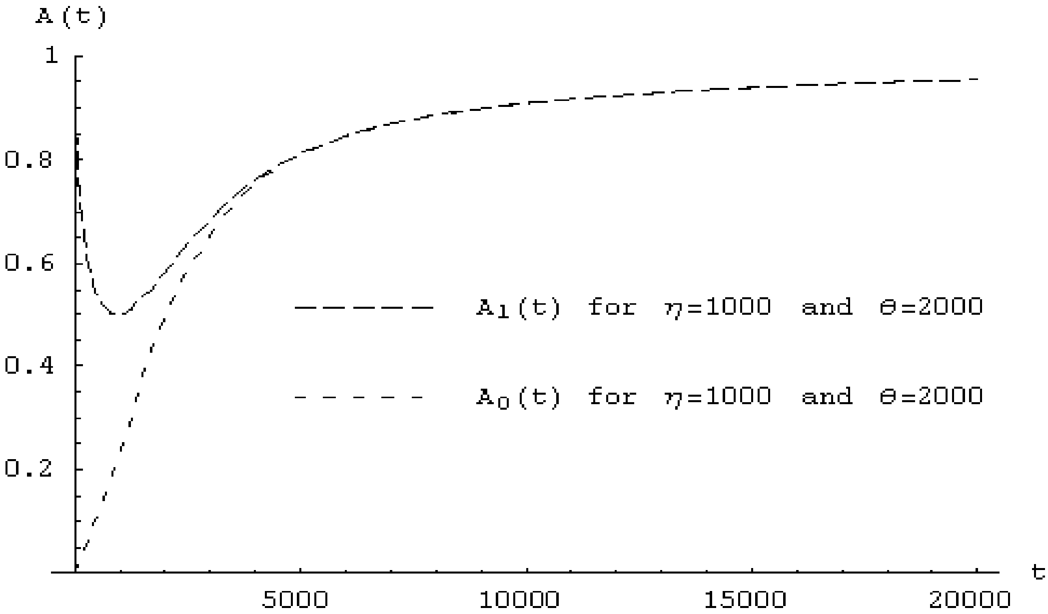

Fig. 2. The availability A(t) for β =0.5, α=1.5 (β < α)

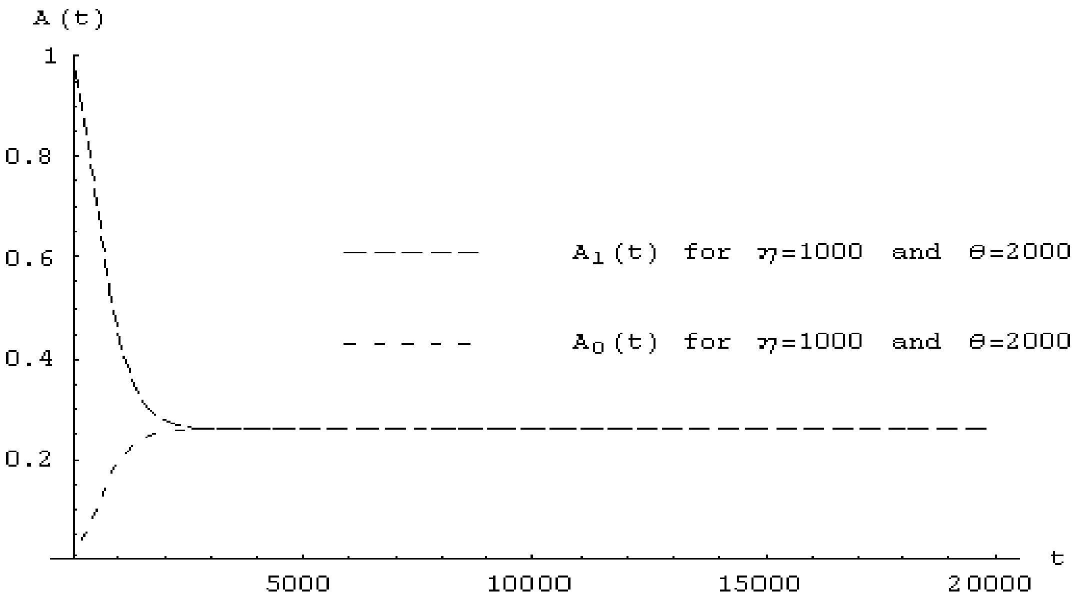

Fig. 3. The availability A(t) for β = α=1.5

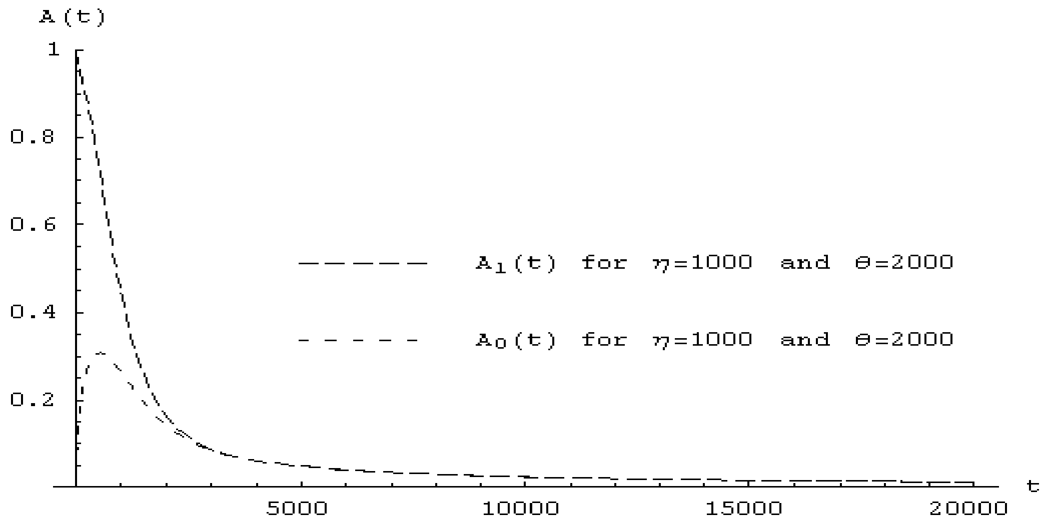

Fig 4. The availability A(t) for β =1.5, α= 0.5 (β > α)

4. The Asymptotic availability

4.1. *Theorem*

$$\lim_{t\to+\infty} A(t) = \lim_{t\to+\infty} \frac{\mu(t)}{\mu(t)+\lambda(t)} \tag{10}$$

Demonstration

It can be assumed that:

$$r(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A(t) \tag{11}$$

Where:

$$r(t) = r_0(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A_0(t), \text{ if } A(t)=A_0(t) \tag{12}$$

And

$$r(t) = r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} A_1(t), \text{ if } A(t)=A_1(t) \tag{13}$$

It may be necessary to prove that:

$$\lim_{t\to+\infty} \frac{\mu(t)+\lambda(t)}{\mu(t)} A(t) = 1 \tag{14}$$

There are four intermediate results can be used to explain this.

- **1st result:**

$$\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} r_0(t) = \lim_{t\to+\infty} r_1(t) \tag{15}$$

Proof :

From Eq. (12), this can be obtained by substituting $A_1(t)$ by Eq.(9)

$$r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} \left(e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} + A_0(t) \right) = \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} + \frac{\mu(t)+\lambda(t)}{\mu(t)} A_0(t) \tag{16}$$

An analogy with Eq. (9) can be deduced:

$$r_1(t) = \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} + r_0(t) \tag{17}$$

Therefore, from Eq. (17), it is necessary, to verify the 1st result, to prove.

$$\lim_{t\to+\infty} \frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} = 0 \tag{18}$$

$$\frac{\mu(t)+\lambda(t)}{\mu(t)} e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} = \left(1+\frac{\lambda(t)}{\mu(t)}\right) e^{-\left(\frac{t}{\eta}\right)^\beta - \left(\frac{t}{\theta}\right)^\alpha} \tag{19}$$

Referring to Eqs. (5) and (6) :

$$\frac{\lambda(t)}{\mu(t)}=\frac{\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}}{\frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}}=\frac{\frac{\beta\eta}{\eta t}\left(\frac{t}{\eta}\right)^{\beta}}{\frac{\alpha\theta}{\theta t}\left(\frac{t}{\theta}\right)^{\alpha}}=\frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)} \qquad (20)$$

Then, Eq. (19) will become:

$$\left(1+\frac{\lambda(t)}{\mu(t)}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}=e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}+\frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}(t)^{\beta-\alpha}e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} \qquad (21)$$

The limit study of Eq. 18 gives:

Figure 1: $\lim\limits_{t\to+\infty} e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}=0$

(22)

And

$$\lim_{t\to+\infty}\frac{\beta}{\alpha}\frac{\theta^{\alpha}}{\eta^{\beta}}(t)^{\beta-\alpha}e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}=0 \qquad (23)$$

So, $\lim\limits_{t\to+\infty} r(t)=\lim\limits_{t\to+\infty} r_0(t)=\lim\limits_{t\to+\infty} r_1(t)$ and the 1st result is verified.

- **2nd result :**

$$r_0(t)=e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv \qquad (24)$$

Proof:

From Eqs. 16 and 17, the $r_0(t)$ function is written as:

$$r_0(t)=\left(1+\frac{\lambda(t)}{\mu(t)}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}} \qquad (25)$$

According to Eqs. (7) and (20), The Eq. (25) will become as follow:

$$r_0(t)=\left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{u}{\eta}\right)^{\beta}+\left(\frac{u}{\theta}\right)^{\alpha}}u^{\alpha-1}du \qquad (26)$$

A change of variables is applied in Eq. (26): $u=t\cdot v\Rightarrow du=tdv$

$$r_0(t)=\left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\theta^{-\alpha}\int_0^t e^{\left(\frac{tv}{\eta}\right)^{\beta}+\left(\frac{tv}{\theta}\right)^{\alpha}}(tv)^{\alpha-1}tdv \qquad (27)$$

$$r_0(t)=\left(1+\frac{\beta}{\alpha}\left(\frac{t}{\eta}\right)^{\beta}\left(\frac{t}{\theta}\right)^{-\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\alpha\left(\frac{t}{\theta}\right)^{\alpha}\int_0^t e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv \qquad (28)$$

Therefore, $r_0(t)=\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\int_0^t e^{\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}}v^{\alpha-1}dv$ and the 2nd Result is verified.

