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CALCULATION OF CONNECTIVITY PROBABILITY IN RECURSIVELY DEFINED RANDOM NETWORKS

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ABSTRACT

In this paper a problem of a construction of new and practically interesting classes of recursively defined networks, including internet type networks, with sufficiently fast algorithms of calculation of connectivity probability is considered. For this aim recursive and asymptotic formulas of connectivity probability calculation are constructed. Asymptotic formulas are based on assumptions that all network arcs are low reliable or there are high reliable and low reliable arcs in considered network. For example in radial-circle scheme radial arcs may be high reliable and circle arcs – low reliable.

1 INTRODUCTION

Recursively defined networks and structures are investigated in reliability theory for a long time. For an example it is possible to consider parallel-sequential connections [1,2] defined as follows. The class K_1 of parallel-sequential connections contains independently working elements A. If the ports $A,B\in K_1$ and their arcs sets do not intersect then the parallel connection $(A||B)\in K_1$ and the sequential connection $(A\rightarrow B)\in K_1$.

Another example is represented in [3] where in frames of logic-probability approach [4] a class of logic forms with independent Boolean random variables is constructed. This class may be considered as recursively defined class K_2 which contains all independent Boolean random variables and if logic forms $A,B\in K_2$, and their sets of variables do not intersect then the conjunction $(A\wedge B)\in K_2$, the disjunction $(A\vee B)\in K_2$ and the negation $\left(\overline{A}\right)\in K_2$. Here we formally put $\left(\overline{\left(\overline{A}\right)}\right)=A$. This class of logic forms has found manifold economical applications in risk and efficiency control.

For recursively defined networks which consist of equally reliable arcs and which describe different physical systems there are methods of a reliability calculation based on a definition of a root of a high power polynomial. These methods have polynomial complexity [5]. Analogous problem is considered [6] for recursively defined networks of "consecutive-k-out-of-n" type.

It is easy to establish that a calculation of the port $A\in K_1$ reliability (a probability that there is a working way between initial and final nodes of A) and a calculation of a reliability of a logic system characterized by a logic form $A\in K_2$ (a probability of an event (A=1)) it is necessary a number of arithmetic operations equal to a number of brackets in the form A representation.

So it is naturally to put a question, is it possible to construct new and practically interesting classes of recursively defined networks, including internet type networks [7], with sufficiently fast algorithms of calculation of connectivity probability.

In this paper this question receives a positive respond. Recursive and asymptotic formulas of connectivity probability calculation are constructed. Asymptotic formulas are based on assumptions that all arcs are low reliable or there are high reliable and low reliable arcs in considered network. For example in radial-circle scheme radial arcs may be high reliable and circle arcs – low reliable.

It is worthy to remark that there are results [8] - [10] mathematically interesting and rich in content to construct upper bounds of connectivity probabilities in networks of general type. These bounds are based on a construction of a maximal system of network frameworks. But in special recursively defined networks it is possible to obtain fast algorithms of reliability calculation without construction of network frameworks. Such approach allows accelerating reliability calculations significantly.

2 RECURSION CREATED BY GLUEING OF NETWORKS IN SINGLE NODE

Suppose that $\Gamma_1',...,\Gamma_l'$ are networks with no intersected sets of arcs and B′ is a family of networks consisting of sequences of independent copies of $\Gamma_1',...,\Gamma_l'$. Consider a recursively defined class B (of networks) with a family of generators $B' \subset B$ and assume that $r(\Gamma)=0$, $\Gamma \in B'$.

Suppose that networks $\Gamma \in B$, $\Gamma' \in B'$ have finite sets of nodes U, U′ and no intersected sets of arcs W, W′. Then the network $\Gamma \overset{u}{\otimes} \Gamma'$ constructed by a glueing of networks Γ, Γ′ in a single node also belongs to class B and $r\left(\Gamma \overset{u}{\otimes} \Gamma'\right) = r(\Gamma)+1$. Here $r(\Gamma)$ may be considered as a number of glueings generating the network Γ and $r(\Gamma)$ does not exceed a number of arcs in Γ.

