ASYMPTOTIC ANALYSIS OF LATTICE RELIABILITY

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ABSTRACT

Asymptotic formulas for connection probabilities in a rectangular lattice with identical and independent arcs are obtained.. For a small number of columns these probabilities may be calculated by the transfer matrices method. But if the number of columns increases then a calculation complexity increases significantly. A suggested asymptotic method allows to make calculations using a sufficiently simple geometric approach in a general case.

INTRODUCTION

A calculation of connection probabilities in a random graph is a complex problem. In a general case it demands a number of arithmetical operations which increases as a geometric progression with a number of arcs in the graph [1], [2]. So this problem is very important in the reliability theory. It attracts special interest of physicists [3], [4] if we consider a random lattice with identical arcs.

A main approach to this problem solution is is in an application of the transfer matrices. In this method it is necessary to obtain recurrent formulas (by a length of the lattice). But a dimension of the transfer matrices increases sufficiently fast with a width of the lattice.

So an idea to construct an alternative approach origins. In this paper this approach is based on a suggestion that a work probability or a failure probability of arcs is small. These assumptions allow to obtain asymptotic formulas in a form of sums of work probabilities for ways or of failure probabilities for cross sections with minimal numbers of arcs.

A determination of such asymptotes becomes sufficiently simple though bulky enumerative problem of the graph theory.

1 MAIN DESIGNATIONS

Suppose that $\Gamma=\{U,W\}$ is the no oriented graph with the finite nodes set U, with the finite arcs set $W=\{w=(u,v),\ u,v\in U\}$ and with the fixed initial and final nodes $u_0,v_0\in U$. Denote by $\mathcal{R}=\{R_1,\ldots,R_m\}$ the set of all acyclic ways R between the nodes u_0, v_0 and the set $\mathcal{L}=\{L\}$ of all cross sections which are defined by the formulas

$\mathcal{A}=\{A\subset U,\ u_0\in A, v_0\notin A\}$, $L=L(A)=\{w=(u,v),\ u\in A, v\in U\setminus A\}$,

$\mathcal{L}=\{L(A),\ A\in\mathcal{A}\}$. An each arc $w\in W$ works with the probability p, $0<p<1$, $\overline{p}=1-p$, independently on all other arcs.

Denote $P_\Gamma=P_\Gamma(p_w,\ w\in W)$ the probability that there is a working way between the nodes u_0, v_0 in the graph Γ and designate by $\overline{P}=1-P_\Gamma$ the failure probability of this graph. Suppose that U_R is the event that all arcs in the way R work and V_L the event that all arcs in the cross section L fail. From the definition is it easy to obtain that

$$P_\Gamma=P\left(\bigcup_{R\in\mathcal{R}}U_R\right),\ \overline{P_\Gamma}=P\left(\bigcup_{L\in\mathcal{L}}V_L\right) \qquad (1)$$

2 ASYMPTOTIC FORMULAS

From the first equality in (1) obtain:

$$\sum_{i=1}^{m} P\left(U_{R_i}\right) - \sum_{1\le i<k\le m}^{m} P\left(U_{R_i}U_{R_k}\right) \le P_\Gamma \le \sum_{i=1}^{m} P\left(U_{R_i}\right).$$

Consequently if the condition $p(h)\to 0$, $h\to 0$, is true then

$$P_\Gamma \sim \sum_{i=1}^{m} P\left(U_{R_i}\right) = \sum_{i=1}^{m} \prod_{w\in R_i} p(h),\ h\to 0, \qquad (2)$$

And the relative error of the asymptotic formula (2) is

$$A_\Gamma = \left| \frac{P_\Gamma}{\sum_{i=1}^{m} P\left(U_{R_i}\right)} - 1 \right| \le mp(h) \to 0\,, h\to 0\,. \qquad (3)$$

Denote $\mathcal{L}_1 = \{L_1,\ldots, L_n\}$ the set of all minimal (by a number of arcs) cross sections from the family $\mathcal{L}$. The second formula in (1) and the family $\mathcal{L}_1$ definition lead to the equality

$$\overline{P_\Gamma} = P\left(\bigcup_{L\in\mathcal{L}_1} V_L\right). \qquad (4)$$