- **3rd result :**

According to the shape parameters β and α, the $r_0(t)$ function should satisfy the two following inequalities:

a) If **$\beta\le\alpha$**, then

$$\frac{\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right]\le r_0(t)\le\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \qquad (29)$$

Or

b) If $\beta\ge\alpha$ then

$$\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right]\le r_0(t)\le\frac{\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}}\right] \qquad (30)$$

Proof:

a) For **$\beta\le\alpha$**

$$\forall v\in]0,1],\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha-1}\le\beta\left(\frac{t}{\eta}\right)^{\beta}v^{\beta-1}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha-1} \qquad (31)$$

And $\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha} \leq \left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}$ (32)

By referring to the second result (24) and the two above mentioned inequalities (31) and (32), so, the $r_0(t)$ function can be put under the form of the following inequality:

$$\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}v^{\alpha-1}dv \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq$$
$$\int_0^1\left(\beta\left(\frac{t}{\eta}\right)^{\beta}v^{\beta-1}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha-1}\right)e^{\left(\left(\frac{t}{\eta}\right)^{\beta}(v)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}dv \quad (33)$$

The calculations of exponential integral allow to express the inequality (33) as follow:

$$\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}\right]_0^1 \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq \left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}\right]_0^1 \quad (34)$$

$$\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}-1\right] \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq \left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}-1\right] \quad (35)$$

So, $\frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \leq r_0(t) \leq \left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]_0^1$, the first inequality (29) is satisfied.

b) For β≥ α

A similar development and demonstration is used for this case also

$$\forall v \in]0,1],\ \beta\left(\frac{t}{\eta}\right)^{\beta}v^{\beta-1}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha-1} \leq \left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha-1} \quad (36)$$

And

$$\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha} \leq \frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha} \quad (37)$$

According to the second result (24) and the two above mentioned inequalities (36) and (37), so, the $r_0(t)$ function can be put under the form of the following inequality:

$$\int_0^1\left(\beta\left(\frac{t}{\eta}\right)^{\beta}v^{\beta-1}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha-1}\right)e^{\left(\left(\frac{t}{\eta}\right)^{\beta}(v)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}dv \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq$$
$$\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)\int_0^1 e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}v^{\alpha-1}dv \quad (38)$$

The calculations of exponential integral allow to express the inequality (38) as follow:

$$\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}v^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}v^{\alpha}\right)}\right]_0^1 \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)v^{\alpha}}\right]_0^1 \quad (39)$$

$$\left[e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}-1\right] \leq e^{\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}r_0(t) \leq \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\frac{\beta}{\alpha}\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}-1\right] \quad (40)$$

$$\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \leq r_0(t) \leq \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[e^{\left(\frac{\beta}{\alpha}-1\right)\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \quad (41)$$

So, $\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \leq r_o(t) \leq \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right]$, the second inequality (30) is satisfied.

- **4th result :**

$$\lim_{t\to+\infty} r_o(t) = 1 \qquad (42)$$

Proof:

a) $\beta \leq \alpha$

By referring to the third result "inequality (29)", to prove the fourth result, it can be sufficient to show that the limits:

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1 \qquad (43)$$

And

$$\lim_{t\to+\infty} \left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1 \qquad (44)$$

Then

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\alpha\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(\ \left(\frac{t}{\eta}\right)^{\beta}\ \ \left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = \lim_{t\to+\infty} \frac{\alpha\left(\frac{t}{\theta}\right)^{\alpha}\left(\frac{\beta\theta^{\alpha}}{\alpha\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}{\alpha\left(\frac{t}{\theta}\right)^{\alpha}\left(\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}\left[1-e^{\left(\ \left(\frac{t}{\eta}\right)^{\beta}\ \ \left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \qquad (45)$$

$$\lim_{t\to+\infty} \frac{\left(\frac{\beta\theta^{\alpha}}{\alpha\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}{\left(\frac{\theta^{\alpha}}{\eta^{\beta}}t^{(\beta-\alpha)}+1\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1[1-0] = 1 \qquad (46)$$ And

$$\lim_{t\to+\infty} \left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = [1-0] = 1 \qquad (47)$$

So, $\lim_{t\to+\infty} r_o(t) = 1$ if $\beta \leq \alpha$

b) $\beta \geq \alpha$

In the same way as explained in the previous case, according to inequality (30), to prove the fourth result, it can be sufficient to show that the limits:

$$\lim_{t\to+\infty} \left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1 \qquad (48)$$

And

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1 \qquad (49)$$

Then

$$\lim_{t\to+\infty} \left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = [1-0] = 1 \qquad (50)$$

And

$$\lim_{t\to+\infty} \frac{\left(\beta\left(\frac{t}{\eta}\right)^{\beta}+\alpha\left(\frac{t}{\theta}\right)^{\alpha}\right)}{\beta\left(\left(\frac{t}{\eta}\right)^{\beta}+\left(\frac{t}{\theta}\right)^{\alpha}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = \lim_{t\to+\infty} \frac{\beta\left(\frac{t}{\eta}\right)^{\beta}\left(1+\frac{\alpha\eta^{\beta}}{\beta\theta^{\alpha}}t^{(\alpha-\beta)}\right)}{\beta\left(\frac{t}{\eta}\right)^{\beta}\left(1+\frac{\eta^{\beta}}{\theta^{\alpha}}t^{(\alpha-\beta)}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] \qquad (51)$$

$$\lim_{t\to+\infty} \frac{\left(1+\frac{\alpha\eta^{\beta}}{\beta\theta^{\alpha}}t^{(\alpha-\beta)}\right)}{\left(1+\frac{\eta^{\beta}}{\theta^{\alpha}}t^{(\alpha-\beta)}\right)}\left[1-e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}-\left(\frac{t}{\theta}\right)^{\alpha}\right)}\right] = 1[1-0] = 1 \qquad (52)$$

So, $\lim_{t \to +\infty} r_o(t) = 1$ and the 4th result (42) is proven also for α≤ β

Finally, the theorem (10) ensues therefore of results 1 and 4

The availability A(t) is plotted together with The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function in figure 5 for β<α, figure 6 for β=α and figure 7 for β>α.

The three figures show that the availability A(t) " with its two solution A_0(t) and A_1(t) " and the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function have tendency to converge towards the same limit when the time t is more important.

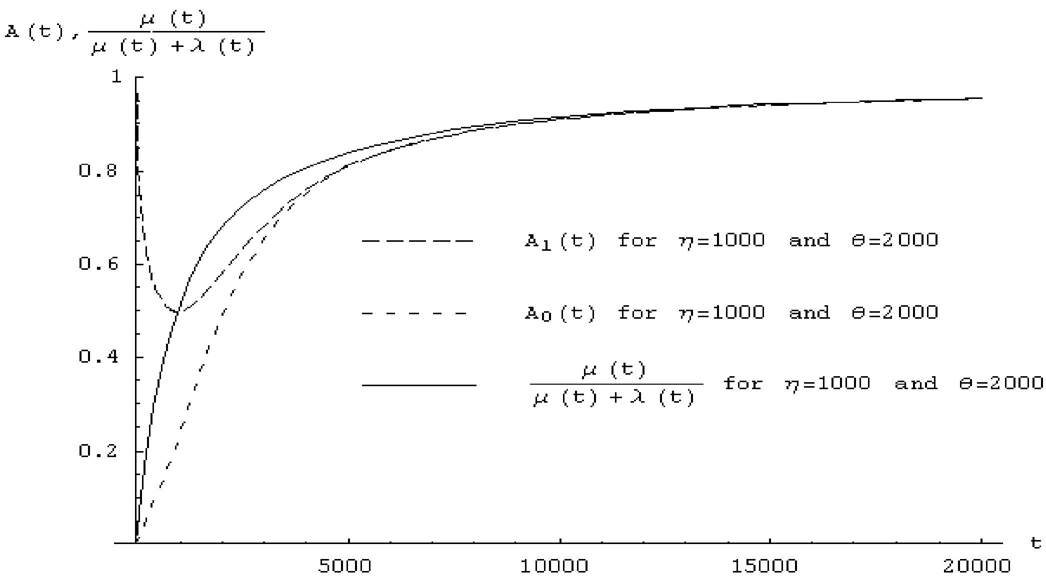

Fig.5. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for β =0.5, α=1.5 (β < α)

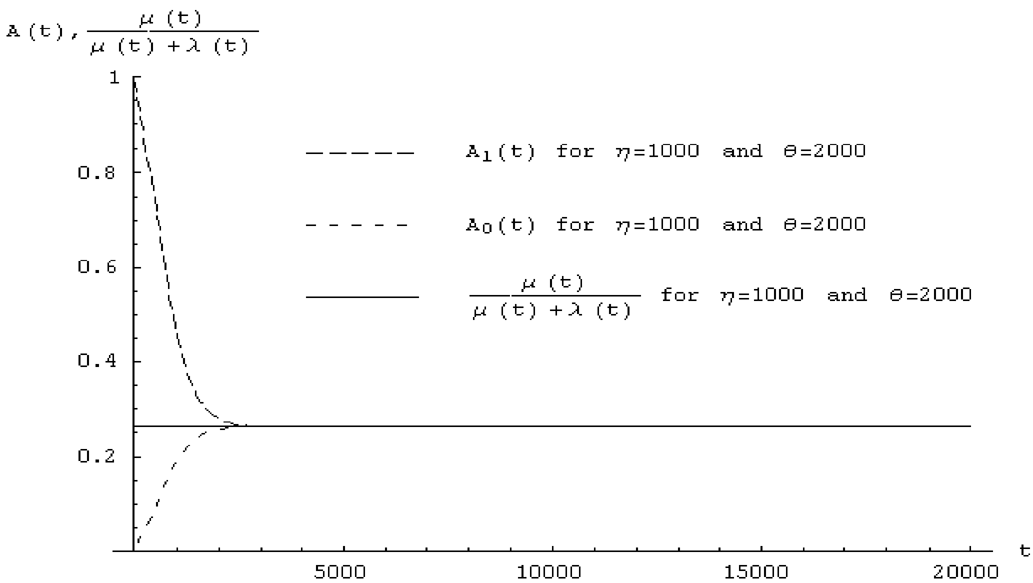

Fig.6. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for A(t) for β = α=1.5