Example 1. The network Γ is radial-circle with n nodes $1,\ldots,n$ on a circle with a center 0 if it has the nodes set $U=\{0,1,\ldots,n\}$ and the arcs set $W=\{(01),\ldots,(0n),\ (12),(23),\ldots,(n\text{-}1)n,(n1)\}$. Suppose that the networks $\Gamma_1',...,\Gamma_l'$ are radial-circle. Construct from them the family of generators B′ and introduce the recursively defined class of networks B obtained by a glueing procedure $\Gamma \overset{u}{\otimes} \Gamma'$, $\Gamma \in B$, $\Gamma' \in B'$ with the reserve that their joint node is a center of the radial-circle network Γ′. The network $\Gamma \in B$ may be considered following [7] as a network of internet type.

We shall interest in the probability $\pi(\Gamma)$ of the network $\Gamma \in B$ connectivity. It is the probability that there is a family of working arcs in the network Γ which are connected with all its nodes and arcs of this family create a connected deterministic network.

Lemma 1. For $\Gamma \in B$, $\Gamma' \in B'$ the following recursive formula is true:

$$\pi\left(\Gamma \overset{u}{\otimes} \Gamma'\right) = \pi(\Gamma)\pi(\Gamma'). \tag{1}$$

Proof. Indeed if the network $\Gamma \overset{u}{\otimes} \Gamma'$ is connected then the networks Γ, Γ′ are connected also. Vice versa if the networks Γ, Γ′ are connected then the network $\Gamma \overset{u}{\otimes} \Gamma'$ is connected also. From the class B definition we have that the probability of the network $\Gamma \overset{u}{\otimes} \Gamma'$ connectivity equals the product $\pi(\Gamma)\ \pi(\Gamma')$.

Denote $N(\Gamma)$ a number of arithmetical operations necessary to calculate the connectivity probability $\pi(\Gamma)$, $\Gamma \in B$, $N' = \sum_{1 \le j \le l} N\left(\Gamma_j'\right)$.

Theorem 1. For any $\Gamma \in B$ the inequality

$$N(\Gamma) \le N' + r(\Gamma) \tag{2}$$

is true.

Proof. It is obvious that for $\Gamma \in B'$ the inequality (2) takes place. Suppose that the formula (2) is true for the network $\Gamma \in B$, check this formula for the superposition $\Gamma \overset{u}{\otimes} \Gamma'$, $\Gamma' \in B'$. From the definition of N' and from the recursive formula (1) obtain that

$$N\left(\Gamma\overset{u}{\otimes}\Gamma'\right)\le N(\Gamma)+1\le N'+r(\Gamma)+1=N'+r\left(\Gamma\overset{u}{\otimes}\Gamma'\right).$$

Remark 1. Theorem 1 gives linear by a number of arcs complexity of a connectivity probability calculation for networks from recursively defined class B.

Remark 2. Analogously to a definition of a connectivity probability $\pi(\Gamma)$ it is possible to introduce a probability $\psi(\Gamma)$ that there is closed working way through all nodes of Γ. Repeating the inequality (2) proof it is possible to obtain the inequality

$$K(\Gamma)\le K'+r(\Gamma), \tag{3}$$

where $K(\Gamma)$ is a number of arithmetical operations necessary to calculate $\psi(\Gamma)$, $\Gamma\in$B, $K'=\sum_{1\le j\le l}K\left(\Gamma'_j\right)$. A formulation of $\psi(\Gamma)$ calculation problem is similar to the travelling salesman problem.

For the network $\Gamma\in$B analogously to Floid and Steinberg problem [11] find a family of probabilities $P_\Gamma=\{P_\Gamma(u_0,v_0) : u_0\neq v_0, u_0,v_0\in U\}$ which consists of $l(\Gamma)(l(\Gamma)-1)/2$ elements. Here $l(\Gamma)$ is a number of nodes in the network Γ. For a definition of the family P_Γ it is convenient to use the recursive formulas

$$P_{\Gamma\overset{u}{\otimes}\Gamma'}(u_0,v_0)=\begin{cases}P_\Gamma(u_0,v_0),\ u_0,v_0\in U,\\ P_{\Gamma'}(u_0,v_0),\ u_0,v_0\in U',\\ P_\Gamma(u_0,v_0)P_{\Gamma'}(u_0,v_0),\ u_0\in U,\ v_0\in U'.\end{cases} \tag{4}$$

Denote $N(\Gamma)$ a number of arithmetical operations necessary to calculate P_Γ, $\Gamma\in$B, $N=\sum_{1\le j\le l}N\left(\Gamma'_j\right)$.