From the formula (4) using an induction by n obtain the inequalities

$$\sum_{i=1}^{n} P\left(V_{L_i}\right) - \sum_{1\le i<k\le n} P\left(V_{L_i}V_{L_k}\right) \le \overline{P_\Gamma} \le \sum_{i=1}^{n} P\left(V_{L_i}\right).$$

So if the condition $\overline{p}(h)\to 0$, $h\to 0$, is true then

$$P_\Gamma \sim \sum_{i=1}^{n} P\left(V_{L_i}\right) = \sum_{i=1}^{n} \prod_{w\in L_i} \overline{p}(h),\ h\to 0, \qquad (5)$$

And the relative error of the asymptotic formula (5) is

$$\overline{A}_\Gamma = \left| \frac{\overline{P}_\Gamma}{\sum_{i=1}^{n} P\left(V_{L_i}\right)} - 1 \right| \le n\overline{p}(h) \to 0\,, h\to 0\,. \qquad (6)$$

3 LOW RELIABLE ARCS

Consider the finite lattice with the size $(n\_+n+n_+)\times(m\_+m+m_+)$ and fix the initial node (0,0) and the final node (n,m) in the internal rectangular S. The nodes $(-n,-m\_)$, $(n+ n_+,\ m +m_+)$ are extreme for the rectangular S' which contains S. Suppose that $p(h)= h$, l_i is the number of arcs in the way R_i , then $P(U_{R_i}) = h^{l_i}$ and from the formula (2) obtain

$$P_\Gamma \sim \sum_{i=1}^{m} h^{l_i} \sim ah^b\,,$$

where $b = \min_{1\le i\le m} l_i$, a is the number of the ways which contain b arcs . It is easy to obtain obvious that $b=m+n$, $a = C_{m+n}^{m}$.

4 HIGH RELIABLE ARCS

Suppose that $\overline{p}(h) = h$, l_i is the number of arcs in the cross section L_i, then $P(V_{L_i}) = h^{l_i}$ and from the formula (2) obtain

$$\overline{P}_\Gamma \sim \sum_{i=1}^{m} h^{l_i} \sim ch^d\,,$$

where $d = \min_{1\le i\le m} l_i$, c is the number of cross sections which have d arcs.

Consider the following cases represented on figures with the same numbers:
1) $n\_= m\_= n_+= m_+=0$; 2) $n\_= m\_=m_+=0$, $n_+>0$; 3) $n\_= m\_=0$, $n_+>0$, $m_+>0$;

4) $m\_=m_+=0$, $n_+>0$, $n\_>0$; 5) $m\_=0$, $n\_>0$, $n_+>0$, $m_+>0$; 6) $n\_>0$, $m\_= n_+=0$, $m_+=0$;
7) $m\_>0$, $n\_>0$, $n_+>0$, $m_+>0$.

In the case 1) internal and external rectangular coincide: $S=S'$, in the cases 2) - 7) the inclusion $S\subset S'$ takes place.

Remark that listed cases do not describe all possible situations. For example an analog of the condition $n\_= m\_= m_+=0$, $n_+>0$ (see the case 2) may be the condition $n\_= m\_= n_+=0$, $m_+>0$. But it is simple to check that all possible arrangements may be reduced to listed ones after a replacement of + by – and visa versa or after a tumbling of the lattice S on ninety degrees to the left or to the right.

The considered lattice may be interpreted as an oriented graph in which the arcs (u,v), (v,u) belong or do not belong to the graph simultaneously. So from the Ford - Falkerson theorem about an equality of a maximal flow and a minimal ability to handle of cross sections [5, гл. I] it is easy to obtain the inequality $d \le \min(a,b)$ where a is the number of arcs outgoing from the initial node and b is number of arcs incoming to the final node. This inequality in the listed cases transforms into the formulas:

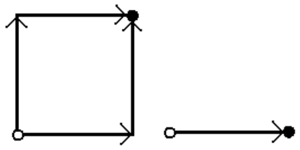

Figure 1. On the left $d\le 2$, $m>0$, on the right $d\le 1$, $m=0$.