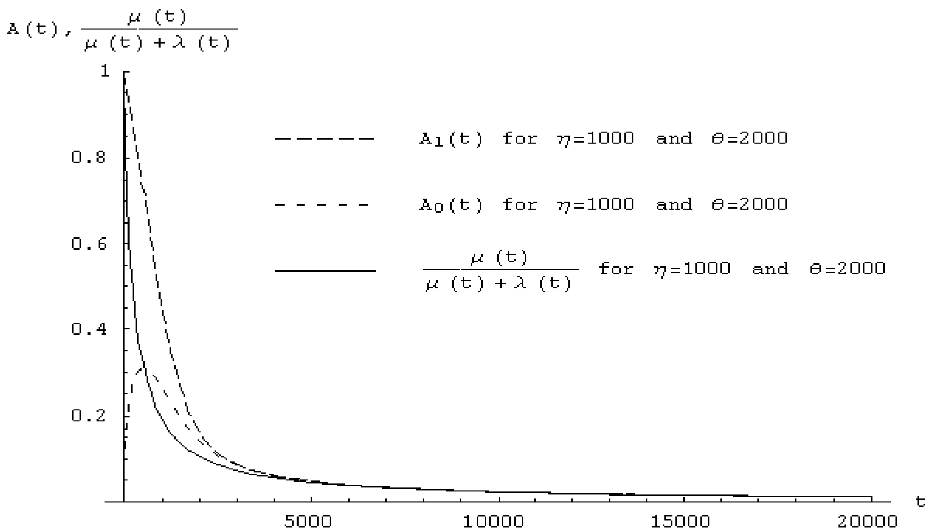

Fig.7. The availability A(t) and $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ for A(t) for β =1.5, α=0.5 (β > α)

4.2. *Asymptotic availability evaluation*

According to (10), the asymptotic availability is defined by:

$$A_\infty = \lim_{t\to+\infty} A(t) = \lim_{t\to+\infty} \frac{\mu(t)}{\mu(t)+\lambda(t)} \tag{53}$$

$$A_\infty = \lim_{t\to+\infty} \frac{1}{1+\frac{\lambda(t)}{\mu(t)}} \tag{54}$$

The study of the limit of the function will be done according to three following cases:

1st case: β<α

$$\lim_{t\to+\infty} \frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty} \frac{\beta}{\alpha}\frac{\theta^\alpha}{\eta^\beta} t^{\beta-\alpha} = 0 \tag{55}$$

Then,

$$A_\infty = \lim_{t\to+\infty} \frac{1}{1+\frac{\lambda(t)}{\mu(t)}} = \frac{1}{1+0} = 1 \tag{56}$$

The converge of the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function, when the time t is more important, to the A_∞= **1** with the sensitivity of the scale parameters (η<θ, η=θ or η>θ) is shown in Fig.8.

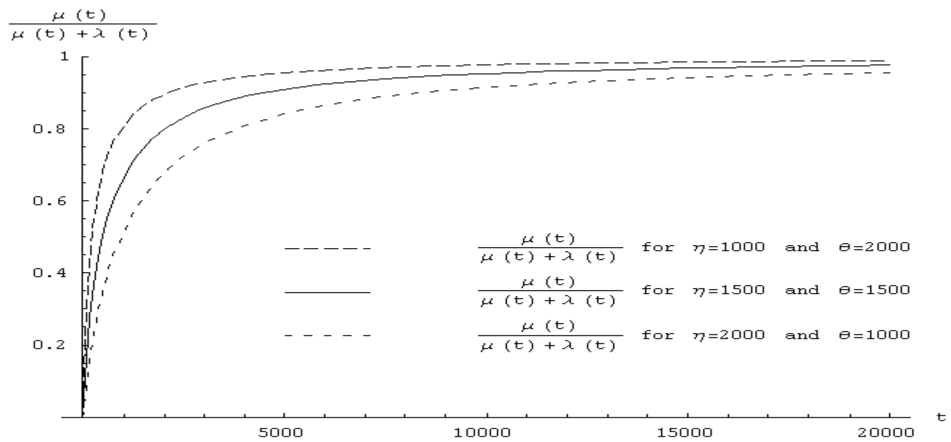

Fig.8. The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function limit studies for β=0.5, α=1.5 (β < α).

2nd case: β=α

$$\lim_{t\to+\infty}\frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty}\frac{\beta\,\theta^{\alpha}}{\alpha\,\eta^{\beta}}t^{\beta-\alpha} = \left(\frac{\theta}{\eta}\right)^{\beta} \quad (57)$$

In this case, the asymptotic availability is defined to be equal to

$$A_{\infty} = \frac{1}{1+\left(\frac{\theta}{\eta}\right)^{\beta}} = \frac{1}{1+\left(\frac{\eta}{\theta}\right)^{-\beta}} = \frac{\theta^{-\beta}}{\theta^{-\beta}+\eta^{-\beta}} \quad (58)$$

Particular cases:

β=α=1: the exponential models

- $A_{\infty} = \frac{\mu}{\lambda+\mu}$ (59)

With $\mu = \frac{1}{\theta}$ and $\lambda = \frac{1}{\eta}$

if η=θ, then

- $A_{\infty} = \frac{1}{2}$ (60)

Fig. 9 shows the asymptotic availability plotted with □=□ with the sensitivity of the scale parameters (η<θ, η=θ or η>θ).

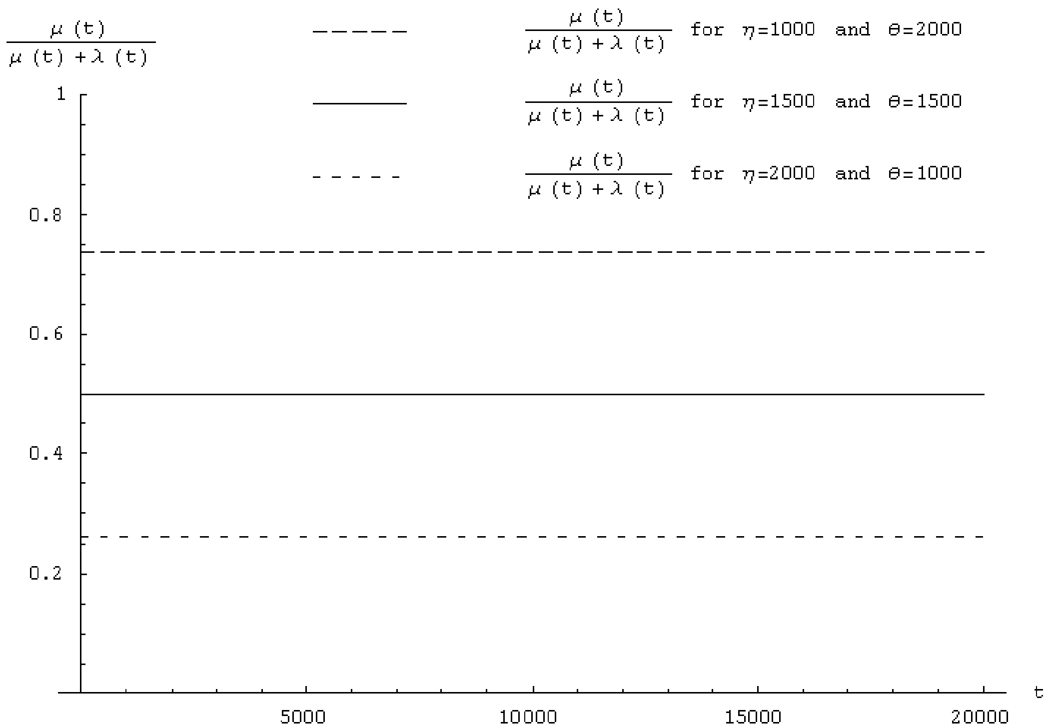

Fig.9. The function $\dfrac{\mu(t)}{\mu(t)+\lambda(t)}$ if β= α=1

3rd case: β>α

$$\lim_{t\to+\infty}\frac{\lambda(t)}{\mu(t)} = \lim_{t\to+\infty}\frac{\beta\,\theta^{\alpha}}{\alpha\,\eta^{\beta}}t^{\beta-\alpha} = +\infty \quad (61)$$

$$\lim_{t\to+\infty}\frac{1}{1+\frac{\lambda(t)}{\mu(t)}} = \frac{1}{1+\infty} = 0 \quad (62)$$

The converge of the $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function, when the time t is more important, to the A_{∞}= **0** with the sensitivity of the scale parameters (η<θ, η=θ or η>θ) is shown in Fig.10.

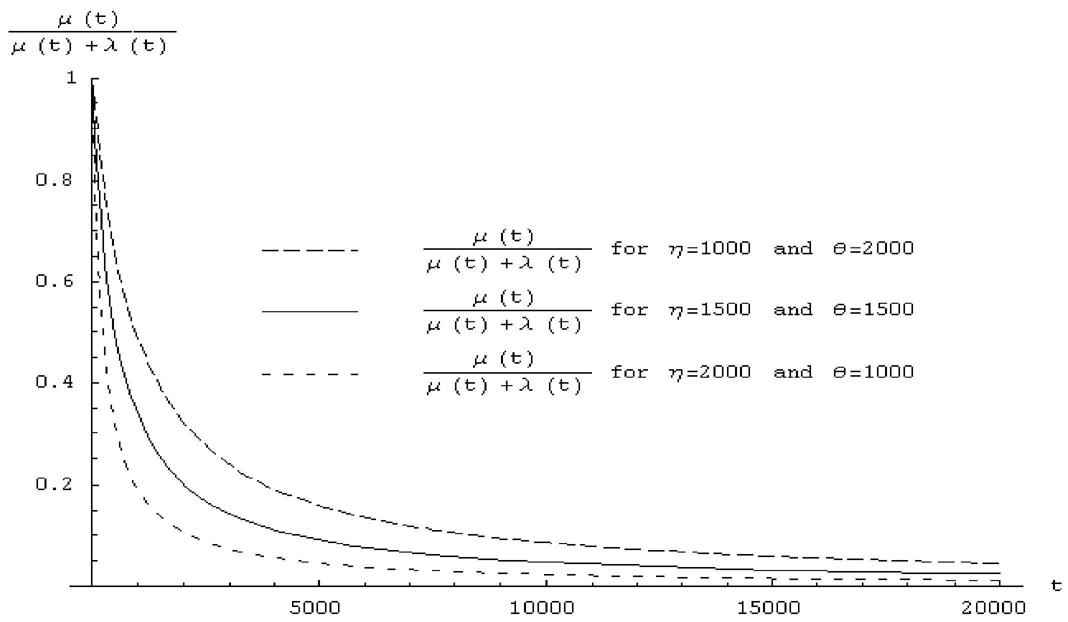

Fig.10. The $\frac{\mu(t)}{\mu(t)+\lambda(t)}$ function limit studies for β=1.5, α=0.5 (β>α).