Theorem 2. For any $\Gamma\in$B the following inequality is true:

$$N(\Gamma)\le\frac{l(\Gamma)(l(\Gamma)-1)}{2}+N. \tag{5}$$

From the formulas (5) we have that

$$\lim_{l(\Gamma)\to\infty}\frac{2N(\Gamma)}{l(\Gamma)(l(\Gamma)-1)}=1. \tag{6}$$

So in asymptotic $l(\Gamma)\to\infty$ a calculation of $P_\Gamma(u_0,v_0)$ a for single pair of nodes $u_0\neq v_0$ it is necessary no more than unit arithmetical operation·

Proof. It is obvious that for $\Gamma\in$B´ the inequality (5) is true. Suppose that this inequality takes place for the network $\Gamma\in$B, prove this inequality for the superposition $\Gamma\overset{u}{\otimes}\Gamma'$, where $\Gamma'\in$B´, for a simplicity denote $l=l(\Gamma)$, $l'=l(\Gamma')$, $L=l(\Gamma\overset{u}{\otimes}\Gamma')=l+l'-1$.

Then from the formula (4) obtain

$$N\left(\Gamma\overset{u}{\otimes}\Gamma'\right)\le N(\Gamma)+(l-1)(l'-1)\le N'+\frac{l(l-1)}{2}+(l-1)(l'-1)\le N'+\frac{L(L-1)}{2}.$$

3 ASYMPTOTIC FORMULAS

Consider Example 1 in which the set of generators consists of radial-circle schemes Γ satisfying the following asymptotic condition. Suppose that each arc $w\in W$ independently on other arcs work with the probability p_w, $0<p_w<1$, $w\in W$ and for $h\to 0$ reliabilities of circle arcs

$$\begin{aligned}&p_{u_1,u_2}=p_{u_1,u_2}(h)\to 0,\ \ p_{u_2,u_3}=p_{u_2,u_3}(h)\to 0,\dots,\\ &p_{u_{n-1},u_n}=p_{u_{n-1},u_n}(h)\to 0,\ \ p_{u_n,u_1}=p_{u_n,u_1}(h)\to 0.\end{aligned} \tag{7}$$

The reliabilities of radial arcs $(u*,u_1),\ldots,(u*,u_n)$ are positive and do not depend on the parameter h

$$p_{u*,u_1} = const > 0,\ldots, p_{u*,u_n} = const > 0. \tag{8}$$

Remark that the conditions (7), (8) are taken from manifold observations of real systems in which reliabilities of radial arcs are significantly larger than reliabilities of circle arcs.

Theorem 3. If the conditions (7), (8) are true than

$$\pi(\Gamma) \to \prod_{i=1}^{n} p_{u*,u_i},\ h \to 0. \tag{9}$$

Proof. Denote the event A that the network Γ is connected. Then there is a family of working arcs $W* \subseteq W$ so that a set of their edges coincides with U and for any pair of nodes from U there is a way R with arcs from $W*$. Define the event $B \subseteq A$ that all radial arcs $(u*,u_1),\ldots,(u*,u_n)$ work. Introduce the event C that there is failing radial arc and the event D that there is working circle arc. It is obvious that

$$A \setminus B \subseteq C \cap D$$

and consequently

$$P(A) - P(B) \le P(C)P(D). \tag{10}$$

From the events A,B,C,D definition obtain that for $h \to 0$ we have

$$P(A) = \pi(\Gamma),\ P(B) = \prod_{i=1}^{n} p_{u*,u_i}\ \ P(C \cap D) = P(C)P(D) \le P(D) \le \sum_{i=1}^{n} p_{u*,u_i} \to 0. \tag{11}$$

Put the formulas (11) into the inequality (10) and obtain the asymptotic formula (9).

