COMPARISON ANALYSIS OF RELIABILITY OF NETWORKS WITH IDENTICAL EDGES

G.Tsitsiashvili

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640041, Russia, Vladivostok, Radio str. 7, IAM, FEB RAS,

e-mail: guram@iam.dvo.ru,

ABSTRACT

Efficient and fast algorithms of parameters calculation in Burtin-Pittel asymptotic formula for networks with identical and high reliable edges are constructed. These algorithms are applied to a procedure of a comparison of networks obtained from a radial-circle network by a cancelling of some edges or their collapsing into nodes or by a separate reservation of these edges.

1 INTRODUCTION

A problem of a calculation of connection probabilities between nodes of a random graph is important and complicated problem in the probability theory. In general case it demands a number of arithmetical operations which increases as a geometric progression dependently on a number of the graph edges (Barlow et al. 1962), (Ushakov et al. 1985). So a construction of special methods of its complexity decreasing is actual in the reliability theory.

For random graphs with identical and high reliable edges there is the Burtin-Pittel asymptotic formula (see for example (Getsbakh, 2000)) which expresses a probability of a disconnection between initial and final edges via a minimal number d of edges in the graph cuts and a number c of cuts with d edges. Apparently one of the most convenient methods of a reliability calculation for graphs with identical edges is based on topological invariants (private communication of I. Gertsbakh). This method allows to calculate a network reliability directly as a polynomial of an edge reliability using the Monte-Carlo method. It allows to calculate integer coefficients c, d also. But the Monte-Carlo method is approximate and may give with a small probability a unit mistake in the coefficients definition. Such mistake may influence significantly on an accuracy of the Burtin-Pittel asymptotic formula.

This circumstance makes necessary to return to accuracy algorithms of Ford and Fulkerson (Ford et al. 1965) to calculate main asymptotic parameter d. In some cases these algorithms may be significantly simplified by a construction of some sufficient conditions in the Ford-Fulkerson problem. These conditions allow to define an influence of different connections – horizontal and vertical on a network reliability. As a result it is possible to find significant changes of the network reliability by its structural transformations like a cancelling of edges or their collapsing into nodes or their separate reservation. These considerations are made on an example of a radial-circle network with a few concentric circles.

Suppose that $\Gamma = \{U, W\}$ is no oriented graph with the finite set U of nodes and the finite- set $W = \{w = (u, v),\, u, v \in U\}$. Assume that each edge $w \in W$ fails with the probability $\overline{p}_w = 1 - p_w$, $0 < \overline{p}_w < 1$, independently on all other edges. Denote $\overline{P}_\Gamma$ the disconnection (or failure probability) of the graph Γ.

Theorem 1. If $\overline{p}_w(h) = h$, $w \in W$, then the Burtin-Pittel formula is true:

$$\overline{P}_\Gamma \sim ch^d, \quad h \to 0. \qquad (1)$$

2 NECESSARY VOLUME OF SEPARATE RESERVE

Suppose now that $\overline{p}_w(h) \equiv q = const$, $0<q<1$, $w \in W$, and each graph edge has k-fold reserve. Such scheme of a separate reservation is analyzed detailed in [1]. Denote the graph Γ with k-fold reserve of each edge by Γ_k and put $N(\varepsilon)=\min\left(n:\overline{P}_{\Gamma_k}<\varepsilon\right)$, $0<\varepsilon<1$.

Theorem 2. The following asymptotic formula takes place:

$$N(\varepsilon) \sim \frac{\ln \varepsilon}{d \ln q},\ \varepsilon \to 0. \tag{2}$$