5. Conclusion

In this paper, the presented work extended the classic availability model to a new asymptotic availability model when the failure and repair rates are distributed according to the Weibull model. The analysis of asymptotic behavior of the system according to the developed model allowed to extract the following result:

The asymptotic availability depends only on the shape parameters of the Weibull models β and α. The scale parameters η and θ do not have an influence in the limit of the availability.

- If $b < a$ then, the system is fully available
- If $b > a$ the system resides in the down state, then, it is unavailable
- If $b = a$, in this case, the asymptotic behavior of the system is analogous to a system governed by the exponential model.

Thus, the future plan includes the research on a novel approach, which will be the combination of two different models (Weibull, Gamma,) or (Weibull, lognormal).

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OBJECT ORIENTED COMMONALITIES IN UNIVERSAL GENERATING FUNCTION FOR RELIABILITY AND IN C++

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Abstract

The main idea of Universal Generating Function is exposed in reliability applications. Some commonalities in this approach and the C++ language are discussed.

Keywords: *Universal Generating function (UGF), C++, reliability.*

INTRODUCTION

Usually, binary systems are considered in the reliability theory. However, this approach does not describe systems with several levels of performance sufficiently. Analysis of multi-state systems forms now a special branch of the reliability theory.

For analysis of such systems consisting of multi-state subsystems/elements, one can use the method of Universal Generating Functions (UGF), which is described below.

1. GENERATING FUNCTION

One frequently uses an effective tool in probabilistic combinatorial analysis: the method of generating functions. For a distribution function of a discrete random variable ξ such that $\Pr\{\xi = k\} = p_k$ for any natural *k*, the generating function has the form

$$\varphi(x) = \sum_k p_k x^k$$

Advantages of using a generating function are well established in this field, and we list a few of those:

(1) For many discrete distributions (e.g., binomial, geometrical, Poisson), there are compact forms of generating functions, which allows one to get analytical solutions quickly and easily.

(2) Moments of statistical distributions can be written in convenient forms. For example, the mathematical expectation of random variable ξ can be found as

$$E\{\xi\} = \frac{\partial}{\partial x}\varphi(x)\Big|_{x=1}.$$

(3) If there are *n* independent random variables ξ_1, ξ_2, ..., ξ_n with the respective generating functions $\varphi_1(x)$, $\varphi_2(x)$,..., $\varphi_n(x)$, then the following generation function can be written for the convolution of these distributions:

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$$\varphi(x) = \prod_{j=1}^{n} \varphi_j(x) .$$

where $\varphi_j(x) = \sum_k p_{jk} x^k$, and p_{jk} is the probability that *j*-th random variable takes value *k*.

2. COMPUTER ALORITHM FOR CALCULATION PRODUCT OF GF'S

Let us present a generating function as a set of *objects*. Each object corresponds to a term in the generating function polynomial. It means that object is a pair of two values: the first is the coefficient, i.e. probability, *p*, and the second is the power of the argument, *a*, i.e. the corresponding random variable.

Consider a computational algorithm for calculation of the convolution of two distributions. One makes the following formal operations.

♦ Take two sets of objects: set $\{(p_{11}, a_{11}), (p_{12}, a_{12}),...,$ $(p_{1k}, a_{1k})\}$ for generating function $\varphi_1(x)$, and set $\{(p_{21}, a_{21}), (p_{22}, a_{22}),..., (p_{2m}, a_{2m})\}$ for generating function $\varphi_2(x)$.

♦ Find all cross "interactions" of objects of the first set with all objects of the second set, using the following rule:

[Interacting objects: (p_{1k}, a_{1k}) and (p_{2m}, a_{2m})] →
[Resulting object: $(p_{1k} p_{2m}; a_{1k} + a_{2m})$].

♦ For all resulting objects with different a_{1k_1} for object-1 and a_{2m_2} for object-2, but such that $a_{1k_1} + a_{2m_2} = a$, one forms a new final resulting object: $(\sum p_{1k_1} p_{2m_2};\ a)$. The total set of such final resulting objects gives us the needed solution: from here we can get probabilities for any *a*.

3. UNIVERSAL GENERATING FUNCTION

We have described a formalized procedure on sets of objects interaction coresponding to product of polynomials. But in practice, we meet a number of situations when this operation is not enough. Consider the following simple examples.

Example 1. Assume that there is a series connection of two (statistically independent) capacitors (Fig. 1).

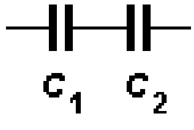

Fig. 1. Series connection of two capacitors.

Assume that c_1 and c_2 are random with discrete distributions: $p_{1k}=\Pr\{c_1=k\}$ and $p_{2j}=\Pr\{c_2=j\}$. One is interested in distribution of total capacity. It is impossible to find the solution with the help of a common generating function. However, there is a possibility to use formal algorithm, described above with the use of corresponding operations over the elements of the objects. The following procedure can be suggested:

♦Take two sets of objects, S_1 and S_2:

$$S_1=\{(p_{11},c_{11}),(p_{12},c_{12}),\ldots,(p_{1k},c_{1k})\}$$

and

$$S_2=\{(p_{21},c_{11}),(p_{22},c_{22}),\ldots,(p_{2m},c_{2m})\},$$

where k is the number of discrete values of the first capacitor, and m is the same for the second one. Here the first element of the object is the probability and the second element is the respective capacity.

♦Find all cross "interactions", Ω, of objects of set S_1 with all objects of set S_2, using the following rule:

$$\Omega\ \{(p_{1i},c_{1i}),(p_{2j},c_{2j})\} = (p^*_{ij};c^*_{ij}).$$

Here p^*_{ij} is the resulting probability calculated in accordance with the multiplication rule (under assumption of independence) as

$$p^*_{ij}=\Omega_{(p)}\{p_{1i},p_{2j}\}=p_{1i}p_{2j}\ ,$$

where $\Omega_{(p)}$ is the rule of interaction of parameters p, which in this particular case is multiplication.

Value of c^*_{ij} is the resulting capacity calculated in accordance with the harmonic sum rule for capacities:

$$c^*_{ij}=\Omega_{(c)}\{c_{1i},c_{2j}\}=\left(c_{1i}^{-1}+c_{2j}^{-1}\right)^{-1},$$

where $\Omega_{(c)}$ is the rule of interaction of parameters c.

♦ Assume that in result we obtain all $R=km$ possible resulting objects of kind $(p^*;c^*)$. Let us order all these resulting pairs in increase of value of c^*: $(p^*_1;c^*_1)$, ..., $(p^*_R;c^*_R)$. For some resulting pairs with numbers, say, i, $i+1,\ldots,i+j$ values of c^* can be the same and equal some C. We converge such objects into a single aggregated object with parameters: $(\sum_{i\le s\le i+j}p^*_s;C)$.

The total set of such final resulting objects gives us the needed solution.

The procedure can be easily expanded on a series connection of several independent capacitors.

$$\Omega^{SER}_{(p)}\{p_{1i},\ p_{2j},\ \ldots,\ p_{nr}\}=p_{1i}\cdot p_{2j}\cdot\ldots\cdot p_{nr}\ ,$$

and

$$\Omega^{SER}_{(c)}\{c_{1i},\ c_{2j},\ \ldots,c_{nr}\}=\left[c_{1i}^{-1}+c_{2j}^{-1}+\ldots+c_{nr}^{-1}\right]^{-1}.$$

<u>Example 2</u>. Pipeline consists of n series sections (pipes). Section j is characterized by random capacity, for which each value v is realized with some probability p. In this case,

$$\Omega^{PAR}_{(p)}\{p_{1i},\ p_{2j},\ \ldots,\ p_{nr}\}=p_{1i}\cdot p_{2j}\cdot\ldots\cdot p_{nr}\ ,$$

and

$$\Omega_{(c)}^{SER}\{v_{1i}, v_{2j}, ..., v_{nr}\} = \min\{v_{1i},\ v_{2j},\ ...,\ v_{nr}\},$$

Example 3. One measures a sum of values, each summand of which is random. With probability p_{js} value j is measured with standard deviation (STD) equal to σ_{js}. In this case, using notation similar to above, one has:

$$\Omega_{(p)}^{PAR}\{p_{1i},\ p_{2j},\ ...,\ p_{nr}\} = p_{1i}\cdot p_{2j}\cdot ...\cdot p_{nr},$$

and

$$\Omega_{(c)}\{\sigma_{1i},\ \sigma_{2j},\ ..., \sigma_{nk_n}\} = \sqrt{\sigma_{1i}^2 + \sigma_{2j}^2 + ... + \sigma_{nr}^2}\ .$$

Examples can be continued and not necessarily with probabilistic parameters.

4. FORMAL DESCRIPTION OF THE METHOD OF UNIVERSAL GENERATING FUNCTIONS

After these simple examples, let us begin with formal description of the Method of Universal Generating Function (UGF[7]). For a more vivid presentation, let us use special terminology to distinguish the UGF from the common generation function. This will relieve us from using traditional terms in a new sense, which may lead to some confusion. Moreover, we hope that this new terminology can help us, in a mnemonic sense, to remember and perhaps even to explain some operations.

In the ancient Roman army, a *cohort* (C) was the main combat unit. Each cohort consisted of *maniples* (M), which were independent and sometimes specialized combat units with several soldiers of different profiles. Several cohorts composed a *legion* (L). The use of this essentially military terminology appears to be convenient in this essentially peaceful mathematical application. A legion is close by its sense to a generating function, a cohort is close to a term of the generating function written in the form of expanded polynomial, and a maniple is close to a parameter of each term.

Starting with polynomial multiplication, in our approach, we will consider less restrictive operations (not only multiplication of terms) and more general parameters. For instance, multiplication of polynomials assumes getting products of coefficients and summation of powers. In our case, we will expand on such restrictive limits on operations.