Remark 3. Repeating Theorem 3 proof it is easy to spread its statement onto a network consisting of center and circle nodes in which center is connected with all circle nodes and radial arcs reliabilities satisfy the condition (8). Circle nodes may be connected with each other by arbitrary (not only circle) arcs which satisfy the condition (7).

Remark 4. Suppose that the network Γ with the nodes set U and with the arcs set W satisfies the following condition. There is the subset W_1 in the set W which creates connected sub network Γ_1 of the network Γ with the nodes set $U_1=U$. And a deleting of any arc from the network Γ_1 makes it disconnected or the equality $U_1=U$ becomes not true. Then if the reliabilities p_w, $w \in W$, satisfy the conditions

$$p_w = const > 0, w \in W_1,\ p_w = p_w(h) \to 0,\ h \to 0,\ w \in W \setminus W_1,$$

then we have

$$\pi(\Gamma) \to \prod_{w \in W_1} p_w,\ h \to 0.$$

Consider the port Γ with fixed initial and final nodes u^*,v^*, finite set of nodes U and finite set of arcs W. Suppose that the set W consists of no intersected subsets W_1, W_2 and for any $w \in W_1$, $p_w(h) \equiv p_w > 0$ and for any $w \in W_2$ $p_w(h) \to 0$, $h \to 0$. Denote $\Re = \{R_1,\ldots R_n\}$ a family of all acyclic ways between the nodes u^*, v^*.

Theorem 4. If the following formulas are true

$$R_1 \subseteq W_1,\ R_2 \cap W_2 \ne \text{Щ}\ldots, R_n \cap W_n \ne \text{Щ} \tag{12}$$

then the probability that the nodes u^*,v^* are connected in the port Γ satisfies the formula

$$P_\Gamma(u^*,v^*) \to \prod_{w \in R_1} p_w,\ h \to 0. \tag{13}$$

Proof. Denote U_R the event that all arcs from the way R work then

$$P_\Gamma(u^*,v^*) \quad P \underset{1\ \ i\ \ m}{\quad} U_{R_i} \quad ,$$

$$\sum_{i=1}^{m} P(U_{R_i}) - \sum_{1\le i<k\le m} P(U_{R_i}U_{R_k}) \le P_\Gamma(u^*,v^*) \le \sum_{i=1}^{m} P(U_{R_i}), \tag{14}$$

From the sets W_1, W_2 definition for $h\to 0$

$$\begin{aligned} &P(U_{R_1}) \equiv \prod_{w\in R_1} p_w,\ P(U_{R_k}) = 0(1),\ 1<k\le m,\\ &P(U_{R_i}U_{R_k}) = 0(1),\ 1\le i\ne k\le m,\ h\to 0. \end{aligned} \tag{15}$$

The formulas (14), (15) lead to (13).

Remark 5. If the following conditions are true

$$R_1 \subseteq W_1,\ 1\le i\le p<n,\ R_j\cap W_2 \ne Щ,\ p<j\le n, \tag{16}$$

then the connection probability of the nodes u^*,v^* in the port Γ satisfies the formula

$$P_\Gamma(u^*,v^*) \to P\left(\bigcup_{1\le i\le p} U_{R_i}\right),\ h\to 0. \tag{17}$$

Remark 6. Consider the radial-circle scheme described in Example 1 and satisfying the conditions (7), (8). Here the set W_1 consists of radial arcs and the set W_2 – from circle arcs. If the node u^* coincides with the center and v^* coincides the circle node u_i, $i\in\{1,\ldots,n\}$ then the way R_1 consists of the arc (u_*,u_i). If the nodes $u_*=u_i$, $v_*=u_k$, $1\le i\ne k\le m$ are circle then the way R_1 consists of the arcs (u_*, u_i), (u_*, u_k). Consequently in the conditions *(7), (8)* we obtain the formulas

$$P_\Gamma(u_*,u_i) \to p_{u_*,u_i},\ P_\Gamma(u_i,u_k) \to p_{u_*,u_i}p_{u_*,u_k},\ h\to 0,\ 1\le i\ne k\le n. \tag{18}$$