Proof: In the theorem 2 conditions the asymptotic formula (1) has the form

$$\overline{P}_{\Gamma_k} \sim cq^{kd},\ k \to \infty. \tag{3}$$

Consequently for any $\delta>0$ there is $N_1 = N_1(\delta)$ so that for $k > N_1(\delta)$ the following inequality is true:

$$cq^{kd}(1-\delta) \le \overline{P}_{\Gamma_k} \le cq^{kd}(1+\delta). \tag{4}$$

Fix δ and suppose that $\varepsilon < \overline{P}_{\Gamma_{N_1(\delta)}}$ then from the formula (4) obtain the inequalities

$$cq^{N(\varepsilon)d}(1-\delta) \le \varepsilon \le cq^{(N(\varepsilon)-1)d}(1+\delta)$$

and so

$$\ln c + N(\varepsilon) d \ln q + \ln(1-\delta) \le \ln \varepsilon \le \ln c + (N(\varepsilon)-1) d \ln q + \ln(1+\delta).$$

Consequently obtain

$$\frac{\ln c + \ln(1-\delta)}{\ln \varepsilon} + \frac{N(\varepsilon) d \ln q}{\ln \varepsilon} \ge 1 \ge \frac{\ln c + \ln(1+\delta) - d\ln q}{\ln \varepsilon} + \frac{N(\varepsilon) d \ln q}{\ln \varepsilon}$$

and as a result the formula (2) takes place.

Remark that the asymptotic formula (1) for the failure probability $\overline{P}_\Gamma$ includes the constants c,d whereas the asymptotic formula (2) for the necessary volume of the separate reserve $N(\varepsilon)$ includes only the single constant d. To define the constant d it is sufficient to use the Ford-Falkerson algorithm (Ford et al. 1965) with a complexity proportional a cube of the edges number (Kormen et al. 2004). An accuracy calculation of the constant c is much heavier and reduces to NP - complete problem.

Theorem 3. The following inequality is true:

$$N(\varepsilon) \le \frac{|\ln(\varepsilon/n)|}{|\ln q|} + 1. \tag{5}$$

Proof. Indeed the probability P_{Γ_k} satisfies the inequality

$$P_{\Gamma_k} \ge \left(1-q^k\right)^n \ge 1 - nq^k$$

which leads to the formula (5).

Denote by Γ^k the parallel connection of k independent copies of the graph Γ in initial and in final nodes and designate $M(\varepsilon)=\inf\left(k: P_{\Gamma^k} \ge 1-\varepsilon\right)$ the necessary volume of Γ block reserve. If the graph Γ contains r acyclic ways from the initial to the final node and the minimal number of edges in these ways is l then $M(\varepsilon) \ge \frac{1-\varepsilon}{rp^l}$. Suppose that the graph Γ is a sequential connection of n edges then it is easy to obtain that

$$M(\varepsilon) \ge \frac{1-\varepsilon}{p^n}. \tag{6}$$

In the formula (6) a low bound of $M(\varepsilon)$ is an increasing by n geometric progression. Analogous low bound may be obtained for a port Γ in which initial node u is connected with m nodes on the first floor, each node of the first floor is connected with s among m nodes of the second floor, and so on. All nodes on the l-th floor are connected with final node v, all edges are directed from u to v. Such graph is called channel graph and is analyzed detailed in (Harms et al. 1995). The formulas (5), (6) allow to establish a strong asymptotic difference between the separate reserve and the block reserve. Comparison theorems of reliabilities of ports with these reserves have been established in (Barlow et al, 1962).

3 RADIAL-CIRCLE SCHEME

Consider a radial-circle graph Γ with m concentric circles, l radiuses and with independent edges which have failure probability $\overline{p}_w = h$.

3.1 Influence of structural changes on asymptotic of failure probability

Consider a probability $\overline{P}_\Gamma$ of a disconnection between the center u and the node v on internal circle of Γ. Say that circle edges provide horizontal connections and radial edges provide vertical connections in the graph Γ. Our problem is to analyze an influence of horizontal connections on the probability $\overline{P}_\Gamma$ if edges are high reliable and their failure probability $h \to 0$.

For a simplicity suppose that $l > 3$ then it is easy to find $d = 3$, $c = 1$ and the single cut with $d = 3$ edges consists of edges which end in the node v. As a result obtain that $\overline{P}_\Gamma \sim h^3$, $h \to 0$.