Let's denote legion j by L_j. This legion includes v_j different cohorts, C_{jk}:

$$L_j = \left(C_{j1}, C_{j2}, ..., C_{jv_j}\right).$$

The number of cohorts within different legions might be different. However, in our approach, maniples, which consist of a cohort, must be similar by its structure.

Each cohort C_{jk} is composed of some maniples, M, each of which represents different parameters, special characteristics, and auxiliary attributes. Each cohort consists of <u>the same set</u> of maniples:

$$C_{jk} = \left(M_{jk}^{(1)}, M_{jk}^{(2)}, ..., M_{jk}^{(s)}\right).$$

[7] UGF might be also read as Ushakov's Generating Function ☺.

To make description of the method more transparent, let us start with the examples of two legions, L_1 and L_2: each of which consists of the following cohorts, L_1=(C_{12},C_{12},C_{13}) and L_2=(C_{21},C_{22}), and each cohort C_{jk} includes two maniples $M_{jk}^{(1)}$ and $M_{jk}^{(2)}$,i.e. C_{jk}=($M_{jk}^{(1)}$, $M_{jk}^{(2)}$). Denote the operation of legion interaction by Ω_L. This operator is used to obtain the resulting legion L_{RES}. In this simple case, one can write:

$$L_{RES} = \Omega_L \{L_1, L_2\}. \tag{1}$$

This interaction of legions produces six pairs of interactions between different cohorts, which generate the following resulting cohorts:

$$C_{RES-1} = \Omega_C \{C_{11}, C_{21}\}, C_{RES-2} = \Omega_C \{C_{11}, C_{22}\},$$

$$C_{RES-3} = \Omega_C \{C_{12}, C_{21}\},\ C_{RES-4} = \Omega_C \{C_{12}, C_{22}\},$$

$$C_{RES-5} = \Omega_C \{C_{13}, C_{21}\},\ C_{RES-6} = \Omega_C \{C_{13}, C_{22}\}.$$

Here $\Omega_C \{\bullet\}$ denotes the interaction of cohorts.

Interaction of cohorts consists of interaction between its costituent maniples. All cohorts contain maniples of the same types though with individual values of parameters. Let us take, for instance, resulting cohort C_{RES-5}, which is obtained as interaction of cohorts C_{13} and C_{21}. In turn, interaction of these particular cohorts consists in interaction of their corresponding maniples:

$$M_{RES-5}^{(1)} = \Omega_M^{(1)} \{M_{13}^{(1)}, M_{21}^{(1)}\}$$

$$M_{RES-5}^{(2)} = \Omega_M^{(2)} \{M_{13}^{(2)}, M_{21}^{(2)}\}$$

The rules of interaction between maniples of different types, i.e. $\Omega_M^{(1)} \{M_{1i}^{(1)}, M_{2j}^{(1)}\}$ and $\Omega_M^{(2)} \{M_{1i}^{(2)}, M_{2j}^{(2)}\}$ are (or might be) different.

Interaction of n legions can be written as:

$$L = \Omega_L (L_1, L_2, ..., L_n).$$

Operator Ω_L denotes a kind of "n-dimensional Cartesian product" of legions and special final "reformatting" of the resulting cohorts (like converging polynomial terms with the equal power for a common generating function). Since each legion j consists of ν_j cohort, the total number of resulting cohorts in the final legion (after all legion interaction) is equal to

$$\nu = \prod_{1 \le j \le n} \nu_j .$$

Number ν corresponds to the total number of cohorts' interactions.

5. IMPLEMENTING UGF PHILOSOPHY IN COMPUTER LANGUAGE C++

We would like use the UGF (Universal Generating Function) philosophy in an analysis tool and perform reliability calculations for real-world systems. Because we are talking about an (reliability) engineering discipline, all philosophies present the need to be converted into numerical results and predictions. Thus, the UGF philosophy begs an implementation! The implementation task is to identify objects (maniple, cohort, legion) and program all interactions between them. Unfortunately, we run into a combinatoric explosion of possible interactions for a sysem consisting of a large number of (atomic) units. Even moderm computers are not able to enumerate astronomically large (2^{1000}) number of interaction states in system consisting of 1000 binary atomic units. Fortunately, for a class of frequently occuring practical systems, the situation is not as hopeless as it may first appear. For a system to be useful in engineering, it may only fail very infrequently. In a highly reliable system, the failure probability of all atomic units much smaller that the system failure probability. This fact makes most of the interactions exceedingly rare and they can be systematically ignored in an approximation scheme that retains only the dominant contributions.

Let us proceed to find an approximate implementation of the UGF philosophy for highly relaible systems in a system simulator. It should be reasonably easy to identify an atomic unit in reliability theory as a maniple. Independence of the maniples corresponds to statistical independence of the atomic units. A cohort is defined to be a collection of maniples. The same definition holds in the context of reliability theory, where the collection is defined by a failure criterion. In a series system, each atomic unit is assumed to provide distinct and critical functionality. This maps on to the notion of specialized combat units. In a parallel system, all atomic units are statistically identical. This improves survival probability during operation, either in the military or in system reliability! Thus, we may identify a subsystem in reliability engineering as a cohort in UGF formalism.

Interactions between the objects are identified in the simulator by their natural reliability names. *k-out-of-n* combinations are of primary interest. But this class includes the two most frequently appearing reliability structures: series (*n-out-of-n*) and parallel (1-*out-of-n*). In fact, probability of failure of a parallel system is negligible (higher order in numerical smallness) with an additional assumption of high availability of the atomic units. Obviously a series system can be made up of distinct units providing separate functionality to the system.

As an illustration let us consider a system S of two subsystems *A* and *B* in series. Let *A* be atomic and *B* be composed of two atomic units *X* and *Y* in parallel. One possible C++ coding for this (simple) system is

B=*Parallel*(X,*Y*); S = *Series*(A,*B*);

Properties (MTBF, MTTR etc.) of all atomic units are specified at the start of analysis. Operations like Series and Parallel are C++ member functions for the instances of class “unit”. We will not specify unit composition rules in this work. Most of these rules can be found in standard textbooks on reliability engineering. Interested readers may find the remaining ones (involving switching time and PEI) in Chakravarty and Ushakov (2000, 2002).

It remains to identify the “legion”. The preceding paragraphs almost suggest that a legion be identified with the entire system in reliability theory, where the system is further assumed to be represented by its generating function. We would like to note that that this analogy cannot be taken literally sometimes. It is common for a real world reliability system to have deeper hierarchies (e.g., system, equipment shelves, equipment racks, electornic cards) like modern day militaries. In such an elaborate system, we still identify the atomic units as maniples. At the other end, we identify the entire system as a "legion"! All intermediate stages in the hierarchy are considered generalized "cohorts".

In Chakravarty and Ushakov (2000) implementation, any subsystem can be composed from other subsystems at the next lower level of hierarchy (or atomic units which are always at the lowest level). A newly formed subsystem provides an effective reliability description of all units that compose this subsystem. This composition can be continued indefinitely to obtain an effectiveness measure for the entire system. They have shown that this can be recast as an approximation from a system generating function when all atomic units satisfy binary failure criteria (on/off) they are statistically independent, the system itself is highly reliable and reliability design of the system consists of hierarchical blocks.

6. RELIABILITY ANALYSIS OF GLOBALSTAR™ GATEWAYS

Globalstar is a low-earth-orbit (LEO) based telephony system with global coverage. The gateways make its ground segment that connect to the orbiting satellites. The gateways are cpmlex systems with more than a thousand components (e.g., electronic cards). Ushakov (1998), Chakravarty and Ushakov (2002) used the UGF approach for the reliability (performance) analysis of Globalstar™ gateways (fixed ground segment of a low earth orbit satellite communications system). Given the prominence of object oriented abstractions and operations in Globalstar design, it should not be surprising that the reliability analysis naturally fits into the UGF philosophy. Further, these ideas can be naturally implemented in the computer using an object oriented language.

Because of the object oriented nature of system reliability design in Globalstar (interaction between objects like system, racks, shelves, cards are triggered by failure, switching of failed units and changing user demand), Ushakov (1998) proposed that a system reliability simulator should be coded in an object oriented computer language like C++. Later, Chakravarty and Ushakov (2002) implemented a simulator for the Globalstar™ Gateway in C++.

In Chakravarty and Ushakov implementation for Globalstar, C++ objects are in one-to-one correspondence with reliability objects. An object is specified by mean time between failures (MTBF), mean time to repair/replace (MTTR) and an effectiveness weight (partial effectiveness index: PEI). By definition, PEI=1 for binary atomic units. All failure distributions are implicitly assumed to be Exponential. If failed units were to be automatically swapped, a switching time was also assigned by Chakravarty and Ushakov (2000). Even small switching time is important because it changes a parallel system "on paper" to a series system with small MTTR. This may have dramatic effect overall on system reliability.

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SOME INFERENCES ON THE RATIO AVERAGE LIFETIME/TESTING TIME IN ACCEPTANCE SAMPLING PLANS FOR RELIABILITY INSPECTION

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Abstract

In this paper we construct effective single sampling plans for reliability inspection, when the distribution of failure times of underlying objects obey a Weibull law. To this purpose we use the index average lifetime (E (T)/testing time (T) for two values of E(T) - acceptable and non acceptable ones - and known shape parameter (K) of the Weibull cdf. We derive also a relationship between this index and reliability function R(t) of the assumed statistical law. A numerical illustrations is provided in the case of Rayleigh cdf - that is for a Weibull shape k = 2.

Key words: cdf - cumulative distribution function, two-parameter Weibull law, sampling plan, average lifetime, testing time, Rayleigh case.

1. Introduction

From the SQC (Statistical Quality Control) perspective, reliability is considered a dynamical quality characteristic since the performances of a given technical entity are put into light if the underlying element, component or system is in a functional / operational state, performing a prescribed mission for a specified period of time.

Static quality characteristics (such as hardness, length, pressure, volume a.s.o.) are observed and measured directly in units independent of time.

Metaphorically speaking, reliability is viewed as one of the special features of the general concept of quality. Vasiliu (1980, [7, page 26]) defines reliability as „the dimension in time of quality". Two decades later, Yang and Kapur (1997, [9, page 340]) state that „reliability is quality over time".

Anyway, no matter how good is a design, how performing is the production process, how careful is handled and exploited a technical system there is no way to stop its final decay. After a certain period of time - which may be short or quite long - every human made object sooner or later will fail. This event (failure) is due to natural causes (wear-out phenomenon) or to some „artificial" ones as for instance the use of the item in inappropriate conditions (aggressive environment, intensive operational tasks, lack of adequate maintenance actions, mishandling etc.).