4 NUMERICAL EXPERIMENT

Consider the radial-circle scheme with 6 circle nodes and with arcs reliabilities

p(01)=0.993515, p(02)=0.99727, p(03)=0.995938, p(04)=0.980191,
p(05)=0.98099, p(06)=0.990262, p(12)=0.0132823, p(23)=0.00489211,
p(34)=0.010295, p(45)=0.00573119, p(56)=0.0180407, p(61)=0.0034061.

Denote by π*, π** estimates of connectivity probability obtained by the asymptotic formula (9) and by the Monte-Carlo method with 100000 random realizations of the radial-circle scheme. Numerical experiment shows that

$$\pi^* = 0.955411,\ \pi^{**} = 0.95552,\ \left|\frac{\pi^{**}}{\pi^*}-1\right| = 1.14*10^{-4},\ \left|\frac{1-\pi^{**}}{1-\pi^*}-1\right| = 0.0025,$$

and calculation time for π^* is more than 10000 times smaller than calculation time for π^{**}.

Suppose now that

p(01)=0.91, p(02)=0.92, p(03)=0.93, p(04)=0.94, p(05)=0.95, p(06)=0.96,
p(12)=0.0132823, p(23)=0.00489211, p(34)=0.010295,
p(45)=0.00573119, p(56)=0.0180407, p(61)=0.0034061.

Then numerical experiment gives us the estimates

$$\pi^* = 0.667475,\ \pi^{**} = 0.67149,\ \left|\frac{\pi^{**}}{\pi^*}-1\right| = 0.00597957,$$

with the same difference of calculation times for the estimates π*, π**.

Denote $\overline{P}^* = \left\|\overline{P}^*(u_i,u_j)\right\|_{i,j=0}^{6}$ and put $\overline{P}^{**} = \left\|\overline{P}^{**}(u_i,u_j)\right\|_{i,j=0}^{6}$ the matrices of the connection probabilities estimates using the formula (18) and using the Monte-Carlo method with 100000 random realizations. Numerical experiment shows that calculation time of the matrix $\overline{P}^*$ is approximately 20000 times smaller than the calculation time of the matrix $\overline{P}^{**}$. Denote $A = \left\|A(u_i,u_j)\right\|_{i,j=0}^{6}$, where

$$A(u_i,u_j) = \left|\overline{P}^{**}(u_i,u_j)/\overline{P}^*(u_i,u_j)-1\right|.$$

The matrixes $\overline{P}^{*}, \overline{P}^{**}, A$ are represented only by over diagonal elements because they are symmetric and have zero diagonal elements. Calculation results are following:

P*=

–	0.99351	0.99727	0.995938	0.980191	0.98099	0.990262
–	–	0.99080	0.989479	0.973834	0.974628	0.98384
–	–	–	0.993219	0.977515	0.978312	0.987559
–	–	–	–	0.976209	0.977005	0.98624
–	–	–	–	–	0.961558	0.970646
–	–	–	–	–	–	0.971437
–	–	–	–	–	–	–

,

P**=

–	0.99403	0.99740	0.99602	0.9801	0.98201	0.99026
–	–	0.99147	0.99014	0.9742	0.97622	0.98435
–	–	–	0.99348	0.9776	0.9795	0.9877
–	–	–	–	0.9762	0.97822	0.98634
–	–	–	–	–	0.96278	0.9706
–	–	–	–	–	–	0.97254
–	–	–	–	–	–	–

,

A=

–	0.00051836	0.00013036	0.000082334	0.00008264	0.0010398	0.00000202
–	–	0.00067349	0.000667682	0.00039589	0.0016332	0.00051822
–	–	–	0.000262692	00.00015849	0.0012144	0.00014320
–	–	–	–	0.00003128	0.0012434	0.00010185
–	–	–	–	–	0.0012713	0.00004729
–	–	–	–	–	–	0.00113531
–	–	–	–	–	–	–

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