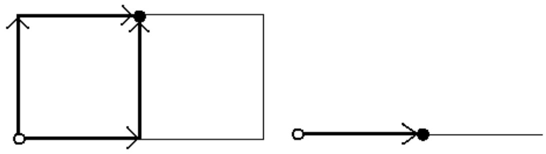

Figure 2. On the left $d\le 2$, $m>0$, on the right $d\le 1$, $m=0$.

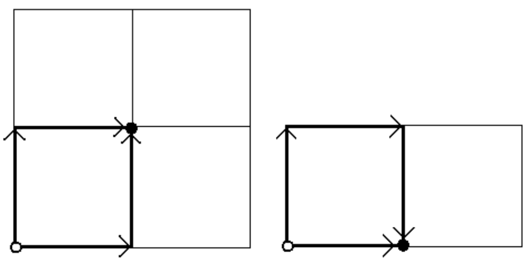

Figure 3. $d=2$, on the left $m>0$, on the right $m=0$.

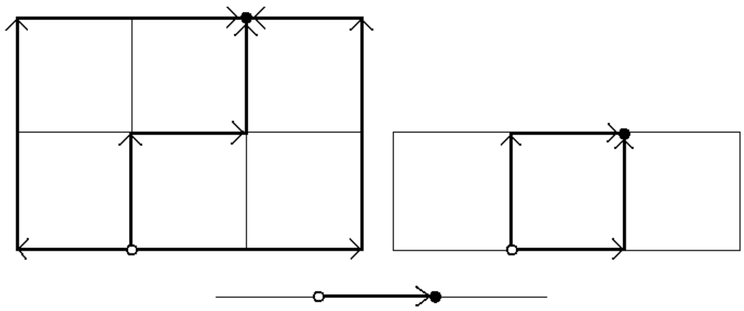

Figure 4. On the left above; $d\le 3$, $m>1$, on the right above $d\le 2$, $m=1$, below $d\le 1$, $m=0$.

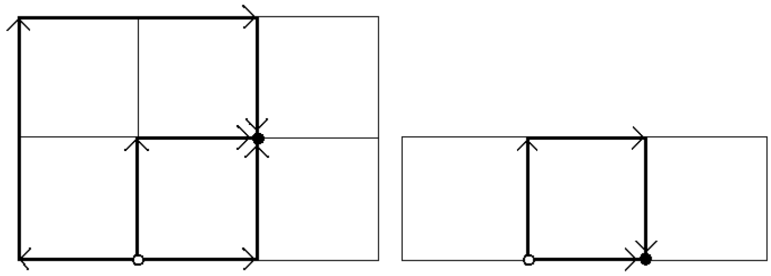

Figure 5. On the left $d\le 3$, $m>0$, on the right $d\le 2$, $m=0$.

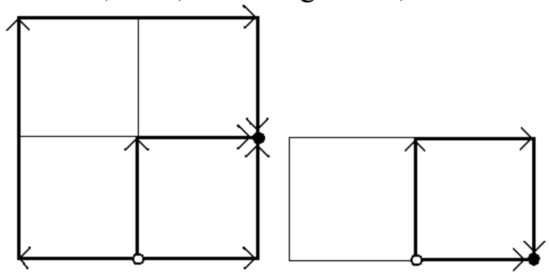

Figure 6. On the left $d\le 3$, $m>0$, on the right $d\le 2$, $m=0$.