A failure occurs in a random manner and usually after a certain period of time when the system was operating supposedly, satisfactorily.

Since we do not know the exact moment when a specified object will fail, we are forced to judge in terms of probabilities and averages involving the time elements as one of the main parameters. The failure behavior of that specific object has to be modeled and hence we are facing to the problem of choosing the most suitable class of life distributions describing this time-to-failure phenomenon.

Nevertheless, we may speak about the so-called „static reliability" where the time element is not instantly (or explicitly) involved (see Blischke and Murthy, 2000 [2, page 173 - 177]).

We refer here to the so-called „stress-strength models" where reliability is regarded as the capacity of item's strength (x) to resist to the action of stress (y). Actually a measure of reliability in this model is R = Prob {x > y} where usually, both x and y are random variables. If this probability is greater than 50% we could expect a desirable reliability of the underlying entity.

In batch inspection procedures if the characteristic of interest is reliability (or durability) of underlying items we must take into account their failure behavior (where time element is the main parameter) in order to construct suitable sampling acceptance plans from economical point of view.

In this paper, we shall present some new results on the index average lifetime (durability)/testing time in the construction of acceptance sampling plans for reliability inspection, when time-to-failure distribution is a two parameter Weibull one.

2. Various approaches of reliability inspection

It is well-known that a very general approach for batch inspection - no matter the nature of quality characteristic investigated - is that called **attributive** one. All practical procedures have been already standardized - see the document MILSTD105 E „Sampling procedures and tables for inspection by attributes" (see Kirkpatrick, 1970 [5, page 354 - 415] where the variant D is entirely reproduced). The simplicity of attributive method lies in the fact that products are classified into categories: conforming and defective (nonconforming) ones regarding some specified criteria. In the case of reliability/durability inspection, this attributive approach ignores the very nature of failure behavior of inspected objects and this could lead to a larger sample (or samples) to be tested: if the items are quite expensive and since the specific test in this case is destructive, the procedure appears to be non-economic.

It is important to notice that the attributive approach ignores in the case of reliability/durability inspection, the following elements: a) what kind of samples we use for inspection: complete ones or censored ones; b) distributional assumption for time-to-failure; c) sampling is with replacement or non replacement; d) testing conditions are normal or accelerated ones; e) items are reparable or non-reparable (if they are non restoring, then E(T) is just the mean durability and $\overline{T}$ (sample mean) is computed with $(t_i)_{1\le i\le n}$ values where t_i is the time to first - and last! - failure of the $i^{\underline{th}}$ item submitted to the test; it is senseless to speak in this case about MTBF - Mean Time Between Failures); f) what is the relationship between testing time (T_0) and the actual operating life of those items.

More useful are in such special case methods based on average operating time or on hazard rate associated to the failure time model specific for each peculiar instance.

The document MILSTD 781 Reliability test: exponential distribution (U.S. Dept. of Defense, Washington D. C., 1984) use the ratio E (T) / T_0 where E (T) is the average lifetime (durability) of underlying objects and T_0 is the testing time. In the exponential case $F(t;\theta) = 1 - \exp(-t/\theta)$, $t \geq 0$, $\theta > 0$, F being the cdf (cumulative distribution function) of the representative variable (T), the mean-value of T is $E(T) = \theta$ and therefore the inference is done straight forwardly on the distributional parameter (details are given in Cătuneanu-Mihalache, 1989 [3], Vodă-Isaic Maniu [8]).

We shall examine now this ratio E (T) / T_0 in the case of a Weibull distribution.

3. The ratio E (T) / T_0 in the case of a Weibull distribution

Let T_w be a two-parameter Weibull distribution with the following cdf

$$T_w : F(t;\theta,k) = 1 - \exp(-t^k/\theta), \quad t \geq 0, \quad \theta, k > 0 \tag{1}$$

The corresponding reliability function is $R_t = \exp(-t^k/\theta)$ and the theoretical mean-value is

$$E(T) = \theta^{1/k} \cdot \Gamma(1 + 1/k) \text{ where } \Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \tag{2}$$

is the well-known Gamma function (see Isaic-Maniu, 1983 [6, page 21]). We have hence

$$\theta = \left[\frac{E(T)}{\Gamma(1 + 1/k)}\right]^k \tag{3}$$

and consequently we get

$$R(t) = \exp\left\{-\left[\frac{t \cdot \Gamma(1 + 1/k)}{E(T)}\right]^k\right\} \tag{4}$$

By taking natural logarithms, we have

$$\ln R(t) = -\left[\frac{t \cdot \Gamma(1 + 1/k)}{E(T)}\right]^k \tag{5}$$

and finally

$$\frac{E(T)}{t} = [-\ln R(t)]^{-1/k} \cdot \Gamma(1 + 1/k) \tag{6}$$

Therefore, the ratio E(T)/t depends on the shape parameter (k) of Weibull's cdf and on its reliability function. If we fix t = T_0 and considering k to be known, we have either to estimate R (T_0) or to fix lower acceptable bound for it.

From (5) we can deduce

$$\ln E(T) = -\frac{1}{k} \cdot \ln[-\ln R(T_0)] + \ln \Gamma(1 + 1/k) + \ln T_0 \tag{7}$$

and taking into account a formula given in Abramowitz-Stegun (1979, [1, page 82]), namely

$$\ln \Gamma(1 + 1/k) \approx -\ln(1 + 1/k) + \frac{1 - C}{k} \tag{8}$$

where C is the Euler-Mascheroni's constant (≈ 0.57721).

If we approximate now ln (1+1/k) as 1/k (let us recall the inequalities $x(1+x)^{-1} \le \ln(1+x) \le x, \quad 0 \le x < 1$) then, if k > 1 the relationship (7) becomes a very simple estimation equation for the shape parameter if it is not known.

4. Construction of acceptance sampling plans

We shall start with the following assumptions:

(1) the items subjected to inspection are non-reparable;

(2) the failure time distribution is a two-parameter Weibull one with known shape parameter;

(3) we use only one sample with no replacement, its size has to be determined;

(4) there is fixed an acceptable average lifetime $[E(T)]_1$ corresponding to a given risk α (usually $\alpha = 0.05$ or 5%), that is we wish to accept a lot with such average value with $1 - \alpha = 0.95$ probability;

(5) there is fixed a non-acceptable average lifetime $[E(T)]_2$ corresponding to a given risk β (usually $\beta = 0.10$ or 10%), that is we with to reject a lot with such average value with $1 - \beta = 0.90$ probability;

(6) there is fixed a testing time T_0 smaller than the actual operating life of the underling items.

Therefore, the sampling plan will be the system of objects $\{(n, A) \mid T_0\}$ where n and A are respectively the sample size and acceptance number which has to be determined and T_0 is the previously fixed testing time.

The decision on the lot is taken as follows: submit to the specific reliability/durability test a sample of size n drawn randomly from a lot of size N (n < N), during a period of units of T_0 (usually, T_0 is given in hours); record then the number (d) of failed elements in the interval $[0, T_0]$; if $d \le A$, then the lot is accepted - otherwise, that is if $d \ge R = A + 1$, the lot is rejected (here, R = A + 1 is the so-called „rejection number").

The elements n and A are determined via the OC - function (Operative Characteristic) of the plan which has the expression

$$L(p) = \sum_{d=0}^{A} \frac{1}{d!} (np)^d e^{-np} \tag{9}$$

where $d! = 1 \cdot 2 \cdot \ldots \cdot A$ and p is the defective fraction of the lot given by

$$p = 1 - \exp\left(-t^k/\theta\right), \quad t \ge 0, \quad \theta, k > 0 \tag{10}$$

and d is the number of failed elements during the testing period T_0 (see for other details US-MIL-HDBK-781 „Reliability Test Methods, Plans and Environments for Engineering Development. Qualification and Production" and Grant and Leavenworth, 1988 [4]).

Choosing two values for p (p_1 and p_2) for which $L(p_1) = 1 - \alpha = 0.95$ and $L(p_2) = 1 - \beta = 0.10$ and using the ratios $[E(T)]_1/T_0$ and $[E(T)]_2/T_0$ we obtain a system which provides the elements of the plan, n and A.

In table 1 we present some values for n and A in the Rayleigh case, that is if k = 2, the input data being (in order to ease the computations) the following quantities: $100T_0/[E(T)]_1$ for which $L(p_1) = 0.95$ and $100T_0/[E(T)]_2$ for which $L(p_2) = 0.10$ (the first figure is given in parentheses).

We do notice that in this approach it is avoided the knowledge of R (T_0) since the input elements are only T_0 and $[E(T)]_{1,2}$ which are fixed previously taking into account the specific case at hand.

Table 1

Elements of the single sampling plan {(n, A) | given T_0} for the input ratios $100T_0$ / $[E(T)]_{1,2}$

A	n: Values of $100T_0/[E(T)]_2$ for which $L(p_2) = 0.10$			
	100	50	25	15
0	3	12	46	130
	(15)	(7.5)	(3.8)	(2.2)
1	6	21	80	224
	(30)	(15)	(7.5)	(4.5)
2	8	30	110	305
	(40)	(19)	(9.9)	(5.9)
3	11	35	139	383
	(42)	(22)	(11)	(6.8)