Suppose now that all circle edges have single reliability and so these edges may be collapsed into nodes. Then the graph Γ transforms in to a sequential connection of m parallel connections of l edges. Simple calculations show that then $d = l$, $c = m$ and consequently $\overline{P}_\Gamma \sim mh^l$, $h \to 0$.

Suppose that all circle edges have zero reliability and so may be cancelled. The there is a single way between the nodes u,v which contains l edges. Consequently $d = 1$, $c = m$ and so $\overline{P}_\Gamma \sim mh$, $h \to 0$.

Suppose that two circle edges connected with the node u have r-fold reserve, $2r + 1 < l$. Then we obtain $d = 2r + 1$, $c = 1$, $\overline{P}_\Gamma \sim h^{2r+1}$, $h \to 0$.

Consequently manipulations with a replacement of circle edges (which characterize horizontal connections) by absolutely reliable or absolutely unreliable or by r-fold reserves may influence significantly on the probability $\overline{P}_\Gamma$.

3.2 Accelerated algorithm of the constant d calculation

The constant d calculation in general case has a cubic complexity and it is sufficiently large for all possible pairs of nodes. So a problem is to simplify this algorithm for some families of graphs. In (Tsitsiashvili, 2010) such simplification is made for a rectangle on a lattice. In this paper a radial-circle graph Γ with m concentric circles and l radiuses is considered.

Using the Ford-Falkerson theorem about an equality of maximal flow and minimal ability to handle of cuts (Ford et al. 1962) it is easy to obtain the inequality

$$d \le \delta, \ \delta = \min(\alpha_u, \alpha_v). \tag{7}$$

Here α_u is the number of edges connected with the initial node u and α_v is the number of edges connected with the final node v. From the Ford-Falkerson theorem and from the inequality (7) we have that

$$d = \delta$$

if there are d independent (no intersected by edges) ways between the nodes u,v.

Use this sufficient condition to calculate the constant d. Denote C_u, C_v the circles which contain the nodes u,v and put R_u, R_v radiuses which contain u,v appropriately. Without a restriction of a generality suppose that C_u borders C_v.

The constant d calculation. Suppose that $l > 2$.

1) If the nodes u,v belong to an external circle then $\delta = 3$ and between u,v there are the following independent: two ways on the internal circle and a way via the center (Fig. 1) consequently $d = 3$.

2) If the nodes u,v belong to an internal circle then $\delta = 4$ and between u,v there are the following independent ways: two of them on the same circle, one way via the center and one way via external circle (Fig. 2). In this case we have $d = 4$.

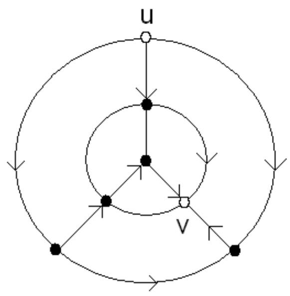

Figure 1.

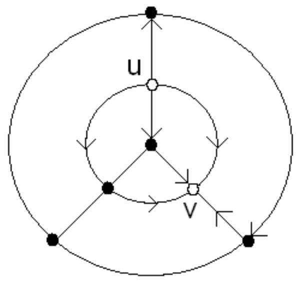

Figure 2.

3) Suppose that u belongs to external circle and v to internal circle then $\delta = 3$. If u,v belong to different radiuses then between u,v there are the following independent ways: the first way is from u along R_u to C_v and then to v, the second way is along C_u to R_v and then to v, the third way is along C_u to a radius which does not coincide with R_u, R_v then to the center and then along R_v to v (Fig. 3a) consequently $d = 3$. If u,v belong to the same radius then there are the following independent ways between u,v: the radial way, two ways along the external circle to some radius then to the center and then along radius to v (Fig. 3b), consequently $d = 3$.

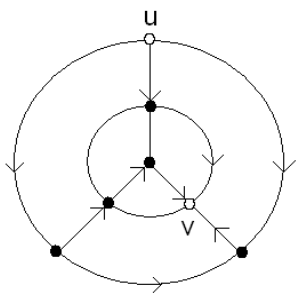

Figure 3a.