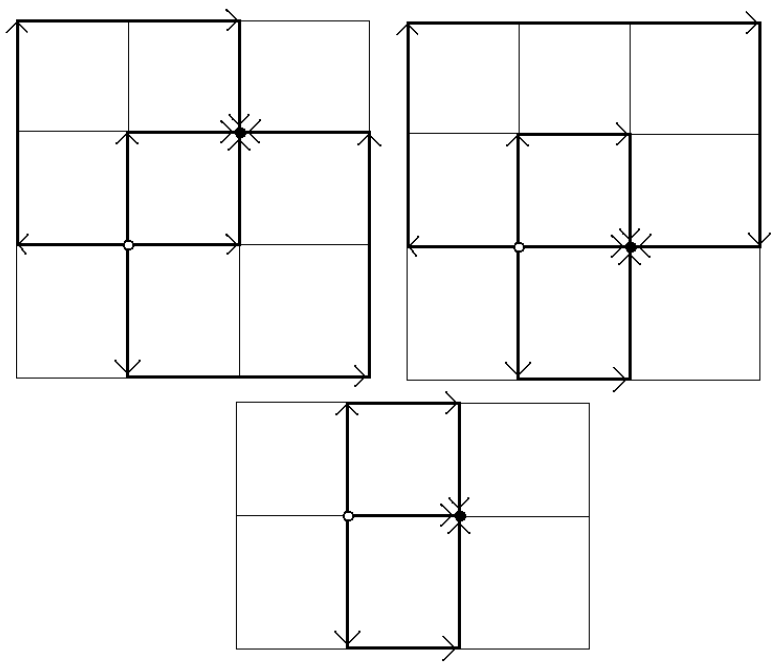

Figure 7. On the left above $d\le 4$, $m>0$, on the right above $d\le 4$, $m=0$, $m_+ + m\_>2$, below $d\le 3$, $m=0$, $m_+ + m\_=2$.

Then choosing load arcs as marked on these figures and unload arcs as all others it is possible to transform obtained inequalities into equalities.:

1), 2), $d=1+I(m>0)$; 3) $d=2$; 4) $d=3I(m>0)+2I(m=1)+I(m=0)$; 5), 6) $d=3I(m>0)+2I(m=0)$;

7) $d=4I(m>0)+4I(m=0, m_- + m_+>2)+3I(m=0, m_- + m_+=2)$.

Calculate now the asymptotic constant c. For this purpose show on the following figures all possible types of cross sections with minimal number of arcs.

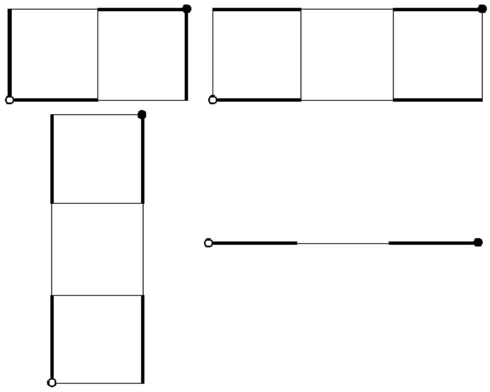

Figure 1a. On the left above *m*>0, on the right above *m*=1, on the left below *m*>0, *n*=1, on the right below *m*=0