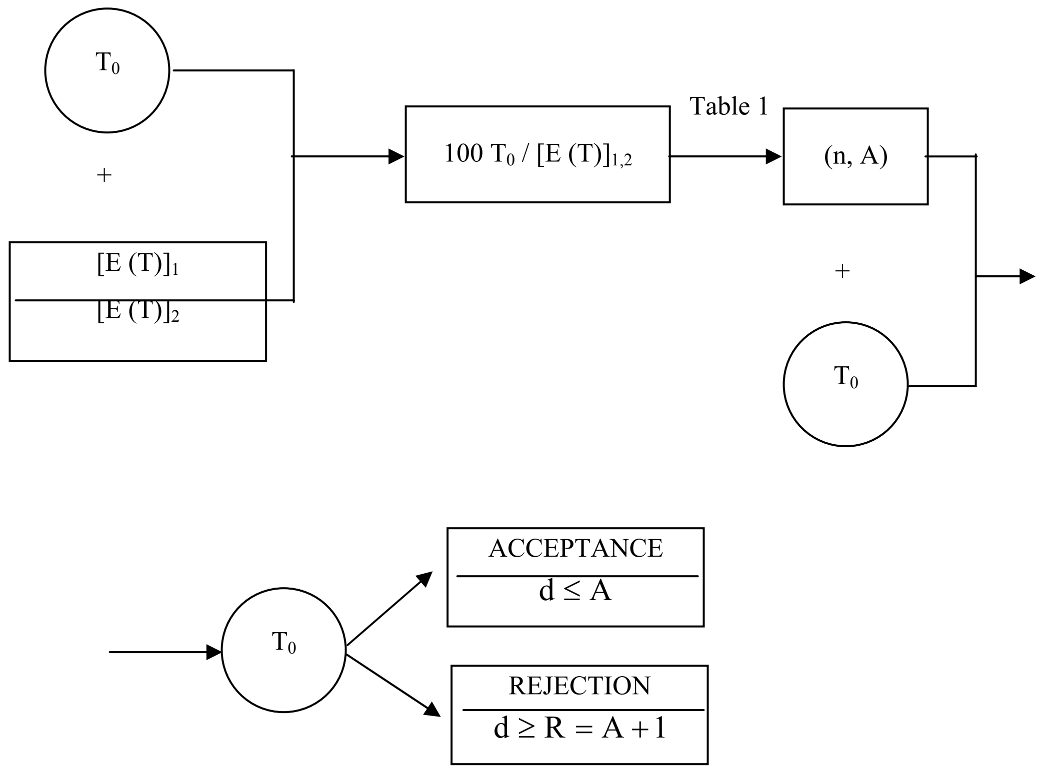
Example: Assume that we have an acceptable durability $[E(T)]_1$ = 5000 hours and a non-acceptable one as $[E(T)]_2$ = 1000 hours. Testing time was fixed at the value T_0 = 500 hours (the usual risks are $\alpha = 0.05$ and $\beta = 0.10$).

Therefore, to find the plan, we evaluate

$$\frac{100\,T_0}{[E(T)]_2} = \frac{100 \cdot 500}{1000} = 50 \text{ and } \frac{100\,T_0}{[E(T)]_1} = \frac{100 \cdot 500}{5000} = 10$$

In table 1, the nearest value of $100T_0 / [E(T)]_1$ for $100T_0 / [E(T)]_2 = 50$ is 15 and hence for the couple 50 (15) we read n = 21 (sample units) and A = 1 (the acceptance number). The plan is hence {(21, 1) | 500} and as a consequence we shall test n = 21 items on a period of 500 hours and record d - the number of failed elements. If d = 0 or 1, we shall accept the lot - otherwise (that is $d \geq 2$) we shall reject it.

The flow of operations is presented below.



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AN ACCURACY OF ASYMPTOTIC FORMULAS IN CALCULATIONS OF A RANDOM NETWORK RELIABILATY

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INTRODUCTION

In this paper a problem of asymptotic and numerical estimates of relative errors for different asymptotic formulas in the reliability theory are considered. These asymptotic formulas for random networks are similar to calculations of Feynman integrals.

A special interest has analytic and numerical comparison of asymptotic formulas for the most spread Weibull and Gompertz distributions in life time models. In the last case it is shown that an accuracy of asymptotic formulas is much higher.

1. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY LOGARITHM

Consider the nonoriented graph Γ with fixed initial and final nodes and with the arcs set W. Define $\mathcal{R}=\{R_1,...,R_n\}$ as the set of all acyclic ways between the initial and final nodes of the graph Γ. Designate P_R the probability of the way R work. Then in the condition

$$p_w \sim \exp\left(-c_w h^{-d_w}\right),\ h\to 0,\ w\in W,$$

we have:

$$P_R \sim \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right),$$

where $C(R)=\sum\limits_{w:d_w=D(R)} c_w$ and $D'_R<D_R$ is a next by a quantity after $D_R=\max\limits_{w\in R} d_w$ element in the set $\{d_w, w\in R\}$, $C'(R)=\sum\limits_{w:d_w=D'(R)} c_w$. If in the way R this element is absent we put then $D'_R=-\infty$, $C'(R)=0$.

Denote $D_\Gamma=\min\limits_{R\in\mathcal{R}} D_R$ and designate $\mathcal{R}_1=\{R:D_R=D_\Gamma\}$, $\mathcal{R}_2=\mathcal{R}\setminus\mathcal{R}_1$, then the probability P_Γ of the graph Γ work satisfies the formulas

$$P_\Gamma \sim P_\Gamma^1+P_\Gamma^2,\ P_\Gamma^i \sim \sum_{R\in\mathcal{R}_i} P_R = \sum_{R\in\mathcal{R}_i} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right),\ i=1,2.$$

By the definition

$$P_\Gamma^1 \sim \sum_{R\in\mathcal{R}_1} \exp\left(-C(R)h^{-D_\Gamma} - C'(R)h^{-D'_R}(1+o(1))\right)\sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\sum_{R\in\mathcal{R}_1}\exp\left(-C'(R)h^{-D'_R}(1+o(1))\right),$$

where $C_\Gamma=\min\limits_{R\in\mathcal{R}_1} C_R$ and $D'_R<D_\Gamma$, $R\in\mathcal{R}_1$, so

$$P_\Gamma^1 \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right),$$

where $D'_\Gamma=\min\limits_{R\in\mathcal{R}_1:C(R)=C_\Gamma} D'_R<D_\Gamma$, $C'_\Gamma=\min\limits_{R\in\mathcal{R}_1:C(R)=C_\Gamma, D'_R=D'_\Gamma} C'(R)$.

And consequently $D_R < D_\Gamma$, $R \in \mathcal{R}_2$, $P_\Gamma^2 = o\left(P_\Gamma^1\right)$:

$$P_\Gamma^2 \sim \sum_{R\in\mathcal{R}_2} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_R}(1+o(1))\right) \sim \sum_{R\in\mathcal{R}_2} \exp\left(-C(R)h^{-D_R}(1+o(1))\right) =$$
$$= \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \sum_{R\in\mathcal{R}_2} \exp\left(-C_\Gamma h^{-D_\Gamma} - C(R)h^{-D_R}(1+o(1))\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \sum_{R\in\mathcal{R}_2} \exp\left(-C(R)h^{-D_R}(1+o(1))\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \exp\left(-C''_\Gamma h^{-D''_\Gamma}(1+o(1))\right),$$

where

$$D''_\Gamma = \min_{R\in\mathcal{R}_2} D_R > D_\Gamma, \quad C''_\Gamma = \min_{R\in\mathcal{R}_2} C(R).$$

So we have:

$$P_\Gamma \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right)\left(\exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right) + \exp\left(-C''_\Gamma h^{-D''_\Gamma}(1+o(1))\right)\right) \sim$$
$$\sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \exp\left(-C'_\Gamma h^{-D'_\Gamma}(1+o(1))\right).$$

As a result obtain that

$$\ln P_\Gamma \sim -C_\Gamma h^{-D_\Gamma}\left(1 + Ah^{\Delta_\Gamma}(1+o(1))\right), \ \Delta_\Gamma = D_\Gamma - D'_\Gamma > 0, \ A = C'_\Gamma / C_\Gamma.$$

And consequently

$$\frac{\ln P_\Gamma}{-C(\Gamma)h^{-D_\Gamma}} - 1 \sim Ah^{\Delta_\Gamma}. \tag{1}$$

2. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY

Assume that $P\left(U_p\right)$ is the probability of the event U_p that all arcs $w_1^p, ..., w_{m_p}^p$ of the way R_p work. Then we have

$$P_\Gamma = P\left(\bigcup_{p=1}^{n} U_p\right). \tag{2}$$

Suppose that the probability of the arc $w \in W$ work equals $\exp\left(-c_w h^{-d_w}\right)$, $h > 0$, where c_w, d_w are some positive numbers and for arcs $w' \neq w''$ the constants $d_{w'} \neq d_{w''}$. So we have

$$P\left(U_p\right) = \exp\left(-\sum_{j=1}^{m_p} c_{w_j^p} h^{-d_{w_j^p}}\right).$$

Assume that the enumeration of the arcs in the way R_p satisfies the inequalities

$$d_{w_1^p} > d_{w_2^p} > ... > d_{w_{m_p}^p}.$$

Denote $D^p = \left(d_{w_1^p}, ..., d_{w_{m_p}^p}\right)$ and introduce on the vectors set $\left\{D^p,\ 1 \le p \le n\right\}$ the following order relation. Say that $D^p \succ D^q$, if for some $k \le \min\left(m_p, m_q\right)$ the first $k-1$ components of these vectors coincide and the k component in the vector D^p is larger than in the vector D^q. If there is not such k and in the vectors D^p, D^q all first $\min\left(m_p, m_q\right)$ components coincide then $D^p \succ D^q$ for $m_p < m_q$.

Remark that for some $p \ne q$ the arcs sets $\left\{w \in R_p\right\}, \left\{w \in R_q\right\}$ can not satisfy the inclusion $\left\{w \in R_p\right\} \subseteq \left\{w \in R_q\right\}$. In the opposite case there is the node u_* in which the ways R_p R_q diverge by the arcs $\left(u_*, u_p\right)$ $\left(u_*, u_q\right)$. But as the arc $\left(u_*, u_p\right) \in \left\{w \in R_q\right\}$ then the way R_q has a cycle. This conclusion contradicts with the assumption that the way R_q is acyclic.