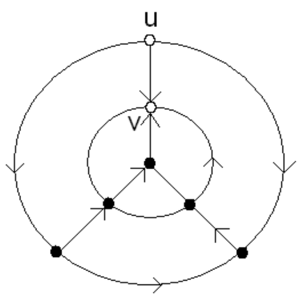

Figure 3b.

4) Suppose that the nodes u,v belong to internal circles then $\delta = 4$. Consider the case when these nodes belong to different radiuses.

If $l = 3$ then it is possible to construct the cut between u,v which consists of edges with ends on C_u and place on different radiuses (Fig. 4a) consequently $d \le 3$. It is easy to prove that between u,v there are not cuts with two edges consequently $d = 3$.

If $l > 3$ then $\delta = 4$ and it is possible to find four independent ways between u,v (Fig. 4b) consequently $d = 4$.

If the nodes u,v belong to the same radius then analogously it is possible to obtain that for $l = 3$ we have $d = 3$, and for $l > 3$ we have $d = 4$.

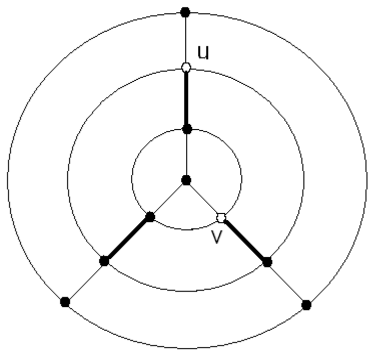

Figure 4a.

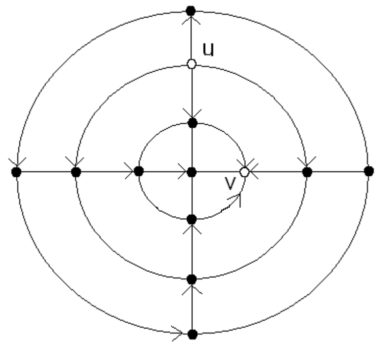

Figure 4b.

The constant c calculation.

Suppose that the node v is the center and the node u is on the m_1-th concentric (from the center) circle, $m_1 \le m$. If $l = 2$ then $c = m_1$. If $l = 3$ then for $m_1 < m$ we have $c = m_1$ and for $m_1 = m$ we have $c = m_1 + 1$. Suppose that $l = 4$ then for $m_1 < m$ we have $c = m_1 + 1$ and for $m_1 = m$ we have $c = 1$. Suppose that $l > 4$ then $c = 1$.

4 CONCLUSION

Remark that an application of Burtin-Pittel asymptotic formula for networks with identical and high reliable edges is sufficiently complicated procedure independently on considered (mainly heuristic arguments). So it is necessary to find some more arguments for the example of radial-circle network with a few concentric circles and radial edges with positive and fixed reliability and low reliable circle edges. This network may be considered using statements from (Tsitsiashvili et al. 2010)].

Consider the port Γ with fixed initial and final nodes u, v and the finite sets of nodes U and edges W. Suppose that the set W consists of no intersected subsets W_1, W_2 where $p_w(h) \equiv p_w > 0,\ w \in W_1$, and for $w \in W_2$ we have $p_w(h) \to 0,\ h \to 0$. Denote $\mathcal{R} = \{R_1, ..., R_n\}$ the set of all acyclic ways between u, v.

Theorem 4. If $R_1 \subseteq W_1$, $R_2 \cap W_2 \ne \varnothing, ..., R_n \cap W_2 \ne \varnothing$ then the connection probability

$$P_\Gamma(u,v) \to \prod_{w \in R_1} p_w\ ,\ h \to 0\ .$$

Using this theorem it is easy to obtain that $P_\Gamma(u,v)$ may be approximated by a product of reliabilities of radial edges which belong to acyclic way between the nodes u, v. An accuracy and a performance of this approximation for the network with a single circle is discussed in (Tsitsiashvili et al. 2010).

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