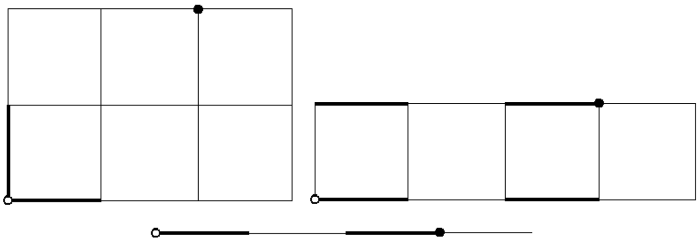

Figure 2a. On the left above *m*>0, on the right above *m*=1, below *m*=0

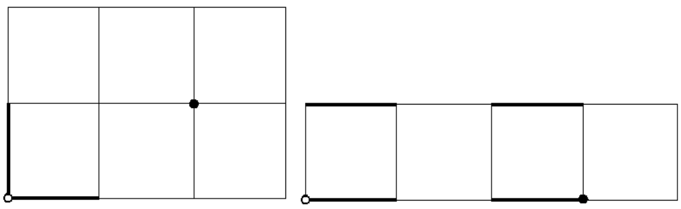

Figure 3a. On the left *m*>0, on the right *m*=0

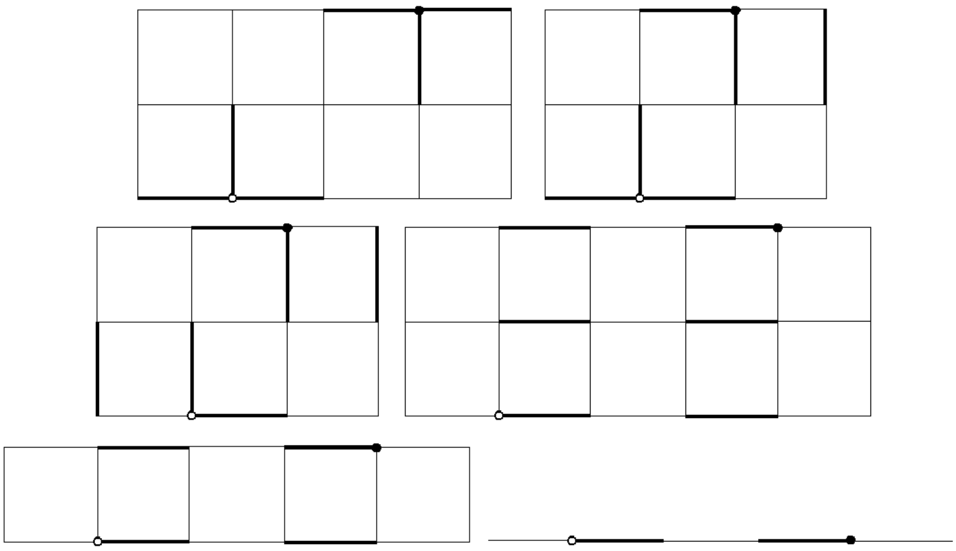

Figure 4a. Overhead to the left *m*>0, to the right *m*>0, n_+=1, middle to the left *m*>0, n_+= n_-=1, to the right *m*=2, bottom to the left *m*=1, to the right *m*=0

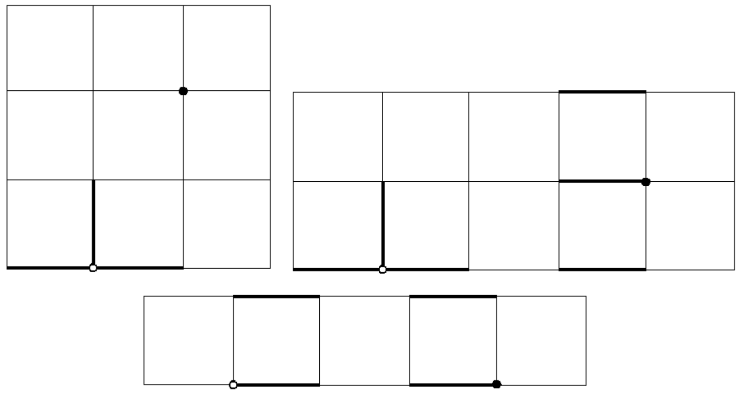

Figure 6a. Overhead to the left $m>0$, to the right $m=1$, $m_+=1$, bottom $m=0$, $m_+=1$

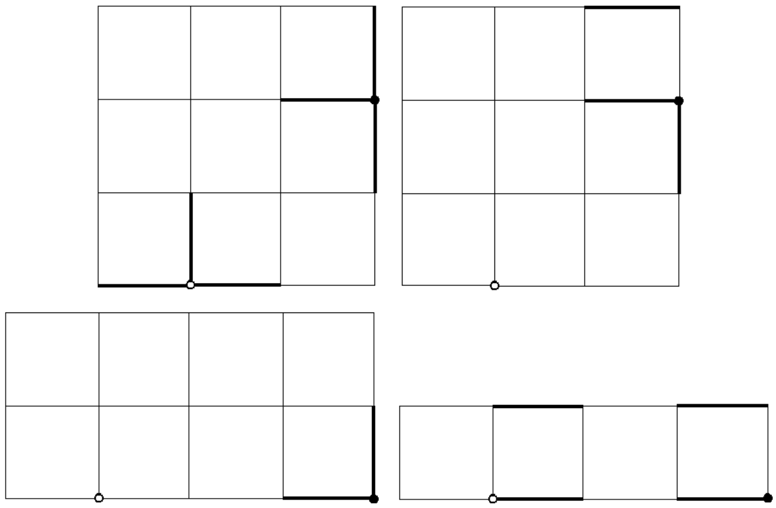

Figure 5a. Overhead to the left $m>0$, to the right $m>0$, $n_+=1$, bottom to the left $m=0$, $m_+>1$, to the right $m=0$, $m_+=1$