So as the quantities d_w are different then $D^p \ne D^q$, $p \ne q$. As a result we obtain the order relation on the vectors set $\left\{D^1, ..., D^n\right\}$, and if $D^p \succ D^q$, $h \to 0$, so $P\left(U_q\right) = o\left(P\left(U_p\right)\right)$. It is not difficult to check that this relation is transitive. Consequently the order relation on the set $\left\{D^1, ..., D^n\right\}$ is linear. Assume that the enumeration of the vectors D^p satisfies the formula $D^1 \succ ... \succ D^n$. From the formula (2) we have

$$P_\Gamma^* - \sum_{1 \le p < q \le m} P\left(U_p U_q\right) \le P_\Gamma \le P_\Gamma^*\ ,\ P_\Gamma^* = \sum_{p=1}^{m} P\left(U_p\right). \tag{3}$$

As the inclusion $\left\{w \in R_p\right\} \subseteq \left\{w \in R_q\right\}$ is not true for $p \ne q$ so in the way R_p there is an arc which does not belong to the way R_q. Consequently we have

$$P\left(U_q\right) = o\left(P\left(U_p\right)\right),\ 1 \le p < q \le m\ ,\ \sum_{1 \le i < j \le m} P\left(U_p U_q\right) = o\left(P\left(U_2\right)\right) \tag{4}$$

The formulas (3), (4) give us the following asymptotic expansion for P_Γ with the first and the second members of the smallness:

$$P_\Gamma \sim P_\Gamma^* \sim P(U_1)\ ,\ P_\Gamma - P(U_1) \sim P(U_2)\ ,\ P(U_2) = o(P(U_1))\ ,\ h \to 0\ . \tag{5}$$

3. AN APPLICATION TO LIFE TIME MODELS

Suppose tha τ_w are independent random variables and characterize life times of the arcs $w \in W$. Denote Denote $p_w(h) = P(\tau_w > t)$ and designate the life time of the graph Γ by

$$\tau_\Gamma = \min_{R \in \mathcal{R}} \max_{w \in R} \tau_w\ .$$

If $h = 1/t$ then we have with $t \to \infty$ the Weibull distributions of the arcs life times and the formula

$$\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma) t^{D_\Gamma}} - 1 = g(t) \sim A t^{-\Delta_\Gamma} = G(t) \tag{6}$$

If $h = \exp(-t)$, $t \to \infty$, then we have the Gompertz distributions of the arcs life times and the formula (1) transforms into

$$\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma)\exp(D_\Gamma t)} - 1 = g_1(t) \sim G_1(t) = G(\exp(t)), \tag{7}$$

so $G_1(t) = o(G(t))$.

Consequently for the Gompertz distributions the convergence rate in the asymptotic (7) is much faster than for the Weibull distributions in (6).

If $h = 1/t$, $t \to \infty$, then for the Weibull distributions of the arcs life times the formula (5) transforms into

$$P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right), \tag{8}$$

$$\frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right)} - 1 = f(t) \sim F(t) = o(1),\ F(t) = \exp\left(-\sum_{j=1}^{m_2} c_{w_j^2} t^{d_{w_j^2}} + \sum_{j=1}^{m_1} c_{w_j^1} t^{d_{w_j^1}}\right).$$

If $h = \exp(-t)$, $t \to \infty$, then for the Gompertz distributions of the arcs life times the formula (5) transforms into

$$P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} \exp\left(d_{w_j^1} t\right)\right),\ \frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m_1} c_{w_j^1} \exp\left(d_{w_j^1} t\right)\right)} - 1 = f_1(t) \sim F_1(t) = F(\exp(t)), \tag{9}$$

so $F_1(t) = o(F(t))$.

Consequently for the Gompertz distributions the convergence rate in the asymptotic (9) is much faster than for the Weibull distributions in (8).

For $h = 1/t$ denote $\left|P_\Gamma^* - P_\Gamma\right| / P_\Gamma = A(t)$, and for $h = \exp(-t)$ designate $\left|P_\Gamma^* - P_\Gamma\right| / P_\Gamma = A_1(t)$. It is clear that $A_1(t) = A(\exp(t))$ tends to zero for $t \to \infty$ much faster than $A(t)$.

From this section we see that the Gompertz distributions of the arcs life times (these distributions are preferable in life time models of alive [1] and of complex information [2] systems), give much more accuracy asymptotic formulas than the Weibull distributions. These both distributions are limit for a scheme of a minimum of independent and identically distributed random variables.

4. RESULTS OF NUMERICAL EXPERIMENTS FOR BRIDGE SCHEMES

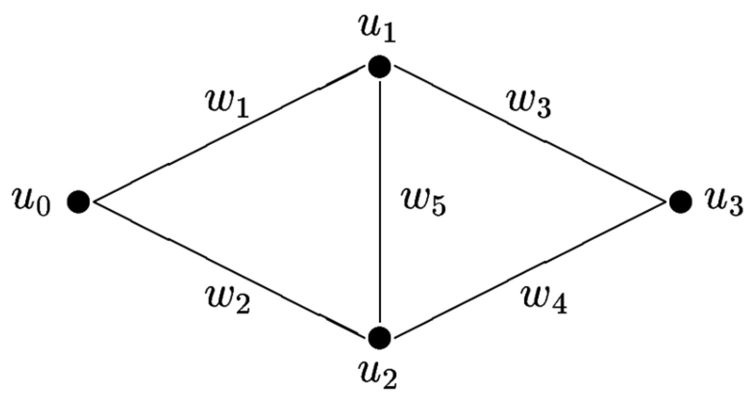

Fig.1 The bridge scheme.

Consider the bridge scheme Γ represented on the Fig. 1 with the parameters $d_1 = 0.02$, $d_2 = 0.09$, $d_3 = 0.5$, $d_4 = 0.72$, $d_5 = 0.2$. Calculate the functions $f(t), f_1(t), A(t), A_1(t), g(t), g_1(t)$.

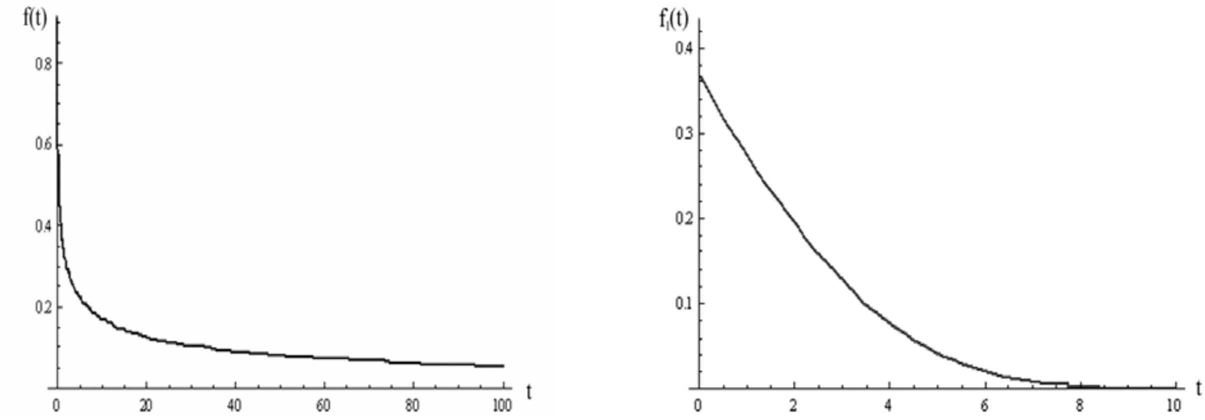

Fig.2 The relative errors $f(t)$ and $f_1(t)$ in the reliability P_Γ calculations

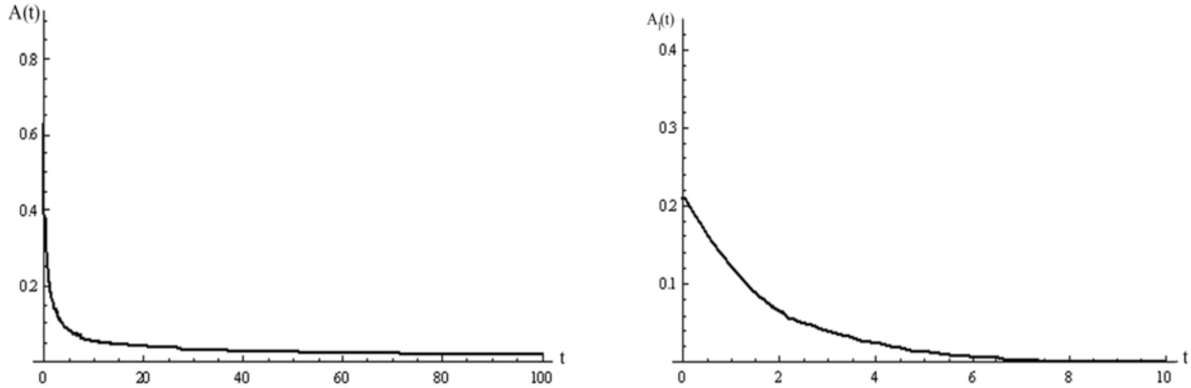

Fig.3 The relative errors $A(t)$ and $A_1(t)$ in the reliability P_Γ calculations

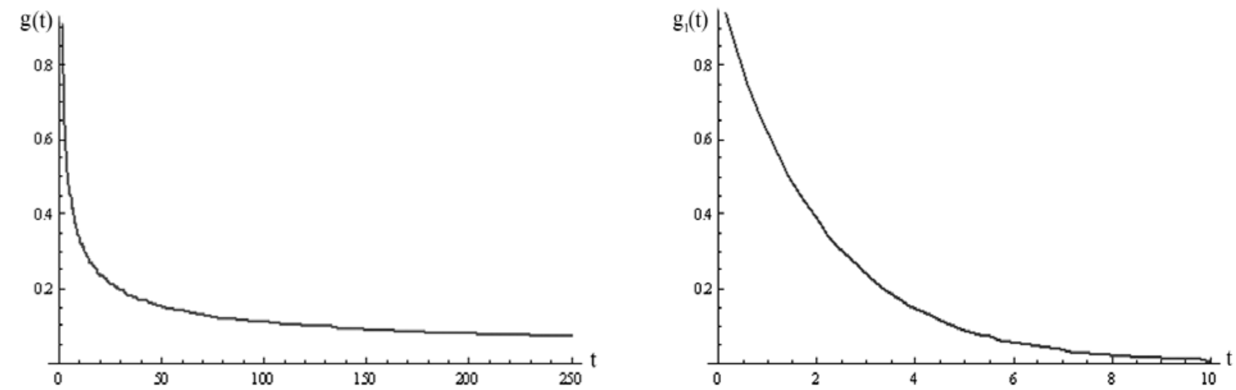

Fig.4 The relative errors $g(t)$ and $g_1(t)$ in $\ln P_\Gamma$ calculations.

The results of the numerical experiments represented above show that a transition from the Weibull to the Gompertz distribution decreases significantly relative errors in calculations of the reliability and its logarithm. The asymptotic estimate P_{Γ}^{*} of the reliability P_{Γ} is better than $P(U_1)$. The relative error of the $\ln P_{\Gamma}$ calculation is larger than the relative error of the P_{Γ} calculation. But a complexity of the $\ln P_{\Gamma}$ calculation is smaller.

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