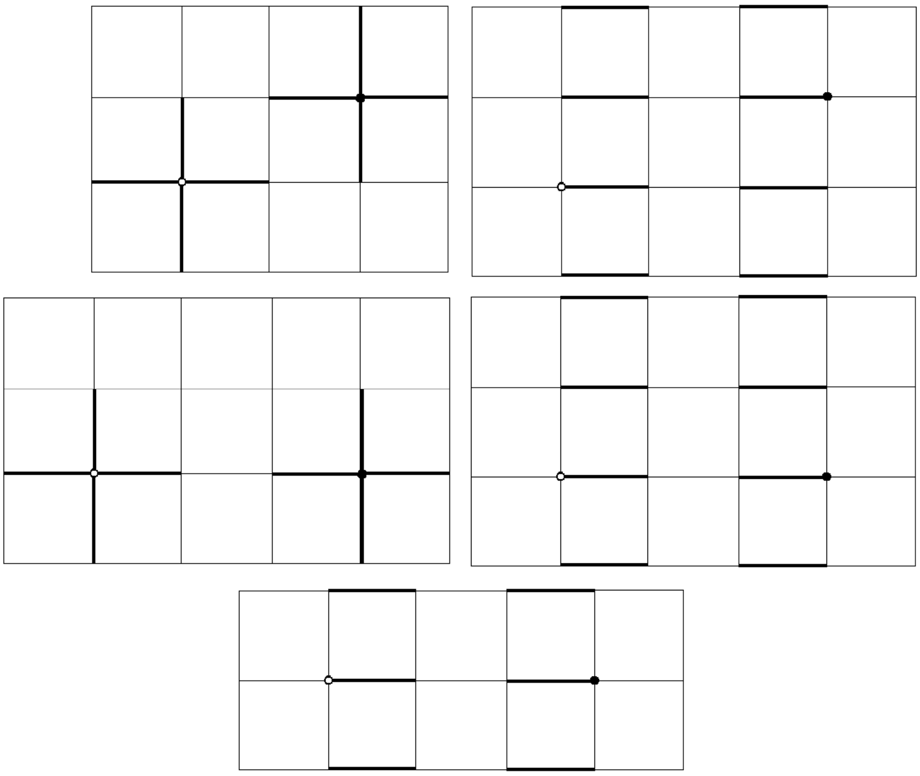

Figure 7a. Overhead to the left $m>0$, to the right $m=1$, $m_+ + m_-=1$, middle to the left $m=0$, to the right $m=0$, $m_+=2$, $m_-=12$, bottom $m=0$, $m_-=m_+=1$

Using these figures it is easy to obtain the following equalities:

1) $c=2I(m>0)+nI(m=0,1)+mI(n=1)$; 2) $c=I(m>0)+nI(m=0,1)$; 3) $c=1+nI(m=0,m_+=1)$;

4) $c=I(m\bullet\ 2)[2I(n_+, n_- > 1)+ 3I(n_+>1, n_-= 1$ или $n_+=1, n_-> 1)+ 4I(n_+= n_-= 1)]+nI(m=0,1,2)$;

5) $c=2I(m>0)+ I(m=0)[1+ nI(m_+=1)]$; 6) $c=I(m>0)+ nI(m=0)$;

7) $c=2I(m>1)+ I(m=1)[2+ nI(m_++m_-=2)]+ I(m=0)[2I(m_++m_->2)+ nI(m_++m_-=2,3)]$.

CONCLUSION

As a result an initial asymptotic problem of a connection probabilities calculation is divided into a few of comparably simple geometric – combinatorial problems. A main difficulty of this solution is in a choice of this division.

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